The Atiyah-Patodi-Singer index theorem

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To my mother, Isla Louise Melrose

PREFACE

This text is a somewhat expanded version of lecture notes written during, and directly after, a course at MIT in the Spring of 1991. Most of the participants had attended a course the preceding Fall on pseudodifferential operators on compact manifolds without boundary, including the Hodge theorem and the construction of the fundamental solution for the wave equation. Approximately this level of sophistication is assumed of the reader. The intention of the second course was to embed the Atiyah-Patodi-Singer index theorem in an analytic framework analogous to that provided by the theory of pseudodifferential operators for the Atiyah-Singer theorem. Since this treatment leads to a variety of current research topics it is presented here in the hope that it will be of use to a wider audience.

There are many people to thank. Foremost I am grateful to the members of the audience of the course for their tolerance and enthusiasm. I am especially grateful to Paolo Piazza for his comments during the course and also as a collaborator in work related to this subject. Others from whom I have learnt in this way are Xianzhe Dai, Charlie Epstein, Dan Freed, Rafe Mazzeo and Gerardo Mendoza. To the last of these I would like to take this opportunity to apologize for my part, whatever that was, in the somewhat mysterious non-appearance of the paper [64] on which a considerable part of Chapter 6, and indeed the general 'b-philosophy,' is based.

More generally I am happy to acknowledge the influence, through conversation, on my approach to this subject of Michael Atiyah, Ezra Getzler, Lars Hörmander, Werner Müller, Bob Seeley, Iz Singer and Michael Taylor. I am indebted to Antônio sá Barreto, Xianzhe Dai, Charlie Epstein and Maciej Zworski who tolerated my neglect of other projects during the process of writing, to Tanya Christiansen, Xianzhe Dai, Andrew Hassell, Lars Hörmander, Gerd Grubb, Rafe Mazzeo, Paolo Piazza and Lorenzo Ramero for comments on the manuscript and especially to Jillian Melrose for her forbearance. To Judy Romvos special thanks for turning my rather crude scrawlings into the original lecture notes.

Danville, Vermont. March 8, 1992

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Introduction and the proof

The Atiyah-Patodi-Singer index theorem (APS theorem) is used in this text as a pivot (or maybe an excuse) to discuss some aspects of geometry and analysis on manifolds with boundary. This volume does *not* contain a general treatment of index theorems even though they are amongst the most basic analytic-geometric results one can find. The power of such theorems in applications largely lies in their simplicity and generality. In particular the statement of the APS theorem is quite simple. In practice a great deal of effort, by many people, has gone into simplifying the proofs. This has lead to, and been accompanied by, a much wider understanding of the analytic framework in which they are centred. In fact, from an analytic perspective, index theorems can be thought of as much as testing grounds, for methods and concepts, as ends in themselves. The Atiyah-Singer theorem, which is the boundaryless precursor to the APS theorem, is intimately connected to the theory of pseudodifferential operators. This volume *is* intended to place the APS theorem in a similar context, the 'b' category and related calculus of b-pseudodifferential operators on a compact manifold with boundary.

The basic approach adopted here is to 'state' and 'prove' the APS theorem immediately, being necessarily superficial on a variety of points. The subsequent nine chapters consist largely in the fleshing out of this proof. Just as the initial discussion is brief, the later treatment is discursive and aims at considerably more than the proof of the index theorem alone. The proof given here is *direct* in two senses. The written proof itself is quite straightforward, given some conceptual background, and in particular the terms in the final formula come out directly in the course of the proof. The model here is Getzler's proof ([35]) of the Atiyah-Singer theorem for Dirac operators on a compact manifold without boundary.

The second sense in which the proof is direct is closely connected to the main thesis of this text. Namely that the APS theorem *is* the Atiyah-Singer theorem in the b-category, which is to say the category of compact manifolds with boundary with metrics having complete cylindrical ends. These metrics are called here (exact) b-metrics. This is by no means a radical position (since it is at least implicit in the original papers) but it is a position taken with some fervour. One consequence of this approach is the suggestion that there are other such theorems, especially on manifolds with corners. It is hoped that the context into which the APS theorem is placed will allow it to be readily understood and, perhaps more importantly, generalized. Of course extensions and generalizations already have been made, see in particular the work of Bismut and Cheeger [16], Cheeger [26], Moscovici and Stanton [69], Müller [70] and Stern [85]; see also Wu [91] and Getzler [36].

