UNIFORM POLYHEDRA

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Abstract. We develop a theory of metric polyhedra, including locally infinite dimensional ones. Motivated by algebraic topology, we focus on their uniform properties (i.e., those preserved by homeomorphisms that are uniformly continuous in both directions) but in doing so we also study their metric and Lipschitz properties. On the combinatorial side, (the face posets of) simplicial or cubical complexes do not suffice for this, and we have to rework some basic PL topology into a purely combinatorial machinery (with all homeomorphisms eliminated in favor of combinatorial isomorphisms) based on posets and their canonical subdivision (which is just the poset of all order intervals of the given poset, ordered by inclusion). Antecedents of this approach to PL topology are found in van Kampen's 1929 dissertation and in modern Topological Combinatorics.

Our main results establish, in particular, close but troubled relations between uniform polyhedra and uniform ANRs, and appear to provide a satisfactory solution to an open-ended problem raised by J. R. Isbell in a series of publications in 1959-64.

1. Introduction

Here is a brief summary of the main results; a more informal discussion follows.

We consider three notions of geometric realization of a countable poset (in particular, of a simplicial or cubical complex) by a separable metrizable uniform space: one coming from an explicit embedding into the unit cube of the functional space c_0 (generalizing a construction of Shtan'ko–Shtogrin [38]), another obtained by gluing together standard simplices via quotient uniformity (akin to the usual geometric realization of a simplicial set which involves quotient topology), and a third employing path metric (resembling geometric polyhedral complexes used in Geometric Group Theory). All three notions are shown to be equivalent (Theorems 3.6 and 3.11). The geometric realization of a locally infinite dimensional poset (even a simplicial complex) may fail to be complete; however the remainder is uniformly a Z-set in the completion (Lemma 3.29).

Geometric realization is promoted to a functor from monotone maps between countable posets to uniformly continuous maps between uniform spaces, which is shown to preserve pullbacks and those pushouts that remain pushouts upon barycentric subdivision (Theorem 3.18). In particular, the functor respects joins, and mapping cylinders of simplicial maps. Here the join of posets can refer to any of the two well-known distinct notions, and the join and mapping cylinder of metrizable uniform spaces are as in [32]. To have arbitrary pushouts one has to deal with *preposets*, which are a non-transitive generalization of posets, also known as acyclic digraphs. Fortunately, up to uniform homotopy, everything boils down to posets and even to *complete quasi-lattices*, that is,

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posets where every set has either the least upper bound or no upper bound whatsoever (Theorem 4.5).

We construct a countable poset whose geometric realization is not a uniform ANR (Example 4.23), and a countable preposet whose geometric realization is not even uniformly locally contractible (Example 4.2). Nevertheless, geometric realizations of complete quasi-lattices are shown to be uniform ANRs (Corollary 4.20), and these are what we call *uniform polyhedra*. This is arguably the hardest result of the paper; the case of simplicial complexes is somewhat easier (Theorem 3.32). Conversely, if X is a uniform ANR (for instance, the loop space of a compact polyhedron), we show that $X \times \mathbb{R}$ is uniformly homotopy equivalent to a uniform polyhedron (Theorem 4.34). We also show that a separable metrizable uniform space is a uniform ANR if and only if it is uniformly ε-homotopy dominated by a uniform polyhedron for each $\varepsilon > 0$ (Theorem 4.33; in this case, geometric realizations of simplicial complexes do suffice).

Among other results, every separable metrizable complete uniform space X is uniformly homeomorphic to the limit of an inverse sequence of uniform polyhedra (Theorem 4.30), and if X is a uniform ANR, the bonding maps may be chosen to be (non-uniform) homotopy equivalences (Theorem 4.36).

Remark 1.1. An examination of the proofs reveals that uniform polyhedra endowed with the path metric are in fact Lipschitz ANRs. We do not know whether uniform polyhedra, and especially geometric realizations of simplicial complexes, could be 1-Lipschitz ANRs with respect to some compatible metric. At the very end of the paper we include a sketchy argument towards the conjecture that every Lipschitz ANR is uniformly homotopy equivalent to a uniform polyhedron (no crossing with $\mathbb{R}$ involved here).

1.A. What is going on in this paper

It should be emphasized that since topological and uniform notions agree on compact spaces (recall that continuous maps with compact domain are uniformly continuous), the theory of uniform polyhedra is not supposed to say anything new about compact polyhedra.[1] Moreover, there is nothing deep about finite-dimensional uniform polyhedra: no matter how one tries to define them, he will most likely succeed, and end up with just an equivalent form of what Isbell himself did in a few pages in [21].

A key difficulty in the infinite-dimensional case can be seen from the following example. Let Δ^n be the standard n-simplex in $\mathbb{R}^{n+1}$, that is the intersection of the first octant with the hyperplane $\sum x_i = 0$. Then Δ^n has a constant (i.e. independent of n) edge length in Euclidean, or l_1, or l_∞ metric. However, the distance from the barycenter of Δ^n (at $(\frac{1}{n+1}, \dots, \frac{1}{n+1})$) to the barycenter of a facet of Δ^n (at $(0, \frac{1}{n}, \dots, \frac{1}{n})$) tends to zero as $n \to \infty$, in either metric.

[1]The combinatorial techniques whose development it forces (see §2) do have applications to compact polyhedra in other contexts (see e.g. [33; §2]).

Thus barycentric subdivision does not work uniformly; whether there exists *any* uniform subdivision into simplices (whose iterations have arbitrarily small simplices) seems to be a difficult geometric problem, perhaps most interesting for the l_1 metric.

1.2. From simplicial complexes to preposets. We take a detour: the basic idea is to use the "canonical subdivision" which when applied to simplicial complexes produces *cubical* complexes (versions of this construction are well-known in Geometric Group Theory and in Topological Combinatorics), and *then* use the l_∞ metric one each cube. This yields a combinatorially controlled uniform structure, since when applied again to cubical complexes, the canonical subdivision still produces cubical complexes. It however takes some effort to resolve the apparent conflict of this "cubical" uniform structure with some basic PL constructions such as cone, join and mapping cylinder, which are manifestly "non-cubical". To this end we further subdivide the cubes into simplices, without introducing new vertices. These simplices are asymmetric, but come with a natural order on their vertices; every such simplex is isometric to the "standard skew n-simplex" for some n, by which we mean the subset $\{(x_1,\dots,x_n)\mid 0\le x_1\le\dots\le x_n\le 1\}$ of $\mathbb{R}^n$ with the l_∞ metric. One can now use the asymmetric simplices as separate building blocks, making sure their vertex orderings agree whenever they overlap; globally, this amounts to having a *dagged* simplicial complex, that is a simplicial complex whose 1-skeleton is endowed with the structure of a dag (=directed acyclic graph, that is a directed graph with no directed cycles). The binary relation on the vertices of a dag determined by the directed edges of the dag is a generalization of partial order; its transitive closure *is* a partial order, and for this reason we call such relation a *pre-partial order*.

Unfortunately, dagged simplicial complexes are plagued by the very same problem that we intended to avoid: for each $\varepsilon>0$ there exists an n such that every point of the standard skew n-simplex is ε-close to some point in its boundary (see Example 3.34).

To avoid this problem, it is natural to consider those dagged simplicial complexes that are *flag complexes* (i.e., where every subcomplex isomorphic to the boundary of a simplex of some dimension $d>1$ lies in an actual copy of the d-simplex). Note that these are determined by their 1-skeleton. On the other hand, the vertices of every simplex in a dagged simplicial complex are totally ordered by the underlying pre-partial order relation.[2] Thus a flag, dagged simplicial complex can be alternatively described as the *dagged* order complex (consisting of all nonempty finite chains) of a *preposet* (i.e. a pre-partially ordered set). It is well-known that a finite poset generally cannot be reconstructed from its usual order complex; however, every preposet (so in particular every poset) is trivially reconstructible from its dagged order complex. Moreover, the dagged order complex construction constitutes an isomorphism between the category of preposets and their order-preserving maps, and the category of dagged simplicial

[2]Indeed, let us consider *sinks* in the dagged simplex, that is vertices whose all incident edges are directed inwards. If there were no sinks, we could start at any vertex and move along directed edges until we exhaust all vertices in the simplex and thus find a cycle — which cannot be. Given any sink, we consider the simplex spanned by the remaining vertices and argue by induction.

complexes and their simplicial maps restricting to direction-preserving maps on the 1-skeleton. For this reason we may work with preposets themselves instead of their dagged order complexes.

Dagged order complexes of finite posets are essentially the same as *star complexes* of van Kampen's dissertation [26] and *cone complexes* of McCrory [30] (see also [1]). While most topologists are scarcely aware of the existence of this alternative language for PL topology, it is in fact widely known and used, in a disguised form, in Topological Combinatorics, dating back at least to Björner [6].

1.3. From preposets to CQLs. We find it convenient to distinguish the combinatorial notion of the order complex of a preposet (which is just a simplicial complex, without any metric or topology) from the *geometric realization* of a preposet, which is the (rectilinear, rather than abstract) order complex endowed with the uniform structure obtained by gluing together the standard skew simplices. (The reader who is not very comfortable with uniform structures may assume for the purposes of this subsection that the geometric realization is endowed with a specific metric, namely, the path metric.)

Unfortunately, preposets do not quite live up to our expectations, as it turns out that there exists a preposet X whose geometric realization is not uniformly locally contractible (more specifically, for each $\varepsilon > 0$ it contains an essential loop of diameter $\leq \varepsilon$), and in particular it is not a uniform ANR (see Example 4.2).

Restricting to posets helps but not too much. The geometric realization of every poset P is uniformly locally contractible, that is, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that every two δ-close uniformly continuous maps from an arbitrary metric space into $|P|$ are uniformly ε-homotopic with values in $|P|$ (Theorem 4.1). On the other hand, there exists a poset Y whose geometric realization does not satisfy the Hahn property (more specifically, for each $\varepsilon > 0$ it contains an embedded sphere of some dimension that is essential in $|Y|$, but null-homotopic in the ε-neighborhood of $|Y|$, which we assume to be embedded in some uniform ANR), and in particular $|Y|$ is not a uniform ANR (see Example 4.23).

The more surprising is that the geometric realization of CQLs, that is posets which are complete quasi-lattices, and in particular of (the posets of nonempty faces of) simplicial and cubical complexes do turn out to be uniform ANRs (Corollary 4.20). Note that this excludes some simplicial posets, also known as pseudo-complexes in the sense of Hilton–Wylie (where each simplex is embedded, but different simplices may have more than one face in common in the geometric realization) as well as some cubical posets.

Recall that our original reason to subdivide the l_∞ cubes into skew simplices was to make sure that the class of our complexes is closed under basic operations such as join and mapping cylinder (or, to put it bluntly, under finite homotopy colimits). We must now admit that the mapping cylinder of a monotone map between CQLs need not be a CQL — nor even a poset (Corollary 3.23). While the mapping cylinder of a simplicial map between simplicial complexes, or more generally of a *closed* map between posets, is a poset (Corollary 3.23), monotone maps that arise naturally in practice, for instance, as approximations to uniformly continuous maps (Theorem 4.22) or as bonding maps between nerves of covers (see 4.16 and Lemma 4.31) are normally not closed. Besides, the homotopy colimit of closed maps between CQLs need not be a CQL (for instance, the double mapping cylinder of the diagram $pt \leftarrow pt \sqcup pt \to pt$).

However, geometric realizations of finite homotopy colimits of monotone maps between CQLs are uniformly homotopy equivalent to geometric realizations of CQLs (see Theorem 4.5). This is really good news, because the geometric realization of the poset Y is not uniformly homotopy equivalent to the geometric realization of any CQL (see Theorem 4.24), and the geometric realization of the preposet X is not uniformly homotopy equivalent to the geometric realization of any poset (see Theorem 4.3).

1.B. Isbell's problem

We include some quotes from Isbell's book and his earlier papers.

> "**Research Problem** B_2. Infinite-dimensional polyhedra. There is a large problem here, namely the systematic investigation of topological and uniform realizations of abstract simplicial complexes. One important paper in the literature (Dowker [1952]) has examined this problem, not from a categorical viewpoint. Dowker's work tends to confirm, what many successful applications suggest, that for topology J. H. C. Whitehead's realization by CW-complexes has strong claims to preference. Its definition is as simple as could be: [...] But Dowker's work highlights the point that the suitability of CW-complexes for homology and homotopy is not conclusive; many realizations are topologically distinct but homotopy equivalent.
> By now substantial experience in uniform spaces supports the pretensions of uniform complexes, in the finite-dimensional case only. (In any case they are homotopy equivalent (topologically) with CW-complexes; Dowker [1952].) In general they are not satisfactory, e.g. because they lack subdivisions. One can save the subdivisions, or any sufficiently narrow requirement, by tailoring a definition to fit. (Kuzminov and Švedov [1960] define a realization for which IV.6 [the covers by the stars of vertices in iterated barycentric subdivisions form a basis of the uniformity] is always valid; but all their applications are in the finite-dimensional case.) The real problem holding up progress is, what applications can be made of infinite-dimensional polyhedra in the general theory of uniform spaces? It would probably be beside the point to carry out a formal investigation of realizations with no specific applications in mind." [23] (1964)

Comments: (i) As stated, the problem is quite vague, but some clarification on what kind of infinite-dimensional uniform complexes are sought here can be inferred from Isbell's previous comments in his earlier papers (quoted below).

(ii) The covers by the stars of vertices in iterated *canonical* subdivisions do form a basis of the uniformity of our uniform polyhedra (see Theorem 3.12).

(iii) We note that another reason for the widespread acceptance of the CW topology was Milnor's theorem [34] (1959) that spaces of maps between CW-complexes are homotopy equivalent to CW-complexes. We do now have a polyhedral version of this result (see [32; Theorem 4.22] and Theorem 4.34); moreover, it is almost (i.e. after crossing with $\mathbb{R}$) up to *uniform* homotopy.

> "It should be noted that the theorem [that Isbell's finite-dimensional uniform simplicial complexes are complete uniform ANRs] as stated is trivially false for arbitrary uniform complexes, since some of them are incomplete. It is false for many complete ones also. It seems likely that strong results might be gotten by using some suitable uniformity for a complex, different from the one defined by $\max|x_\alpha - y_\alpha|$, though not necessarily different for finite-dimensional complexes." [21] (1959)

Comments: indeed, with our adjusted uniformity, the theorem is now extended to infinite-dimensional simplicial complexes (Theorem 3.32).

> "I should like to repeat the remark from [[10] and [21]] that the uniform complexes are clearly not the right concept for the infinite-dimensional case. The finite-dimensionality in 7.2 [that every residually finite-dimensional complete uniform space is an inverse limit of finite-dimensional uniform simplicial complexes] and 7.3 [that every residually finite-dimensional complete uniform ANR is uniformly homotopy dominated by a finite dimensional uniform simplicial complex] may very likely appear for no better reason than that we do not have the right uniformity for the complexes." [22] (1961)

Comments: indeed, with our adjusted uniformity, the two mentioned results are now extended to infinite-dimensional simplicial complexes (Theorems 4.30 and 4.32).

1.C. Motivations

The author's basic reason for starting out this project was that he could no longer afford being ignorant of algebraic topology of Polish spaces, for it holds back progress in geometric topology of compacta, including some questions about manifolds. Spaces of interest include:

(i) $B\mathbb{Z}_p$ and the homeomorphism group of a manifold (to be discussed in a moment);
(ii) the space of topological knots in $\mathbb{R}^3$ (cf. [31; §1.A, (1)]);
(iii) the loop space of a compactum (cf. [31; Remark to Corollary 8.8]);
(iv) the 2-point configuration space of a compactum (cf. [31; §1.A, (2)]).

Another basic source of motivation for understanding the topology of Polish spaces comes from foundations of mathematics (homotopy type theory and positive set theory).

To be a bit less speculative, let us mention some applications of the results of the present paper that are expected to appear elsewhere.

1.4. The difficulty of the Hilbert–Smith Conjecture is caused, in the first place, by the lack of appropriate invariants. In particular, it is well-known that the additive group $\mathbb{Z}_p$ of p-adic integers has cohomological dimension ≤ 2 with respect to Čech cohomology, either ordinary or extraordinary [43]. However, there is a notion of a classifying space of a topological group that is well-defined up to *uniform* homotopy equivalence, and it is not hard to compute[3] that $\mathbb{Z}_p$ has infinite cohomological dimension with respect

[3]and this is how one can make sense of the more interesting of the two mutually contradictory calculations mentioned in [43]

to *Pontryagin* complex K-theory. The latter is defined as follows. We show (Theorem 4.30) that every Polish (=separable metrizable complete) uniform space X is the limit, in the category of uniform spaces, of an inverse sequence of uniform polyhedra P_i (which cannot be chosen finite-dimensional in general!), and since these uniform polyhedra are uniform ANRs (Theorem 3.32 suffices here), it follows from [32; Theorem 5.14] that the *Pontryagin cohomology* $h^n(X) := \operatorname{dirlim} h^n(P_i)$ is well-defined for every cohomology theory h^*.

1.5. Expected applications to Polish *topological* spaces include:

(i) a computation of the "commutator" of direct and inverse limits for cohomology groups of nerves P_{ij} of compacta K_i in a locally compact Polish topological space — with applications to embedding theory (cf. [31; §1.A, (2)]);

(ii) a theory of *very strong shape* of Polish topological spaces which aims to 'correct' strong shape (only in the non-compact case) by using non-discretely indexed families of nerves, so as to replace non-separable simplicial mapping telescopes by ones with a Polish topology. The effect of very strong shape is that it eliminates the dependence of the uncountable $\lim^1$ and higher derived limits on postulates independent of ZFC, and appears to provide an adequate solution (for Polish topological spaces only) to Skliarienko's problem of shape invariance of the Steenrod–Sitnikov homology for paracompact topological spaces, which in his own opinion "will be a test... for the strong shape theory itself" [39].

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2. Combinatorics of combinatorial topology

2.A. Cone complexes

2.1. Binary relations and preposets. By a *preposet* we mean a set $\mathcal{P}$ endowed with a binary relation $\prec$ that is *strictly acyclic* in the sense that there exists no sequence $x_0, \dots, x_n \in \mathcal{P}$, with n being a nonnegative integer, such that $x_0 \prec x_1 \prec \dots \prec x_n \prec x_0$. In particular, $\prec$ is anti-reflexive (i.e. $x \not\prec x$ for any $x \in \mathcal{P}$). We also write $x \prec y$ as $y \succ x$.

For an anti-reflexive relation $\prec$, its *inclusive* counterpart $\preceq$ is defined by $x \preceq y$ iff $x \prec y$ or $x = y$; it is reflexive (i.e. $x \preceq x$ for each $x \in \mathcal{P}$). For a reflexive relation $\preceq$, its *exclusive* counterpart $\prec$ is defined by $x \prec y$ iff $x \preceq y$ and $x \neq y$; it is anti-reflexive. The operations of inclusive/exclusive counterpart constitute mutually inverse bijections between the set of all reflexive relations on a set $\mathcal{P}$ and the set of all anti-reflexive relations on $\mathcal{P}$.

The inclusive counterpart $\preceq$ of a strictly acyclic relation $\prec$ is characterized by being reflexive and *acyclic* in the sense that if $x_0, \dots, x_n \in \mathcal{P}$, with n being a nonnegative integer, satisfy $x_0 \preceq x_1 \preceq \dots \preceq x_n \preceq x_0$, then $x_0 = \dots = x_n$. In particular, $\preceq$ is anti-symmetric (i.e. $x \preceq y$ and $y \preceq x$ imply $x = y$ for each $x, y \in \mathcal{P}$). By the above, a preposet may be equivalently viewed as a set $\mathcal{P}$ endowed with a binary relation $\preceq$ that is reflexive and acyclic.

The *transitive closure* of a binary relation $\prec$ on a set $\mathcal{P}$ is the relation $\prec\!\!\prec$ on $\mathcal{P}$ defined by $x \prec\!\!\prec y$ iff there exist $z_1, \dots, z_n \in \mathcal{P}$ for some nonnegative integer n such that $x \prec z_1 \prec \dots \prec z_n \prec y$; it is transitive (i.e. $x \prec\!\!\prec y$ and $y \prec\!\!\prec z$ imply $x \prec\!\!\prec z$).

A binary relation $<$ on a set $\mathcal{P}$ is called a *strict partial order* if it is anti-reflexive and transitive. It is easy to see that $<$ is a strict partial order iff its inclusive counterpart $\leq$ is a *partial order*, that is, is reflexive, anti-symmetric and transitive. A set endowed with a (strict or non-strict) partial order is called a *poset*.

It is easy to see that a binary relation is (strictly) acyclic iff its transitive closure is a (strict) partial order. Thus every poset is a preposet, and for every preposet $P = (\mathcal{P}, \prec)$, its transitive closure $\langle P \rangle := (\mathcal{P}, \prec\!\!\prec)$ is a poset. So one may view preposets not just as a generalization of posets, but more specifically as posets endowed with an additional structure.

Given a poset $P = (\mathcal{P}, \leq)$, one defines the *covering* relation $\prec$ on P by $p \prec q$ if $p < q$ and there exists no $r \in \mathcal{P}$ with $p < r < q$. Clearly, $\prec$ is the minimal strictly acyclic relation whose transitive closure is $<$.

2.2. Conical maps. Let $P = (\mathcal{P}, \leq)$ and $Q = (\mathcal{Q}, \preceq)$ be preposets. An order preserving or *conical* map between them is a map $f \colon \mathcal{P} \to \mathcal{Q}$ such that $v \leq w$ implies $f(v) \preceq f(w)$ for all $v, w \in \mathcal{P}$. It is called a *conical embedding* if the converse implication holds as well. Every conical embedding is obviously injective, but not every injective conical map is a conical embedding. An *isomorphism* of preposets is a conical bijection whose inverse is conical, or equivalently a surjective conical embedding.

We say that Q is a *subpreposet* of P (or a *subposet* of P if P itself is known to be a poset), and write $Q \subset P$, if $\mathcal{Q}$ is a subset of $\mathcal{P}$ and the inclusion $\mathcal{Q} \hookrightarrow \mathcal{P}$ is an embedding of Q into P.

The *dual* of a preposet $P = (\mathcal{P}, \leq)$ is the preposet $P^* := (\mathcal{P}, \geq)$.

2.3. Unary operations: cone, dual cone, boundary, coboundary. Let P be a preposet. The *cone* CP over P is obtained by adjoining to P an additional element, denoted $\hat{1}$, which is set to be greater than every element of P. The *dual cone* $C^*P := (C(P^*))^*$ is obtained by adjoining to P an additional element, denoted $\hat{0}$, which is set to be less than every element of P.

The *(co)boundary* ∂P (resp. $\partial^* P$) of P is defined if P has a greatest (least) element. In that case P is of the form CQ (resp. C^*Q), and ∂P (resp. $\partial^* P$) is set to be equal to Q. It is easy to see that if the (co)boundary exists, then it is unique — not just up to isomorphism but as a subpreposet of P. The notion of (co)boundary is directly related to (co)boundary in (co)homology, cf. [14; figure on p. 26]. It also agrees with a notion of combinatorial manifold with (co)boundary, which arises in combinatorial PL transversality (in the sequel to this paper).

Example 2.4 (2^S and Δ^S)**.** Let S be a set (possibly infinite). The relation of inclusion on the set $\mathit{2}^S$ of all subsets of S is a partial order. The resulting *subset poset* $2^S = (\mathit{2}^S, \subset)$ is isomorphic to its own dual (by taking the complement). The poset Δ^S of all nonempty subsets of S will be called a *(combinatorial) simplex* or the *S-simplex*, or the *n-simplex* (notation: Δ^n) in the case where S is $[n+1] := \{0, 1, \dots, n\}$. If $T \subset S$ is non-empty, Δ^T is called a *face* of Δ^S. Faces that are 0-simplices (i.e. singletons) are also called *vertices*, and faces that are 1-simplices are also called *edges*. Note that $\partial\Delta^S = \partial(\partial^* 2^S)$ is isomorphic to its own dual.

Example 2.5 (face poset of polytope)**.** Let B be a convex polytope in some Euclidean space $\mathbb{R}^d$; that is, B is the convex hull of a finite set of points. The relation of inclusion on the set $\mathcal{F}_B$ of all (non-empty) faces of B is a partial order, so we have the *face poset* $F_B = (\mathcal{F}_B, \subset)$. If σ is a rectilinear simplex in $\mathbb{R}^d$, that is, the convex hull of an affinely independent set S of points, then F_σ is isomorphic to the combinatorial simplex Δ^S from the preceding example. If two convex polytopes are affinely equivalent, clearly their face posets are isomorphic; but not vice versa.

The boundary face poset ∂F_B of the convex polytope B is well-known to be dual to the boundary face poset $\partial F_{B^\star}$ of the polar $B^\star$, provided that the interior of B contains the origin (see [20], [44]). The *polar* is defined by $B^\star = \{x \in \mathbb{R}^d \mid (x, y) \le 1 \text{ for all } y \in B\}$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^d$; if B contains the origin, $B^\star$ is a convex polytope such that $(B^\star)^\star = B$ (see [20], [44]). We recall that the polar of a (rectilinear) n-simplex is an n-simplex; the polar of an n-cube is an n-cross-polytope; the polar of an icosahedron is a dodecahedron; in $\mathbb{R}^4$, the polar of a 24-cell is a 24-cell, and the polar of a 120-cell is a 600-cell (see [20], [44]).

Example 2.6 (rectilinear simplicial complex)**.** Let $\mathcal{K}$ be a rectilinear simplicial complex, that is, a finite set of rectilinear simplices in some $\mathbb{R}^d$ such that

(i) if $\sigma \in \mathcal{K}$ and τ is a face of σ, then $\tau \in \mathcal{K}$;
(ii) if $\sigma, \tau \in \mathcal{K}$ then $\sigma \cap \tau \in \mathcal{K}$.

The simplices of $\mathcal{K}$ form a poset $K = (\mathcal{K}, \subset)$ with respect to inclusion. It is not hard to see that two rectilinear simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ are isomorphic (i.e. are related by a simplicial bijection) if and only if the posets $K = (\mathcal{K}, \subset)$ and $L = (\mathcal{L}, \subset)$ are isomorphic.

