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INVARIANT KNOTS AND SURGERY IN CODIMENSION 2

bv Santiago LOPEZ DE MEDRANO

In the first part of this paper we study the problem of finding an invariant knot for an involution of a homotopy sphere Σ^{4k}. By an involution (T, Σ^n) we understand a fixed point free involution $T : \Sigma^n \to \Sigma^n$, smooth or p.l., of a homotopy sphere Σ^n. Reference [17] contains the properties of these involutions that will be needed. In the second part we use the experience obtained in the study of invariant knots to suggest the lines along which future research in the study of codimension 2 problems could be carried out, and we state a few results, which are only the initial steps in this direction.

Conversations with Drs. F. González Acuña and Mauricio Gutiérrez were very helpful in the elaboration of the ideas presented in this paper.

1. Invariant Knots.

An *invariant knot* for an involution (T, Σ^n) is an embedded (locally flat, in the p.l. case) homotopy sphere $\Sigma^{n-2} \subset \Sigma^n$ which is invariant under T(i.e. $T(\Sigma^{n-2}) = \Sigma^{n-2}$),

Σ^{n-2}),and a *trivial invariant knot* is one that is trivial as a knot, i.e. one that bounds an embedded disc $D^{n-1} \subset \Sigma^n$. In this last definition no relation between D and T is required, but it can be assumed that $D \cap TD = \Sigma^{n-2}$ if $n \geqslant 6$, by the fibering theorem ([5]).

We want to consider the problem of finding an invariant knot for a given involution (T, Σ^n). For $n \geqslant 7$, n not a multiple of 4, this can be solved using the Browder-Livesay theory and its developments ([6], [17]), and for $n \geqslant 7$ we can solve the problem of finding trivial invariant knots. Browder and Livesay defined an invariant $\sigma(T, \Sigma^n)$ which lies in the following groups :

$$\sigma(T, \Sigma^n) \in \begin{cases} \mathbf{Z} & \text{for} \quad n \equiv 3 \text{ mod. } 4 \\ \mathbf{Z}_2 & \text{for} \quad n \equiv 1 \text{ mod. } 4 \\ 0 & \text{for} \quad n \text{ even.} \end{cases}$$

and using this invariant and some of its properties, another invariant $\rho(T, \Sigma^n)$ can be defined for $n \neq 4$ with values in the groups

$$\rho(T, \Sigma^n) \in \begin{cases} \mathbf{Z}_2 & \text{for} \quad n \equiv 3 \text{ mod. } 4 \\ 0 & \text{for} \quad n \equiv 3 \text{ mod. } 4 \end{cases}$$

The results are :

THEOREM 1. ([17]). — *For* $n \geqslant 7$, $n \not\equiv 0$ (mod. 4), (T, Σ^n) *admits an invariant knot if, and only if* $\rho(T, \Sigma^n) = 0$. *For* $n \geqslant 7$, (T, Σ^n) *admits a trivial invariant knot if, and only if,* $\sigma(T, \Sigma^n) = 0$ *and* $\rho(T, \Sigma^n) = 0$.

All values of the invariant σ can be realized both in the p.l. and in the smooth cases, and all values of the invariant ρ can be realized in the p.l. case and for n odd in the smooth case, but known examples with non-zero value of ρ in the smooth case are scarce for n even. In any case, this shows that there are plenty of examples of involutions that do not admit invariant knots, and, for $n \equiv 3$ mod. 4, of involutions that admit invariant knots but do not admit trivial ones.

The case $n = 4k$ is the only one that cannot be reduced to the Browder-Livesay theory, and is the one that we shall study in this section. We shall present all the ideas and proofs, including a direct definition of the invariant ρ for this case, so that only ocassional references to the theory of involutions are needed. These ideas appear also in [17], but have been refined and simplified for this presentation to make it as self-contained as possible, and in view of the generalization given in section 2.

So far we know that (T, Σ^{4k}) admits a trivial invariant knot if, and only if, $\rho(T, \Sigma^{4k}) = 0$. The general form of Theorem 1 suggests that this condition is also necessary for the existence of an invariant knot, but it could still be possible that (T, Σ^{4k}) admits an invariant knot, even if it doesn't admit a trivial one, just as in the case mentioned above of an involution (T, Σ^{4k+3}). We shall see what happens.

It is convenient to rephrase the problem in terms of the quotient spaces : if (T, Σ^n) is an involution, the quotient $Q^n = \Sigma^n/T$ is called a *homotopy projective space*. As the terminology suggests, it can be shown ([17], IV.3.1) that Q^n is homotopy equivalent to real projective space P^n, and the homotopy equivalence is essentially unique. We can reformulate the problem of finding an invariant knot as follows : given a homotopy projective space Q^n, find an embedded homotopy projective space $Q^{n-2} \subset Q^n$, such that the embedding induces an isomorphism of fundamental groups. From Levine's unknotting theorem ([13]) it follows that the problem of finding a trivial invariant knot for (T, Σ^n) is equivalent to that of finding an embedded $Q^{n-2} \subset Q^n$ so that the complement $Q^n - Q^{n-2}$ has the homotopy type of S^1, as is case for the standard embedding $P^{n-2} \subset P^n$.