A review of the proof below, annotated with references to the intervening chapters to make it complete, can be found in §9.1. Towards the end of this Introduction there is an outline of the content of the later chapters.

1. The Atiyah-Singer index theorem.

Consider the Atiyah-Singer index theorem on a compact manifold without boundary. The version for Dirac operators is necessarily proved along the way to the APS theorem. It can be written in brief ([11])

$$\operatorname{ind}(\eth_E^+) = \int_X \mathrm{AS}\,. \tag{In.1}$$

Here $\eth_E^+$ is a twisted Dirac operator, with coefficient bundle E, on the compact even-dimensional spin manifold X and AS is the Atiyah-Singer integrand. This is the volume part (form component of maximal degree) of the product of a characteristic class on X, the $\widehat{A}$ genus, and the Chern character of E :

$$\mathrm{AS} = \mathrm{Ev}_{\dim X}\left(\widehat{A}(X)\cdot \mathrm{Ch}(E)\right). \tag{In.2}$$

Here $\mathrm{Ev}_{\dim X}$ evaluates a form to the coefficient of the volume form of the manifold which it contains. A fundamental feature of (In.1) is that the left side is analytic in nature and the right side is topological, or geometric. One point in favour of Getzler's proof of the index formula is that it is *not* necessary to understand the properties of (In.2) independently, i.e. the theory of characteristic classes is not needed to derive the formula (although it certainly helps to understand it). The left side of (In.1) is, by definition,

$$\operatorname{ind}(\eth_E^+) = \dim\operatorname{null}(\eth_E^+) - \dim\operatorname{null}(\eth_E^-)$$

where $\eth_E^+$ and $\eth_E^-$ act on $\mathcal{C}^\infty$ sections of the appropriate bundles, $\eth_E^-$ being the adjoint of $\eth_E^+$, and the finite dimensionality of the null spaces follows by ellipticity. The direct proof of (In.1) simplifies the original proof of Atiyah and Singer ([11], [12], [72]) and the modifications by Patodi ([73]), Gilkey ([37]) and Atiyah, Bott and Patodi ([5]). Full treatments of proofs along these lines can be found in Berline, Getzler and Vergne [20], Freed [33], Hörmander [48], Roe [77] and Taylor [88].

2. The Atiyah-Patodi-Singer index theorem.

The APS theorem ([8]-[10]) is a generalization of (In.1) to manifolds with boundary. There are two, complementary, ways of thinking about a compact manifold with boundary. The most familiar way is to think of

X as half of a compact manifold *without* boundary by doubling across the boundary. The other approach is to think of the boundary as *at infinity* or at least impenetrable. The latter approach is the one adopted here, whereas in [8]-[10] both approaches are used and the celebrated Atiyah-Patodi-Singer boundary condition (see the discussion in §3 below) comes from the interplay between them. More precisely this means that

$$\begin{gathered} ð^+_E \text{ is a twisted Dirac operator with} \\ \text{respect to an exact } b\text{-spin structure on } X^{2n}. \end{gathered} \tag{In.3}$$

The reason that X is assumed to be even-dimensional is that in case $\dim X$ is odd and $\partial X = \emptyset$ the index vanishes. There *is* an index theorem in the odd dimensional case but, for Dirac operators, it is relatively simple. For the moment, the notion of a spin structure is left undefined, as is the Dirac operator associated to it.

The exact b-metrics on a compact manifold with boundary are complete Riemann metrics on the interior which make the neighbourhood of the boundary into an asymptotically cylindrical end. More precisely, an exact b-metric is a Riemann metric which takes the form

$$g = \left(\frac{dx}{x}\right)^2 + h \tag{In.4}$$

near ∂X, with h a smooth 2-cotensor which induces a Riemann metric on the boundary and $x \in \mathcal{C}^\infty(X)$ a defining function for the boundary. It is important to emphasize that this notion of a b-metric is taken seriously below. For example, the frame bundle of a metric of this type is a smooth principal bundle up to the boundary. An exact b-spin structure is simply a spin structure for an exact b-metric, i.e. a refinement of the frame bundle to a principal Spin bundle, where $\mathrm{Spin}(2n)$ is the non-trivial double cover of $\mathrm{SO}(2n)$.