Example 2.7 (rectilinear cone complex)**.** Let $\mathcal{P}$ be a *rectilinear cone complex*[4], that is, a family of subcomplexes of a rectilinear simplicial complex $\mathcal{K}$ such that

(i) for each $\sigma \in \mathcal{P}$, all maximal simplices of σ (i.e. those simplices of σ that are not faces of other simplices of σ) share a common vertex v_σ;
(ii) for each $\sigma \in \mathcal{P}$, the set ∂_σ of all simplices of σ disjoint from v_σ is a union of elements of $\mathcal{P}$;

[4] These objects arose in van Kampen's dissertation [26] and also in the work of M. M. Cohen and E. Akin (see [1; p. 456]) and were further studied by McCrory [30].

(iii) if $\sigma, \tau \in \mathcal{P}$ and $v_\sigma = v_\tau$, then $\sigma = \tau$.

The cones (i.e. the elements) of $\mathcal{P}$ form a poset $P = (\mathcal{P}, \subset)$ with respect to inclusion.

McCrory's observation that every finite poset is isomorphic to a poset of this form will be reviewed in Proposition 2.25(a) below. For now, let us confine ourselves to face posets of convex polytopes. If B is a convex polytope in $\mathbb{R}^d$, then there exists a rectilinear simplicial complex $B^\flat$ in $\mathbb{R}^d$ such that each face of B is a union of simplices of $B^\flat$. Namely, $B^\flat$ has one vertex v_σ in the interior of each (nonempty) face σ of B, and for to every nonempty chain $\sigma_1 \subset \dots \subset \sigma_n$ of faces of B, the rectilinear simplex with vertices $v_{\sigma_1}, \dots, v_{\sigma_n}$ is contained in $B^\flat$. Now the family of all subcomplexes of $B^\flat$ triangulating the (non-empty) faces of B is a rectilinear cone complex $\mathcal{P}$, and the poset $P = (\mathcal{P}, \subset)$ is isomorphic to F_B.

Proposition 2.8. *Let $\mathcal{P}$ be a rectilinear cone complex.*

(a) If $\sigma \in \mathcal{P}$, then every vertex of σ is v_ρ for some $\rho \in \mathcal{P}$ with $\rho = \sigma$ or $\rho \subset \partial_\sigma$.

(b) For all $\sigma, \tau \in \mathcal{P}$, either $\sigma = \tau$, or $\sigma \subset \partial_\tau$, or $\tau \subset \partial_\sigma$, or else $\sigma \cap \tau \subset \partial_\sigma \cap \partial_\tau$.

(c) For all $\sigma, \tau \in \mathcal{P}$, the intersection $\sigma \cap \tau$ is a union of cones of $\mathcal{P}$.

This will follow trivially from Proposition 2.25(a), but a direct proof contributes to our understanding of rectilinear cone complexes.

Proof. (a). If $v \neq v_\sigma$, then $v \in \partial_\sigma$. By (ii), there exists a cone ρ of $\mathcal{P}$ contained in ∂_σ and containing v. By finiteness we may assume that ρ contains no other cone with this property. Then $v \notin \partial_\rho$ by (ii). Hence $v = v_\rho$. $\square$

(b). We will show that either $\tau = \sigma$, or $\tau \subset \partial_\sigma$, or else $\sigma \cap \tau \subset \partial_\tau$. If $v_\tau = v_\sigma$, then $\tau = \sigma$ by (iii). If $v_\tau \neq v_\sigma$ and $v_\tau \in \sigma$, then by (a), $v_\tau = v_\rho$ for some $\rho \in \mathcal{P}$, $\rho \subset \partial_\sigma$. Then $\tau = \rho$ by (iii), hence $\tau \subset \partial_\sigma$. Finally, if $v_\tau \notin \sigma$, then $\sigma \cap \tau \subset \partial_\tau$. $\square$

(c). By (b) we may assume that $\sigma \cap \tau \subset \partial_\sigma \cap \partial_\tau$. Let A be a simplex of $\sigma \cap \tau$. Similarly to the proof of (a), A is contained in some cone λ of $\mathcal{P}$ contained in ∂_σ such that $A \notin \partial_\lambda$, and in some cone μ of $\mathcal{P}$ contained in ∂_τ such that $A \notin \partial_\mu$. Then by (b), $\lambda = \mu$. $\square$

Example 2.9 (rectilinear cone precomplex)**.** A family $\mathcal{P}$ of subcomplexes of a rectilinear simplicial complex $\mathcal{K}$ is said to be a *rectilinear cone precomplex* if

(i) $\mathcal{K}$ is a *flag complex*, that is, every subcomplex of $\mathcal{K}$ isomorphic to the boundary of a simplex of some dimension $d > 1$ is contained in a subcomplex of $\mathcal{K}$ isomorphic to the d-simplex;

(ii) each $\sigma \in \mathcal{P}$ is a *full subcomplex* of $\mathcal{K}$, that is, if σ contains the boundary of some simplex τ of $\mathcal{K}$, then σ contains τ;

(iii) there exist a bijection $v\colon \mathcal{P} \to \mathcal{K}^{(0)}$ between elements of $\mathcal{P}$ and vertices of $\mathcal{K}$ and a function $\mu\colon \mathcal{K}^{(0)} \to \mathbb{N} = \{0, 1, \dots\}$ such that for each $\sigma \in \mathcal{P}$, the vertices of $\mathcal{K}$ that lie in σ are precisely $v(\sigma)$ along with all vertices w of $\mathcal{K}$ connected to $v(\sigma)$ by an edge in $\mathcal{K}$ and satisfying $\mu(w) < \mu(v(\sigma))$.

The cones (i.e. the elements) of $\mathcal{P}$ form a preposet $P = (\mathcal{P}, \preceq)$ with respect to the relation $\sigma \preceq \tau$ iff $v(\sigma) \in \tau$. We show in Proposition 2.25(b) below that every finite preposet is isomorphic to a preposet of this form. The function μ has its origins in PL Morse theory.[5]

2.10. Cones. Let $P = (\mathcal{P}, \preceq)$ be a preposet. The *cone* $\lceil p \rceil$ (resp. the *dual cone* $\llcorner p \lrcorner$) of the element p of $\mathcal{P}$ is the preposet $(\mathcal{C}, \leq)$, where $\mathcal{C}$ is the subset of $\mathcal{P}$ consisting of all $q \in \mathcal{P}$ such that $q \preceq p$ (resp. $q \succeq p$), and $r \leq s$ iff $r \preceq s$. The duality is expressed by $\llcorner p \lrcorner = (\lceil p^* \rceil)^*$. We may also write $\lceil p \rceil_P$ and $\llcorner p \lrcorner^P$ to emphasize the preposet P. The definitions of $\lceil p \rceil$ and $\llcorner p \lrcorner$ are in agreement with the previously defined cone and dual cone over a poset; namely, $\lceil p \rceil$ is the cone over $\partial \lceil p \rceil$, and $\llcorner p \lrcorner$ is the dual cone over $\partial^* \llcorner p \lrcorner$.

If $\mathcal{P}$ is a rectilinear simplicial complex and $\preceq$ is the inclusion relation (see Example 2.6), the cone of a rectilinear simplex $\sigma \in \mathcal{P}$ is the face poset F_σ, viewed as the poset of all simplices of the subcomplex of $\mathcal{P}$ triangulating σ (in fact, this subcomplex does happen to be a 'cone' in the terminology of Rourke–Sanderson [36; 2.8(7)]); whereas the dual cone of σ is isomorphic to what is known as the 'dual cone' of σ in PL topology (see [36; 2.27(6)]).

2.11. Cone complexes and precomplexes. By a *cone complex* we mean a countable poset where every cone is finite. A *cone precomplex* is a preposet whose transitive closure is a cone complex. A cone precomplex P such that the dual preposet P^* is also a cone precomplex is called *locally finite.*

Remark 2.12. Cone precomplexes other than cone complexes arise in practice as triple deleted prejoins of cone complexes and as mapping cylinders of non-closed conical maps between cone complexes (see Corollary 3.23); the non-closed conical maps in turn arise in practice as diagonal maps $P \to P \times P$ and more importantly as bonding maps between nerves of coverings.

Lemma 2.13. *(a) Every preposet injects conically into a simplex.*

(b) A preposet is a poset iff it is isomorphic to a subposet of a simplex.

Proof. (b). Since every simplex is itself a poset, every preposet embedded into a simplex is a poset. Conversely, every poset $P = (\mathcal{P}, \geq)$ is isomorphic to the poset of cones of P ordered by inclusion of their underlying sets, which is a subposet of $2^{\mathcal{P}} \setminus \{\emptyset\}$. ☐

(a). This follows from (b) since every preposet injects conically into its transitive closure. ☐

2.14. Subcomplex. Let $P = (\mathcal{P}, \preceq)$ be a preposet. A *subprecomplex* (resp. *dual subprecomplex*) of P is a subpreposet $Q = (\mathcal{Q}, \preceq)$ of P such that $p \prec q \in \mathcal{Q}$ (resp. $p \succ q \in \mathcal{Q}$) implies $p \in \mathcal{Q}$. When P itself is known to be a poset, we shorten 'subprecomplex' to *subcomplex*. Note that if Q is a subprecomplex of P, then $\langle Q \rangle$ is a subcomplex of $\langle P \rangle$.

[5]Our motivating sources include [5] and [30; Example on p. 275]. For comprehensive treatments of PL Morse theory see Kearton–Lickorish [27] (along with references to Kosiński and Kuiper therein), Forman [17] (along with elaborations in [35] and [28]) and Bestvina [4].

On the other hand, every subpreposet Q of P *(dually) generates* the subpreposet $\lceil Q\rceil$ (resp. $\lfloor Q\rfloor$) of P with underlying set consisting of $\bigcup_{q\in\mathcal{Q}}\lceil q\rceil$ (resp. $\bigcup_{q\in\mathcal{Q}}\lfloor q\rfloor$). If P is a poset, $\lceil Q\rceil$ and $\lfloor Q\rfloor$ coincide with the smallest subcomplex and the smallest dual subcomplex containing Q.

From now on we often do not distinguish between a preposet $P=(\mathcal{P},\preceq)$ and its underlying set $\mathcal{P}$ (by an abuse of notation).

2.15. Complete quasi-lattice. A *complete quasi-lattice* (CQL) is a poset P such that every non-empty $Q\subset P$ that has an upper bound in P (i.e. a $p\in P$ such that $Q\subset\lceil p\rceil$) also has a least upper bound in P (i.e. an upper bound $p\in P$ such that $\lfloor p\rfloor$ contains all upper bounds of Q in P).

Lemma 2.16. *A poset P is a CQL iff every non-empty subset of P that has a lower bound in P also has a greatest lower bound in P.*

Proof. By symmetry, it suffices to prove the 'only if' assertion. If P is a CQL and a subset $Q\subset P$ has a lower bound in P, then the set L of all lower bounds of Q in P is nonempty and has an upper bound in P (specifically, any element of Q will do). Then there exists the greatest lower bound u of L in P, that is, $\lceil u\rceil$ contains L and $\lfloor u\rfloor$ contains all upper bounds of L, in particular, all of Q. By definition, u is the greatest lower bound of Q. $\square$

Corollary 2.17. *(a) Every simplex is a CQL.*

(b) Every subcomplex of a CQL is a CQL.

Proof. (a). The greatest lower bound of a set of A of nonempty subsets $S_\alpha\subset S$ is their intersection $\bigcap_\alpha S_\alpha$, if it is nonempty; else A has no lower bounds. $\square$

(b). Let P be a CQL and Q its subcomplex. If the greatest lower bound of a subset $S\subset Q$ exists in P, then it belongs to Q, since Q contains all its lower bounds. $\square$

2.18. Atoms. An element σ of a poset P is called an *atom* of P, if $\lceil\sigma\rceil=\{\sigma\}$. The set of all atoms of P will be denoted $A(P)$. A poset P is called *atomic*, if every its element is the least upper bound of some subset of $A(P)$. It is easy to see that $A(\lceil\sigma\rceil)=A(P)\cap\lceil\sigma\rceil$. Hence every element σ of an atomic poset is the least upper bound of $A(\lceil\sigma\rceil)$.

In the case of atomic posets Lemma 2.13 admits an amplification:

Lemma 2.19. *If P is an atomic poset, then the formula $\sigma\mapsto A(\lceil\sigma\rceil)$ defines an embedding of P into $\Delta^{A(P)}$.*

Proof. It is easy to see that the formula defines a conical map $f\colon P\to 2^{A(P)}$. Since each $\sigma\in P$ is the least upper bound of $A(\lceil\sigma\rceil)$, the latter is nonempty (whence the image of f is in $\Delta^{A(P)}$). If $A(\lceil\tau\rceil)\subset A(\lceil\sigma\rceil)$, then the least upper bound σ of $A(\lceil\sigma\rceil)$ is an upper bound of $A(\lceil\tau\rceil)$. Hence its least upper bound τ satisfies $\tau\le\sigma$ (whence f is an embedding). $\square$

Lemma 2.20. *An atomic poset P is a CQL if and only if every $R \subset A(P)$ that has an upper bound in P has a least upper bound in P.*

Proof. Only the "if" direction needs a proof. Suppose we are given an $S \subset P$ that has an upper bound in P. Then so does $R := A(\lceil S \rceil)$. Hence by our assumption R has a least upper bound ρ. Since P is atomic, every $p \in S$ is the least upper bound of $A(\lceil p \rceil) = A(\lceil S \rceil) \cap \lceil p \rceil$. Since ρ is an upper bound of R, it is an upper bound of its subset $A(\lceil p \rceil)$ for each $p \in S$. However p is the least upper bound of the same set, so ρ is an upper bound p. Thus ρ is an upper bound of S. If ρ' is another upper bound of S, then ρ' is an upper bound of $A(\lceil S \rceil)$. But ρ is the least upper bound of the same set, so $\rho' \geq \rho$. Thus ρ is the least upper bound of S. $\square$

2.21. Simplicial complex. A *simplicial* poset is a CQL where every cone is isomorphic to a simplex. (Compare [3].) A simplicial cone complex is abbreviated to a *simplicial complex*. The cones of a simplicial complex K are thus called its *simplices*. Clearly, every simplicial complex is atomic.

Theorem 2.22. *A poset is simplicial iff it is isomorphic to a subcomplex of a simplex.*

In particular, it follows that a cone complex is a simplicial complex iff it is isomorphic to a subcomplex of a simplex.

Proof. The 'if' assertion is straightforward. Every cone of a simplex is a simplex. A subcomplex of a simplex is a CQL by Corollary 2.17.

Conversely, let K be a simplicial complex. Let us consider the embedding $f\colon K \to \Delta^{A(K)}$ constructed in Lemma 2.19. If $T \subset A(\lceil \sigma \rceil)$, then the least upper bound σ of $A(\lceil \sigma \rceil)$ is an upper bound of T, hence its least upper bound τ exists and satisfies $\tau \leq \sigma$. Therefore $A(\lceil \tau \rceil) = A(K) \cap \lceil \tau \rceil$ contains T, moreover τ is the least upper bound of $A(\lceil \tau \rceil)$, as well as of T. Since $\lceil \tau \rceil$ is isomorphic to a simplex, this implies $T = A(\lceil \tau \rceil)$. So $f(\tau) = T$, whence the image of f is a subcomplex of $\Delta^{A(K)}$. $\square$

If K is a finite simplicial complex, it is isomorphic to a subcomplex of a finite-dimensional simplex by the proof of Theorem 2.22. The latter is in turn isomorphic to the poset of nonempty faces of some rectilinear simplex Δ in some Euclidean space (see Example 2.5). Hence K is isomorphic to the poset of simplices of some rectilinear subcomplex of Δ. We have proved

Corollary 2.23. *Every finite simplicial complex is isomorphic to the poset of inclusions of simplices of some rectilinear simplicial complex.*

2.24. Barycentric subdivision. Let $P = (\mathcal{P}, \preceq)$ be a preposet. A *chain* in P is a $\mathcal{Q} \subset \mathcal{P}$ that is a totally ordered by $\prec$ (that is, for each $p, q \in \mathcal{Q}$ either $p \prec q$ or $p \succeq q$; note that this already implies that $\preceq$ is transitive on $\mathcal{Q}$). The poset $P^\flat$ of all nonempty finite chains of P ordered by inclusion is a subcomplex of $\Delta^{\mathcal{P}}$ and so a simplicial poset (a simplicial complex if P is countable); it is called the *barycentric subdivision* of P.

Proposition 2.25. *(a) [30] Every finite cone complex is isomorphic to the poset of cones of a rectilinear cone complex.*

(b) Every finite cone precomplex is isomorphic to the preposet of cones of a rectilinear cone precomplex.

Proof. (a). Let $P = (\mathcal{P}, \preceq)$ be the given cone complex, and let $p \in P$. The poset $\lceil p \rceil^\flat$ of all nonempty finite chains of P contained in $\lceil p \rceil$ is a subcomplex of the barycentric subdivision $P^\flat$. In turn, $P^\flat$ is a subcomplex of $\Delta^{\mathcal{P}}$. The latter is isomorphic to the poset of nonempty faces of some rectilinear simplex Δ in some Euclidean space (see Example 2.5). Hence $P^\flat$ is isomorphic to the poset of simplices of some rectilinear subcomplex $\mathcal{K}$ of Δ. Then the rectilinear subcomplexes of $\mathcal{K}$ corresponding to $\lceil p \rceil^\flat$, $p \in P$, under this isomorphism, clearly form a rectilinear cone complex, whose poset of cones is P. □

(b). Let P be the given cone precomplex, and let us inject it into a finite simplex Δ^n by the proof of Lemma 2.13(a). Then $P^\flat$ is embedded in $(\Delta^n)^\flat$. Every vertex v of $(\Delta^n)^\flat$ is a chain $\{\sigma_v\}$ of length one, where $\sigma_v \in \Delta^n$. Let $\mu\colon (\Delta^n)^\flat \to [n+1] = \{0, 1, \dots, n\}$ send v to the dimension of the simplex σ_v. Then for $\sigma, \tau \in P$, if $\sigma \prec \tau$ then $\mu(\{\sigma\}) < \mu(\{\tau\})$. Also it is easy to see that $P^\flat$ is a flag complex for every cone precomplex P. The rest of the proof repeats that of (a). □

2.26. Flag complex and full subcomplex. Let P be a preposet. A subcomplex Q of P is called *full* in P if $\partial \lceil q \rceil \subset Q$ and $\partial \lceil q \rceil \neq \emptyset$ imply $q \in Q$. A *flag complex* is a simplicial complex K such that every subcomplex of K that is isomorphic to the boundary of a simplex of dimension > 1 is the boundary of some simplex of K. Obviously, every full subcomplex of a flag complex is a flag complex.

It is easy to see that the barycentric subdivision of every cone precomplex is a flag complex; and that if Q is embedded in P, then $Q^\flat$ is a full subcomplex of $P^\flat$. Using these facts, it is easy to prove

Proposition 2.27. *Let K be a simplicial complex and L a subcomplex of $K^\flat$.*

(a) L is a flag complex iff it is the barycentric subdivision of some cone precomplex injected in K.

(b) L is a full subcomplex of $K^\flat$ iff it is the barycentric subdivision of some cone complex embedded in K.

2.28. Simplicial maps. A map f between simplicial posets is called *simplicial* if it takes each simplex $\lceil \sigma \rceil$ onto (not just into) $\lceil f(\sigma) \rceil$. Clearly, every isomorphism of simplicial posets is simplicial. Every map of sets $f\colon S \to T$ induces a simplicial map $\Delta^f\colon \Delta^S \to \Delta^T$. If K is a subcomplex of Δ^S and L is a subcomplex of Δ^T, it is easy to see that every simplicial map $K \to L$ is a restriction of Δ^f for some $f\colon S \to T$.

If $f\colon P \to Q$ is a conical map between preposets, it sends every nonempty finite chain of P into a nonempty finite chain of Q. The resulting map $f^\flat : P^\flat \to Q^\flat$ is the restriction to a subcomplex of the simplicial map $\Delta^f\colon \Delta^P \to \Delta^Q$, so it is simplicial.

2.B. Some basic PL topology

Let $P = (\mathcal{P}, \leq)$ and $Q = (\mathcal{Q}, \leq)$ be preposets.

The *prejoin* $P + Q$ is the preposet $(\mathcal{P} \sqcup \mathcal{Q}, \preceq)$, where $p \preceq q$ iff either $p \leq q$ and both $p, q \in P$; or $p \leq q$ and both $p, q \in Q$; or $p \in P$ and $q \in Q$. Clearly, the prejoin of two posets is a poset. Note that $CP \simeq P + pt$ and $C^*P \simeq pt + P$.

The *product* $P \times Q$ is the preposet $(\mathcal{P} \times \mathcal{Q}, \preceq)$, where $(p, q) \preceq (p', q')$ iff $p \leq p'$ and $q \leq q'$. It is easy to see that $2^S \times 2^T \simeq 2^{S \sqcup T}$ naturally in S and T.

The *join* $P * Q := \partial^*(C^*P \times C^*Q)$ is obtained from $(C^*P) \times (C^*Q)$ by removing the bottom element $(\hat{0}, \hat{0})$. Thus $C^*(P * Q) \simeq C^*P \times C^*Q$, whereas $P * Q$ itself is the union $C^*P \times Q \cup P \times C^*Q$ along their common part $P \times Q$.

From the above, $\Delta^S * \Delta^T \simeq \Delta^{S \sqcup T}$ naturally in S and T. It follows that the join of simplicial complexes $K \subset \Delta^S$ and $L \subset \Delta^T$ is isomorphic to the simplicial complex $\{\sigma \cup \tau \subset S \sqcup T \mid \sigma \in K \cup \{\emptyset\}, \tau \in L \cup \{\emptyset\}, \sigma \cup \tau \neq \emptyset\} \subset \Delta^{S \sqcup T}$.

The join and the prejoin are related via barycentric subdivision: $(P + Q)^\flat \simeq P^\flat * Q^\flat$. Indeed, a nonempty finite chain in $P + Q$ consists of a finite chain in P and a finite chain in Q, at least one of which is nonempty. Note that in contrast to prejoin, join is commutative: $P * Q \simeq Q * P$. Prejoin is associative; in particular, $C(C^*P) \simeq C^*(CP)$.

Remark 2.29. In the case where P and Q are finite simplicial complexes, the above mentioned isomorphism

$$P * Q \simeq C^*P \times Q \underset{P \times Q}{\cup} P \times C^*Q$$

can be regarded as a combinatorial form of the well-known (cf. [40; 4.3.20]) homeomorphism

$$X * Y \cong (pt * X) \times Y \underset{X \times Y}{\cup} X \times (pt * Y),$$

where $X = |P|$ and $Y = |Q|$. However it does not quite fit in the familiar simplicial realm even in this case, for C^*P and $P \times Q$ are no longer simplicial complexes.

2.30. An explicit embedding. We illustrate the above by describing an explicit embedding of any n-polyhedron P in $\mathbb{R}^{2n+1}$, which seems to have appeared only recently[6] [33]. Given a triangulation K of P, let S_i be the subposet of K consisting of all i-simplices (thus every pair of elements in S_i is incomparable). The conical map $K \to S_0 + \dots + S_n$ is clearly an embedding. Then we also have an embedding $K^\flat \hookrightarrow S_0 * \dots * S_n$. The finite set $|S_0|$ embeds in the unit interval $|pt * pt|$, so P embeds in $|pt * pt * S_1 * \dots * S_n| = |C(pt * S_1 * \dots * S_n)|$. We have $C(pt * S_1 * \dots * S_n) = C(pt) \times CS_1 \times \ldots \times CS_n$. Each $|CS_i|$ embeds in $|C(pt * pt)|$, and $C(pt * pt) = C(pt) \times C(pt)$. Thus P embeds in the $(2n+1)$-cube $|C(pt)|^{2n+1}$.

[6] We note that there is a well-known explicit embedding of an n-polyhedron in $\mathbb{R}^{2n+1}$ based on the fact that every set of at most $m+1$ distinct points on the "moment curve" $\gamma(\mathbb{R}) \subset \mathbb{R}^m$, $\gamma(t) = (t, t^2, \dots, t^m)$, is affinely independent – which fact is proved using the Vandermonde determinant.

2.31. Star and link. If P is a preposet and $\sigma \in P$, we define the *star* $\mathrm{st}(\sigma, P) = \lceil \llcorner\sigma\lrcorner \rceil$ and the *link* $\mathrm{lk}(\sigma, P) = \partial^* \llcorner\sigma\lrcorner$. Thus if P is a poset, $\mathrm{st}(\sigma, P)$ is a subcomplex of P and $\mathrm{lk}(\sigma, P)$ is a dual subcomplex of P.

If K is a simplicial complex and $\sigma \in K$, then $\mathrm{lk}(\sigma, K)$ is isomorphic to the subcomplex of K consisting of all $\tau \in \mathrm{st}(\sigma, K)$ disjoint from σ; indeed, an isomorphism is given by $\sigma \sqcup \tau \mapsto \tau$. It follows that $\mathrm{st}(\sigma, K) \simeq \ulcorner\sigma\urcorner * \mathrm{lk}(\sigma, K)$ for every *simplicial* complex K.

Given $\sigma \in P$ and $\tau \in Q$, clearly $\llcorner(\sigma, \tau)\lrcorner^{P\times Q} \simeq \llcorner\sigma_1\lrcorner \times \llcorner\sigma_2\lrcorner$; applying the coboundary, we obtain

$$\mathrm{lk}((\sigma, \tau), P \times Q) \simeq \mathrm{lk}(\sigma, P) * \mathrm{lk}(\tau, Q).$$

Remark 2.32. In the case where P and Q are finite simplicial complexes, the latter isomorphism can be regarded as a combinatorial form of the well-known (cf. [40; 4.3.21]) homeomorphism

$$\mathrm{lk}((x, y), X \times Y) \cong \mathrm{lk}(x, X) * \mathrm{lk}(y, Y),$$

where $|P| = X$ and $|Q| = Y$ are compact polyhedra. However it does not quite fit in the familiar simplicial realm even in this case, for $P \times Q$ is no longer a simplicial complex.