Browder's embedding theorem

The best way to attack the problem is to use the methods of the proof of Browder's embedding theorem (in fact, there is a theorem that says that this is the best possible way : [17], Theorem VI.1) which we proceed to describe.

Let M^m be a closed manifold (smooth or p.l.) and $N^n \subset M^m$ a submanifold with normal bundle ξ. Then, given a homotopy equivalence $f : M' \to M$ we would like to find inside M' a manifold N' homotopy equivalent to N. We have to state this problem in a more precise form, and sometimes we have to consider also the complements of the submanifolds. For this purpose, it is natural to introduce the following definitions :

DEFINITION. — Let $f : M' \to M$ be a homotopy equivalence and N a submanifold of M. We say that f is *weakly h-regular at* N if

(i) f is t-regular at N, and

(ii) if $N' = f^{-1}(N)$, $f|N' : N' \to N$ is a homotopy equivalence.

If, further, we have

(iii) $f|M' - N' : M' - N' \to M - N$ is a homotopy equivalence, then we say that f is *strongly h-regular at* N.

("Homotopy equivalence" will mean "simple homotopy equivalence", whenever the distinction is relevant).

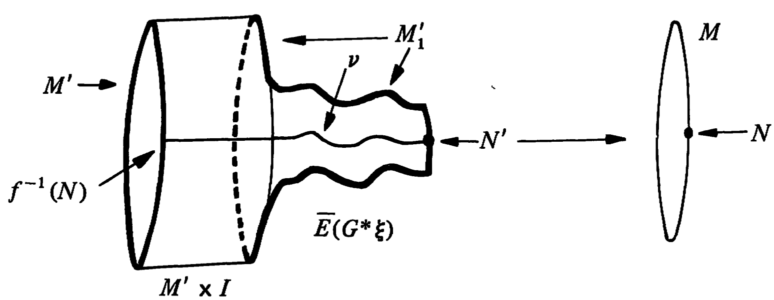
The problem now is, when is a homotopy equivalence $f : M' \to M$ homotopic to one that is weakly, or strongly, h-regular at N? If we make f t-regular at N, and we consider the map $g = f|f^{-1}(N) : f^{-1}(N) \to N$, it is easy to see that g is a normal map in a natural way, whose normal cobordism class depends only on the homotopy class of f, and defines a surgery obstruction $\theta(g)$ in the appropriate group. $\theta(g)$ is the obstruction to obtaining a homotopy equivalence, normally coborant to g, so $\theta(g) = 0$ is a necessary condition for making f weakly h-regular at N. Browder's embedding theorem says that, under some circumstances, this condition is sufficient for making f strongly h-regular at N.

Browder's Embedding Theorem ([3]). *Assume that both M and $M - N$ are 1-connected and $n \geqslant 5$. Then, if $\theta(g) = 0$, f is homotopic to a map strongly h-regular at N.*

Actually a more general situation is covered by this theorem, where instead of the pair (M, N) one gives only the homotopy theoretical information which is called a "normal system" or a "Poincaré embedding", and the manifold N' can be specified from the begining within its normal cobordism class. Also, if instead of assuming $\theta(g) = 0$, one assumes that g is normally cobordant to a homotopy equivalence to cover the small dimensions, we only have to ask $m \geqslant 5$. Wall has generalized this theorem to the case where $\pi_1(M - N) \approx \pi_1(M)$ (induced by the inclusion), which is always the case when $m \geqslant n + 3$, and has described the obstruction groups in the general situation ([21]). In all these results, the final conclusion is strong h-regularity, which is more than we can hope for in our problem when $\rho \neq 0$.

We describe the proof of this theorem only for $m = 4k$, for simplicity, the other cases requiring only minor modifications. Since $\theta(g) = 0$, g is normally cobordant to a homotopy equivalence $g_1 : \overline{N' \to N}$. If $G : \overline{V \to N}$ is the normal cobordism, we can glue $M' \times I$ and $\overline{E}(G^*\xi)$ along $\overline{E}(g^*\xi) \times \{1\}$, where $\overline{E}(G^*\xi)$ denotes the total space of the closed disc bundle of $G^*\xi$, etc., and where $\overline{E}(g^*\xi) \times \{1\}$ has been identified with a tubular neighborhood of $f^{-1}(N) \times \{1\}$ in $M' \times \{1\}$, thus obtaining a normal cobordism between f and a new normal map $f_1 : M_1' \to M$, such that $f^{-1}(N) = N'$. (This trick will be refered to as the normal cobordism extension lemma).

Now f_1 restricts to the homotopy equivalence $g_1 : N' \to N$, but is not itself a homotopy equivalence. We correct this by doing surgery on the complement of N' in M_1'. Let $X = M - U$, where U is an open tubular neighborhood of N in M and $X_1' = M_1' - U'$, where U' is an open tubular neighborhood of N' in M_1'.