One important property of an exact b-spin structure (which exists precisely when a spin structure exists) is that it induces a spin structure on the boundary. The corresponding Dirac operator on the boundary will be denoted $ð_{0,E}$. A useful assumption, which will be removed later, is

$$ð_{0,E} \text{ is invertible.} \tag{In.5}$$

In fact $ð_{0,E}$ is elliptic and self-adjoint so (In.5) just means that its null space reduces to $\{0\}$. As a consequence of (In.5), $ð^+_E$ is Fredholm on its natural domain (the Sobolev space defined by the metric) and the APS theorem states that

$$\operatorname{ind}(ð^+_E) = \int_X \mathrm{AS} - \frac{1}{2}\eta(ð_{0,E}). \tag{In.6}$$

Here the Atiyah-Singer integrand, AS, is the same as before, manufactured from differential-geometric information in the spin structure and auxiliary bundle by local operations. On the other hand the η-invariant is a global object constructed from $\eth_{0,E}$, so fixed purely in terms of boundary data. In fact it is a spectral invariant of $\eth_{0,E}$. This decomposition into a 'local interior' and a 'global boundary' term is fundamental to the utility of the result.

3. Boundary conditions versus b-geometry.

The boundary of a compact manifold with boundary always has a collar neighbourhood, i.e. a neighbourhood of the form $[0,r)_x \times \partial X$, say for $r > 1$. An exact b-metric (In.4) is (or gives the manifold) a cylindrical end if, on the collar, h is simply the pull-back of a metric on ∂X. Often (although not so often here) the end is considered as unbounded in that $t = \log x$ is introduced as a variable, putting the boundary at $t = -\infty$. The manifold $X_1 = X \setminus ([0,1] \times \partial X)$ is diffeomorphic to X.

The spin bundles ${}^{\pm}S$ on X can be identified on the collar neighbourhood of the boundary, since they can be identified over the boundary, by an isomorphism with the spinor bundle of the boundary. Then (see §3.11) the Dirac operator becomes

$$\eth^+ = M_-^{-1} \cdot \left(x\frac{\partial}{\partial x} + \eth_0 \right) \cdot M_+, \tag{In.7}$$

where $M_\pm$ are the isomorphisms between the spinor bundles ${}^{\pm}S$ on the collar and S_0, the spinor bundle of ∂M and and $\eth_0$ is the Dirac operator on the boundary. The null space of $\eth^+$ acting on distributions on the collar can then be examined in terms of the eigen-decomposition for $\eth_0$, which is self-adjoint. Thus the solutions of $\eth^+ u_+ = 0$ are superpositions of the special solutions

$$u = x^{-z} M_+^{-1} v, \ \eth_0 v = zv.$$

This solution is square-integrable with respect to the metric if and only if $z < 0$.

The APS boundary condition for the Dirac operator restricted to the region $x \ge 1$, i.e. $t \ge 0$ is

$$Q_+ \left(M_+ u_{\restriction x=1} \right) = 0, \tag{In.8}$$

where Q_+ is the orthogonal projection onto the non-negative eigenspace of $\eth_0$. This projector is a pseudodifferential operator and $\eth^+$ with the boundary condition (In.8) can be considered as an elliptic boundary problem. In particular there is an associated Fredholm operator and the analysis can be carried out using the theory of elliptic boundary problems introduced by Calderón ([25]) and developed further by Seeley [81] (see also Boutet de Monvel [21] and Grubb [41]).

With the APS boundary condition, the appearance in (In.6) of the eta invariant is a little less striking, since it represents just a part of the information in the projection Q_+. The condition (In.8) reflects the square-integrability of an extension of the solution into $t < 0$, i.e. to the whole of the original manifold X. It will not be encountered below[1]. As already noted, the invertibility of the operator $\eth^+$, which in general for an exact b-metric is not quite as simple as (In.7), is attacked directly and its generalized inverse and the associated heat kernels are shown to be elements of the appropriate space of b-pseudodifferential operators.