2.33. Canonical subdivision. If $P = (\mathcal{P}, \leq)$ is a preposet and $a, b \in \mathcal{P}$ are such that $a \leq b$, the *interval* $[a, b]$ is the subposet $\llcorner a\lrcorner \cap \ulcorner b\urcorner = \{c \in P \mid a \leq c \leq b\}$ of P. We say that an interval $[a, b]$ is *pre-included* in an interval $[c, d]$ and write $[a, b] \Subset [c, d]$ if $a \in [c, d]$ and $b \in [c, d]$ When P is a poset, this is just the usual inclusion relation. In general, $\Subset$ is a reflexive acyclic relation on the set of all intervals of P (the latter set is really just the relation $\leq$ which officially is a subset of $\mathcal{P} \times \mathcal{P}$). We define the *canonical subdivision* $P^\#$ of P to be the preposet of all intervals of P ordered by pre-inclusion.

We note that the canonical subdivision of every poset is an atomic poset. If P is a CQL, then so is $P^\#$. Clearly, $(P^*)^\# \simeq P^\#$ and $(P \times Q)^\# = P^\# \times Q^\#$ for all preposets P and Q. It follows that $(C^*(P * Q))^\# = (C^*P)^\# \times (C^*Q)^\#$ and in particular, $(P * Q)^\# = (C^*P)^\# \times Q^\# \cup P^\# \times (C^*Q)^\#$.

Remark 2.34. In the case where P is a poset, the operation of canonical subdivision is known (under different names) in Topological Combinatorics (see Babson, Billera and Chan [3] and references there, and Živaljević [45; Definition 7]), as well as in pure Combinatorics (see [29] and references there). Geometric versions of this construction, mostly restricted to the case where P is a simplicial complex, are also known in algebraic topology (see [15]), in combinatorial geometry (see [8]) and in geometric group theory (see [5]).

2.35. Cubical complexes. The poset $I^S := (2^S)^\#$ is called a *cube* or the *S-cube*; or the n-cube (notation: I^n) if $S = [n]$. Note that

$$I^S = (C^*\Delta^S)^\# \simeq (C^*(\mathop{*}_S \Delta^0))^\# = (\prod_S C^*\Delta^0)^\# = \prod_S (C^*\Delta^0)^\# = \prod_S I^1.$$

In particular, $I^{S \sqcup T} \simeq I^S \times I^T$ by an isomorphism natural in S and T.

A *cubical* poset is a CQL where every cone is isomorphic to a cube. (Compare [3].) A 'cubical cone complex' is abbreviated to a *cubical complex*. Cones of a cubical complex are thus called its *cubes*. By the above, the product of two cubical complexes is a cubical complex. Every cubical complex is atomic, since every cube is.

In contrast to Theorem 2.22, not every cubical complex is isomorphic to a subcomplex of a cube. For instance, the simplicial complex $\partial\Delta^2$, which also happens to be a cubical complex, is not isomorphic to any subcomplex of any cube, as it contains 'a cycle of odd length'.

Lemma 2.36. *(a) If K is a simplicial complex, then $(C^*K)^\#$ is a cubical complex.*

(b) If Q is a cubical complex and $\sigma \in Q$, then $\operatorname{lk}(\sigma, Q)$ is a simplicial complex and $\operatorname{st}(\sigma, Q) \simeq \lceil\sigma\rceil \times (C^* \operatorname{lk}(\sigma, Q))^\#$.

Proof. (a). By Theorem 2.22, K is isomorphic to a subcomplex of some simplex Δ^S. Then $(C^*K)^\#$ is isomorphic to a subcomplex of the cube $(C^*\Delta^S)^\# \simeq I^S$. □

(b). Given a $\tau \in \operatorname{lk}(\sigma, Q)$, we have $(\lceil\tau\rceil, \lceil\sigma\rceil) \simeq (I^T, I^S)$ for some sets $S \subset T$. Then $\operatorname{lk}(\sigma, \lceil\tau\rceil)$ can be identified with the poset, embedded in I^T and consisting of all intervals strictly containing $[\emptyset, S]$ — that is, of all intervals $[\emptyset, S \cup R]$, where $\emptyset \neq R \subset T \setminus S$. Hence it is isomorphic to $\Delta^{T\setminus S}$.

We have $\operatorname{st}(\sigma, \lceil\tau\rceil) \simeq \lceil\sigma\rceil \times I^{T\setminus S}$, whereas $I^{T\setminus S} \simeq (C^*\Delta^{T\setminus S})^\# \simeq (C^* \operatorname{lk}(\sigma, \lceil\tau\rceil))^\#$. The resulting isomorphism $\operatorname{st}(\sigma, \lceil\tau\rceil) \simeq \lceil\sigma\rceil \times (C^* \operatorname{lk}(\sigma, \lceil\tau\rceil))^\#$ is natural in τ, which implies the second assertion.

To complete the proof of the first assertion, we note that each cone of $\operatorname{lk}(\sigma, Q)$ is of the form $\operatorname{lk}(\sigma, \lceil\tau\rceil)$, which in turn has been shown to be isomorphic to a simplex. Now $\operatorname{lk}(\sigma, Q)$ is a dual subcomplex of the CQL Q, and hence itself a CQL. □

2.37. Van Kampen duality. The definition of join: $C^*(P * Q) = C^*P \times C^*Q$, along with $(C^*P)^* \simeq C(P^*)$ imply that $C((P * Q)^*) \simeq (C(P^*) \times C(Q^*))$, or equivalently,

$$P * Q \simeq (\partial(C(P^*) \times C(Q^*)))^*$$

for any preposets P and Q. In the case where P_λ are posets, this formula was known already to E. R. van Kampen (1929), cf. [30; Proposition 1.2]. It implies, for instance, that the boundary of the n-cube is dual (as a poset) to the boundary of the n-cross-polytope (compare Example 2.5):

$$\mathop{\ast}_{i=1}^{n} \partial I^1 \simeq (\partial I^n)^*.$$

Theorem 2.38 ([3] (see also [13], [7; 5.22], [25; Fig. 2]; compare [19])**.** *If K is a finite simplicial complex, there exists a finite cubical complex Q such that $\operatorname{lk}(v, Q)$ is isomorphic to K for every vertex v of Q.*

The 'mirroring' construction of Theorem 2.38 has a complex analogue, where the $(\mathbb{Z}/2)^n$ symmetry is replaced by an $(S^1)^n$ symmetry; it is known as the 'moment-angle complex' (see [8]). Both constructions are special cases of the 'polyhedral product' [11].

Proof. K can be identified with a subcomplex of the $(n-1)$-simplex $\Delta^{[n]}$ for some n. Then C^*K is identified with a subcomplex of $2^{[n]}$. The 'folding' conical map $f\colon I \to 2^{[1]}$

can be multiplied by itself to yield a conical map $F\colon I^n \to 2^{[n]}$. Let $Q = F^{-1}(C^*K)$. Then Q is a subcomplex of I^n and contains the set $F^{-1}(\hat{0})$ of all vertices of I^n. For each vertex $v = [S,S]$ in this set, define $J_v\colon 2^{[n]} \to I^n$ by $T \mapsto [T \setminus S, T \cup S]$. Then J_v is a conical map, $J_v(2^{[n]}) = \llcorner v \lrcorner$, and the composition $2^{[n]} \xrightarrow{J_v} I^n \xrightarrow{F} 2^{[n]}$ is the identity. Hence $F|_{\llcorner v \lrcorner}$ is an isomorphism. Therefore $\operatorname{lk}(v, Q) \simeq \operatorname{lk}(\hat{0}, C^*K) = K$. $\square$

2.39. Simple complexes. A *simple poset* is a CQL P such that for every $\sigma \in P$ and every $\tau \in \operatorname{lk}(\sigma, P)$, the poset $\operatorname{lk}(\sigma, \ulcorner\tau\urcorner)$ is isomorphic to a simplex. This includes simplicial and cubical posets as special cases.

Lemma 2.40. *(a) If K is a simplicial poset, then C^*K is a simple poset.*

(b) If P is a simple poset, $\sigma \in P$, then $\operatorname{lk}(\sigma, P)$ is a simplicial poset.

(c) If P is a CQL, then it is a simple poset iff $P^\#$ is a cubical poset.

(d) If P is a simple poset, then P^* is a simple poset.

A version of the 'only if' implication in (c) is found in [3].

Proof. (a). Since K is a CQL, then so is C^*K. If $\tau \in C^*K$, then either $\tau \in K$ or $\tau = \hat{0}$. If further $\sigma \in \operatorname{lk}(\tau, C^*K)$, then $\operatorname{lk}(\tau, \ulcorner\sigma\urcorner_{C^*K})$ is isomorphic to the simplex $\operatorname{lk}(\tau, \ulcorner\sigma\urcorner_K)$ in the first case and to the simplex $\ulcorner\sigma\urcorner$ in the second case. $\square$

(b). Similar to the proof of the first assertion of Lemma 2.36(b). $\square$

(c). Given a $\sigma \in P$, by (b), $\llcorner\sigma\lrcorner \simeq C^*K$ for some simplicial poset K. Hence $\llcorner\sigma\lrcorner^\#$ is a cubical poset by Lemma 2.36(a). Thus cones of $P^\#$ are cubes, and each $\llcorner\sigma\lrcorner^\#$ is a CQL. Now suppose that $P^\#$ contains a lower bound for a collection of intervals $[\sigma_\lambda, \tau_\lambda]$. Since P is a CQL, $\bigcap_\lambda \llcorner\sigma_\lambda\lrcorner = \llcorner\sigma\lrcorner$ for some $\sigma \in P$. Hence $\bigcap_\lambda [\sigma_\lambda, \tau_\lambda] = \bigcap [\sigma, \tau_\lambda]$, which is an interval since $\llcorner\sigma\lrcorner^\#$ is a CQL. Thus $P^\#$ is a CQL.

Conversely, suppose that $P^\#$ is a cubical poset. Since P is a poset, $\llcorner\sigma\lrcorner^\#$ is a subcomplex of $P^\#$ for each $\sigma \in P$, and in particular a cubical poset. Then by Lemma 2.36(b), each $\operatorname{lk}([\sigma,\sigma], \llcorner\sigma\lrcorner^\#)$ is a simplicial poset. On the other hand, it is isomorphic to $\operatorname{lk}(\sigma, P)$ since P is a poset. Thus P is simple. $\square$

(d). Both the hypothesis and the conclusion are equivalent to saying that P is a CQL and every interval $[\sigma, \tau] \in P$ considered as a subposet is isomorphic to the dual cone over a simplex ($\Leftrightarrow$ the cone over the dual of a simplex). $\square$

2.41. Affine polytopal complexes. An *affine polytopal complex* (compare [4; last remark to Definition 2.1]) is a countable complete quasi-lattice K where to every $\sigma \in K$ there is associated an isomorphism of $\ulcorner\sigma\urcorner$ with the poset of non-empty faces of a convex polytope P_σ (compare Example 2.5) so that every face inclusion $\ulcorner\tau\urcorner \subset \ulcorner\sigma\urcorner$ is realized by an affine isomorphism between P_τ and the face of P_σ corresponding to τ. This includes cubical and simplicial complexes as special cases. Every affine polytopal complex is atomic since the poset of nonempty faces of every convex polytope is. Affine polytopal complexes are not equivalent to 'cell complexes' in the sense of Rourke–Sanderson [36], who additionally require linear embeddability into some Euclidean space. For instance, the cubulation of the Möbius band into 3 squares is an affine polytopal complex (and a cubical complex) but not a cell complex in the sense of [36] (see [8]).

A convex d-polytope is called *simple* if every its vertex is incident to precisely d edges (which is the minimal possible number). In particular, this includes all finite-dimensional simplices and cubes. Every k-face of a simple d-polytope is clearly itself a simple k-polytope (by downward induction on k); hence it is incident to precisely $d-k$ of $(k+1)$-faces (by counting their edges). Hence its link in the d-polytope is the boundary of a $(d-k-1)$-polytope with $d-k$ vertices, which can only be the simplex. Thus every affine polytopal complex whose polytopes are simple is a simple poset. In particular, Lemma 2.40(c) has the following consequence:

Theorem 2.42. *If K is an affine polytopal complex whose polytopes are simple, then $K^\#$ is a cubical complex.*

This is a combinatorial form of a well-known geometric construction (see [8]).

2.43. Infinite product and join. Similarly to the case of two factors one defines the *product* $\prod_{\lambda\in\Lambda} P_\lambda$ of an arbitrary family of preposets P_λ. Similarly, their *join* is defined by $\mathop{\ast}_{\lambda\in\Lambda} P_\lambda = \partial^* \prod_{\lambda\in\Lambda} C^*P_\lambda$. Note the *van Kampen duality* $\mathop{\ast}_{\lambda\in\Lambda} P_\lambda \simeq (\partial \prod_{\lambda\in\Lambda} CP^*_\lambda)^*$.

If we are given some basepoints $b_\lambda \in P_\lambda$, there is the *pointed weak product* $\prod^w_{\lambda\in\Lambda} P_\lambda$, which is embedded in $\prod_{\lambda\in\Lambda} P_\lambda$ and is the union of $(\prod_{\lambda\in\Phi} P_\lambda) \times (\prod_{\lambda\in\Lambda\setminus\Phi} \{b_\lambda\})$ over all finite $\Phi \subset \Lambda$. It also has the basepoint $(b_\lambda)_{\lambda\in\Lambda}$.

The *weak join* $\mathop{\ast}^w_{\lambda\in\Lambda} P_\lambda$ is by definition $\partial^*(\prod^w_{\lambda\in\Lambda} C^*P_\lambda)$, where all the basepoints are taken at $\hat{0}$. Thus the weak join is the union of all finite subjoins. It can be identified with $\bigcup_{\lambda\in\Lambda}(P_\lambda \times \prod^w_{\kappa\in\Lambda\setminus\{\lambda\}} C^*P_\kappa)$. The *weak van Kampen duality* reads $\mathop{\ast}^w_{\lambda\in\Lambda} P_\lambda \simeq (\partial \prod^w_{\lambda\in\Lambda} CP^*_\lambda)^*$, where all basepoints are taken at $\hat{1}$.

The *weak Λ-simplex* $\Delta^\Lambda_w := \mathop{\ast}^w_\Lambda \Delta^0$ is identified with the poset $\partial^* 2^\Lambda_w$ of all nonempty finite subsets of Λ; it has no greatest element whenever Λ is infinite. Neither has the *weak Λ-cube* $I^\Lambda_w := (2^\Lambda_w)^\# \simeq \prod^w_\Lambda I^1$, where all basepoints are taken at $[\hat{0}, \hat{0}]$. In contrast, the *co-weak Λ-cube* $I^\Lambda_c := \prod^w_\Lambda I^1$, where all basepoints are taken at $[\hat{0}, \hat{1}]$, has a greatest element. Moreover, by virtue of the weak van Kampen duality, $\partial I^\Lambda_c \simeq (S^\Lambda_w)^*$, where the *weak Λ-sphere* $S^\Lambda_w = \mathop{\ast}^w_\Lambda \partial I^1$. Somewhat reminiscent of the mirroring construction (in the proof of Theorem 2.38), S^Λ_w contains copies of Δ^Λ_w, and therefore $(I^\Lambda_c)^*$ contains copies of 2^Λ_w. In particular, $(I^\Lambda_c)^\# \simeq ((I^\Lambda_c)^*)^\#$ contains copies of $(2^\Lambda_w)^\# = I^\Lambda_w$.

3. Geometric realization

3.A. Geometric realization via embedding

Rectilinear geometric realization of a *finite* preposet has been described in Proposition 2.25. Lemma 2.13 yields a realization of an arbitrary preposet within some *combinatorial* simplex. This however avoids the issue of sensible geometric realization of the simplex itself. Let us now address it. In the case of finite simplicial complexes, the following construction yields essentially the same result as in [38]. (The author found it while being unaware of [38].)

3.1. Geometric realization. If S is a set, the functional space $[0,1]^S$ of all maps $f\colon S \to [0,1]$ is endowed with the metric $d(f,g) = \sup_{s\in S}|f(s) - g(s)|$. Note that the underlying uniform space of $[0,1]^S$ is just $U(S,[0,1])$, where S endowed with the discrete uniformity. The subset $\{0,1\}^S$ of $[0,1]^S$ may be identified with the power set $\mathit{2}^S$, by associating to every subset $T \subset S$ its *characteristic function* $\chi_T\colon S \to [0,1]$, defined by $\chi_T(T) = 1$ and $\chi_T(S\setminus T) = 0$. Note that if S is finite, $[0,1]^S$ is just the usual $|S|$-dimensional cube with the l_∞ metric, and $\{0,1\}^S$ is the set of its vertices.

Let us recall the embedding of a poset into a simplex given by Lemma 2.13(b). Given a poset $P = (\mathcal{P}, \le)$, we identify every $p \in P$ with the cone $\ulcorner p \urcorner$, viewed as an element of $\mathit{2}^{\mathcal{P}} = \{0,1\}^{\mathcal{P}} \subset [0,1]^{\mathcal{P}}$. Then the *geometric realization* of P is a subspace $|P| \subset [0,1]^{\mathcal{P}}$, defined to be the union of the convex hulls of all nonempty finite chains of P. Note that the cube vertex at the origin, $\{0,0,\dots\}$ is never in $|P|$ since $\ulcorner p \urcorner$ is never empty.

More generally, given a preposet $P = (\mathcal{P}, \prec)$, following Lemma 2.13(a) we inject it into its transitive closure $\langle P\rangle = (\mathcal{P}, \prec\!\prec)$, and identify every $p \in P$ with the cone $\ulcorner p \urcorner_{\prec\prec}$ in the transitive closure, viewed again as an element of $\mathit{2}^{\mathcal{P}} = \{0,1\}^{\mathcal{P}} \subset [0,1]^{\mathcal{P}}$. Then the *geometric realization* of P is a subspace $|P| \subset [0,1]^{\mathcal{P}}$, defined to be the union of the convex hulls of all nonempty finite chains of P. Every chain of P is a chain of $\langle P\rangle$, hence $|P| \subset |\langle P\rangle|$.

Since $[0,1]^{\mathcal{P}}$ is complete, the closure $\overline{|P|}$ of $|P|$ in $[0,1]^{\mathcal{P}}$ is uniformly homeomorphic to the completion of $|P|$. Note that each convex hull in the definition of $|P|$ is compact, and therefore separable. Hence if P is countable, $|P|$ is separable; consequently $\overline{|P|}$ is a Polish uniform space, that is, a separable metrizable complete uniform space.

3.2. Generalized geometric realization. Let P be a preposet, and fix an injection $j\colon P \to 2^S$ for some S. The underlying set $\mathit{2}^S$ of the poset 2^S is identified, as before, with $\{0,1\}^S \subset [0,1]^S$. Then the *geometric j-realization* of P is a subspace $|P|_j \subset [0,1]^S$, defined to be the union of the convex hulls of the j-images of all nonempty finite chains in P. We note four basic examples:

- The injection j_P of Lemma 2.13 yields the *standard* geometric realization $|P|_{j_P} = |P|$.
- If P has a least element, then there is a more economical embedding $j'_P\colon P \hookrightarrow 2^{\partial^* P}$, $p \mapsto \ulcorner p \urcorner \cap \partial^* P$. We call $|P|' := |P|_{j'_P}$ the *reduced* geometric realization of P.
- On the other hand, if P is an atomic poset, then Lemma 2.19 yields a more economical embedding $a_P\colon P \hookrightarrow \Delta^{A(P)}$. We call $|P|^\bullet := |P|_{a_P}$ the *atomic* geometric realization of P. Note that $a_P\colon \Delta^\Lambda \to \Delta^\Lambda$ is the identity.
- Finally if $P = C^*Q$, where Q is an atomic poset, then a_Q extends to $a'_P\colon P \hookrightarrow 2^{A(P)}$. We still call $|P|^\bullet := |P|_{a'_P}$ the *atomic* geometric realization of P. Note that $a'_{2^\Lambda}\colon 2^\Lambda \to 2^\Lambda$ is the identity.

3.3. Geometric realization of cone precomplex. If $P = (\mathcal{P}, \leq)$ is a cone complex, then $\mathcal{P}$ is countable, and the cone $\lceil p \rceil$ of every $p \in P$ is finite. Then $j_P(P) \subset \Delta^{\mathcal{P}}$ lies in the weak $\mathcal{P}$-simplex $\Delta^{\mathcal{P}}_w$, that is the set of all nonempty finite subsets of $\mathcal{P}$ (see 2.43). If C is a finite chain of $\Delta^{\mathcal{P}}$, its convex hull lies in $[0,1]^{\sup C} \times \{0\}^{\mathcal{P} \setminus \sup C}$. Consequently, $|\Delta^{\mathcal{P}}_w|^\bullet$ lies in $q_0 := ([0,1], 0)^{(\mathcal{P}^+, \infty)}$, where $\mathcal{P}^+ = \mathcal{P} \cup \{\infty\}$ is the one-point compactification of the discrete space $\mathcal{P}$. Since q_0 is complete, $\overline{|\Delta^{\mathcal{P}}_w|^\bullet}$ also lies in q_0. More generally, if P is a cone precomplex, its transitive closure $\langle P \rangle$ is a cone complex, and $\overline{|P|}$ lies in $\overline{|\langle P \rangle|} \subset \overline{|\Delta^{\mathcal{P}}_w|^\bullet} \subset q_0$.

It is not hard to see that q_0 itself is identified with $\overline{|2^{\mathcal{P}}_w|^\bullet}$. On the other hand, note that $\overline{|\Delta^{\mathbb{N}}_w|^\bullet} \subsetneqq \overline{|\Delta^{\mathbb{N}}|^\bullet}$, since $|\Delta^{\mathbb{N}}|^\bullet$ includes convex hulls of finite chains of infinite subsets of $\mathbb{N}$. In fact, since there are uncountably many of infinite subsets of $\mathbb{N}$, $|\Delta^{\mathbb{N}}|^\bullet$ is not separable.

If P is a cone complex that is *Noetherian*, i.e. contains no infinite chain (which could only be ascending since all cones are finite), then $|P|$ is closed in $[0,1]^{\mathcal{P}}$ and hence complete. It follows that $|P|$ is complete also for every Noetherian cone *precomplex* P, that is a preposet whose transitive closure is a Noetherian cone complex.

Example 3.4. If P is a poset and $j\colon P \to 2^S$ is an injection but not an embedding, then $|P|_j$ need not be isometric to $|P|$. Indeed, let P be the subposet of $2^{\{a,b,c\}}$ with elements $\emptyset$, $\{a\}$, $\{a,b\}$, $\{a,b,c\}$, $\{c\}$. Let $j\colon P \to 2^{\{a,b,c\}}$ re-embed $\{c\}$ onto $\{b\}$ and fix the other elements. Let C be the chain $\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}\}$ and let D be the chain $\{\emptyset, \{c\}, \{a,b,c\}\}$ in P. A point $x \in |C| \subset |P|$ has coordinates (x_a, x_b, x_c) for some numbers $1 \geq x_a \geq x_b \geq x_c \geq 0$, and a point $y' \in |D|$ has coordinates (y_1, y_2) for some numbers $1 \geq y_2 \geq y_1 \geq 0$. Then the image y of y' in $|P|$ has coordinates (y_2, y_2, y_1), and the image y_j of y' in $|P|_j$ has coordinates (y_2, y_1, y_2). Setting $(x_a, x_b, x_c) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{4})$ and $(y_1, y_2) = (\frac{3}{4}, \frac{1}{2})$, we obtain $d(x,y) = \frac{1}{2}$ and $d(x, y_j) = \frac{1}{4}$.

Example 3.5. Here is a simpler example of the same kind. Let P be the subposet of $2^{\{a,b\}}$ with elements $\emptyset$, $\{a\}$ and $\{b\}$. Let $j\colon P \to 2^{\{a,b\}}$ re-embed $\{a\}$ onto $\{a,b\}$ and fix the other elements. Let C be the chain $\{\emptyset, \{a\}\}$ and let D be the singleton chain $\{\{b\}\}$ in P. Let $x \in |C| \subset |P|$ have coordinates $(x_a, x_b) = (0, \frac{1}{2})$; then its image x_j in $|P|_j$ has coordinates $(\frac{1}{2}, \frac{1}{2})$. The point $y \in |D| = \{y\}$ has coordinates $(1, 0)$. Hence $d(x,y) = 1$ and $d(x_j, y) = \frac{1}{2}$.

Theorem 3.6. *If P is a poset and $j\colon P \to 2^S$ is an embedding, then $|P|_j$ is isometric to $|P|$.*

This trivially implies that if P is a preposet and $j\colon P \to 2^S$ is an injection that factors through an embedding of the transitive closure, then $|P|_j$ is isometric to $|P|$.

Proof. We first consider the case where P is the totally ordered n-element poset $[n] = (\{1, \dots, n\}, \leq)$, where $\leq$ has the usual meaning. To avoid confusion, we consider the standard embedding $j_{[n]}$ of $[n]$ in $2^{\{1,\dots,n\}}$, that is, $j_{[n]}(i) = \{1, \dots, i\}$. Then each $j_{[n]}(i) \in 2^{\{1,\dots,n\}}$ is identified with the vertex $(0, \dots, 0, 1, \dots, 1)$ ($n-i$ zeroes, i ones) of the simplex $|[n]| \subset [0,1]^n$. Similarly each $P_i := j(i) \in 2^S$ is identified with a point of $\{0,1\}^S \subset$

$[0,1]^S$. Then $jj_{[n]}^{-1}$ extends uniquely to an affine map $\Phi_j\colon |[n]| \to |[n]|_j$. A point $x = (x_1,\dots,x_n) \in [0,1]^n$ lies in $|[n]|$ if and only if $1 = x_1 \ge x_2 \ge \dots \ge x_n \ge 0$. Let us write $\hat{x} = \Phi_j(x)$. It is easy to see that $\hat{x}(s) = x_i$ for each $s \in P_i \setminus P_{i-1}$, where $P_0 = \emptyset$, and $\hat{x}(s) = 0$ for $s \notin P_n$.

Given another point $y \in |[n]|$, since each $P_i \setminus P_{i-1}$ is nonempty, it follows that $d(\hat{x},\hat{y}) = \sup_{s\in S} |\hat{x}(s) - \hat{y}(s)|$ equals $\max_{i\in[n]} |x_i - y_i| = d(x,y)$. Thus Φ_j is an isometry, which completes the proof of the case $P = [n]$.