Since we can assume that f_1 sends U'_1 onto U as a bundle map, and X'_1 onto X, we have a normal map $h = f_1 | X'_1 : X'_1 \to X$, and since $h | \partial X'_1$ is a homotopy equivalence, we can try to make h a homotopy equivalence, by doing surgery on the interior of X'_1. The obstruction to doing this, being the index of the intersection form on ker h_*, can be identified with the obstruction to making f_1 a homotopy equivalence. But this obstruction is 0, since f_1 is normally cobordant to the homotopy equivalence f. Therefore we can find a normal cobordism, rel. boundary, between h and a homotopy equivalence, and this cobordism, together with $U'_1 \times I$, gives a normal cobordism between f_1 and a homotopy equivalence $f_2 : M'_2 \to M$ which is strongly h-regular at N. Since f and f_2 are normally cobordant and the normal cobordism is odd dimensional, we can turn it into an h-cobordism, and therefore $M' = M'_2$ and f is homotopic to f_2, so the theorem is proved.

The invariant ρ.

We want to consider the case $M = P^{4k}, N = P^{4k-2}$. In this case $\pi_1(M) = \mathbf{Z}_2$ and $X = M - U$ is a closed tubular neighborhood of the P^1 that links P^{4k-2} in P^{4k}. Therefore X is the total space of the non-orientable $(4k - 1)$-disc bundle over $S^1 = P^1$, so it is non-orientable and $\pi_1(X) = \mathbf{Z}$. In another description, X is the mapping torus of the orientation reversing diffeomorphism $D^{4k-1} \to D^{4k-1}$.

Let $f : Q^{4k} \to P^{4k}$, $k > 1$, be a homotopy equivalence, t-regular at P^{4k-2} and $g = f | f^{-1}(P^{4k-2})$. It is shown in [17], Theorem 1, IV.3.3, that $\theta(g) = 0$ (and this is the only place where we shall use the Browder-Livesay theory ; there is a cohomological proof of the same fact in [20]), so we can apply the normal cobordism extension lemma to obtain a normal map $f_1 : M_1 \to P^{4k}$, normally cobordant to f, such that $f_1^{-1}(P^{4k-2}) = Q^{4k-2}$ and $g_1 = f_1 | Q^{4k-2} : Q^{4k-2} \to P^{4k-2}$ is a homotopy equivalence, and such that f_1 sends a tubular neighborhood U_1 of Q^{4k-2} in M_1 onto U as a bundle map, and $X_1 = M_1 - U_1$ onto X. Let $h = f_1 | X_1$. To carry out the next step in the proof of Browder's embedding theorem in our case, we should have $\theta(h) = 0$, but this will not always be the case. Therefore, we define.

$$\rho(Q^{4k}) = \theta(h)$$

To show ρ is well defined, let $f_1' : M_1' \to P^{4k}$ be another normal map with the same properties as f_1, and let h' be the corresponding map. If $F : W \to P^{4k}$ is a normal cobordism between f_1 and f_1', t-regular at P^{4k-2}, and $V = F^{-1}(P^{4k-2})$, we can turn $F|V$ into an h-cobordism because $L_{4k-1}(\mathbf{Z}_2, -) = 0$ ([20], [21]). But, by the normal cobordism extension lemma (for manifolds with boundary this time) we can assume that V itself is an h-cobordism, by changing F through a normal cobordism, rel. boundary. Since we can further assume that F sends a tubular neighborhood of V in W onto U by a bundle map, and Y, the complement of that neighborhood, onto X, $F|Y : Y \to X$ is a normal cobordism, rel. boundary, between h an h', so $\theta(h) = \theta(h')$ and ρ is well defined.

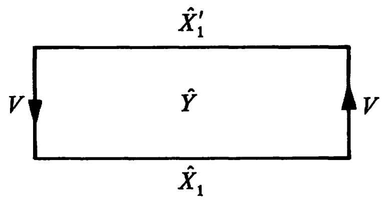
Therefore, if $\rho(Q^{4k}) = 0$ we can proceed as in the proof of Browder's embedding theorem, and obtain a homotopy equivalence $f_2 : Q_2^{4k} \to P^{4k}$ which is normally cobordant to f and strongly h-regular at P^{4k-2}. Since $L_{4k+1}(\mathbf{Z}_2, -) = 0$ ([20], [21]) we can turn a normal cobordism between f and f_2 into an h-cobordism, and therefore $Q_2^{4k} = Q^{4k}$ and f_2 is homotopic to f. In other words, (T, Σ^{4k}) admits the trivial invariant knot Q^{4k-2}. It is not difficult to see that a trivial invariant knot for (T, Σ^{4k}) induces a homotopy equivalence $f : Q^{4k} \to P^{4k}$, strongly h-regular at P^{4k-2} ([17], Theorem VI.1) and therefore $\rho(T, \Sigma^{4k}) = \rho(Q^{4k})$, being the obstruction to strong h-regularity, is the obstruction to the existence of a trivial invariant knot for (T, Σ^{4k}). We have then proved the second part of Theorem 1 for $n = 4k$ with our new definition of ρ, and also that this definition must coincide with the original one. To study the case $\rho \neq 0$ we need a detailed description of the surgery obstruction $\theta(h)$.

The surgery obstruction.

The surgery obstruction $\theta(h)$ can be described using the methods of [2] (see also [21]). Let $h : X_1 \to X$ be a normal map such that $h|\partial X_1$ is a homotopy equivalence, and let $D = D^{4k-1}$ be a fibre of $X \to S^1$. By the fibering theorem ([5]) we can assume that $h^{-1}(\partial D)$ is a homotopy sphere. Make h t-regular at D and let $W = h^{-1}(D)$.

W is a framed manifold with boundary $h^{-1}(\partial D)$, so it is framed cobordant, rel. boundary, to a disc D', and by the normal cobordism extension lemma we can assume that $h^{-1}(D) = D'$. Let $\hat{X}_1$ and $\hat{X}$ be the manifolds obtained from X_1 and X by cutting along (i.e. by removing a tubular neighborhood of) D' and D, respectively. Then h induces a normal map $\hat{h} : \hat{X}_1 \to \hat{X}$. Since $\hat{X}$ is a disc, $\theta(\hat{h}) = 1/8$ (Index X_1). We claim that the mod. 2 class of $\theta(\hat{h})$ is the surgery obstruction of h. This is because :

(a) $\theta(\hat{h})$ mod. 2 depends only on the normal cobordism class of h. For if $H : Y \to X$ is a normal cobordism, rel. boundary, between h and another normal map $h' : X_1' \to X$ such that $h'^{-1}(D)$ is a disc, we can again assume that $H^{-1}(\partial D)$ is an h-cobordism. If $V = H^{-1}(D)$ and $\hat{Y}$ is obtained from Y by cutting along V, then $\hat{Y}$ can be considered as a (normal) cobordism, rel. boundary between $\hat{X}_1'$ and $V \cup \hat{X}_1 \cup V$. (V gets the same orientation twice, because Y is non-orientable).



Therefore $\theta(\hat{h}') = \theta(\hat{h}) + 2\theta(H|V)$.

(b) If $\theta(\hat{h})$ is even h is normally cobordant, rel. boundary, to a homotopy equivalence. This is because we can construct a cobordism like the above Y with any value of $\theta(H|V)$ (using the normal cobordism extension lemma), and by choosing it properly we can assume $\theta(\hat{h}') = 0$. But that means that we can perform surgery on the interior of $\hat{X}_1'$ to obtain a disc, which amounts to performing surgery on the interior of X_1' to make it homotopy equivalent to X.

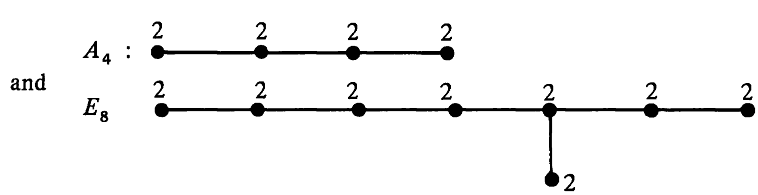
(c) If h is a homotopy equivalence, then $\theta(\hat{h}) = 0$. Because we can assume from the beginning that $h^{-1}(D) = D'$, by the fibering theorem ([5]), and then $\hat{X}_1$ is a disc.

We can further say that a normal map with non-zero obstruction is normally cobordant to one with $X_1 = X \# M_0$ (connected sum along the boundary), where M_0 is the Milnor manifold obtained by plumbing along E_8 ([4], [10]).

Now let $f : Q^{4k} \to P^{4k}$ be a homotopy equivalence, weakly h-regular at P^{4k-2}. $Q^{4k} - f^{-1}(P^{4k-2})$ is not necessarily homotopy equivalent to X, i.e., to S^1, but anyway it must be quite simple ; in particular, it must have the same homology groups as X. The question now is whether such a simple manifold can carry a non-zero surgery obstruction or not ; or in other words, whether we can or cannot "simplify" $X \# M_0$ enough. Now, the fact that the surgery obstruction of a normal map $X_1 \to X$ doesn't change if we add to X_1 two copies of M_0 can be interpreted as follows : we can move one of the copies around an orientation reversing loop, and it will come back as $-M_0$, so we can cancel it with the other copy of M_0 by surgery. For the map $X \# M_0 \to X$, if we could somehow split M_0 into two equal parts, and move one of the parts around the loop so it comes back with the opposite orientation, we could expect to simplify $X \# M_0$ by surgery, and hopefully get something that looks like the complement of a Q^{4k-2} in a Q^{4k}. This is in principle what we shall do next.

Cracking.