We now resume the proof of the general case, where $P = (\mathcal{P},\le)$ is an arbitrary poset. Given a finite chain C in P, we may represent C as the image of the poset $[k]$, where k is the cardinality of C, under the (unique) isomorphism $c\colon [k] \to C$. Consider the compositions $c\colon [k] \xrightarrow{c} P \subset 2^{\mathcal{P}}$ and $jc\colon [k] \xrightarrow{c} P \xrightarrow{j} 2^S$. These extend to the isometries $\Phi_c\colon |[k]| \to |[k]|_c \subset |P|$ and $\Phi_{jc}\colon |[k]| \to |[k]|_{jc} \subset |P|_j$. The compositions $|[k]|_c \xrightarrow{\Phi_c^{-1}} |[k]| \xrightarrow{\Phi_{jc}} |[k]|_{jc}$ agree with each other for different c, and thus combine into a map $\Phi_j\colon |P| \to |P|_j$ that is an isometry on the convex hull of every finite chain of P.

To complete the proof, it suffices to show that all $x, y \in |P|$ satisfy $d(\Phi_j(x),\Phi_j(y)) = d(x,y)$. This will follow once we prove that $d(\Phi_j(x),\Phi_j(y))$ does not depend on j. We may assume without loss of generality that P has the least element $\hat{0}$ and the greatest element $\hat{1}$; for if P has no least (resp. greatest) element, then $S \notin j(P)$ (resp. $\emptyset \notin j(P)$), and therefore j extends to an embedding of CP (resp. C^*P) in 2^S defined by $\hat{1} \mapsto S$ (resp. $\hat{0} \mapsto \emptyset$). Let A (resp. B) be some chain in P containing $\hat{0}$ and $\hat{1}$, whose convex hull contains x (resp. y). We consider the (unique) isomorphisms $a\colon [m] \to A$ and $b\colon [n] \to B$, where m is the cardinality of A and n is the cardinality of B. Thus $a(1) = \hat{0} = b(1)$ and $a(m) = \hat{1} = b(n)$.

Let $\prec$ be the covering relation of the subposet $A \cup B$ of P. (That is, $x, y \in A \cup B$ satisfy $x \prec y$ iff $x < y$ and there exists no $z \in A \cup B$ such that $x < z < y$.) Let $(k_1,l_1),\dots,(k_r,l_r)$ be all pairs in $[m]\times[n]$ such that either $a(k_i) = b(l_i)$ or $a(k_i) \notin B$, $b(l_i) \notin A$ and $a(k_i) \prec b(l_i)$, where each $k_i \le k_{i+1}$. Let $Z \subset [r]$ be the set of indices i such that $a(k_i) = b(l_i)$. It is easy to see[7] that each $k_i < k_{i+1}$ and each $l_i < l_{i+1}$. Similarly let $(k'_1,l'_1),\dots,(k'_{r'},l'_{r'})$ be all pairs in $[m]\times[n]$ such that either $a(k'_i) = b(l'_i)$ or $a(k'_i) \notin B$, $b(l'_i) \notin A$ and $a(k'_i) \succ b(l'_i)$; we may assume that each $k'_i < k'_{i+1}$ and each $l'_i < l'_{i+1}$. Let Z' be the set of indices i such that $a(k'_i) = b(l'_i)$. We note that $k_1 = l_1 = k'_1 = l'_1 = 1$, $k_r = k'_{r'} = m$ and $l_r = l'_{r'} = n$.

[7]Indeed, suppose that $k_i = k_{i+1}$ (and $l_i \ne l_{i+1}$). The cases (1) $i, i+1 \in Z$; (2) $i \in Z$ and $i+1 \notin Z$; (3) $i \notin Z$ and $i+1 \in Z$ are ruled out for trivial reasons. In the remaining case (4) $i, i+1 \notin Z$ we have either $l_i < l_{i+1}$ or $l_i > l_{i+1}$. Then either $a(k_{i+1}) = a(k_i) < b(l_i) < b(l_{i+1})$ or $a(k_i) = a(k_{i+1}) < b(l_{i+1}) < b(l_i)$. Hence either $a(k_{i+1}) \not\prec b(l_{i+1})$ or $a(k_i) \not\prec b(l_i)$, which is a contradiction. Thus $k_i < k_{i+1}$. Next suppose that $l_i \ge l_{i+1}$. Then $a(k_i) < a(k_{i+1}) \le b(l_{i+1}) \le b(l_i)$, so $i \notin Z$. Hence $a(k_i) < a(k_{i+1}) < b(l_{i+1}) \le b(l_i)$, so $a(k_i) \not\prec b(l_i)$, which is a contradiction.

It is easy to see[8] that $a(k_i+1) \not\leq b(l_{i+1}-1)$ for each i. Let us write $A_i = ja(i)$ and $B_i = jb(i)$. Since j is an embedding, we obtain that $A_{k_i+1} \not\subset B_{l_{i+1}-1}$. On the other hand, since $k_i < k_{i+1}$, we have $A_{k_i+1} \subset A_{k_{i+1}} \subset B_{l_{i+1}}$; and similarly $A_{k_i} \subset B_{l_{i+1}-1}$. Thus $A_{k_i+1} \setminus A_{k_i}$ has a nonempty intersection with $B_{l_{i+1}} \setminus B_{l_{i+1}-1}$. In other words, the set Σ of all pairs (κ, λ) such that $A_\kappa \setminus A_{\kappa-1}$ has a nonempty intersection with $B_\lambda \setminus B_{\lambda-1}$ includes the set Δ of all pairs of the form (k_i+1, l_{i+1}). By symmetry, Σ also includes the set Δ' of all pairs of the form (k'_{i+1}, l'_i+1).

We claim that for each $(\kappa, \lambda) \in \Sigma$ there exist a $(k,l) \in \Delta$ and a $(k',l') \in \Delta'$ such that $k \leq \kappa \leq k'$ and $l' \leq \lambda \leq l$. Indeed, if $a(\kappa) \leq b(\lambda-1)$, then $A_\kappa \subset B_{\lambda-1}$; in particular, $(\kappa,\lambda) \notin \Sigma$. If $a(\kappa)$ and $b(\lambda-1)$ are incomparable, let i be the maximal number such that $\kappa > k_i$. Then $\kappa \leq k_{i+1}$, so $\lambda - 1 < l_{i+1}$. Finally, if $a(\kappa) > b(\lambda-1)$, let i be the maximal number such that $\kappa > k_i$. We claim that still $\lambda - 1 < l_{i+1}$. Suppose on the contrary that $\lambda - 1 \geq l_{i+1}$. Then $a(\kappa) > b(\lambda-1) \geq b(l_{i+1}) \geq a(k_{i+1})$. On the other hand, $\kappa \leq k_{i+1}$ by our choice of i, and so $a(\kappa) \leq a(k_{i+1})$, which is a contradiction. This completes the proof of the assertion on (k,l); and the assertion on (k',l') is proved similarly.

We have $x = \Phi_a(\alpha)$ and $y = \Phi_b(\beta)$ for some $\alpha = (\alpha_1, \dots, \alpha_m) \in |A|$ and some $\beta = (\beta_1, \dots, \beta_n) \in |B|$. Let us denote $\Phi_j(x) = \Phi_{ja}(\alpha)$ by $\hat\alpha$ and $\Phi_j(y) = \Phi_{jb}(\beta)$ by $\hat\beta$. We have $d(\hat\alpha, \hat\beta) = \sup_{s\in S} |\hat\alpha(s) - \hat\beta(s)|$. Here $\hat\alpha(s) = \alpha_\kappa$ for each $s \in A_\kappa \setminus A_{\kappa-1}$ (where $A_0 = \emptyset$) and $\hat\alpha(s) = 0$ for $s \notin A_m$; similarly, $\hat\beta(s) = \beta_\lambda$ for each $s \in B_\lambda \setminus B_{\lambda-1}$ (where $B_0 = \emptyset$) and $\hat\beta(s) = 0$ for $s \notin B_n$. Hence $d(\hat\alpha, \hat\beta) = \max_{(\kappa,\lambda)\in\Sigma} |\alpha_\kappa - \beta_\lambda|$.

Next we recall that $1 = \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m \geq 0$ and $1 = \beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq 0$. In particular, $k \leq \kappa \leq k'$ implies $\alpha_k \geq \alpha_\kappa \geq \alpha_{k'}$; and $l' \leq \lambda \leq l$ implies $\beta_{l'} \geq \beta_\lambda \geq \beta_l$. Hence $\alpha_k - \beta_l \geq \alpha_\kappa - \beta_\lambda \geq \alpha_{k'} - \beta_{l'}$, which implies $|\alpha_\kappa - \beta_\lambda| \leq \max\{|\alpha_k - \beta_l|, |\alpha_{k'} - \beta_{l'}|\}$. Thus

$$d(\hat\alpha, \hat\beta) = \max_{(k,l)\in\Delta\cup\Delta'} |\alpha_k - \beta_l|.$$

The right hand side does not depend on j; therefore so does the left hand side, that is, $d(\Phi_j(x), \Phi_j(y))$. ∎

Corollary 3.7. *If Q is a subpreposet of a preposet P, then $|Q|$ admits a natural isometric embedding in $|P|$.*

Proof. If P and Q are posets, then by Theorem 3.6, $|Q| = |Q|_{j_Q}$ is isometric to $|Q|_j \subset |P|_{j_P} = |P|$, where j is the composition $Q \subset P \xrightarrow{j_P} 2^{\mathcal{P}}$.

In the general case, the transitive closure $\langle Q\rangle$ is a subposet of $\langle P\rangle$, and it is easy to see that the image of the isometric embedding $|Q| \subset |\langle Q\rangle| \to |\langle P\rangle|$ lies in $|P|$. ∎

Corollary 3.8. *If P is a preposet, $|P^*|$ is isometric to $|P|$.*

[8] Indeed, if $a(k_i+1) = b(l_{i+1}-1)$, then $k_i+1 = k_j$ and $l_{i+1}-1 = l_j$ for some j; hence $i < j < i+1$, which is a contradiction. Suppose that $a(k_i+1) < b(l_{i+1}-1)$. Since $<$ is the transitive closure of $\prec$, there exist $\kappa \geq k_i+1$ and $\lambda \leq l_{i+1}-1$ such that $a(\kappa) \prec b(\lambda)$. If $a(\kappa) \in B$, then $k_i < \kappa = k_j$ for some j such that $b(l_j) = a(k_j) = a(\kappa) < b(\lambda)$. Hence $l_j < \lambda < l_{i+1}$, and therefore $i < j < i+1$, which is a contradiction. Thus $a(\kappa) \notin B$, and similarly $b(\lambda) \notin A$. Hence $k_i < \kappa = k_j$ and $l_{i+1} > \lambda = l_j$ for some j; hence $i < j < i+1$, which is a contradiction.

Proof. First assume that P is a poset. By Theorem 3.6, there exists an isometry $\Phi_j\colon |P| \to |P|_j$, where $j\colon P \to 2^{\mathcal{P}}$ is defined by $j(p) = \mathcal{P} \setminus \llcorner p \lrcorner$. The isomorphism $\varphi\colon 2^{\mathcal{P}} \to (2^{\mathcal{P}})^*$ defined by $\varphi(S) = \mathcal{P} \setminus S$ extends to a self-isometry Φ of $[0,1]^{\mathcal{P}}$, taking $|P|_j$ onto $|P^*|$. Indeed we have $\varphi j(p) = j_{P^*}(p^*)$, where $j_{P^*}\colon P^* \to 2^{\mathcal{P}}$ is the standard embedding, $j_{P^*}(p^*) = \ulcorner p^* \urcorner = \llcorner p \lrcorner^*$.

If P is a preposet, we apply the above construction to its transitive closure $\langle P\rangle$. Given a finite chain of P, viewed as an embedding $c\colon [n] \to \langle P\rangle \subset 2^{\mathcal{P}}$, we have $|[n]|_c \subset |P| \subset |\langle P\rangle|$. Clearly, the isometry $|\langle P\rangle| \to |\langle P\rangle|_j \to |\langle P\rangle^*| = |\langle P^*\rangle|$ takes $|[n]|_c$ onto $|[n]^*|_{c^*}$. ☐

Remark 3.9. We recall that $2^{\mathcal{P}}_w$ is identified with a subcomplex of $(I_c^{\mathcal{P}})^*$ (see 2.43). The completed geometric realization $\overline{|I_c^{\mathcal{P}}|}$ therefore contains an isometric copy of $\overline{|2^{\mathcal{P}}_w|}$. The latter is in turn isometric to the completed *atomic* geometric realization $\overline{|2^{\mathcal{P}}_w|^\bullet} = q_0$. This isometry extends to an isometry between $\overline{|I_c^{\mathcal{P}}|}$ and $Q_0 := ([-1,1],0)^{(\mathcal{P}^+,\infty)}$.

Theorem 3.10. *Let P and Q be preposets.*

(a) $|P \times Q|$ is uniformly homeomorphic to $|P| \times |Q|$.

(b) $|P + Q|$ is uniformly homeomorphic to $|P| * |Q|$

In particular, $|CP|$ and $|C^*P|$ are uniformly homeomorphic to $C|P|$.

(c) $|P * Q|$ is uniformly homeomorphic to $|P| * |Q|$.

Proof. (a). The injections $P \to 2^{\mathcal{P}}$ and $Q \to 2^{\mathcal{Q}}$ as in Lemma 2.13 yield $|P| \subset [0,1]^{\mathcal{P}}$ and $|Q| \subset [0,1]^{\mathcal{Q}}$, where $P = (\mathcal{P}, \preceq)$ and $Q = (\mathcal{Q}, \leq)$. The injection $P \times Q \to 2^{\mathcal{P}} \times 2^{\mathcal{Q}} \simeq 2^{\mathcal{P} \sqcup \mathcal{Q}}$ yields $|P \times Q| \subset [0,1]^{\mathcal{P} \sqcup \mathcal{Q}}$. Meanwhile, $|P| \times |Q|$ lies in $[0,1]^{\mathcal{P}} \times [0,1]^{\mathcal{Q}}$, which may be identified with $[0,1]^{\mathcal{P} \sqcup \mathcal{Q}}$. To see that $|P \times Q| = |P| \times |Q|$ under this identification, it suffices to consider the case where P and Q are nonempty finite totally ordered sets. This case (and the more general case where P and Q are finite) follows using that a chain in $2^{\mathcal{P}} \times 2^{\mathcal{Q}} = 2^{\mathcal{P} \sqcup \mathcal{Q}}$ lies in $P \times Q$ if and only if it projects onto a chain in P and onto a chain in Q. ☐

(b). Consider the injection $P + Q \to 2^{\mathcal{P} \sqcup pt \sqcup \mathcal{Q}}$ defined by $\sigma \mapsto \ulcorner \sigma \urcorner$ if $\sigma \in P$, and by $\sigma \mapsto \ulcorner \sigma \urcorner \cup pt$ if $\sigma \in Q$. This yields $|P+Q| \subset [0,1]^{\mathcal{P} \sqcup pt \sqcup \mathcal{Q}}$, so that $|P|$ lies in $[0,1]^{\mathcal{P}} \times \{0\} \times \{0\}$ and $|Q|$ in $\{1\} \times \{1\} \times [0,1]^{\mathcal{Q}}$. It is easy to see that $|P+Q|$ is the union of $|P|$, $|Q|$ and all straight line segments with one endpoint in $|P|$ and another in $|Q|$. (Beware that these segments alone cover $|P+Q|$ only if both P and Q are nonempty.) Thus $|P+Q|$ is the independent rectilinear join of $|P|$ and $|Q|$, as defined in [32; §3.B]. Hence by [32; Theorem 3.19], $|P+Q|$ is uniformly homeomorphic to $|P| * |Q|$. ☐

(c). From definition, $P * Q$ is the subpreposet $C^*P \times Q \cup P \times C^*Q$ of $C^*P \times C^*Q$. By the proof of part (b), $|C^*P|$ is the rectilinear cone $c|P|$, as defined in [32; §3.B]. Then by part (a), $|C^*P \times C^*Q|$ is uniformly homeomorphic to $c|P| \times c|Q|$; whereas $|P * Q|$ is uniformly homeomorphic to its subspace $c|P| \times |Q| \cup |P| \times c|Q|$. Write $|P| = X$ and $|Q| = Y$ for the sake of brevity. Then, Lemmas 3.18, 3.16 and 3.11 in [32] yield uniform

homeomorphisms

$$cX \times Y \cup X \times cY \to cX \times Y \underset{X\times Y}{\cup} X \times cY \to CX \times Y \underset{X\times Y}{\cup} X \times CY \to X * Y,$$

where each of the amalgamated unions in the middle is defined as a pushout in the category of uniform spaces (and so is endowed with the quotient uniformity). $\square$

3.B. Geometric realization via quotient

Given a collection of preposets $P_\alpha = (\mathcal{P}_\alpha, \le)$, their *disjoint union* $\bigsqcup_\alpha P_\alpha$ is their coproduct in the category of preposets; more explicitly, it is the preposet $(\bigsqcup_\alpha \mathcal{P}_\alpha, \preceq)$, where $p_\alpha \in P_\alpha$ and $p_\beta \in P_\beta$ satisfy $p_\alpha \preceq p_\beta$ iff $\alpha = \beta$ and $p_\alpha \le p_\beta$. We note that disjoint union does not commute with geometric realization unless the index set is finite, because every infinite disjoint union of non-discrete uniform spaces is easily seen to be non-metrizable.

Theorem 3.11. *Let P be a poset, and let $P_\sqcup$ be the disjoint union of all nonempty finite chains of P. Let $\pi\colon P_\sqcup \to P$ be determined by the inclusions $C \subset P$, where $C \in P^\flat$. Then*

(a) $|\pi|\colon |P_\sqcup| \to |P|$ is a quotient map (in the category of uniform spaces);

(b) if d is the standard metric on $|P_\sqcup|$, then

$$d_\infty(x,y) = \inf_{n\in\mathbb{N}} \inf_{\substack{x_1,\dots,x_{n-1}\in|P| \\ (x_0:=x,\ x_n:=y)}} \sum_{i=0}^{n-1} d(\pi^{-1}(x_i), \pi^{-1}(x_{i+1}))$$

is a metric on $|P|$.

Theorem 3.11(a) implies that $|P|$ is a quotient space of $|P_\sqcup|$ (in the category of uniform spaces). This is reminiscent of the definition of geometric realization of semi-simplicial sets, and of the well-known characterization of the topology of a CW-complex as the topology of a quotient (in the category of topological spaces!) of the disjoint union of its cells.

Theorem 3.11(b) is reminiscent of the definition of geometric polyhedral complexes used in metric geometry and in geometric group theory (see [7], [9]).

Proof. This is based on the technique of quotient maps of finite type (see [32; §3.A]) and on the proof of Theorem 3.6 above.

Write $q = |\pi|$, and let d stand for the usual metric on $|P_\sqcup|$ and on P. Clearly, q is surjective. Given $x, y \in |P|$, let $d_n(x,y) = \inf_{x=x_0,\dots,x_n=y} \sum d(q^{-1}(x_i), q^{-1}(x_{i+1}))$ and $d_\infty(x,y) = \inf_{n\in\mathbb{N}} d_n(x,y)$. It is easy to see that d_∞ is a pseudo-metric on $|P|$ (while each d_n need not satisfy the triangle axiom) and that the identity maps $(|P|, d_n) \xrightarrow{\text{id}} (|P|, d_\infty) \xrightarrow{\text{id}} (|P|, d)$ are uniformly continuous for each n. If $(|P|, d) \xrightarrow{\text{id}} (|P|, d_n)$ is uniformly continuous for some n, then on the one hand, d_∞ is uniformly equivalent to d, and on the other hand, by [32; Lemma 3.2(d′)], d_∞ induces the quotient uniformity on $|P|$. Thus it suffices to show that $(|P|, d) \xrightarrow{\text{id}} (|P|, d_3)$ is uniformly continuous.

Suppose that $P = (\mathcal{P}, \leq)$, and let $\hat{P} := C^*CP$ (with additional elements $\hat{0}$ and $\hat{1}$). The standard geometric realization $|P| \subset 2^{\mathcal{P}}$ lies in the reduced geometric realization $|\hat{P}|' \subset 2^{\mathcal{P}\cup\{\hat{1}\}}$ (where $\hat{0}$ is identified with $\emptyset \in 2^{\mathcal{P}\cup\{\hat{1}\}}$).

Pick some $x, y \in |P|$, and let $A \subset P$ and $B \subset P$ be any chains whose convex hulls contain x and y respectively. We have unique isomorphisms $a\colon [m] \to \hat{A}$ and $b\colon [n] \to \hat{B}$, where $a(1) = \hat{0} = b(1)$ and $a(m) = \hat{1} = b(n)$. Let $(k_1, l_1), \dots, (k_r, l_r)$ and $(k'_1, l'_1), \dots, (k'_{r'}, l'_{r'})$, and Z and Z' be as in the proof of Theorem 3.6. We also recall the notation $\alpha_i = x(s)$ for some $s \in a(i) \setminus a(i-1)$, and $\beta_j = y(t)$ for some $t \in b(j) \setminus b(j-1)$, where $2 \leq i \leq m$ and $2 \leq j \leq n$. Observe that this does not depend on the choices of s and t.

We now define an 'intermediary' chain $C \subset P$, viewed as an isomorphism $c\colon [q] \to \hat{C}$. The inductive construction starts with $c(1) = \hat{0}$, and in the event that $c(i) = \hat{1}$ it terminates with $q = i$. Suppose that $c(i) = a(k)$ for some $k < m$; if $k \neq k_j$ for any j, then let $c(i+1) = a(k+1)$; if $k = k_j$ for some $j \notin Z$, then let $c(i+1) = b(l_j)$. Similarly, suppose that $c(i) = b(l)$ for some $l < n$; if $l \neq l'_j$ for any j, then let $c(i+1) = b(k+1)$; if $l = l'_j$ for some $j \notin Z'$, then let $c(i+1) = a(k'_j)$. Finally, if $c(i) = a(k) = b(l) < \hat{1}$, then we are free to set either $c(i+1) = a(k+1)$ or $c(i+1) = b(l+1)$.

Next we define an $x' \in |A'|_{a'}$ and a $y' \in |B'|_{b'}$, where $A' = A \cap C$ and $B' = B \cap C$, viewed as isomorphisms $a'\colon [m'] \to \hat{A}'$ and $b'\colon [n'] \to \hat{B}'$. Given an $s \in \mathcal{P}$, we have $s \in a'(m') = \ulcorner\hat{1}\urcorner = \mathcal{P} \cup \{\hat{1}\}$, and $s \notin a'(1) = \ulcorner\hat{0}\urcorner = \emptyset$. Hence $s \in a'(i) \setminus a'(i-1)$, where $2 \leq i \leq m'$. Pick some $h_i \in [m]$ so that $a'(i-1) \leq a(h_i - 1)$ and $a(h_i) \leq a'(i)$. We must be more specific for $i = 1$ and $i = m$, and we set $h_2 = 2$ (which is the least among all possible choices) and $h_{m'} = m$ (which is the greatest among all possible choices). Since $h_i \geq 2$, we may set $x'(s) = \alpha_{h_i}$. Let $\alpha'_i = x'(s)$ for any $s \in a'(i) \setminus a'(i-1)$, where $2 \leq i \leq m'$; clearly this does not depend on the choice of s. Thus $\alpha'_2 = \alpha_2$ and $\alpha'_{m'} = m$. Since $x \in |A|_a$, we have $\alpha_2 = 1$ and $\alpha_m = 0$. Therefore $\alpha'_2 = 1$ and $\alpha'_{m'} = 0$, whence $x' \in |A'|_{a'}$. We can similarly define a $y' \in |B'|_{b'}$ and consequently β'_j where $2 \leq j \leq n'$.

Let us estimate $d(x, x')$ from above. Suppose that $s \in a'(i) \setminus a'(i-1)$, where $2 \leq i \leq m'$. If both $a'(i)$ and $a'(i-1)$ belong to $\hat{C}$, then $x'(s) = x(s)$. Else we have $a'(i-1) = a(k_j)$ and $a'(i) = a(k'_{j'})$, where $1 \leq j < r$ and $1 < j' \leq r'$. Moreover, by the construction of C we have $l'_{j'-1} < l_j \leq l'_{j'} < l_{j+1}$. By definition, both $x(s)$ and $x'(s)$ belong to $[\alpha_{k_j+1}, \alpha_{k'_{j'}}]$. Since $l'_{j'-1} + 1 \leq l_{j+1}$, we have $\alpha_{k'_{j'}} - \alpha_{k_j+1} \leq \alpha_{k'_{j'}} - \beta_{l'_{j'-1}+1} + \beta_{l_{j+1}} - \alpha_{k_j+1}$. However $|\alpha_{k'_{j'}} - \beta_{l'_{j'-1}+1}| \leq d(x, y)$ and $|\beta_{l_{j+1}} - \alpha_{k_j+1}| \leq d(x, y)$ by the proof of Theorem 3.6. Thus $|x(s) - x'(s)| \leq 2d(x, y)$. We have proved that $d(x, x') \leq 2d(x, y)$.

We have $d_3(x, y) \leq d(x, x') + d(x', y') + d(y', y)$. By the above, $d(x, x') \leq 2d(x, y)$, and similarly $d(y', y) \leq 2d(x, y)$. By the triangle axiom, $d(x', y') \leq d(x', x) + d(x, y) + d(y, y') \leq 5d(x, y)$. Hence $d_3(x, y) \leq 9d(x, y)$. Thus $(|P|, d) \to (|P|, d_3)$ is uniformly continuous. ∎

Corollary 3.12. *If P is a preposet, then $|P^\#|$ is uniformly homeomorphic to $|P|$ by a homeomorphism $h\colon |P^\#| \to |P|$.*

Moreover, if P is a poset, then $d_\infty(x, y) = 2d_\infty(h(x), h(y))$ for all $x, y \in |P^\#|$.

Proof. Consider the map of sets $f\colon P^\# \to |P| = |P|_{jP}$ defined by sending an element $[\sigma,\sigma] \in P^\#$ into the vertex $|\{\sigma\}|_{jP}$ of $|P|_{jP}$, and an element $[\sigma,\tau] \in P^\#$ with $\sigma < \tau$ into the central point of the straight line segment connecting the vertices $|\{\sigma\}|_{jP}$ and $|\{\tau\}|_{jP}$. A finite chain C of $P^\#$ is of the form $[\sigma_1,\tau_1] \Subset \cdots \Subset [\sigma_n,\tau_n]$, where $\sigma_1 \le \cdots \le \sigma_n \le \tau_n \le \cdots \le \tau_1$. By collapsing all the equality signs in the latter string of inequalities we obtain a sting of strict inequalities, which represents a chain $\tilde{C}$ of P (of length $\ge n$). Then $f(C) \subset |\tilde{C}|_{jP}$; hence f extends by linearity to a map $h\colon |P^\#| \to |P|$.