We now describe a process that is, in a sense, the opposite of plumbing. Recall ([4], [10]) that by the process of plumbing we can associate to a weighted graph, such as



a parallelizable $4k$-manifold with boundary, as follows : for every vertex (with weight 2) take a copy of the tangent closed disc bundle of S^{2k}, and plumb two of these copies together if the corresponding vertices are joined by an edge in the graph. This plumbing of two copies consist in identifying product neighborhoods $D^{2k} \times D^{2k}$ – one in each bundle, and disjoint from any other such neighborhoods where plumbing has been done at a previous stage – with each other by an identification that interchanges the base and fibre factors. The manifold constructed from A_4 will be denoted by A_4 again, and the one constructed from E_8 is, by definition, the Milnor manifold M_0. Let $L = \partial A_4$, and W the manifold obtained from L by removing an open disc. $\Sigma_0 = \partial M_0$ is the generator of $\theta^{4k-1}(\partial\pi)$. The homology groups of these manifolds can be computed : $H_i(A_4) = 0$ for $i \neq 2k$, and $H_{2k}(A_4)$ is free on 4 generators, represented by the 0-sections of the bundles, with respect to which the intersection form has as matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

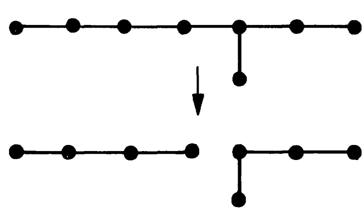
which has index 4 and determinant 5. This last fact implies that

$$H_{2k-1}(L) \approx H_{2k-1}(W) = \mathbf{Z}_5 .$$

All the other homology groups of L and W are trivial, except the top dimensional for L, being an orientable closed manifold. Similarly, $H_i(M_0) = 0$ for $i \neq 2k$, and $H_{2k}(M_0)$ is free on 8 generators $e_1, \ldots, e_8$ with respect to which the intersection from has as matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

We want to show that M_0 is the union of two copies of A_4, glued along W. Symbolically, the proof of this can be viewed as the process of cracking the E_8 into two copies of A_4, by breaking one of the links :



In precise terms, let $e'_1, \ldots, e'_8$ be the elements of $H_{2k}(M_0)$ given by

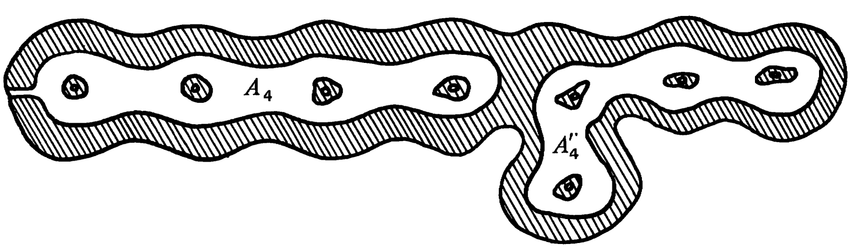
$$e'_i = e_i \quad i \neq 5$$

$$e'_5 = -e_1 + 2e_2 - 3e_3 + 4e_4 - 5e_5 + 4e_6 - 2e_7 + 3e_8$$

These elements do not form a basis of the group ; in fact they generate a subgroup of index 5 of $H_{2k}(M_0)$. The interesting thing about them is that the matrix of intersection numbers $e'_i \cdot e'_j$ is

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

which is clearly equivalent to the block sum of two copies of the matrix of A_4. That is, the link between the fourth and fifth rows and columns has disappeared! If we represent these elements by embedded spheres whose only intersections with each other are those given by this matrix and are transversal, then a regular neighborhood of the union of the spheres representing $e'_1, \ldots, e'_4$ is easily seen (by choosing an adequate Riemannian metric near the intersection points) to be diffeomorphic to A_4 (and we will call it A_4). So is a regular neighborhood of the spheres representing $e'_5, \ldots, e'_8$, and we will denote it by A'_4. We can assume A_4 and A'_4 are disjoint and contained in the interior of M_0, but we will take a small tube joining the boundary of A_4 to the boundary of M_0, and we will consider it as also forming part of A_4.



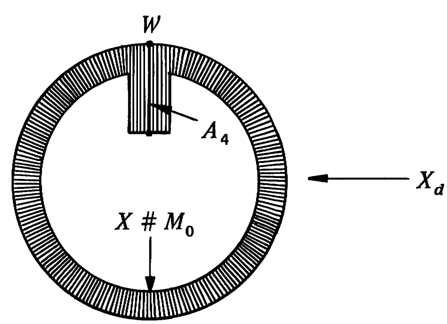
(This picture can be misleading ; the "tube" representing e'_5 really goes all over M_0, but missing the "tubes" representing $e'_1, \ldots, e'_4$ and e'_7).

Let $K = \overline{M_0 - A_4}$. We now show that the inclusion $A'_4 \subset K$ induces an isomorphism of homology groups, which implies that $\overline{K - A'_4}$ is an h-cobordism and, everything being simply connected, that K is diffeomorphic to A'_4. To prove this, first one can see, using Lefschetz duality, excision and universal coefficients, that $H_{2k-1}(K) = 0$. Then the Mayer-Vietoris sequence of $(M_0\ ; A_4, K)$,

$$0 \to H_{2k}(A_4) \oplus H_{2k}(K) \to H_{2k}(M_0) \to H_{2k-1}(W) \to 0$$

shows that $H_{2k}(A_4) \oplus H_{2k}(K)$ can be identified with a subgroup of $H_{2k}(M_0)$ of index 5. Since $H_{2k}(A_4) \oplus H_{2k}(A'_4)$ is contained in this subgroup, and has also index 5 in $H_{2k}(M_0)$, being the subgroup generated by the $\{e_i'\}$, it follows that these two subgroups are equal, and that the inclusion induces an isomorphism $H_{2k}(A_4') \approx H_{2k}(K)$. Since all other groups are trivial, this proves our assertion. Therefore M_0 can be expressed as the union of two copies of A_4, glued along W by an orientation reversing diffeomorphism d.

We shall be interested in the mapping torus of d, which we shall denote by X_d, whose boundary is the mapping torus of a diffeomorphism of S^{4k-2} representing Σ_0. We clearly have a normal map $h_d : X_d \to X$, obtained by collapsing the complement of a collar neighborhood of ∂X_d fibrewise to an S^1. (There is no obstruction to making this map normal, because all the homology of X_d comes from S^1 ; see below). Now $X \# M_0$ has the same boundary as X_d, and in fact it is normally cobordant, rel. boundary, to X_d, since the framed cobordism A_4 from W to a disc induces, by the normal cobordism extension lemma, a normal cobordism, rel. boundary, from X_d to $X \# M_0$ (which now appears as the union of two copies of A_4, joined by a tube, and then glued along W by d).



Therefore $h_d : X_d \to X$ represents the normal cobordism class with non-zero surgery obstruction.

The only thing left to do is to see if X_d looks like the complement of a tubular neighborhood of a Q^{4k-2} in a Q^{4k}. For this to be true it is necessary that the double cover $\widetilde{X}_d$ looks like the complement of a knot. Now $\widetilde{X}_d$ is the mapping torus of d^2, so it can be described as the union of two copies of $W \times I$ glued along one end by d^2 and along the other one by the identity. Therefore we have a Mayer-Vietoris sequence

$$0 \to H_{2k}(\widetilde{X}_d) \to H_{2k-1}(W \times \dot{I}) \to H_{2k-1}(W \times I) \oplus H_{2k-1}(W \times I) \to H_{2k-1}(\widetilde{X}_d) \to 0$$

Identifying both middle groups with $\mathbf{Z}_5 \oplus \mathbf{Z}_5$, it follows that the central homomorphism has as matrix

$$\begin{pmatrix} 1 & 1 \\ d_*^2 & 1 \end{pmatrix}$$

so we have to compute d_*^2. d_* itself must be multiplication by a certain number m. Let $x, y \in H_{2k-1}(W)$ be such that $L(x, y) = 1$, where

$$L : H_{2k-1}(W) \times H_{2k-1}(W) \to \mathbf{Z}_5$$

is the non-degenerate bilinear pairing given by linking numbers ([12]). Since d is orientation reversing we have

$$1 = L(x , y) = - L(d_* x , d_* y) = - L(mx , my) = - m^2 L(x , y) = - m^2$$

Therefore d_*^2 is multiplication by $m^2 = -1$, the above matrix in non-singular, and the central map in the Mayer-Vietoris sequence is an isomorphism. Therefore we have

$$\pi_1(\widetilde{X}_d) = H_1(\widetilde{X}_d) = \mathbf{Z}$$

$$H_i(\widetilde{X}_d) = 0 \quad , \quad i > 1 \quad .$$

(and since m must equal ± 2, the same holds for X_d), and also $\pi_i(X_d) = 0$ for $1 < i < 2k - 1$. So we have shown that we can represent the normal map into X with non-zero surgery obstruction by X_d, which has very little homology, and looks like the complement of a Q^{4k-2} in a Q^{4k}. In fact we can now prove :

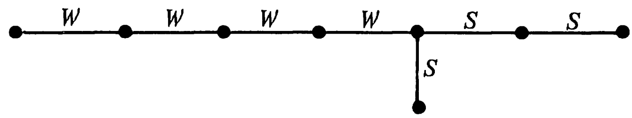
THEOREM 2. — *Every involution* $(T , \Sigma^{4k}) , k > 1$, *admits an invariant knot. In fact, it admits one that is simple and equivarantly fibered.*

For the proof, we only have to do a weak version of the last steps of the proof of Browder's embedding theorem. We had arrived before at a normal map $f_1 : M_1 \to P^{4k}$, such that $f_1^{-1}(P^{4k-2}) = Q^{4k-2}, f_1 | Q^{4k-2}$ is a homotopy equivalence and $f_1 | X_1 = h : X_1 \to X$. The case $\theta(h) = 0$ has already been considered. If $\theta(h) \neq 0$, we know that h is normally cobordant to $h_d : X_d \to X$, rel. boundary, so we get a new normal map $f_2 : Q'^{4k} \to Q^{4k}$, where $Q' = U_1 \cup X_d$. Now $\widetilde{Q}' = \widetilde{U}_1 \cup \widetilde{X}_d$ is clearly a homotopy sphere, because it is simply connected and it is easy to see from the properties of $\widetilde{X}_d$ that it has no homology below the top dimension, so f_2 is a homotopy equivalence, weakly h-regular at P^{4k-2}. The rest of the proof follows as in the case $\theta(h) = 0 : Q' = Q^{4k}$ and f is homotopic to f_2, so (T , Σ^{4k}) admits the invariant knot $\widetilde{Q}^{4k-2}$. The exterior of this knot is $\widetilde{X}_d$, and since $\pi_i(\widetilde{X}_d) \approx \pi_i(S^1)$ for $1 < i < 2k - 1$, the knot is simple, by definition ([14]), and $\widetilde{X}_d / T = X_d$ fibers over S^1, which can be taken as a definition of an "equivariantly fibered" knot.