It is not hard to see that h is a bijection (note that this appears to be known in the case where P is a poset [3], [45]). Indeed, for every chain D of P, $f^{-1}(|D|_{jP})$ can be identified with $D^\#$. So the assertion reduces to the case $P = [n]$, which can be checked directly.

To show that h is a uniform homeomorphism we may assume that P is a poset by considering the transitive closure. Then it suffices to prove that $d_\infty(x,y) = 2d_\infty(h(x),h(y))$ for all $x,y \in |P^\#|$. By the definition of the d_∞ metrics (with respect to the usual metrics d on $P_\sqcup$ and $(P^\#)_\sqcup$; see the statement of Theorem 3.11(b)), it suffices to prove this when $x,y \in |D^\#|$ for some chain D of P. So the assertion again reduces to the case $P = [n]$, which can be checked directly. $\square$

Corollary 3.13. *Let P be a poset, and let $P_\Box$ be the disjoint union of all intervals of P. Let $\rho\colon P_\Box \to P$ be determined by the inclusions $Q \subset P$, where $Q \in P^\#$. Then $|\rho|\colon |P_\Box| \to |P|$ is a quotient map (in the category of uniform spaces).*

Proof. Consider the commutative square

$$\begin{array}{ccc} (P_\Box)_\sqcup & \longrightarrow & P_\Box \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ P_\sqcup & \xrightarrow{\pi} & P. \end{array}$$

Here π' is trivially a quotient map, and π is a quotient map by Theorem 3.11(a). Hence $\pi\pi'$ is a quotient map, and therefore so is ρ. $\square$

Remark 3.14. Corollary 3.13 is, in a sense, easier than Theorem 3.11, for it can also be proved as follows. One first shows that $|r_X|$ (defined in §3.C below) is uniformly continuous without using Corollary 3.13; this can be done by writing an explicit formula for $|r_X|$ in coordinates. Next one observes that $|r_X^\#|\colon |P^{\#\#}| \to |P^\#|$ takes any pair of sufficiently close points onto a pair of points contained in $|\ulcorner q \urcorner|$ for the same interval $q \in P^\#$. It then remains to use [32; Lemma 3.2(d′)] in the same way as it is used in the proof of Theorem 3.11.

Remark 3.15. Corollary 3.13 suffices to show that $h\colon |P^\#| \to |P|$, as defined in the proof of Corollary 3.12, is a uniform homeomorphism. Indeed, writing $P = (S,\le)$, the usual metric on $|P|$ is induced from that on $|2^S|^\bullet$ via the usual embedding $P \hookrightarrow 2^S$, and similarly the usual metric on $|P^\#|$ is induced from that on $|(2^S)^\#|$ (see Lemma 2.13 and Corollary 3.7). So it suffices to show that $h\colon |(2^S)^\#| \to |2^S|^\bullet$ is a uniform homeomorphism. Let $Q = (2^S)^\#$. By Corollary 3.13, $(|Q|,d)$ is uniformly homeomorphic to $(|Q|, d_\infty)$, where d denotes the usual metric on $|Q|$ and on $|Q_\square|$, and d_∞ is as in the statement of Theorem 3.11(b). On the other hand, it is easy to see that $(|Q|, d_\infty)$ is isometric to $(|2^S|^\bullet, 2d)$, where d is the usual l_∞ metric on the cube $|2^S|^\bullet$. Hence $(|Q|, d)$ is uniformly homeomorphic to $(|2^S|^\bullet, d)$.

Remark 3.16. If P is a simple poset (see 2.39), then it is easy to see that the usual metrics d on $|P_\sqcup|$ and on $|P_\square|$ lead to the same d_∞ metric on $|P|$. So Theorem 3.11 for such P can be recovered from Corollary 3.13.

3.17. Geometric realization of a conical map. Given a conical map $f\colon P \to Q$ between posets, it extends uniquely to a map $|f|\colon |P| \to |Q|$ that is affine on every convex hull of a chain. In fact, $|f|$ is clearly 1-Lipschitz on every convex hull of a chain. On the other hand, f lifts uniquely to a conical map $f_\sqcup\colon P_\sqcup \to Q_\sqcup$. Then $|f_\sqcup|$ is 1-Lipschitz (globally), and in particular, uniformly continuous. Since $q\colon |Q_\sqcup| \to |Q|$ is uniformly continuous, so is the composite arrow in the commutative diagram

$$\begin{array}{ccc} |P_\sqcup| & \xrightarrow{f_\sqcup} & |Q_\sqcup| \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow \\ |P| & \xrightarrow{f} & |Q|. \end{array}$$

By Theorem 3.11(a), $p\colon |P_\sqcup| \to |P|$ is a quotient map; in other words, the uniformity of $|P|$ is final with respect to p. Hence $|f|$ is uniformly continuous. We call it the *geometric realization* of f.

It is easy to see that geometric realization of posets and of conical maps determines a functor (also called the *geometric realization*) from the category of posets and conical maps to the category of metrizable uniform spaces and uniformly continuous maps.

Theorem 3.18. *The geometric realization functor preserves pullbacks, as well as those pushouts that remain such upon barycentric subdivision.*

Proof. The assertion is equivalent to the preservation of finite products, finite coproducts (which always remain finite coproducts upon barycentric subdivision), embeddings, and those quotient maps that remain quotient maps upon barycentric subdivision. Finite products were considered in Theorem 3.10(a) and embeddings in Corollary 3.7. The preservation of finite coproducts is obvious.

Finally, let $f\colon P \to Q$ be a quotient map of posets such that $f^\flat\colon P \to Q$ is also a quotient map. In particular[9], $f^\flat$ is surjective, so every chain in Q is the image of a chain in P. Then $|f_\sqcup|\colon |P_\sqcup| \to |Q_\sqcup|$ is a uniformly continuous retraction, and therefore a quotient map. By Theorem 3.11(a), also $q\colon |Q_\sqcup| \to |Q|$ is a quotient map. Then the composite arrow in the preceding commutative diagram is a quotient map. The assertion now follows from and the fact that if a composition $X \to Y \xrightarrow{f} Z$ is a quotient map, then so is f. ∎

[9]In fact, if the simplicial map $f^\flat$ is surjective, then $f^\flat$ and f are quotient maps.

Theorem 3.18 yields an alternative proof of Theorem 3.10(b,c):

Corollary 3.19. *If P and Q are posets, then $|P * Q|$ and $|P + Q|$ are uniformly homeomorphic to $|P| * |Q|$.*

Proof. $P*Q$ is the pushout of the diagram $P\times Q\times I \supset P\times Q\times \partial I \to P\times\{\{0\}\}\sqcup Q\times\{\{1\}\}$, where $I = \Delta^{\{0,1\}}$; $P+Q$ is the pushout of the diagram $P\times Q\times [2] \supset P\times Q\times(\{1\}\sqcup\{2\}) \to P \times \{1\} \sqcup Q \times \{2\}$, where $[2] = (\{1,2\},\leq)$; and $X * Y$ is the pushout of the diagram $X \times Y \times [0,1] \supset X \times Y \times \{0,1\} \to P \times \{0\} \sqcup Q \times \{1\}$. Since $|I|$ and $|[2]|$ are uniformly homeomorphic to $[0,1]$, the assertion follows from Theorem 3.18. $\square$

3.20. Adjunction preposet and mapping cylinder. Let P be a poset, A a subposet of P and $f\colon A \to Q$ a conical map. We define the *adjunction preposet* $P \cup_f Q$ to be the pushout of the diagram $P \supset A \xrightarrow{f} Q$ in the category of preposets.

Given a conical map of posets $f\colon P \to Q$, the *mapping cylinder* $MC(f) = P \times [2] \cup_{f_1} Q$ and the *dual mapping cylinder* $MC^*(f) = P \times [2] \cup_{f_2} Q$, where f_i is the composition $P \times \{i\} \simeq P \xrightarrow{f} Q$. Note that $MC^*(f) \simeq (MC(f^*))^*$ and that there are natural conical bijections (which are not embeddings) $MC(f) \to Q + P$ and $MC^*(f) \to P + Q$.

3.21. Open and closed maps. We call a map of preposets $f\colon P \to Q$ *closed (resp. open)* if $f(\lceil p \rceil) = \lceil f(p) \rceil$ (resp. $f(\lfloor p \rfloor) = \lfloor f(p) \rfloor$) for every $p \in P$. If P and Q are posets, this is equivalent to saying that f is conical and takes (dual) subcomplexes onto (dual) subcomplexes, or that f is continuous and closed (open) with respect to the topology whose open sets are the dual subcomplexes. Clearly, a map between simplicial complexes is closed iff it is simplicial. An example of a non-closed conical map is given by the diagonal map of the 1-simplex Δ^1 into $\Delta^1 \times \Delta^1$.

Lemma 3.22. *Let $P \supset A \xrightarrow{f} Q$ be a partial conical map of posets.*

(a) If A is a subcomplex of P and f is closed, then $P \cup_f Q$ is a poset.

(b) Dually, if A is a dual subcomplex of P and f is open, then $P \cup_f Q$ is a poset.

Proof. Let $p \in P$, $a \in A$ and $q \in Q$. Suppose that $p > a$ and $f(a) > q$. If f is closed, then $q = f(b)$ for some $b < a$, whence $p > b$; and if A is a dual subcomplex of P, then $p \in A$, and so $f(p) > q$.

Now suppose that $p < a$ and $f(a) < q$. If f is open, then $q = f(b)$ for some $b > a$, whence $p < b$; and if A is a subcomplex of P, then $p \in A$, and so $f(p) < q$. $\square$

Corollary 3.23. *Let $f\colon P \to Q$ be a conical map of posets. Then $MC(f)$ is a poset iff f is closed; dually, $MC^*(f)$ is a poset iff f is open.*

Proof. If f is closed, then $MC(f)$ is a poset by Lemma 3.22. If f is not closed, there exist a $p \in P$ and a $q < f(p)$ such that $q \neq f(p')$ for any $p' < p$. Then $(p, 2) > (p, 1) = f(p) > q$ but $(p, 2) \not> q$ in $MC(f)$. The dual assertion follows from $MC^*(f) \simeq (MC(f^*))^*$. $\square$

3.C. Second canonical neighborhood and uniform ARs

3.24. Handle decomposition. Let X be a poset. Then $[\sigma,\tau]<[\rho,\upsilon]$ in $X^\#$ iff $\rho\le\sigma$ and $\tau\le\upsilon$. The dual cone $\llcorner[\sigma,\tau]\lrcorner$ is the poset of all such intervals $[\rho,\upsilon]$; clearly, it is isomorphic to $(\ulcorner\sigma\urcorner)^*\times\llcorner\tau\lrcorner$. Then every cone $\ulcorner[\sigma,\tau]^*\urcorner=(\llcorner[\sigma,\tau]\lrcorner)^*$ of the dual poset $h(X):=(X^\#)^*$ is isomorphic to the product $\ulcorner\sigma\urcorner\times(\llcorner\tau\lrcorner)^*=\ulcorner\sigma\urcorner\times\ulcorner\tau^*\urcorner$.

The maximal cones of $h(X)$ (i.e. those cones that are not properly contained in other cones) are of the form $\ulcorner[\sigma,\sigma]^*\urcorner$, where $\sigma\in X$, and are called the *(canonical) handles* h_σ of the poset X. Each h_σ is isomorphic to the product of the cone $\ulcorner\sigma\urcorner$ of X and the cone $\ulcorner\sigma^*\urcorner$ of X^*. These cones are called the *core* and *cocore* of the handle h_σ.

When $\sigma<\tau$, the intersection $h_\sigma\cap h_\tau$ is clearly the cone $\ulcorner[\sigma,\tau]^*\urcorner$ of $h(X)$. However, when σ and τ are incomparable in X, it may be that $h_\sigma\cap h_\tau$ is nonempty (namely, it is nonempty iff both $\ulcorner\sigma\urcorner\cap\ulcorner\tau\urcorner$ and $\llcorner\sigma\lrcorner\cap\llcorner\tau\lrcorner$ are nonempty). This distinguishes canonical handles from the familiar ones (which are defined in the literature as subcomplexes of the second barycentric subdivision of a simplicial combinatorial manifold).

3.25. Collapsing handles onto cores. Let X be a poset. The *canonical subdivision map* $X^\#\to X$ is defined by $[\lambda,\kappa]\mapsto\kappa$. Let us consider the composition $X^\#\xrightarrow{j_X}(X^*)^\#\xrightarrow{\#}X^*$, where the isomorphism $j_X\colon X^\#\to(X^*)^\#$ is given by $[\sigma,\tau]\mapsto[\tau^*,\sigma^*]$. Then the dual map $r_X\colon h(X)\to X$ to this composition is given by $[\sigma,\tau]^*\mapsto\sigma$. The restriction of r_X to the cone $\ulcorner[\sigma,\tau]^*\urcorner\simeq\ulcorner\sigma\urcorner\times\ulcorner\tau^*\urcorner$ is the projection onto the first factor $\ulcorner\sigma\urcorner$. We note that the projection onto the second factor $\ulcorner\tau^*\urcorner$ is given by the composition $\bar r_X\colon h(X)\xrightarrow{(j_X)^*}h(X^*)\xrightarrow{r_{X^*}}X^*$. We also note that $h_\sigma=r_X^{-1}(\ulcorner\sigma\urcorner)$ and $h_\tau=\bar r_X^{-1}(\ulcorner\tau^*\urcorner)$.

Lemma 3.26. *Let K be a poset.*

(a) $|MC(r_K)|$ is uniformly homeomorphic to $|MC(\mathrm{id}_K)|$ by a homeomorphism that is the identity on K and extends the homeomorphism $|h(K)|\to|K^\#|\to|K|$ given by 3.8 and 3.12.

(b) $|MC^*(r_K)|$ is uniformly homeomorphic to $|MC(\mathrm{id}_{h(K)})|$ by a homeomorphism that is the identity on $h(K)$ and extends the homeomorphism $|K|\to|K^\#|\to|h(K)|$ given by 3.8 and 3.12.

Here $MC^*(r_K)\simeq(MC(r_K^*))^*$, where r_K^* is the composition $K^\#\xrightarrow{j_K}(K^*)^\#\xrightarrow{\#}K^*$.

(a). We define $f\colon|MC(r_K)|\to|MC(\mathrm{id}_K)|$ as required on the top and bottom, and extend it linearly to the convex hull of every chain. A chain of $MC(\mathrm{id}_K)=K\times[2]$ is of the form $B+A$, where $A=(\alpha_1<\dots<\alpha_n)$ is a chain in the domain, and $B=(\beta_1<\dots<\beta_m)$ is a chain in the range, with $\beta_m\le\alpha_1$. From the similar description of chains of $MC(\mathrm{id}_{h(K)})$ we deduce that a chain of $MC(r_K)$ is of the form $D+C$, where C is a chain of $h(K)$ of the form $[\sigma_1,\tau_1]^*<\dots<[\sigma_r,\tau_r]^*$ and D is a chain of K of the form $\rho_1<\dots<\rho_s$, with $\rho_s\le\sigma_1$. It is easy to see that f sends $|h(B)+A|$ onto $|B+A|$ via the join of the uniform homeomorphism $|h(B)|\to|B|$ with $\mathrm{id}_{|A|}$. It follows that f is a uniform homeomorphism. $\square$

(b). We define $f\colon |MC^*(r_K) \to |MC(\mathrm{id}_{h(K)})|$ as required on the top and bottom, and extend it linearly to the convex hull of every chain. A chain of $MC^*(\mathrm{id}_{h(K)}) = h(K)\times[2]$ is of the form $A+B$, where $A = (\alpha_1 < \dots < \alpha_n)$ is a chain in the domain, and $B = (\beta_1 < \dots < \beta_m)$ is a chain in the range, with $\alpha_n \le \beta_1$. It follows that a chain of $MC(r_K)$ is of the form $C + D$, where C is a chain of $h(K)$ of the form $[\sigma_1, \tau_1]^* < \dots < [\sigma_r, \tau_r]^*$ and D is a chain of K of the form $\rho_1 < \dots < \rho_s$, with $[\sigma_r, \tau_r]^* \le [\rho_1, \rho_s]^*$. It is easy to see that f sends $|C + D|$ onto $|C + h(D)|$ via the join of $\mathrm{id}_{|C|}$ and the uniform homeomorphism $|D| \to |h(D)|$. It follows that f is a uniform homeomorphism. □

Theorem 3.27. *Let K be a poset and L a subcomplex of K. Then $|L|$ is a uniform neighborhood retract of $|K|$.*

Proof. Clearly $|\ulcorner h(L)\urcorner_{h(K)}|$ is a uniform neighborhood of $|h(L)|$ in $|h(K)|$. By 3.8 and 3.12, it corresponds to a uniform neighborhood of $|L|$ in $|K|$. On the other hand, let $\partial N(L) = \ulcorner h(L)\urcorner \setminus h(L) \subset h(K)$, and let $N(L) = L \cup MC^*(r_K|_{\partial N(L)}) \subset MC^*(r_K)$. The conical map $MC^*(r_K) \to MC^*(\mathrm{id}_K) \to K$ is a retraction, and sends $N(L)$ into L; hence it restricts to a retraction $N(L) \to L$. Thus $|L|$ is a uniform retract of $|N(L)|$.

Finally, let f be the composition of the uniform homeomorphism $|MC^*(r_K)| \to |MC^*(\mathrm{id}_{h(K)})|$ from Lemma 3.26(b), with the geometric realization of the projection $\pi\colon MC^*(\mathrm{id}_{h(K)}) \to h(K)$. Then f restricts to the identity on $|\partial N(L)|$ and to the uniform homeomorphism $|L| \to |h(L)|$ on $|L|$. A chain of $MC^*(r_K|_{\partial N(L)})$ is of the form $C + D$, where C is a chain of $\partial N(L) \subset h(K)$ of the form $[\sigma_1, \tau_1]^* < \dots < [\sigma_r, \tau_r]^*$ and D is a chain of $L \subset K$ of the form $\rho_1 < \dots < \rho_s$, with $[\sigma_r, \tau_r]^* \le [\rho_1, \rho_s]^*$. Since $\partial N(L)$ is disjoint from $h(L)$, it is easy to see that f sends $|C+D|$ homeomorphically onto $|C+h(D)|$ via the join of $\mathrm{id}_{|C|}$ and the uniform homeomorphism $|D| \to |h(D)|$. It follows that f restricts to a uniform homeomorphism between $N(L)$ and $|\ulcorner h(L)\urcorner_{h(K)}|$. □

3.28. Relative canonical subdivisions. Let K be a poset and let L be a subcomplex of K. In the notation of the proof of Proposition 3.27, let $h(K, L)$ denote $N(L) \cup (h(K) \setminus h(L))$. Then by the proof of Proposition 3.27, $|h(K, L)|$ is uniformly homeomorphic to $|h(K)|$ and hence to $|K|$.

Dually, let $K^\#_L = MC^*(r^*_K) \setminus (\llcorner L \lrcorner \cup \llcorner (K \setminus L)^{\#} \lrcorner)$. This contains $L^\#$ and $K \setminus L$, and $|K^\#_L|$ is uniformly homeomorphic to $|K|$ similarly to the above (using part (a) of 3.27). Note that $(P + Q)^\# \simeq (P^* * Q)^\#_{P^* * \emptyset \cup \emptyset * Q}$, which yields an alternative proof that $|P + Q|$ is uniformly homeomorphic to $|P * Q|$.

We recall from [32] that a uniform space is called *homotopy complete* if there exists a homotopy $h_t\colon \overline{|P|} \to \overline{|P|}$, where $\overline{|P|}$ is the completion of $|P|$, with $h_0 = \mathrm{id}$ and $h_t(\overline{|P|}) \subset |P|$ for $t > 0$.

Lemma 3.29. *Let P be a countable poset. Then $|P|$ is homotopy complete.*

Proof. We first consider the case $P = 2^{\mathbb{N}}_w$. The atomic geometric realization $|2^{\mathbb{N}}_w|^\bullet$ is a dense subspace of $q_0 = \overline{|2^{\mathbb{N}}_w|^\bullet}$ consisting of all functions that are nonzero in only finitely many points. Define $h_t\colon I \to I$, $t \in I = [0, 1]$, by $H_t(s) = \max\{0, 1-(1-s)(1+t)\}$. Next define a homotopy $h_t\colon I^{\mathbb{N}} \to I^{\mathbb{N}}$ by $h_t(f) = H_t f$. Clearly H_t is uniformly continuous, $h_0 = \mathrm{id}$, and $h_t(q_0) \subset |2^{\mathbb{N}}_w|^\bullet$ for $t > 0$. (We may identify q_0 with a subspace of $I^{\mathbb{N}}$ using the inclusion $\mathbb{N} \subset \mathbb{N}^+$.)

In the general case, we note that $|P|$ is uniformly homeomorphic to $|P^\#|$, and $P^\#$ is atomic. Thus we may assume without loss of generality that P is atomic.

Let R be the composition $MC(r_P) \to MC(\mathrm{id}_P) \to P$ (extending the map r_P), and let $H\colon |MC(r_P)| \to |MC(\mathrm{id}_P)|$ be the uniform homeomorphism of Lemma 3.26(a). Define $h_P\colon \overline{|P|} \times I \to \overline{|P|}$ to be the unique extension of $|R|H^{-1}$ over the completion, where $|[2]|$ is identified with $I = [0,1]$ by the affine homeomorphism sending $\{2\}$ to 1. Further let $h^\bullet_P$ be defined similarly to h_P but using *atomic* geometric realizations throughout, provided that P is either atomic or the dual cone over an atomic poset. Then it is easy to check that $h^\bullet_{2^{\mathbb{N}}_w}$ coincides with the homotopy h_t constructed above. On the other hand, the definition of $|P|$ is based on the embedding of $P = (\mathcal{P}, \leq)$ in $\Delta^{\mathcal{P}}_w$, and h_P is the restriction of $h^\bullet_{\Delta^{\mathcal{P}}_w}$, which in turn is the restriction of $h^\bullet_{2^{\mathcal{P}}_w}$. Hence $h_P(\overline{|P|} \times (0,1]) \subset |2^{\mathcal{P}}_w|^\bullet \cap \overline{|P|} = |P|$. ∎

Lemma 3.30. *$|2^{\mathbb{N}}_w|$ is a uniform AR.*

Proof. By Theorem 3.6, the completion of $|2^{\mathbb{N}}_w|$ is isometric to $\overline{|2^{\mathbb{N}}_w|^\bullet} = q_0$, which is known to be a uniform AR (see [32; Corollary 4.10(a)]). By Lemma 3.29 $|2^{\mathbb{N}}_w|$ is homotopy complete, hence by [32; Theorem 4.19] it is itself a uniform AR. ∎

Remark 3.31. Since $2^{\mathbb{N}}_w \simeq C^*(\Delta^{\mathbb{N}}_w)$, by Theorem 3.10(b) (or alternatively by Corollary 3.19), $|2^{\mathbb{N}}_w|$ is uniformly homeomorphic to a cone, and therefore is uniformly contractible. Thus asserting that it is a uniform AR is equivalent (see [32; Lemma 4.35]) to asserting that it is a uniform ANR.

Theorem 3.32. *If P is a simplicial complex, then $|P|$ is a uniform ANR.*

The finite-dimensional case is due to Isbell [21; 1.9], [23; VI.15].

We give two proofs: one based on Theorem 3.27 and another based on the uniform version of Hanner's criterion of ANR'ness [32; Theorem 4.31(b)].

Both proofs start by recalling that by Theorem 2.22, P is isomorphic to a subcomplex of the simplex $\Delta^{\mathbb{N}}_w$. Hence $P^\#$ is isomorphic to a subcomplex of $(\Delta^{\mathbb{N}}_w)^\#$, which in turn is a subcomplex of $(C^*\Delta^{\mathbb{N}}_w)^\# = (2^{\mathbb{N}}_w)^\#$.

Proof by infinite process. For each $n \geq 1$ we further have that $P^{\#n}$ is isomorphic to a subcomplex of $Q^{\#n}$, where Q denotes $2^{\mathbb{N}}_w$. Hence the conical map $r_{Q^{\#n}}\colon h(Q^{\#n}) \to Q^{\#n}$ (see 3.25) sends $\ulcorner h(P^{\#n}) \urcorner$ onto $P^{\#n}$. Clearly $|\ulcorner h(P^{\#n}) \urcorner|$ is a uniform neighborhood of $|h(P^{\#n})|$ in $|h(Q^{\#n})|$, namely the 1-neighborhood in the d_∞ metric. Let r_n denote the composition

$$|Q| \xrightarrow{h_{n+1}} |Q^{\#(n+1)}| \xrightarrow{H} |h(Q^{\#n})| \xrightarrow{|r_{Q^{\#n}}|} |Q^{\#n}| \xrightarrow{h_n^{-1}} |Q|,$$

where h_n is the uniform homeomorphism in Corollary 3.12 and H is the uniform homeomorphism in Corollary 3.8. Then r_n sends the 2^{-n}-neighborhood $h_{n+1}^{-1}H^{-1}(|\ulcorner h(P^{\#n}) \urcorner|)$

of $|P|$ onto $|P|$ and is 2^{-n}-close to the identity. Moreover, $r_n|_{|P|}$ is uniformly 2^{-n}-homotopic to the identity with values in $|P|$, because for every poset K, the composition $|K| \xrightarrow{h_1} |K^\#| \xrightarrow{H} |h(K)| \xrightarrow{r_K} |K|$ is uniformly 1-homotopic to the identity (the homotopy is constructed in coordinates in the proof of Lemma 3.29 and combinatorially in Lemma 3.26(a)). Since by Lemma 3.30, $|Q|$ is a uniform ANR, we infer from [32; Theorem 4.31(b)] that so is $|P|$.[10] $\square$

Combinatorial proof. By Theorem 3.27, we have that $|P^\#|$ is a uniform neighborhood retract of $|(2^{\mathbb{N}}_w)^\#|$. Hence by Corollary 3.12, $|P|$ is a uniform neighborhood retract of $|2^{\mathbb{N}}_w|$. Since $|2^{\mathbb{N}}_w|$ is a uniform ANR by Lemma 3.30, so is $|P|$. $\square$

From [32; Lemma 4.22] we infer

Corollary 3.33. *If (Y,B) is a pair of metrizable uniform spaces and (P,Q) is a pair of simplicial complexes, then $U((Y,B),(|P|,|Q|))$ is a uniform ANR.*

Example 3.34. Consider the poset $[n] = (\{1,\dots,n\},\le)$. Let C_n be the amalgamated union of $C^\#$ for all proper subchains $C \subsetneq [n]$. The canonical map $j_n\colon C_n \to [n]^\#$ is an injection, but not an embedding for $n>1$. Consider the map $j := \sqcup_{n\in\mathbb{N}} j_n$, injecting $P := \bigsqcup_{n\in\mathbb{N}} C_n$ into $Q := \bigsqcup_{n\in\mathbb{N}}[n]^\#$.