Remarks. — The proof of this theorem gives us a direct geometric way of computing the surgery group $L_{4k}(\mathbf{Z}_2 , -) = \mathbf{Z}_2$, since it can be used to prove [17] Theorem 1, IV.3.3 without having to appeal to this computation. Also, it can be used to construct very simple examples of non-standard p.l. involutions : In P^{4k} substitute X by X_d (their boundaries are p.l. homeomorphic) and the involution obtained has $\rho \neq 0$.

The decomposition $M_0 = A_4 \cup_d A'_4$ is interesting in itself, since it shows that M_0 (and also the closed p.l. manifold $\overline{M}_0$, obtained from M_0 by attaching to it the cone on its boundary) is a "twisted double". This is a case not covered by the theorems of Smale [18], Barden [1], Levitt [16] and Winkelnkemper [22], which show that under certain, quite general conditions, a manifold must be a twisted double. Our example is more twisted than any of those covered by these theorems, in the sense that d is orientation reversing.

The process of cracking can be applied to other situations. For example, the E_8 graph can be cracked at other links, giving a decomposition of M_0 as the union of the manifolds obtained by plumbing according to the subgraphs into which E_8 is divided. In the following diagram those links at which this cracking process can be carried out are labeled W(eak), and those at which it cannot be done are labeled S(trong) :



For the weak links, formulas giving the e_i' are very similar to the ones we have given here.

This gives several relations between the boundaries of the plumbed manifolds. For example, we have shown that $L \# \Sigma_0$ is diffeomorphic to $-L$. It is possible that this process could be exploited to complete the classification of highly connected odd dimensional manifolds up to diffeomorphism ([19]).

Another remark can be made about the comparison with the situation of a knot $\Sigma^{4k-2} \subset S^{4k}$. It is proved in [11] that every such knot is cobordant to the trivial knot. If one tried to carry over the proof to the equivariant case, one would have to carry out Kervaire's proof, which can be done, and then apply some equivariant version of the engulfing theorem, as in [14] Lemma 4. But since we know that there are involutions (T, Σ^{4k}) which admit invariant knots, but do not admit trivial ones, it is not true that every invariant knot for a (T, Σ^{4k}) is equivariantly cobordant (with the obvious definition of this term) to a trivial invariant knot. Therefore, there must be something wrong with equivariant engulfing (as could be expected from the fact that the connectivity conditions on the quotient spaces are as bad as possible).

2. Surgery in Codimension 2.

The proof of theorem 2 suggest the general philosophy for dealing with surgery problems in codimension 2 : do not insist on obtaining homotopy equivalences when you are doing surgery on the complement of a submanifold, be happy if you can obtain the correct homology conditions. This has relevance both in the existence problems, as in the existence of invariant knots, and in the classification problems, as in the cobordism classification of knots.

In its simplest form, this approach suggests the following definitions and problems :

A map $f : X \to Y$ is a *homology equivalence* (H-equivalence) if it satisfies the following conditions :

(i) $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

(ii) $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for all i.

A cobordism $(W ; M_0, M_1)$ is an H-cobordism if both inclusions $M_i \subset W$ are H-equivalences. Two H-equivalences $f_i : M_i \to M$ between manifolds are *H-cobordant* if they extend to a map $F : W \to M$, where W is an H-cobordism between the M_i.

PROBLEM 1. — When is a normal map $M' \to M$ normally cobordant to an H-equivalence ?

PROBLEM 2. — When are two normally cobordant H-equivalences H-cobordant ?

Problem 1 is equivalent to the question of which elements in the Wall group can be represented by H-equivalences, so this problem is in a certain sense simpler that the standard surgery problem, since its obstruction cannot be stronger than the standard surgery obstruction. On the other hand Problem 2 is much more complicated than the standard problem of obtaining h-cobordisms, since in the only known non-simply-connected example, that of cobordism of knots, the obstruction groups are not finitely generated ([14]).

In the applications the problems are more complicated to formulate. First of all, we are really interested in the relative case, where manifolds have a boundary, and the restrictions of the maps and cobordisms to the boundaries are homotopy equivalences and h-cobordisms. This is the situation when we consider cobordism classes of knots : two knots are cobordant if, and only if, their exteriors are H-cobordant, rel. boundary, when we consider them together with their normal maps onto the exterior of the trivial knot. This example also suggests that condition (i) in the definition of an H-equivalence could and should be weakened, if not totally forgotten, in the sense that the solutions to Problems 1 and 2 will probably be unaffected by this modification of the definitions. This also seems to be the case in other situations, like in the study of H-cobordism classes of homology spheres ([8]).

The other complication has been already found in the proof of Theorem 2 : we had to make sure that the double covering of the map $h_d : X_d \to X$, and not only h_d itself, was an H-equivalence. In general we can say that $f : X \to Y$ is an H-equivalence with respect to a subgroup G of $\pi_1(Y)$ if the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is an H-equivalence, where $\widetilde{Y} \to Y$ is the covering corresponding to G. (If $G = 0$, this means that f is a (weak) homotopy equivalence). In the applications G is the kernel of $\pi_1(M - N) \to \pi_1(M)$. Another interesting case is when G is the kernel of the orientation map.

When M is orientable, the best possible solution of Problem 1 would be that a normal map is normally cobordant to an H-equivalence if its surgery obstruction lies in the kernel of the homomorphism $L_m(\pi_1(M)) \to L_m(0)$ induced by the orientation map, that is, if its good old index or Kervaire invariant is 0. It this were true the weak h-transversality problem in codimension 2 would be solved whenever the ambient manifold is simply connected. For other forms of Problem 1 there are similar conjectures with equally nice consequences. For the moment we can prove some of these conjectures when the fundamental group is $\mathbf{Z}$, obtaining the following theorem on weak h-regularity :

THEOREM 3. — *Assume (M^m, N^{m-2}) is such that $\pi_1(M - N) = \mathbf{Z}$ and either $\pi_1(M) = 0$ or $\pi_1(M) = \mathbf{Z}_2$. Then, if $m - 2 \geqslant 5$, a homotopy equivalence*

$$f : M' \to M$$

is normally cobordant to a homotopy equivalence weakly h-regular at N if, and only if, the surgery obstruction $\theta(g) = 0$, where $g = f \mid f^{-1}(N)$.

The proof rests on the knowledge of a good number of examples from knot theory and the theory of involutions. When one is trying to do surgery to make the complement of the inverse image of N H-equivalent to $M - N$, one can make it a homotopy equivalence outside the inverse image of a tube representing a generator of $\pi_1(M - N)$. Then one can use these examples to substitute this inverse image by something H-equivalent (with respect to the kernel of $\pi_1(M - N) \to \pi_1(M)$) to the tube, just as we did in the proof of Theorem 2. In this way we get a homotopy equivalence, weakly h-regular at N and normally cobordant to f. When $\pi_1(M) = \mathbf{Z}_2$ there are a few cases when we cannot conclude that this homotopy equivalence is h-cobordant (and therefore homotopic) to f, but under extra hypotheses, which are probably irrelelevant, we can obtain this stronger result. When $\pi_1(M) = 0$ there is no problem.

About Problem 2 we have very little to say. One would hope that there are obstruction groups, similar to Levine's knot cobordism groups, and that these groups depend only on the fundamental group. If this were the case, there would be nice consequences again : many problems of classification of embeddings in codimension 2 up to concordance would be reduced in a large measure to knot cobordism theory, and there would be a geometric interpretation of the periodicity of Levine's groups.

The methods of knot cobordism theory are in most cases too specific to be directly helpful in the general situation. One such method is the use of engulfing to show that every knot is cobordant to a simple knot ([14], Lemma 4) since we have shown in particular that this method cannot work for the case of invariant knots. We have found a proof of this result that only uses surgery (similar proofs have been found independently by Kervaire and Ungoed-Thomas) which works also for invariant knots :

THEOREM 4. — *Every invariant knot for (T, Σ^n) is equivariantly cobordant to a simple invariant knot.*

The proof consists in constructing an (equivariant) H-cobordism between the complement of the knot and the complement of a simple knot, which gradually kills the homotopy groups. The general step goes as follows : If X is the complement of the knot and if we assume $\pi_1(X) \approx \pi_1(S^1)$ for $i < q$ and q is below the middle dimension, we can perform equivariant surgery on the generators of $\pi_q(X)$, obtaining a cobordism W between X and X', rel. boundary. Now both X' and W have some unwanted homology in dimension $q + 1$. However, since $\pi_{q+1}(X') \to H_{q+1}(X')$ is onto, because $H_{q+1}(\mathbf{Z}) = 0$ (See [7], p. 483) we can kill this homology by doing surgery on X', which kills automatically also the extra homology in W, thus obtaining an H-cobordism W' between X and X'', where $\pi_i(X'') = \pi_i(S^1)$ for $i \leqslant q$. This type of proof also works when we consider knots which are invariant under other group actions, and for links in codimension 2 ([9]).

The next step would be to compute the equivariant cobordism classes of invariant knots, which means that we should identify the obstruction to doing the last step (the middle dimension) of the homology surgery process described above. There are further complications because there are in some dimensions examples of two trivial invariant knots which are not equivariantly cobordant ([17], VI.3, Corollary), and in the cases where there is no trivial invariant knot, we don't know if there is a simplest invariant knot to which we could refer all the others. There is the nice circumstance, however, that the equivariant H-cobordism class of the exterior of an invariant knot, and the involution restricted to the invariant knot itself, determine completely the involution.

There is another problem, even more difficult than Problem 2, namely that of deciding when two H-equivalences are h-cobordant. This has to do with the problem of isotopy of embeddings, and one case has been solved in [15].

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Instituto de Matemáticas
Universidad Nacional Autónoma de México
Torre de Ciencias
Ciudad Universitaria
México 20
Mexique