Then the generalized geometric realization $|P|_j$ is not a uniform ANR. Indeed, it follows from Corollary 3.12 that each $|C_n|_{j_n}$ is uniformly homeomorphic to $X_n := \bigcup_{C\subsetneq[n]} |C| \subset |[n]|$. Now $|[n]| = \{(x_0,\dots,x_n) \mid 0 = x_0 \le \dots \le x_n = 1\}$, and X_n consists of all $(x_0,\dots,x_n) \in |[n]|$ such that $x_i = x_{i+1}$ for some i. But each $(x_0,\dots,x_n) \in |[n]|$ satisfies $x_{i+1} - x_i \le \frac{1}{n}$ for some i (by the pigeonhole principle). Hence the $\frac{1}{2n}$-neighborhood of X_n in $|[n]|$ is the entire $|[n]|$. Consequently, for each $\varepsilon > 0$ the ε-neighborhood of $|P|_j$ in $|Q|$ contains $|\bigsqcup_{n\in\mathbb{N}\setminus[m]}[n]^\#|$ for some m, and so does not retract uniformly or even continuously onto $|P|_j$.

4. Uniform polyhedra

4.A. Uniform local contractibility

Theorem 4.1. *If P is a countable poset, then $|P|$ is uniformly locally contractible.*

Proof. Given $x, y \in |P|$ with $d(x,y) < 1$, the proof of Theorem 3.11 above produces $x', y' \in |P|$ such that each of the pairs $\{x,x'\}$, $\{x',y'\}$, $\{y',y\}$ lies in the convex hull of a single chain of P, and $d(x,x')$ and $d(y,y')$ are bounded above by $2d(x,y)$. We shall modify this pair of discontinuous maps $(x,y)\mapsto x'$, $(x,y)\mapsto y'$ into a pair of uniformly continuous maps φ,ψ from the uniform neighborhood $\{(x,y) \mid d(x,y)<\delta\}$ of the diagonal in $|P|\times|P|$ into $|P|$ such that $d(x,\varphi(x,y))$ and $d(y,\psi(x,y))$ are bounded above by $\frac{\varepsilon}{2}$. Given δ-close maps $f,g\colon X\to|P|$, we then define a homotopy $h_t\colon X\to|P|$ by $h_0 = f$, $h_1 = g$, $h_{1/3}(x) = \varphi(f(x),g(x))$ and $h_{2/3}(x) = \psi(f(x),g(x))$ and by linear

[10] A slightly more direct argument instead of Lemma 3.30 uses that the completion q_0 of $|Q|$ is a uniform ANR (see [32; Corollary 4.10(a)]) and that by the proof of Lemma 3.29, r_n extends to a map $q_0 \to |Q|$ which sends the completion of the uniform neighborhood of $|P|$ into $|P|$.

extension to the remaining values of t. Then $h_{1/3}$ and $h_{2/3}$ are uniformly continuous as compositions of uniformly continuous maps, and are $\frac{\varepsilon}{2}$-close to h_0 and h_1, respectively. Since each of h_1, $h_{2/3}$ and $h_{1/3}$ is $(\frac{\varepsilon}{2}+\delta)$-close to h_0, we infer that h_t is a uniformly continuous $(\frac{\varepsilon}{2}+\delta)$-homotopy.

It remains to construct φ and ψ. Pick some $x, y \in |P|$ with $d(x,y) < \delta$, and let $A \subset P$ and $B \subset P$ be some finite chains whose convex hulls contain x and y respectively. Arguing as in the proof of Theorem 3.11, we may enlarge P to $\hat{P} = C^*CP$ and consider the unique isomorphisms $a\colon [m] \to \hat{A} \subset \hat{P}$ and $b\colon [n] \to \hat{B} \subset \hat{P}$. Thus $a(1) = \hat{0} = b(1)$ and $a(m) = \hat{1} = b(n)$. Let $(k_1, l_1), \dots, (k_r, l_r)$ and $(k_1', l_1'), \dots, (k_{r'}', l_{r'}')$, and Z and Z', and α_i, β_i be as in the proof of Theorem 3.6. We recall that $\alpha_i = x(s)$ for any $s \in a(i) \setminus a(i-1)$, $1 < i \le m$, and $\beta_i = y(s)$ for any $s \in b(i) \setminus b(i-1)$, $1 < j \le n$.

The basic problem with the original construction of x', y' in the proof of Theorem 3.11 is that they depend on the choice of A, B. But they should not if φ and ψ are to be continuous; indeed, if A, B are taken to be the *smallest* chains whose convex hulls contain x, y respectively, then a pair $(\tilde{x}, \tilde{y})$ arbitrarily close to (x,y) can give rise to a different pair of chains $(\tilde{A}, \tilde{B})$.

Let δ be such that $\delta \le \frac{\varepsilon}{6}$ and $N := \frac{1}{4\delta} \in \mathbb{Z}$. Let $u_i, u_i' \in [m]$ and $v_i, v_i' \in [n]$ be the maximal numbers such that $\alpha_{u_i} \ge 1 - 4i\delta$, $\alpha_{u_i'} \ge 1-(4i+1)\delta$, $\beta_{v_i} \ge 1-(4i+2)\delta$ and $\beta_{v_i'} \ge 1-(4i+3)\delta$. Thus $\alpha_{u_0} = 1$ and $u_0 \ge 2$, whereas $\alpha_{u_N} = \alpha_{u_N - 1} = 0$ and $u_N = m$. It is easy to see[11] that for each $\kappa \in [m]$ there exists a $\lambda \in [n]$ such that $a(\kappa) \le b(\lambda)$ and $\beta_\lambda \ge \alpha_\kappa - \delta$. Similarly, for each $\lambda \in [n]$ there exists a $\kappa \in [m]$ such that $b(\lambda) \le a(\kappa)$ and $\alpha_\kappa \ge \beta_\lambda - \delta$. Hence each $a(u_i') \le b(v_i)$ and each $b(v_i') \le a(u_{i+1})$. Thus we get an 'intermediary' chain C consisting of:

$$\begin{aligned}\hat{0} &\le a(u_1) \le a(u_1+1) \le \dots \le a(u_1')\\ &\le b(v_1) \le b(v_1+1) \le \dots \le b(v_1')\\ &\le a(u_2) \le a(u_2+1) \le \dots \le a(u_2')\\ &\le \quad \dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

(It should be noted that if $u_i \in Z$, then the inequalities $u_i \le u_i' \le \dots \le u_{i+k} \le u_{i+k}'$ may all happen to be equalities for an arbitrarily large k. This is the only way that it can happen, for it is easy to see[12] that if $a(\kappa) \notin B$, then $\alpha_{\kappa+1} \ge \alpha_\kappa - 2\delta$.)

[11] Indeed, let i be the minimal number satisfying $k_i \ge \kappa$, and let $\lambda = l_i$. If $i = 1$ then $\beta_\lambda = \alpha_\kappa = 1$. Else $\kappa > k_{i-1}$, hence $\beta_\lambda = \beta_{l_i} \ge \alpha_{k_{i-1}+1} - d(x,y) \ge \alpha_\kappa - \delta$. (The first inequality was established in the proof of Theorem 3.6.)

[12] Let i be the minimal number satisfying $k_i \ge \kappa$. By the hypothesis $i \ne 1$. Then $\kappa > k_{i-1}$, hence $\alpha_\kappa \le \alpha_{k_{i-1}+1} \le \beta_{l_i} + d(x,y)$. Next let j be the minimal number satisfying $l_j' \ge l_i$. Then $j \ne 1$ due to $i \ne 1$. Hence $l_i > l_{j-1}'$, so $\beta_{l_i} \le \beta_{l_{j-1}'+1} \le \alpha_{k_j'} + d(x,y)$. Thus $\alpha_\kappa \le \alpha_{k_j'} + 2\delta$. Now if $k_j' > k_i$, then $k_j' > \kappa$ due to $k_i \ge \kappa$. If $k_j' \le k_i$, then $a(k_i) \le b(l_i) \le b(l_j') \le a(k_j') \le a(k_i)$, implying $k_j' = k_i$ and $i \in Z$. The latter implies $k_i \ne \kappa$ in view of the hypothesis. Then $k_i > \kappa$, and so $k_j' > \kappa$ once again. Thus we obtain that $\alpha_{k_j'} \le \alpha_{\kappa+1}$.

If we use this chain C to construct x' and y' as in the proof of Theorem 3.11, the result will no longer depend on the choice of A and B. However, the definition of C now involves the maximum function, which is discontinuous; so an arbitrarily small change in the coordinates of x can lead to a significant (even though bounded above by δ) change in the coordinates of x'.

Thus we need a new construction of x' and y' that would compensate for the discontinuity of the maximum function. We set $x'(s) = 1 - 4i\delta$ for all $s \in a(u_i) \setminus a(u'_{i-1})$, and (not entirely symmetrically) $y'(s) = 1-4i\delta$ for all $s \in b(v_i) \setminus b(v'_{i-1})$. We shall define $x'(s)$ and $y'(s)$ for the remaining values of s by distributing the total jump value 4δ (between e.g. $x'(s)$ and $x'(t)$ for $s \in a(u_i) \setminus a(u'_{i-1})$ and $t \in a(u_{i+1}) \setminus a(u'_i)$, provided that such s and t exist) over all the jumps so as to best approximate the (continuous) uniform distribution. Thus the jump value over $a(j) \setminus a(j-1)$ must be proportional to the step length $\alpha_{j-1} - \alpha_j$ for each $j \in [u_i+1, u'_i+1]$. The total horizontal length of the stairs is δ (from $1-4i\delta$ to $1-(4i+1)\delta$, for instance). Therefore we set $x'(s) = (1-4i\delta)-4((1-4i\delta)-\alpha_j)$ for all $s \in a(j) \setminus a(j-1)$, for each $j \in [u_i+1, u'_i]$. Similarly (but not entirely symmetrically) $y'(s) = (1-4i\delta) - 4((1-(4i+2)\delta) - \beta_j)$ for all $s \in b(j) \setminus b(j-1)$, for each $j \in [v_i+1, v'_i]$. We define $\varphi(x, y) = x'$ and $\psi(x, y) = y'$. We also define $\alpha'_j = x'(s)$ for all $s \in a(j) \setminus a(j-1)$, where $2 \leq j \leq m$, and $\beta'_j = y'(s)$ for all $s \in b(j) \setminus b(j-1)$, where $2 \leq j \leq n$ (beware that this notation is not entirely analogous to that in the proof of Theorem 3.11). Then $\alpha'_2 = \alpha'_{u_0} = 1$ and $\alpha'_{m-1} = \alpha'_{u_N} = 0$, where $u_0 \geq 2$ and $u_N = m$, so $x' \in |A|_a$. Due to the non-symmetric definition of y', also $\beta'_2 = \beta_{v_0} = 1$ and $\beta'_{n-1} = \beta'_{v_{N-1}} = 0$, where $v_0 \geq 2$ and $v_{N-1} \leq n$, so $y' \in |B|_b$.

It is easy to check that x' and y' do not depend on the choice of A and B. When $s \in a(u_i) \setminus a(u'_{i-1})$ we have $x(s) \in [\alpha_{u_i}, \alpha_{u'_{i-1}+1}] \subset [1-4i\delta, 1-(4i-3)\delta]$ whereas $x'(s) = 1-4i\delta$. When $s \in a(u'_i) \setminus a(u_i)$ we have $x'(s) - D = 4[x(s) - D]$, where $D = 1-4i\delta$, so $x'(s) - x(s) = [x'(s) - D] - [x(s) - D] = 3[x(s) - D] \in [0, 3\delta]$. In both cases $|x'(s) - x(s)| \leq 3\delta \leq \varepsilon/2$ as desired. Similarly (but not entirely symmetrically) $|y'(s) - y(s)| \leq 3\delta \leq \varepsilon/2$.

It remains to verify that φ and ψ are uniformly continuous, that is, for each $\zeta > 0$ there exists an $\eta > 0$ such that $d(x, \tilde{x}) < \eta$ and $d(y, \tilde{y}) < \eta$ imply $d(x', \tilde{x}') < \zeta$ and $d(y', \tilde{y}') < \zeta$. By the proof of Theorem 3.11 for each $\theta > 0$ there exists an $\eta > 0$ (namely, $\eta = \theta/5$) such that given $x, \tilde{x} \in |P|$ with $d(x, \tilde{x}) \leq \eta$, there exist $x^*, \tilde{x}^* \in |P|$ such that each of the pairs $\{x, x^*\}$, $\{x^*, \tilde{x}^*\}$, $\{\tilde{x}^*, \tilde{x}\}$ has diameter at most θ and lies in the convex hull of some chain of P. Given $y, \tilde{y} \in |P|$ with $d(y, \tilde{y}) \leq \eta$, we similarly get $y^*, \tilde{y}^*$. Therefore it suffices to consider the case where the pairs $\{x, \tilde{x}\}$ and $\{y, \tilde{y}\}$ lie in the convex hulls of some chains A and B, respectively. Since φ and ψ are well-defined, we may assume that $x', \tilde{x}', y', \tilde{y}'$ are all defined using these A and B. In this case, we set $\eta = \min(\zeta/4, \delta/2)$.

Thus suppose that $d(x, \tilde{x}) < \eta$. In other words, $|\alpha_j(s) - \tilde{\alpha}_j(s)| < \eta$ for all $s \in \mathcal{P}$. Fix some j; by symmetry we may assume that $\alpha_j(s) \geq \tilde{\alpha}_j(s)$. Since $\eta < \delta$, one of the following four cases has to occur for some i:

(i) $1 - 4i\delta > \alpha_j(s) \geq \tilde{\alpha}_j(s) \geq 1 - (4i+1)\delta$;
(ii) $1 - (4i-3)\delta > \alpha_j(s) \geq \tilde{\alpha}_j(s) \geq 1 - 4i\delta$;
(iii) $1 - (4i-3)\delta > \alpha_j(s) \geq 1 - 4i\delta > \tilde{\alpha}_j(s) \geq 1 - (4i+1)\delta$;
(iv) $1 - 4i\delta > \alpha_j(s) \geq 1 - (4i+1)\delta > \tilde{\alpha}_j(s) \geq 1 - (4i+4)\delta$.

In the case (i), we have $\alpha'_j(s) - \tilde{\alpha}'_j(s) = 4(\alpha_j(s) - \tilde{\alpha}_j(s))$. In the case (ii), both $\alpha'_j(s)$ and $\tilde{\alpha}'_j(s)$ equal $1 - 4i\delta$. In the case (iii), $\alpha'_j(s) = 1 - 4i\delta$, whereas $\tilde{\alpha}'_j(s) \in [1 - 4i\delta, 1 - 4i\delta - 4\eta]$. Similarly, in the case (iv), $\tilde{\alpha}'_j(s) = 1 - (4i+4)\delta$, whereas $\alpha'_j(s) \in [1 - (4i+4)\delta + 4\eta, 1 - (4i+4)\delta]$.

In all cases, $\alpha'_j(s) - \tilde{\alpha}'_j(s) \in [0, 4\eta]$. This shows that $d(x', \tilde{x}') \leq 4\eta$. Thus φ is uniformly continuous; the uniform continuity of ψ is verified similarly. ☐

Example 4.2. Given a preposet $P = (\mathcal{P}, \leq)$, we define the "co-deleted prejoin" $P \boxplus P^*$ to be the preposet $(\mathcal{P} \sqcup \mathcal{P}^*, \preceq)$, where $\mathcal{P}^* = \{p^* \mid p \in P\}$ is a just fancy notation for a copy of $\mathcal{P}$, and the relation is defined by

- $p \preceq q$ iff $p \leq q$;
- $p^* \preceq q^*$ iff $p \geq q$;
- $p^* \preceq q$ never holds;
- $p \preceq q^*$ iff either $p \leq q$ or $p \geq q$

for all $p, q \in \mathcal{P}$. Note that $P \boxplus P^*$ need not be a poset even if P is.

Let us define $j\colon P \to (P \boxplus P^*)^\#$ by $j(p) = [p, p^*]$. Obviously, j is a conical embedding, i.e. $p \leq q$ if and only if $j(p) \preceq j(q)$. We claim that $|j|$ is a homotopy equivalence. Indeed, $|j|$ is homotopic to the composition $|P| \xrightarrow{|i|} |P| \xrightarrow{h} |P^\#|$, where h is the uniform homeomorphism. On the other hand, $i^\flat$ is split by the simplicial map $r\colon (P \boxplus P^*)^\flat \to P^\flat$, defined on vertices by $p, p^* \mapsto p$. Given a chain $\sigma = (p_1 < \cdots < p_n) \in P^\flat$ we have $r^{-1}(\ulcorner\sigma\urcorner) = \ulcorner\sigma \boxplus \bar{\sigma}\urcorner$, where $\sigma \boxplus \bar{\sigma}$ denotes the chain $(p_1 < \cdots < p_n < p_n^* < \cdots < p_1^*) \in (P \boxplus P^*)^\flat$. Since r is simplicial, it follows that $|r|\colon |(P \boxplus P^*)^\flat| \to |P^\flat|$ has contractible point-inverses, and therefore (or by Quillen's fiber lemma) is a homotopy equivalence. If k is a homotopy inverse to $|r|$, then $k = k|ri^\flat| \simeq |i^\flat|$, so $|i^\flat|$ is also a homotopy equivalence.

Let K_0 be the preposet of the four sets 0, $\{0,1\}$, $\{0,2\}$ and $\{\{0,1\},\{0,2\}\}$ ordered by $\in$. Thus $|K_0|$ is homeomorphic to S^1. Let $K_{n+1} = K_n \boxplus K_n^*$. Finally let $K = K_0 \sqcup K_1 \sqcup \dots$.

We claim that $|K|$ is not uniformly locally contractible (and in particular is not a uniform ANR). Indeed, by the above we have an embedding $f_n\colon K_0 \to K_n^{\#n}$ such that $|f_n|$ is a homotopy equivalence. In order to use the d_∞ metric, which has been shown to work only for posets, we consider the transitive closure. Let f'_n be the composition $K_0 \xrightarrow{f_n} K_n^{\#n} \subset \langle K_n^{\#n} \rangle \subset \langle K_n \rangle^{\#n}$. Since f'_n is conical, the image of $|f'_n|$ has diameter 1 in the d_∞ metric on $|\langle K_n \rangle^{\#n}|$, hence by Corollary 3.12, the image of the composition $|K_0| \xrightarrow{|f'_n|} |\langle K_n \rangle^{\#n}| \xrightarrow{h_n} |\langle K_n \rangle|$ has diameter 2^{-n} in the d_∞ metric on $|\langle K_n \rangle|$. Since $\mathrm{id}\colon (|\langle K_n \rangle|, d_\infty) \to (|\langle K_n \rangle|, d)$ is 1-Lipschitz, the image of the composition $|K_0| \xrightarrow{|f_n|} |K_n^{\#n}| \xrightarrow{h_n} |K_n|$ has diameter at most 2^{-n} with respect to the usual metric on $|K_n|$. However this composition is not null-homotopic since it is a homotopy equivalence.

We note that the preposet K in Example 4.2 satisfies the following property (P): For each $\varepsilon > 0$ there exists an essential map $S^1 \to |K|$ with image of diameter $< \varepsilon$. On the other hand, since $|K^\flat|$ is a uniform ANR, $|K|$ is a non-uniform ANR, and in particular satisfies the non-uniform homotopy extension property. It follows that every metrizable uniform space that is uniformly homotopy equivalent to $|K|$ satisfies (P) as well. In particular, we get the following

Theorem 4.3. *There exists a countable preposet whose geometric realization is not uniformly homotopy equivalent to a uniform ANR, nor even to a uniformly locally contractible metrizable uniform space.*

4.4. Thickened mapping cylinder. Let $f\colon P \to Q$ be a conical map between countable posets. Let $j_P\colon P \hookrightarrow 2^{\mathbb{N}}$ be the usual embedding $p \mapsto \ulcorner p \urcorner$, where the underlying set of P is identified with a subset of $\mathbb{N}$. Let F be the composition $P \times 2^{\mathbb{N}} \to P \xrightarrow{f \times j_P} Q \times 2^{\mathbb{N}}$ of the projection and the joint map. Finally let $TMC(f)$ be the transitive closure $\langle MC(F)\rangle$. Note that $TMC(f)$ contains canonical copies of $P = P \times \{\emptyset\}$ and $Q = Q \times \{\emptyset\}$.

Theorem 4.5. *Let $f\colon P \to Q$ be a conical map between countable posets. Then $|TMC(f)|$ is uniformly homotopy equivalent to $|MC(f)|$ relative to $|P \sqcup Q|$; and if P and Q are CQLs, then so is $TMC(f)$.*

This follows easily from

Lemma 4.6. *Let $f\colon P \to Q$ be a conical embedding between countable posets. Then*
(a) if P and Q are CQLs, then so is $\langle MC(f)\rangle$;
(b) $|\langle MC(f)\rangle|$ uniformly deformation retracts onto $|MC(f)|$.

Proof. (a). Let $S \subset \langle MC(f)\rangle$. Then $S = S_P \sqcup S_Q$, where $S_P \subset P$ and $S_Q \subset Q$. If S has a lower bound which belongs to P then it has the greatest lower bound (since P is a CQL). So we may assume that every lower bound of S belongs to Q. Then it is easy to see that

(∗) every lower bound of S is also a lower bound of $S_Q \cup f(S_P)$.

If S has a lower bound, then by (∗) so does $S_Q \cup f(S_P)$. Hence $S_Q \cup f(S_P)$ has the greatest lower bound q (since Q is a CQL). By another application of (∗), q is the greatest lower bound of S. ☐

(b). Since f is an embedding, we may identify $\langle MC(f)\rangle$ with a subposet of $Q \times [2]$. Then $h(\langle MC(f)\rangle)$ gets identified with a subposet of $h(Q \times [2]) \simeq h(Q \times [2]^*)$. It is easy to see that the conical map $r_{Q\times[2]^*}\colon h(Q \times [2]^*) \to Q \times [2]^*$ (see 3.25) sends $h(\langle MC(f)\rangle)$ onto $MC^*(f)$. Using this, similarly to the proof of Theorem 3.27 one constructs a uniform retraction of $|h(\langle MC(f)\rangle)|$ onto $|h(MC^*(f))|$, and by using Lemma 3.26 or similarly to the proof of Lemma 3.29 one constructs a uniform homotopy from this retraction to the identity. ☐

4.B. Combinatorics of covers and approximation of maps

In this subsection we shall need basic operations and relations on covers as introduced in [32; 2.5], as well as the following additional notation.

4.7. Nerve. We recall that the *nerve* of a cover $C \subset 2^S$ of a set S is the simplicial poset $N(C) \subset 2^C$, where a subset $B \subset C$ is a simplex of $N(C)$ iff $\bigcap B$ (the intersection of all elements of B) is nonempty. The notion of nerve was introduced by Alexandroff [2]. We note that for a cover C,

- C countable and point-finite iff $N(C)$ is a simplicial complex;
- C is countable and Noetherian iff $N(C)$ is a Noetherian simplicial complex;
- C is countable and star-finite iff $N(C)$ is a locally finite simplicial complex.

4.8. Simplex determined by subset. Given a nonempty $T \subset S$ that is contained in at least one element of C, let $\Delta_C(T)$ denote the element $\{U \in C \mid T \subset U\}$ of $N(C)$. Given an $s \in S$, we write $\Delta_C(s) = \Delta_C(\{s\})$. Note that every element of $N(C)$ belongs to some simplex of $N(C)$ of the form $_\ulcorner\Delta_C(s)_\urcorner$ for some $s \in S$.

4.9. Cover by open stars. If P is a poset, by the *open star* $\mathring{\mathrm{st}}(p, P)$ of a $p \in P$ we mean $|\operatorname{st}(p, P)| \setminus |\operatorname{st}(p, P) \setminus {}^\llcorner p^\lrcorner|$. (We recall that $\operatorname{st}(p, P) = \lceil {}^\llcorner p^\lrcorner \rceil$.)

If P is atomic with atom set Λ, we have the cover $\{\mathring{\mathrm{st}}(\lambda, P) \mid \lambda \in \Lambda\}$ of $|P|^\bullet$ by *open stars of vertices.* It is easy to see that the open star of a vertex $v \in \Lambda$ in $|P|^\bullet$ is precisely the set of points of $|P|^\bullet \subset [0,1]^\Lambda$ whose vth coordinate is nonzero. On the other hand, the set of points of $|P^\bullet|$ whose vth coordinate equals 1 is precisely the dual cone $|{}^\llcorner v^\lrcorner|$ of v. These dual cones of vertices form a cover of $|P|^\bullet$, and it follows that the cover of $|P|^\bullet$ by the open stars of vertices is uniform with Lebesgue number $1 - \varepsilon$ for each $\varepsilon > 0$.

Lemma 4.10. *Let X be a metrizable uniform space and C a countable point-finite uniform open cover of X. Then there exists a uniformly continuous map $\varphi_C \colon X \to |N(C)|$ that sends each $x \in X$ into the interior of $|_\ulcorner\Delta_C(x)_\urcorner|$.*

Furthermore, given a uniform cover D of X, a cover C' of X and a bijection $C \to C'$, denoted $U \mapsto U'$, such that $\operatorname{st}(U', D) \subset U$ for each $U \in C$, then we can choose φ_C so that each $\varphi_C(U')$ lies in the dual cone $|{}^\llcorner U^\lrcorner|$.

Note that the first assertion implies that each $U \in C$ is the preimage of the open star of the vertex $\{U\}$ of $N(C)$.

Proof. Note that there always exist C' and D as specified. For instance, take D to be the cover by all open ε-balls of radius ε, and take each $X \setminus U'$ to be the closed ε-neighborhood of $X \setminus U$, where 3ε is a Lebesgue number of C. Then every subset of X of diameter ε has its ε-neighborhood contained in some $U \in C$; and therefore is itself contained in the U'.

Let λ be a Lebesgue number of D. Then each $d(U',\, X \setminus U) \geq \lambda$. For each $U \in C$ define $f_U \colon X \to [0,1]$ by $f_U(x) = \min(\lambda^{-1} d(x,\, X \setminus U),\, 1)$. Let us consider $\varphi = \prod f_U \colon X \to l_\infty^c$.

Then $\{U \in C \mid f_U(x) > 0\} = \Delta_C(x)$, and since C' is a cover, $\{U \in C \mid f_U(x) = 1\} \neq \emptyset$. Hence $\varphi(X) \subset |N(C)|$. Next, $f_U^{-1}(0) = X \setminus U$, which implies the assertion on C, and $f_U(U') = \{1\}$, which implies the assertion on C'. Finally, $|f_U(x) - f_U(y)| \leq \lambda^{-1} d(x, y)$ for each $U \in C$, so φ is uniformly continuous. □

4.11. Intersection poset and Venn diagram. Given a cover C of a set S, the *intersection poset* $IP(C)$ is the subposet of 2^C consisting of all nonempty $B \subset C$ such that $\bigcap B$ is not contained in any element of $C \setminus B$. The terminology "intersection poset" derives from Lemma 4.12(a) below, which however characterizes $IP(C)$ only up to isomorphism, and not as a subposet of Δ^C.

The *Venn diagram* $VD(C)$ is the subposet of 2^C consisting of all $B \subset C$ such that $\bigcap B$ is not contained in $\bigcup(C \setminus B)$. This is a formalization of the intuitive notion of a "Venn diagram", also known as "Euler diagram", from courses of "abstract mathematics", for it can be argued that $VD(C)$ contains all the combinatorial information on containment of points of X in elements of C (see Lemma 4.12(b) below) — and nothing else (see Lemma 4.14(b) below).

Clearly, $VD(C) \subset IP(C) \subset N(C)$ and $\lceil VD(C) \rceil = \lceil IP(C) \rceil = N(C)$. Note that if C is countable and point-finite, then $IP(C)$ and $VD(C)$ are cone complexes. Clearly $\Delta_C(x)$ belongs to $VD(C)$ for every $x \in S$; in contrast, $\Delta_C(T)$ belongs to $IP(C)$ for every $T \subset S$ that is contained in at least one element of C.

Lemma 4.12. *Let C be a cover of a set S.*

(a) $IP(C)$ is isomorphic to the poset consisting of arbitrary nonempty intersections of elements of C, ordered by reverse inclusion. In particular, $IP(C)$ is a CQL.

(b) $VD(C)$ is isomorphic to the poset consisting of those intersections of elements of C that are of the form $\bigcap \Delta_C(s)$ for some $s \in S$, ordered by reverse inclusion.

Proof. It will be convenient to work in a slightly greater generality. The definitions of $N(C)$, $IP(C)$ and $VD(C)$ generalize straightforwardly for any collection $\varphi\colon C \to 2^S$ of subsets of S (possibly with repeated subsets $\varphi(U) = \varphi(U')$ and with $\bigcup \varphi(C)$ not necessarily covering the whole of S). It is easy to see that

- $N(\varphi) = \Phi^{-1}(\Delta^S)$, where $\Phi\colon \Delta^C \to 2^S$ is defined by $D \mapsto \bigcap \varphi(D)$;
- $IP(\varphi) = \Delta_\varphi(\Delta^S) \setminus \{\emptyset\}$, where $\Delta_\varphi\colon \Delta^S \to 2^C$ is defined by $T \mapsto \varphi^{-1}(\lfloor T \rfloor)$;
- $VD(\varphi) = \Delta_\varphi(A(\Delta^S)) \setminus \{\emptyset\}$, where the subset $A(\Delta^S) = \{\{s\} \mid s \in S\}$ of Δ^S should not be confused with the element S of Δ^S.

We note that the maps Φ and Δ_φ are dual-conical, and restrict to mutually inverse bijections between $IP(\varphi)$ and $\Phi(\Delta^C) \setminus \{\emptyset\}$. In particular, $IP(\varphi)$ is isomorphic to $(\Phi(\Delta^C) \setminus \{\emptyset\})^*$, which implies the first assertion of (a). Similarly, $VD(\varphi)$ is isomorphic to $\Phi(\Delta_\varphi(A(\Delta^S)) \setminus \{\emptyset\})^*$, which yields (b). □

Remark 4.13. It follows from the proof that $IP(\varphi) \simeq IP(\Phi)$ and $VD(\varphi) \simeq VD(\Phi)$.

Lemma 4.14. *Let P be a poset embedded in some Δ^Λ. Let C be the cover $\{\lfloor \lambda \rfloor \cap P \mid \lambda \in \Lambda\}$ of the underlying set of P by the dual cones of vertices.*

(a) $VD(C) \simeq P$.

(b) $IP(C) = VD(C)$ *if and only if for every* $R \subset A(\Delta^\Lambda)$*, the set of all upper bounds of* R *in* P *either is empty or is the dual cone in* P *of a single element of* P.

Proof. (a). By Lemma 4.12(b), $VD(C)$ is isomorphic to the poset consisting of the dual cones ${}^{\llcorner}p^{\lrcorner\Delta^\Lambda} \cap P = {}^{\llcorner}p^{\lrcorner P}$ of all elements $p \in P$, ordered by reverse inclusion. The latter is obviously isomorphic to P. ☐

(b). By Lemma 4.12(a), $IP(C)$ is isomorphic to the poset of all nonempty intersections of the form $\bigcap \sigma$, where $\sigma \subset C$, ordered by reverse inclusion. We have $\bigcap \sigma = P \cap \bigcap_{\lambda \in R_\sigma} {}^{\llcorner}\lambda^{\lrcorner}$, where $R_\sigma = \{\{\lambda\} \mid \lambda \in \Lambda,\ ({}^{\llcorner}\lambda^{\lrcorner} \cap P) \in \sigma\}$. (The subset R_σ of Δ^Λ should not be confused with the element $\bigcup R_\sigma$ of Δ^Λ.) Thus $IP(C)$ is in bijection with the set of all nonempty intersections of the form $P \cap {}^{\llcorner}R^{\lrcorner}$, where $R \subset \Lambda$. By the proof of Lemma 4.12, the same bijection sends $VD(C)$ onto the set of the dual cones ${}^{\llcorner}p^{\lrcorner P}$ of all elements $p \in P$, and the assertion follows. ☐

Corollary 4.15. *Let* P *be an atomic poset, and let* C *be the cover of* P *by the dual cones of its atoms. Then* $IP(C) = VD(C)$ *if and only if* P *is a CQL.*

It is not hard to see that the same assertion is true of the cover of $|P|$ by the geometric realizations of the dual cones of the vertices of P, and of the cover of $|P|$ by the open stars of these vertices.

Proof. Let us embed P in $\Delta^{A(P)}$ as in Lemma 2.19. Lemma 4.14(b) then says that $IP(C) = VD(C)$ if and only if every $R \subset A(P)$ that has an upper bound in P has a least upper bound in P. This proves the "if" assertion, and the "only if" assertion now follows from Lemma 2.20. Alternatively, the "only if" assertion follows from Lemma 4.14(a) and the second assertion of Lemma 4.12(a). ☐

4.16. Canonical bonding map. Let C and D be covers of a set S, and suppose that C star-refines D. We define a map $\varphi^C_D\colon N(C) \to N(D)^\#$ by sending each $\sigma \in N(C)$ into $[\Delta_D(\bigcup \sigma), \Delta_D(\bigcap \sigma)] \in N(D)^\#$. Here $\Delta_D(\bigcup \sigma)$ is non-empty by the star-refinement hypothesis; every vertex of $\Delta_D(\bigcup \sigma)$ is obviously a vertex of $\Delta_D(\bigcap \sigma)$; and $\Delta_D(\bigcap \sigma) \in N(D)$ since $\bigcap \sigma \neq \emptyset$.

Given a $\tau \le \sigma$, we have $\bigcup \tau \subset \bigcup \sigma$, whence $\Delta_D(\bigcup \tau) \ge \Delta_D(\bigcup \sigma)$; and $\bigcap \tau \supset \bigcap \sigma$, whence $\Delta_D(\bigcap \tau) \le \Delta_D(\bigcap \sigma)$. Thus φ^C_D is conical.

Finally, recall that $IP(D)$ contains every element of $N(D)$ of the form $\Delta_D(T)$, where $T \subset S$. Hence $[\Delta_D(\bigcup \sigma), \Delta_D(\bigcap \sigma)]$ belongs to the isomorphic copy of $IP(D)^\#$ in $N(D)^\#$. Thus we may write $\varphi^C_D\colon N(C) \to IP(D)^\#$.

Remark 4.17. Let us discuss some motivation/geometry behind the definition of φ^C_D.

If $V \in \sigma$ then $\{V\} \le \sigma$, so by the above $\Delta_D(V)$ belongs to $[\Delta_D(\bigcup \sigma), \Delta_D(\bigcap \sigma)]$. Hence the map defined on vertices by $\{V\} \mapsto [\Delta_D(V), \Delta_D(V)]$ extends uniquely to a conical map $N(C) \to N(D)^\#$ sending each simplex ${}_{\ulcorner}\sigma_{\urcorner}$ into the cube ${}_{\ulcorner}[\Delta_D(\bigcup \sigma), \Delta_D(\bigcap \sigma)]_{\urcorner}$.

If C strongly star-refines D, then $\Delta_D(\mathrm{st}(V,C)) \in N(C)$, and it is not hard to see that φ^C_D sends each $\mathrm{st}(\{V\}, N(C))$ into the canonical subdivision of the dual cone of $\Delta_D(\mathrm{st}(V,C))$ in the subcomplex $\bigcup\{\ulcorner\Delta_D(x)\urcorner \mid x \in V\}$ of $N(D)$.

Proposition 4.18. *Let C and D be covers of a set S. If C strongly star-refines D, there exists a simplicial map $N(C) \xrightarrow{g} N(D)$ such that $N(C)^\# \xrightarrow{g^\#} N(D)^\#$ is homotopic to the composition $N(C)^\# \xrightarrow{(\varphi^C_D)^\#} N(D)^{\#\#} \xrightarrow{r^\#} N(D)^\#$, where r stands for either $r_{N(D)}$ or $r^*_{N(D)}$, by a conical homotopy $N(C)^\# \times I \to N(D)^\#$ sending each block of the form $\ulcorner\sigma\urcorner^\# \times I$ into the star of $[\Delta_D(\bigcup\sigma), \Delta_D(\bigcap\sigma)]$ in $N(D)^\#$.*

Proof. Given a vertex $\{V\}$ of $N(C)$, the hypothesis furnishes a vertex $\{V'\}$ of $N(D)$ such that $\mathrm{st}(V,C) \subset V'$. For each $\sigma \in N(C)$ and each $V \in \sigma$ we have $\bigcup\sigma \subset V'$, and consequently $\{V'\} \in \ulcorner\Delta_D(\bigcup\sigma)\urcorner$. Hence $V \mapsto V'$ extends to a simplicial map $g\colon N(C) \to N(D)$ sending each σ onto some $\sigma' \subset \Delta_D(\bigcup\sigma) \subset \Delta_D(\bigcap\sigma)$. Then $g^\#$ sends each $[\sigma,\tau]$ onto $[\sigma',\tau']$. The composition $N(C)^\# \xrightarrow{(\varphi^C_D)^\#} N(D)^{\#\#} \xrightarrow{r^\#} N(D)^\#$ sends it onto $[\Delta_D(\bigcup\sigma), \Delta_D(\bigcup\tau)]$ when $r = r_{N(D)}$ and onto $[\Delta_D(\bigcap\sigma), \Delta_D(\bigcap\tau)]$ when $r = r^*_{N(D)}$. The required homotopy is defined by sending each $([\sigma,\tau], \{0,1\})$ onto $[\sigma', \Delta_D(\bigcup\sigma)]$ in the first case and onto $[\sigma', \Delta_D(\bigcap\sigma)]$ in the second case. ☐

Theorem 4.19. *Let Q be a countable CQL. Then $|Q|$ satisfies the Hahn property.*

Proof. Given an $\varepsilon > 0$, let C' be the cover of $|Q^{\#n}|$ by the open stars of vertices, where $2^{-n+1} < \varepsilon$ and $n \geq 1$ so that $Q^{\#n}$ is atomic. Since Q is a CQL, so is $Q^{\#n}$, and therefore $IP(C') = VD(C') \simeq Q^{\#n}$. Let $h_n\colon |Q^{\#n}| \to |Q|$ be the uniform homeomorphism given by Corollary 3.12, and let $C = h_n(C')$. Let δ be the Lebesgue number of C with respect to the d_∞ metric on $|Q|$.

Given a metric space X and a (γ,δ)-continuous map $f\colon X \to |Q|$ for some $\gamma > 0$, let E be the cover of X by $\frac{\gamma}{4}$-balls. Then E star-refines $D := f^{-1}(C)$. Let Φ denote the composition

$$X \xrightarrow{\varphi_E} |N(E)| \xrightarrow{|\varphi^E_D|} |IP(D)^\#| \subset |IP(C)^\#| \xrightarrow{h} |IP(C)| \cong |Q^{\#n}| \xrightarrow{h_n} |Q|,$$

where φ_E is the uniformly continuous map given by Lemma 4.10 and φ^E_D is the canonical bonding map. Given an $x \in X$, by Lemma 4.10 $\varphi_E(x) \in |\ulcorner\Delta_E(x)\urcorner|$. By the definition of φ^E_D we have $\varphi^E_D(\Delta_E(x)) = [\Delta_D(\bigcup\Delta_E(x)), \Delta_D(\bigcap\Delta_E(x))] \subset \ulcorner\Delta_D(\bigcap\Delta_E(x))\urcorner \subset \ulcorner\Delta_D(x)\urcorner$. The latter is identified with $\ulcorner\Delta_C(f(x))\urcorner$, where $\Delta_C(f(x))$ is an element of $IP(C) \simeq Q^{\#n}$, and it follows that $\Phi(x) \in h_n(|\ulcorner\Delta_C(f(x))\urcorner|)$. Now $|\ulcorner\Delta_C(f(x))\urcorner|$ has diameter ≤ 2 with respect to the d_∞ metric on $|Q^{\#n}|$, so $h_n(|\ulcorner\Delta_C(f(x))\urcorner|)$ has diameter $\leq 2^{-n+1}$ with respect to the d_∞ metric on $|Q|$. Since this set contains both $\Phi(x)$ and $f(x)$, we infer that Φ is ε-close to f with respect to the d_∞ metric on $|Q|$. ☐

We define a *uniform polyhedron* to be the geometric realization of a countable CQL. Theorem 4.1, Theorem 4.19, and [32; Theorem 4.30] have the following

Corollary 4.20. *Uniform polyhedra are uniform ANRs.*

Lemma 4.21. *For each $\varepsilon > 0$ there exist an n and a $\delta > 0$ such that for each $\gamma > 0$ there exists an M such that for each $m \geq M$ the following holds. Let P be a preposet and Q a CQL, and $f\colon |P| \to |Q|$ be a (γ, δ)-continuous map. Then there exists a conical map $g\colon P^{\#m} \to Q^{\#n}$ such that the composition $|P| \xrightarrow{h_m^{-1}} |P^{\#m}| \xrightarrow{g} |Q^{\#n}| \xrightarrow{h_n} |Q|$ is ε-close to f.*

Proof. Let $2^{-n+1} < \varepsilon$, $n \geq 1$, let $\delta < 2^{-n-1}$, and let $2^{-M+1} < \gamma/4$, $M \geq 1$.

Let C' be the cover of $|Q^{\#n}|$ by the open stars of vertices (using that $Q^{\#n}$ is atomic due to $n \geq 1$). Since C' has Lebesgue number $\frac{1}{2}$ with respect to the usual metric d on $|Q^{\#n}|$, is also has Lebesgue number $\frac{1}{2}$ with respect to the d_∞ metric, due to $d(x, y) \leq d_\infty(x, y)$. Then $C := h_n(C')$ has Lebesgue number 2^{-n-1} (and therefore also Lebesgue number δ) with respect to the d_∞ metric on $|Q|$.

Let E' be the cover of $|P^{\#m}|$ by the open stars of vertices (using that $P^{\#m}$ is atomic due to $m \geq M \geq 1$). Then E' refines the cover of $|P^{\#m}|$ by balls of radius 2 about every vertex of $P^{\#m}$ with respect to the d_∞ metric on $|P^{\#m}|$. Hence $E := h_m(E')$ refines the cover of $|P|$ by balls of radius 2^{-m+1} (and therefore also that by balls of radius $\gamma/4$) about all points of $|P|$ with respect to the d_∞ metric on $|P|$. We note that the composition $\varphi\colon |P| \xrightarrow{h_m^{-1}} |P^{\#m}| \cong |VD(E)| \subset |N(E)|$ satisfies $\varphi(x) \in |{\ulcorner}\Delta_E(x){\urcorner}|$.

The assertion now follows by the proof of Theorem 4.19. □

From the preceding lemma we infer

Theorem 4.22. *For each $\varepsilon > 0$ there exists an n such that the following holds. Let $f\colon |P| \to |Q|$ be a uniformly continuous map, where P is a preposet and Q is a CQL. Then there exists an M such that for each $m \geq M$ there exists a conical map $g\colon P^{\#m} \to Q^{\#n}$ such that the composition $|P| \xrightarrow{h_m^{-1}} |P^{\#m}| \xrightarrow{g} |Q^{\#n}| \xrightarrow{h_n} |Q|$ is ε-close to f.*

Example 4.23. Let $P_1 = [2]$ and let $P_{i+1} = P_i + [2]$. Finally let $P = \bigsqcup_{n\in\mathbb{N}} P_{2^n+1}$. We claim that $|P|$ does not satisfy the Hahn property (and in particular is not a uniform ANR).

Indeed, let $Q_n = (P_{2^n+1})^{\#n}$, and let C_n be the cover of $|Q_n|$ by the stars of atoms of Q_n. Then $|VD(C_n)| \cong |Q_n| \cong |P_{2^n+1}|$ is a 2^n-sphere; but we shall now show that $|N(C_n)|$ is contractible.

If K is a poset, then $K^{\#n}$ is isomorphic to the poset consisting of non-decreasing sequences $a = (a_1 \leq \dots \leq a_{2^n})$ of elements of K, where $a \geq b$ iff $a_i \leq b_i$ for all odd i and $b_i \leq a_i$ for all even i. Such a sequence a represents an atom of $K^{\#n}$ iff $a_i = a_{i+1}$ for all odd i; and a coatom of $K^{\#n}$ iff $a_i = a_{i+1}$ for all even $i < 2^n$, a_1 is an atom of K and a_{2^n} is a coatom of K. Thus the atoms of $K^{\#n}$ can be identified with non-decreasing sequences $a = (a_2 \leq a_4 \leq \dots \leq a_{2^n-2} \leq a_{2^n})$ of elements of K, and the coatoms of $K^{\#n}$ with non-decreasing sequences $s = (s_1 \leq s_3 \leq \dots \leq s_{2^n-1} \leq s_{2^n+1})$ of elements of K, where s_1 is an atom and s_{2^n+1} a coatom of K; in this notation, $a \leq s$ iff $s_1 \leq a_2 \leq s_3 \leq a_4 \leq \dots \leq a_{2^n} \leq s_{2^n+1}$. If C is the cover of $|K^{\#n}|$ by the open stars of vertices of $K^{\#n}$, then the vertices of $N(C)$ correspond to the atoms of $K^{\#n}$, and a set of vertices of $N(C)$ determines a simplex of $N(C)$ iff the corresponding atoms of $K^{\#n}$ all belong to the cone of some coatom of $K^{\#n}$.

Consider the projection $\pi\colon P_i = [2]+\dots+[2] \to [1]+\dots+[1] \simeq [i]$. By the above, $N(C_n) \subset 2^A$, where A is the set of all non-decreasing sequences $a = (a_2 \le a_4 \le \dots \le a_{2^n})$ of elements of P_{2^n+1}, and $N(C_n)$ consists of all $S \subset A$ such that there exists a non-decreasing sequence $s = (s_1 \le s_3 \le \dots \le s_{2^n+1})$ of elements of P_{2^n+1}, where $\pi(s_1) = 1$ and $\pi(s_{2^n+1}) = 2^n + 1$, and $s_1 \le a_2 \le s_3 \le a_4 \le \dots \le a_{2^n} \le s_{2^n+1}$.

Let $L_k = \{a \in A \mid \pi(a_{2i}) \le 2i \text{ for all } i \le k\}$ and $R_k = \{a \in A \mid \pi(a_{2i}) \ge 2i \text{ for all } i \ge 2^{n-1}+1-k\}$. Thus $A = L_0 \supset L_1 \supset \dots \supset L_{2^{n-1}}$, and $A = R_0 \supset R_1 \supset \dots \supset R_{2^{n-1}}$. Note that $L_{2^{n-1}} \cap R_{2^{n-1}} = \{a \in A \mid \pi(a_{2i}) = 2i \text{ for all } i\}$, which lies in a single simplex, as witnessed by any sequence s with $\pi(s) = (1 \le 3 \le \dots \le 2^n+1)$. For $i = 0,\dots,2^{n-1}$ let N_i be the full subcomplex of $N(C_n)$ spanned by L_i, and for $i = 2^{n-1}+1,\dots,2^n$ let N_i be the full subcomplex of $N(C_n)$ spanned by $L_{2^{n-1}} \cap R_{i-2^{n-1}}$. Thus $N_0 = N(C_n)$; on the other hand, since N_{2^n} is a full simplex, $|N_{2^n}|$ is contractible. We shall now construct a deformation retraction of $|N_i|$ onto $|N_{i+1}|$ for each $i = 0,\dots,2^n-1$.

We first define a retraction $r_k\colon L_{k-1} \to L_k$ by $r_k(a) = a$ if $a \in L_k$, and else by $r_k(a) = b$, where $b_{2i} = a_{2i}$ for $i \ne k$ and $\pi(b_{2k}) = 2k$. (This leaves two possibilities for b_{2k}, among which we choose arbitrarily.) Let $S \in N_{k-1}$ be witnessed by a sequence $s = (s_1 \le s_3 \le \dots \le s_{2^n+1})$ of elements of P_{2^n+1}, where $\pi(s_1) = 1$ and $\pi(s_{2^n+1}) = 2^n+1$. If $\pi(s_{2k+1}) \le 2k$, then all elements of S belongs to L_k, and so $r_k|_S$ is the identity. Else $\pi(s_{2k+1}) \ge 2k+1$, and since all elements of S belong to L_{k-1}, we may assume that $\pi(s_{2k-1}) \le 2k-1$ by modifying s if necessary. Then $r_k(S)$ is a simplex of N_k, and furthermore $S \cup r_k(S)$ is a simplex of N_{k-1}, as witnessed by the same sequence s. Thus r_k extends to a simplicial retraction $R_k\colon N_{k-1} \to N_k$, and furthermore we get a simplicial map $H_k\colon N_{k-1} * N_{k-1} \to N_{k-1}$ that restricts to the identity on the first factor and to R_k on the second factor. It follows that $|R_k|$ is a deformation retraction.

We next similarly define a retraction $r'_k\colon R_{k-1} \to R_k$ by $r'_k(a) = a$ if $a \in R_k$, and else by $r'_k(a) = b$, where $b_{2i} = a_{2i}$ for $i \ne 2^{n-1}+1-k$ and $\pi(b_{2^n+2-2k}) = 2^n+2-2k$. (This leaves two possibilities for b_{2^n+2-2k}, among which we choose arbitrarily.) Then $r'_k(L_{2^{n-1}}) \subset L_{2^{n-1}}$, and then the preceding argument goes through. This completes the proof that $|N(C_n)|$ deformation retracts onto a contractible subspace, and therefore is itself contractible.

Suppose that $|P|$ satisfies the Hahn property. (The following somewhat technical argument can be somewhat simplified by using Lemma 4.31 below, modulo the proof of that lemma.) Since $|P|$ is uniformly locally contractible, there exits an $\varepsilon > 0$ such that every two ε-close maps into $|P|$ are homotopic with respect to the usual metric on $|P|$. Let D_n be the cover of $|P^{\#n}|$ by the open stars of vertices of $P^{\#n}$. Since $P \simeq VD(D_0)$ is a subposet $N(D_0)$, we have an isometric embedding $|P| \subset |N(D_0)|$ with respect to the usual metrics. We recall that the d_∞ metric on $N(D_0)$ is uniformly equivalent to the usual metric d. Let d' denote the d_∞ metric on $|N(D_0)|$ restricted over $|P|$. Let $\delta > 0$ be such that δ-close points in $|P|$ with respect to d' are ε-close with respect to

d. By our assumption there exists a $\gamma > 0$ such that for every $\beta > 0$, every (β, γ)-continuous map into $(|P|, d')$ is δ-close with respect to d' to a continuous map. Now the $\frac{\gamma}{3}$-neighborhood U of $|P|$ in $|N(D_0)|$ with respect to the d_∞ metric on $|N(D_0)|$ admits a discontinuous, $(\gamma/3, \gamma)$-continuous retraction onto $(|P|, d')$. Thus we obtain a continuous map $f \colon U \to (|P|, d')$ such that $f|_{|P|}$ is δ-close to the identity. Then by the above $f|_{|P|}$ is homotopic to the identity.

Let n be such that $2^{-n} \le \gamma/3$. Let $R_n = \ulcorner VD(D_0)^{\#n} \urcorner_{N(D_0)^{\#n}}$. Then $|R_n|$ lies in the 1-neighborhood of $|VD(D_0)^{\#n}|$ with respect to the d_∞ metric on $|N(D_0)^{\#n}|$. Hence the image of $|R_n|$ under the homeomorphism $|N(D_0)^{\#n}| \cong |N(D_0)|$ lies in the 2^{-n}-neighborhood of $|P|$ with respect to the d_∞ metric on $|N(D_0)|$. Thus we obtain a continuous map $g \colon |R_n| \to |P|$ whose restriction over $P^{\#n}$ is homotopic to the homeomorphism $|P^{\#n}| \cong |P|$. Next, each D_{i+1} star-refines D_i, so we have the canonical bonding map $\varphi_{D_i}^{D_{i+1}} \colon N(D_{i+1}) \to N(D_i)^{\#}$. It is easy to see that the composition $P^{\#(n+1)} \simeq VD(D_{i+1}) \xrightarrow{\varphi} VD(D_i)^{\#n} \simeq P^{\#n}$, where φ is the restriction of $\varphi_{D_i}^{D_{i+1}}$, is the identity map. By iterating we obtain a conical map $\varphi_n \colon N(D_n) \to N(D_0)^{\#n}$ that extends the composition $VD(D_n) \simeq P^{\#n} \simeq VD(D_0)^{\#n}$. Since φ_n is conical and $N(D_n) = \ulcorner VD(D_n) \urcorner$, the image of φ_n lies in R_n. Thus we obtain a continuous map $h \colon |N(D_n)| \to |P|$ whose restriction to $|VD(D_n)|$ is homotopic to the homeomorphism $|P^{\#n}| \cong |P|$. In particular, we obtain a continuous map $k \colon |N(C_n)| \to |P_{2^n+1}|$ whose restriction to $|VD(C_n)|$ is homotopic to the homeomorphism $|Q_n| \cong |P_{2^n+1}|$. This yields a continuous retraction of the (2^n+1)-ball onto the boundary 2^n-sphere, which is a contradiction.

We note that the poset P in Example 4.23 satisfies the following property (Q): There exist essential maps $e_n \colon S^{2^n} \hookrightarrow |P|$ such that for each $\varepsilon > 0$ there exists an n, a $\delta > 0$ and a discontinuous, (δ, ε)-continuous extension of e_n over B^{2^n+1}. On the other hand, since $|P^\flat|$ is a uniform ANR, $|P|$ is a non-uniform ANR, and in particular satisfies the non-uniform homotopy extension property. It follows that every metrizable uniform space that is uniformly homotopy equivalent to $|P|$ satisfies (Q) as well. In particular, using that $|P|$ is uniformly locally contractible, we get the following

Theorem 4.24. *There exists a countable poset whose geometric realization is not uniformly homotopy equivalent to a uniform ANR, nor even to a metrizable uniform space satisfying the Hahn property.*

The remainder of this subsection is not used elsewhere in this paper, and could be of interest primarily to the reader who is looking for a class of posets larger than CQLs whose geometric realizations are uniform ANRs.

4.25. Hereditarity. We call a cover D of a metric space M *hereditarily uniform*, if there exists a $\lambda > 0$ such that each $E \subset D$ is a uniform cover of $\bigcup E$ with Lebesgue number λ. Any such λ is a *hereditary Lebesgue number* of the hereditarily uniform cover.

We say that a cover C of a set S *hereditarily star-refines* a cover D of S if for every $E \subset D$, the cover $C \cap (\bigcup E)$ of the subset $\bigcup E \subset X$ star-refines the cover E of $\bigcup E$.

If D is a cover of M by sets of diameters $< \varepsilon$, and C is a hereditarily uniform cover of M with a hereditary Lebesgue number 2ε, then clearly D hereditarily star-refines C.

Lemma 4.26. *Let C and D be covers of a set S. If C hereditarily star-refines D, then φ_D^C sends $N(C)$ into $VD(D)^\#$.*

Proof. The hypothesis implies that for every $x \in C$, every $T \subset \operatorname{st}(x, C)$ such that $x \in T$ satisfies the following property $(*)$: if T lies in $\bigcup E$ for some $E \subset D$, then it lies in some element of E. In particular, $(*)$ is satisfied by any T of the form $\bigcap \sigma$ or $\bigcup \sigma$, where $\sigma \in N(C)$. On the other hand, every element of $N(D)$ of the form $\Delta_D(T)$ where T satisfies $(*)$ clearly belongs to $VD(D)$. □

4.27. Construction of hereditary uniform covers. For a finite-dimensional atomic poset P with atom set Λ, it is easy to construct a hereditarily uniform cover of $|P|$, namely the cover by the sets $U_\lambda = \bigcup_{\sigma \ge \lambda} |H_\sigma|$ composed of the barycentric handles $H_\sigma = \ulcorner \langle \sigma \rangle^* \urcorner \subset (P^\flat)^*$, where $\sigma \in P$. The hereditarity is due to the fact that $|H_\sigma|$ and $|H_\tau|$ are uniformly disjoint when σ and τ are incomparable and P is finite-dimensional. This argument does not apply to canonical handles $h_\sigma = \ulcorner [\sigma, \sigma]^* \urcorner \subset (P^\#)^*$ because $h_\sigma \cap h_\tau$ can be nonempty when σ and τ are incomparable.

Clearly, the preimage of a hereditarily uniform cover under a uniformly continuous map of metrizable uniform spaces is hereditarily uniform. Hence by using Lemma 4.10, we infer that every uniform cover of a residually finite-dimensional metrizable uniform space admits a hereditarily uniform refinement (in fact, one of a finite multiplicity). We conjecture that the hypothesis of residual finite-dimensionality cannot be dropped here.

4.28. Weak hereditarity. We call a cover D of a metric space M *weakly hereditarily uniform*, if there exists a $\lambda > 0$ such that for every $F \subset D$ satisfying $\bigcap F \subset \bigcup (D \setminus F)$, the cover $(D \setminus F) \cap (\bigcap F)$ of the subset $\bigcap F \subset X$ is a uniform cover of $\bigcap F$ with Lebesgue number λ. Any such λ is called a *weak hereditary Lebesgue number* of the weakly hereditarily uniform cover. A hereditarily uniform cover is weakly hereditarily uniform by considering $E = D \setminus F$; and a weakly hereditarily uniform cover is uniform by considering $F = \emptyset$ (in which case $\bigcap F = M$).

We say that a cover C of a set S *weakly hereditarily star-refines* a cover D of S if for every $F \subset D$ satisfying $\bigcap F \subset \bigcup (D \setminus F)$, the cover $C \cap (\bigcap F)$ of the subset $\bigcap F \subset X$ star-refines the cover $(D \setminus F) \cap (\bigcap F)$ of $\bigcap F$. Similarly to the above,

$$\text{hereditary star-refinement} \Rightarrow \text{weak hereditary star-refinement} \Rightarrow \text{star-refinement}.$$

It is easy to see that if D is a cover of M by sets of diameters $< \varepsilon$, and C is a weakly hereditarily uniform cover of M with a weak hereditary Lebesgue number 2ε, then D weakly hereditarily star-refines C.

Beware that the preimage of a weakly hereditarily uniform cover under a uniformly continuous map f of metrizable uniform spaces need not be weakly hereditarily uniform, because $\bigcap F \not\subset \bigcup (D \setminus F)$ does not imply $f^{-1}(\bigcap F) \not\subset f^{-1}(\bigcup (D \setminus F))$.

The proof of Lemma 4.26 works to establish

Lemma 4.29. *Let C and D be covers of a set S. If C weakly hereditarily star-refines D, then φ_D^C sends $N(C)$ into $VD(D)^{\#}$.*

4.C. Approximation of spaces

Theorem 4.30. *Every separable metrizable complete uniform space is the limit of a convergent inverse sequence of geometric realizations of simplicial complexes and uniformly continuous maps.*

The compact case of Theorem 4.30 was proved by Freudenthal [18]; previously, Alexandroff [2] had obtained a close result, namely a version of the compact case of Proposition 4.31 below with mere refinement instead of the star-refinement in the hypothesis, and with every thread, as a point of the inverse limit in the conclusion, replaced by the set of the simplicial neighborhoods of its elements.

In the case of residually finite-dimensional spaces, Theorem 4.30 was proved by Isbell [23; Lemma V.33] (see also [10; Lemma 1.6], [24; Lemma 14], [22; 7.2]), taking into account that his geometric realization is uniformly homeomorphic to ours in the case of a finite-dimensional simplicial complex. In the general case, we use a different notion of geometric realization and different bonding maps, but otherwise our proof is modelled on Isbell's argument. [13]

Theorem 4.30 follows from [32; Theorem 2.37] along with the following

Lemma 4.31. *Let $\{C_n\}$ be a basis of the uniformity of a metrizable complete uniform space X, where each uniform cover C_n is countable and point-finite. Let $N_i = |N(C_i)|$, and let $p_i\colon N_{i+1} \to N_i$ be the composition $|N(C_{i+1})| \xrightarrow{|\varphi_{C_i}^{C_{i+1}}|} |N(C_i)^{\#}| \xrightarrow{h} |N(C_i)|$, where $\varphi_{C_i}^{C_{i+1}}$ is the canonical bonding map, and h is the uniform homeomorphism. Then the inverse sequence $\dots \xrightarrow{p_1} N_1 \xrightarrow{p_0} N_0$ is convergent, and its limit L is uniformly homeomorphic to X.*

Proof. Let $s_i(x)$ denote the simplex $\ulcorner\Delta_{C_i}(x)\urcorner$. Then each $f_i := \varphi_{C_i}^{C_{i+1}}$ sends $s_{i+1}(x)$ into $s_i(x)^{\#}$ for each $x \in X$. Hence each p_i sends $|s_{i+1}(x)|$ into $|s_i(x)|$. Since f_i is conical, $|f_i|$ is 1-Lipschitz with respect to the d_∞ metrics, and therefore p_i is $\frac{1}{2}$-Lipschitz with respect to the d_∞ metrics. Since the diameter of $|s_i(x)|$ is bounded above by 2 in the d_∞ metric, the diameter of $p_i^{i+n}(|s_{i+n}(x)|)$ is bounded above by 2^{1-n}. Since each $|s_i(x)|$ is compact, their inverse limit (with the restrictions of p_i as the bonding maps) is nonempty, and by the above it has zero diameter. Thus it is a single point $\lambda(x) \in L$.

Each N_i is the union of the $|s_i(x)|$ over all $x \in X$, and every $|s_i(x)|$ contains $p_i^\infty(\lambda(x))$. Hence every point of $p_i^{i+n}(N_{i+n})$ is 2^{1-n}-close to a point of $p_i^\infty(L)$. Thus the inverse sequence is convergent (see [32; Lemma 5.6(c)]).

[13] There is a minor error in the proof of step (2) in [23; Lemma V.33], as the Cauchy filter base considered there might consist entirely of the empty sets. This can be remedied as shown in the last paragraph of our proof.

To see that $\lambda\colon X \to L$ is uniformly continuous, it suffices to show that every its coordinate $\lambda_i\colon X \xrightarrow{\lambda} L \xrightarrow{p_i^\infty} N_i$ is uniformly continuous. Indeed, for each $x \in X$ and each $j \geq i$ we have $\lambda_j(x) \in |s_j(x)| = |{}_\ulcorner\Delta_{C_j}(x)_\urcorner|$. For each $V \in C_j$, every $x \in V$ satisfies $V \in \Delta_{C_j}(x)$. Hence $\lambda_j(V) \subset |\operatorname{st}(\{V\}, N(C_j))|$. The diameter of $|\operatorname{st}(\{V\}, N(C_j))|$ is bounded above by 4, hence its image under p_i^j has diameter at most $2^{2-(j-i)}$. Thus for each $\varepsilon > 0$ there exists a $j \geq i$ such that $\lambda_i(C_j) = p_i^j\lambda_j(C_j)$ refines the cover of N_i by ε-balls. Since $\{C_j\}$ is a fundamental sequence of covers of X, we infer that λ_i is uniformly continuous.

Next, given $x, y \in X$ at a distance $\varepsilon > 0$, there exists an $n = n(\varepsilon)$ such that any two elements of C_n containing x and y are disjoint. Then λ_n sends x and y into disjoint closed simplices $|s_n(x)|$ and $|s_n(y)|$ of N_n. It follows that λ is injective and, using the uniform continuity of each p_n^∞, that λ^{-1} is uniformly continuous.

Finally, if $(q_i) \in L$ is a thread of $q_i \in N_i$, and σ_n is the minimal simplex of $N(C_n)$ such that $q_n \in |\sigma_n|$, then $f_n(\sigma_{n+1}) \subset \sigma_n^\#$, moreover, σ_n is the minimal simplex of $N(C_n)$ satisfying the latter property. Hence $\sigma_n = \Delta_{C_n}(\bigcap \sigma_{n+1})$; in particular, $\bigcap \sigma_{n+1} \subset \bigcap \sigma_n$. Let S_n be the closure of $\bigcap \sigma_n$. Then S_n lies in the closure of an element of C_n. Since $\{C_i\}$ is a basis of the uniformity of X, for each $\varepsilon > 0$ there exists an n such that every element of C_n is of diameter at most ε. It follows that the inverse sequence $\dots \subset S_1 \subset S_0$ is Cauchy (see [32; Lemma 5.6(d)]). Since X is complete, so are the S_i's, hence $\dots \subset S_1 \subset S_0$ is convergent (see [32; Lemma 5.6(b)]) and therefore $\bigcap S_i$ is nonempty (see [32; Lemma 5.6(f)]). Since the diameters of S_i tend to zero, $\bigcap S_i$ must be a single point q. Now q lies in the closure of $\bigcap \sigma_n$, and each $q' \in \bigcap \sigma_n$ satisfies $\sigma_n \subset s_n(q')$ and $\lambda_n(q') \in |s_n(q')|$. Since λ_n is continuous, $\lambda_n(q)$ lies in the closed subset $|{}_\ulcorner{}^\llcorner\sigma_n{}^\lrcorner{}_\urcorner|$ of $|N(C_n)|$. Hence $s_n(q) \subset {}_\ulcorner{}^\llcorner\sigma_n{}^\lrcorner{}_\urcorner$, or equivalently $\sigma_n \subset {}_\ulcorner{}^\llcorner s_n(q){}^\lrcorner{}_\urcorner$. Since $\lambda(q)$ is also the inverse limit of the simplicial neighborhoods $|{}_\ulcorner{}^\llcorner s_n(q){}^\lrcorner{}_\urcorner|$, we conclude that $\lambda(q) = (q_i)$. Thus λ is surjective. $\square$

Lemma 4.32. *Every separable uniform ANR is uniformly ε-homotopy dominated by the geometric realization of a simplicial complex, for each $\varepsilon > 0$.*

The following proof is based on Theorem 4.30; the reader who feels that this is a bit of an overkill will easily devise a more elementary proof (cf. [23; proof of 7.3]) based on Lemma 4.10 and [32; Lemmas 4.29(a) and 4.26(a)].

Proof. By [32; Theorem 4.19], the given uniform ANR is uniformly ε-homotopy equivalent to its completion X, for each $\varepsilon > 0$. By Theorem 4.30, X is the limit of a convergent inverse sequence of geometric realizations P_i of simplicial complexes, and uniformly continuous bonding maps p_i. By [32; Corollary 5.17] there exists a k and a uniformly continuous retraction $r_{[k,\infty]}\colon P_{[k,\infty]} \to X$. For each $l \geq k$ let r_l and $r_{[l,\infty]}$ denote the restrictions of $r_{[k,\infty]}$ over P_l and over $P_{[l,\infty]}$, respectively. Let $p_{[k,\infty]}^\infty\colon X \times I \to P_{[k,\infty]}$ be obtained by combining the maps $p_i^\infty\colon X \to P_i$. Then for each $l \geq k$, the composition $r_{[l,\infty]}p_{[l,\infty]}^\infty\colon X \times I \to X$ is a uniformly continuous homotopy between $X \xrightarrow{p_l^\infty} P_k \xrightarrow{r_l} X$

and id_X. Moreover, for each $\varepsilon > 0$ there exists a k such that $r_{[l,\infty]}p^{\infty}_{[l,\infty]}$ is an ε-homotopy. Thus X is uniformly ε-homotopy dominated by P_l. □

From Lemma 4.32 and [32; Corollary 4.31] we immediately obtain

Theorem 4.33. *A separable metrizable uniform space is a uniform ANR if and only if it is uniformly ε-homotopy dominated by the geometric realization of a simplicial complex, for each $\varepsilon > 0$.*

Theorem 4.34. *If X is a uniform ANR, then $X \times \mathbb{R}$ is uniformly homotopy equivalent to a uniform polyhedron.*

Proof. By Lemma 4.32 we are given uniformly continuous maps $d\colon |K| \to X$ and $u\colon X \to |K|$, where K is a simplicial complex, such that the composition $X \xrightarrow{u} |K| \xrightarrow{d} X$ uniformly homotopic to the identity by an homotopy $h\colon X \times I \to X$. We now perform a uniform version of Mather's trick (see [16]): $X \times \mathbb{R}$ is uniformly homotopy equivalent to the double mapping telescope of

$$\dots \to X \xrightarrow{du} X \xrightarrow{du} X \to \dots,$$

which is in turn uniformly homotopy equivalent to the double mapping telescope of

$$\dots \to X \xrightarrow{d} |K| \xrightarrow{u} X \xrightarrow{d} |K| \xrightarrow{u} X \to \dots,$$

which is in turn uniformly homotopy equivalent to the double mapping telescope of

$$\dots \to |K| \xrightarrow{ud} |K| \xrightarrow{ud} |K| \to \dots.$$

Since $|K|$ is a uniform ANR, by [32; Lemma 4.26] it is uniformly locally contractible. Then by Theorem 4.22, ud is uniformly homotopic to the composition $|K| \xrightarrow{h_m^{-1}} |K^{\#m}| \xrightarrow{|f|} |K^{\#n}| \xrightarrow{h_n} |K|$ for some conical map $f\colon K^{\#m} \to K^{\#n}$, where $m \ge n$ for the sake of definiteness. On the other hand, we have the conical map $\#(m-n)\colon K^{\#m} \to K^{\#n}$ (see 3.25), whose geometric realization is uniformly homotopic to the uniform homeomorphism $h_m\colon |K^{\#m}| \to |K^{\#n}|$ (see Lemma 3.26). Thus $X \times \mathbb{R}$ is uniformly homotopy equivalent to the geometric realization of the double mapping telescope of

$$\dots \xleftarrow{\#(m-n)} K^{\#n} \xrightarrow{f} K^{\#m} \xleftarrow{\#(m-n)} K^{\#n} \xrightarrow{f} \dots.$$

By Theorem 4.5, the latter is uniformly homotopy equivalent to the geometric realization of the thickened double mapping telescope, which is a uniform polyhedron. □

Remark 4.35. Similar arguments (with double mapping telescopes not of individual nerves but of their mapping telescopes) also show that if X is a complete uniform ANR, then $X \times \mathbb{R}$ is the limit of a convergent inverse sequence $\dots \xrightarrow{q_1} Q_1 \xrightarrow{q_0} Q_0$ of geometric realizations of countable preposets and uniformly continuous maps such that each q_i is a uniform homotopy equivalence.

Theorem 4.36. *Every complete uniform ANR is the limit of a convergent inverse sequence $\dots \xrightarrow{q_1} Q_1 \xrightarrow{q_0} Q_0$ of uniform polyhedra and uniformly continuous maps such that each q_i is (non-uniformly) a homotopy equivalence.*

Proof. Let X be the given complete uniform ANR. By Theorem 4.30, X is the limit of a convergent inverse sequence $\dots \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0$ of uniform polyhedra and uniformly continuous maps. Suppose that we have constructed a finite sequence $n_0, n_1, n_2, \dots, n_k$ and a finite chain of uniform polyhedra and uniformly continuous maps $P_{n_k} \subset Q_k \xrightarrow{q_{n-1}} \dots \xrightarrow{q_1} P_{n_1} \subset Q_1 \xrightarrow{q_0} P_{n_0} \subset Q_0$ such that the composition $X \xrightarrow{p^\infty_{n_i}} P_{n_i} \subset Q_i$ is a homotopy equivalence for each $i \le k$.

Since P_{n_k} is uniformly locally contractible, there exists an $\varepsilon > 0$ such that every two ε-close uniformly continuous maps $Y \to P_{n_k}$ are uniformly homotopic. Let δ be such that p^∞_i sends δ-close points into $\varepsilon/2$-close points. Since the inverse sequence is convergent and X satisfies the Hahn property (see [32; Lemma 4.29(a)]), there exists an $m \ge n$ and a map $r\colon P_m \to X$ such that the composition $X \xrightarrow{p^\infty_m} P_m \xrightarrow{r} X$ is δ-close to the identity. Then the composition $X \xrightarrow{p^\infty_m} P_m \xrightarrow{r} X \xrightarrow{p^\infty_{n_k}} P_{n_k}$ is $\varepsilon/2$-close to $p^\infty_{n_k}$. Since the inverse sequence is convergent, there exists an $l \ge m$ such that the composition $P^l \xrightarrow{p^l_m} P_m \xrightarrow{r} X \xrightarrow{p^\infty_{n_k}} P_{n_k}$ is ε-close to $p^l_{n_k}$. Let d be the composition $P_l \xrightarrow{p^l_m} P_m \xrightarrow{r} X$, let $u = p^\infty_l$, and let f be the composition $P_l \xrightarrow{d} X \xrightarrow{u} P_l$. Then the composition $P_l \xrightarrow{f} P_l \xrightarrow{p^l_{n_k}} P_{n_k}$ is uniformly homotopic to $p^l_{n_k}$. This yields a uniformly continuous map $F\colon MC(f) \to P_{n_k}$ that restricts to $p^l_{n_k}$ on each of the two copies of P_l in $MC(f)$. Let $n_{k+1} = l$. Applying to the maps $X \xrightarrow{u} P_l \xrightarrow{d} X$ the construction in the proof of Theorem 4.34, we obtain the double mapping telescope Q_l which is the geometric realization of a preposet homotopy equivalent to X, via the composition $X \xrightarrow{u} P_l \subset Q_{k+1}$. The partial map $Q_{k+1} \supset P_l \xrightarrow{p^l_{n_k}} P_{n_k}$ now extends (using F) to a total uniformly continuous map $q_{k+1}\colon Q_{k+1} \to P_{n_k}$, and we are done with the inductive step.

The assertion now follows using that inverse limit is unchanged upon passage to an infinite subsequence. ☐

Remark 4.37. In trying to prove (or disprove) that every uniform ANR X is homotopy equivalent a uniform polyhedron, an obvious strategy would be to examine "Siebenmann's variation on West's proof that compact ANRs finite types" (as elaborated upon by Edwards, Chapman and Ferry).

Let us discuss here another possible strategy: to replace $\mathbb{R}$ with a uniformly contractible space such as $|\mathbb{Z}|$ (where $\mathbb{Z}$ is ordered in the usual way) or $|\Delta^{\mathbb{Z}}_w|$. This brings in higher homotopies in the picture.

If $d\colon |K| \to X$ and $u\colon X \to |K|$ are as in the proof of Theorem 4.34, let f be the composition $|K| \xrightarrow{d} X \xrightarrow{u} |K|$, and let $H\colon X \times I \to X$ be a uniform homotopy between $X \xrightarrow{u} |K| \xrightarrow{d} X$ and the identity. Thus $G\colon |K| \times I \xrightarrow{d\times \mathrm{id}_I} X \times I \xrightarrow{h} X \xrightarrow{u} |K|$ is a uniform homotopy between f and ff. More generally, define H_n inductively to be the composition $X \times I^n \xrightarrow{h\times \mathrm{id}_{I^{n-1}}} X \times I^{n-1} \xrightarrow{h_{n-1}} X$, and let G_n be the composition $|K| \times I^n \xrightarrow{d\times \mathrm{id}_{I^n}} X \times I^n \xrightarrow{h_n} X \xrightarrow{u} |K|$. For instance, G_2 is a 2-homotopy bounded by the two homotopies (G followed by $G(f \times \mathrm{id}_I)$; and G followed by fG) between f

and fff, corresponding to the two bracketings $(ff)f$ and $f(ff)$. By Theorem 4.22, we may approximate G_n by the composition $|K \times I^n| \xrightarrow{h_{\lambda_n}^{-1}} |(K \times I^n)^{\#\lambda_n}| \xrightarrow{|F_n|} |K|$ for some conical map $F_n\colon (P \times I^n)^{\#\lambda_n} \to P$.

Since λ_n is likely to be unbounded as $n \to \infty$, we have to modify our $X \times |\Delta_w^{\mathbb{Z}}|$ by replacing every $X \times |\Delta^p|$, $p \in \Delta_w^{\mathbb{Z}}$ (thus p is a nonempty finite subset of $\mathbb{Z}$) with $F_p := (X \times |\Delta^p|)^{\#\lambda_{|p|}}$ along with iterated mapping cylinders of partial maps $F_p \supset F_q^{\#(\lambda_{|p|}-\lambda_{|q|})} \to F_q$, where the unmarked arrow is the $2^{\lambda_{|q|}-\lambda_{|p|}}$-Lipschitz identity map, for each $q < p$; and $X^\#$ is same as X but with the metric stretched by a factor of 2 (thus $|P^\#|$ is isometric to $|P|^\#$, where both $|P|$ and $|P^\#|$ are endowed with the d_∞ metric). For the uniform deformation retraction of $X \times |\Delta_w^{\mathbb{Z}}|$ onto $X \times \{0\}$ (using that $\Delta_w^{\mathbb{Z}} \simeq \{0\} * \Delta_w^{\mathbb{Z}\setminus\{0\}}$) to remain uniform after the modification, the function λ_n must grow linearly as $n \to \infty$; indeed if $\lambda_n = \lambda n$, then the modified deformation retraction will be 2^λ-Lipschitz.

But for λ_n to grow linearly, d, u and H must be Lipschitz. It can be seen from the proof of Lemma 4.10 that u is indeed Lipschitz by construction (with values in the usual, rather than d_∞ metric). For d and H to be Lipschitz, X must be a Lipschitz ANR. To avoid technical difficulties such as comparing the usual and d_∞ metrics on geometric realizations, by the Lipschitz category we understand what is sometimes called "Lipschitz at small scale", i.e. a map f between metric spaces is *Lipschitz* if there exist $\lambda, \mu > 0$ such that $d(x,y) \le \mu$ implies $d(f(x), f(y)) \le \lambda d(x,y)$.

The above argument is likely to yield the following assertion; we leave the details to the interested reader.

Conjecture 4.38. *Every Lipschitz ANR is uniformly homotopy equivalent to a uniform polyhedron.*

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