4. Preliminaries to the proof.

Let ${}^{\pm}S$ be the two spinor bundles over the compact, even dimensional, exact b-spin manifold, X. The idea, used already in [5] and in a related manner by McKean and Singer [58] and dating back, in other contexts, at least to Minakshisundarum and Pleijel [68] is to consider the heat kernels

$$\exp(-t\eth_E^-\eth_E^+),\ \exp(-t\eth_E^+\eth_E^-), \tag{In.9}$$

where the Dirac operator is

$$\eth_E^+ : \mathcal{C}^\infty(X; {}^+S \otimes E) \longrightarrow \mathcal{C}^\infty(X; {}^-S \otimes E)$$

and $\eth_E^-$ is its adjoint. Suppose for the moment that $\partial X = \emptyset$ and consider the Atiyah-Singer theorem. Both $\eth_E^-\eth_E^+$ and $\eth_E^+\eth_E^-$ are elliptic, self-adjoint and non-negative so the heat kernels (In.9) are, for $t > 0$, smoothing operators. The fact that 0 is an isolated spectral point of both means that

$$\begin{aligned} \lim_{t\to\infty} \exp(-t\eth_E^-\eth_E^+) &= \pi_{\operatorname{null}(\eth_E^+)} \\ \lim_{t\to\infty} \exp(-t\eth_E^+\eth_E^-) &= \pi_{\operatorname{null}(\eth_E^-)} \end{aligned} \tag{In.10}$$

where π_N is orthogonal projection onto the finite dimensional subspace $N \subset L^2(X; L)$, for the appropriate bundle L. The convergence in (In.10) is exponential, within smoothing operators. The trace functional, just the sum of the eigenvalues of a finite rank operator, extends continuously to smoothing operators, so from (In.10) it follows that

$$\lim_{t\to\infty} \operatorname{Tr}[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)] = \operatorname{ind}(\eth_E^+). \tag{In.11}$$

[1] The expunging of the APS boundary condition, in explicit form, is a 'feature' of this proof which is fundamental, although not universally welcomed.

The extension of the trace to smoothing operators is described by Lidskii's theorem. Namely smoothing operators are those with $\mathcal{C}^\infty$ Schwartz kernels ([79]), i.e. they can be written as integral operators

$$Ku(x) = \int_X K(x,x')u(x') \text{ with } K \in \mathcal{C}^\infty.$$

Then

$$\operatorname{Tr} K = \int_X K(x,x). \tag{In.12}$$

The single most important property of the trace functional is that it vanishes on commutators:

$$\operatorname{Tr}[K_1, K_2] = 0, \tag{In.13}$$

as follows readily from (In.12). This remains true if K_1 is a differential operator, provided K_2 is smoothing.

At least formally consider (since the operators act on different bundles)

$$\begin{aligned}
&\frac{d}{dt}\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big)\\
&= -\big(\eth_E^-\eth_E^+\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\eth_E^+\eth_E^-\big)\\
&= -[\eth_E^-, \eth_E^+\exp(-t\eth_E^-\eth_E^+)].
\end{aligned} \tag{In.14}$$

Here the identity

$$\exp(-t\eth_E^+\eth_E^-)\eth_E^+ = \eth_E^+\exp(-t\eth_E^-\eth_E^+), \tag{In.15}$$

which follows from the uniqueness of solutions to the heat equation, has been used. With the trace taken in (In.14), (In.13) and (In.11) together give the remarkable identity of McKean and Singer

$$\operatorname{ind}(\eth_E^+) = \operatorname{Tr}\left[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\right] \;\forall\; t > 0. \tag{In.16}$$

The formulæ (In.1) and (In.2) arise from a clear understanding of the behaviour of the heat kernels as $t \downarrow 0$, i.e. from the *local index theorem* (proved by Gilkey [37] and Patodi [72]):

$$\operatorname{AS}(x) = \lim_{t\downarrow 0}\operatorname{tr}\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big)(x,x), \tag{In.17}$$

where the 'little' trace, tr, is just the trace functional on the bundles ${}^{\pm}S \otimes E$. The direct proof of (In.17) is Getzler's rescaling argument. The final formula (In.1) arises by applying (In.12) to (In.17).

So, to prove (In.6), in which the η-term should appear as a *defect*, it is natural to look at the heat kernels (In.9) when $\partial X \neq \emptyset$. The *fundamental problem* with the generalization of the proof outlined above is that, when $\partial X \neq \emptyset$ in the 'b-' setting,

$$\exp(-t\eth_E^-\eth_E^+),\ \exp(-t\eth_E^+\eth_E^-) \text{ are not trace class.}$$

Indeed the Atiyah-Patodi-Singer boundary condition was introduced to replace these by trace class operators. There is however a direct generalization of the statement that these exponentials are smoothing operators, which they still are in the interior. Namely there is a calculus of pseudodifferential operators ([61], [64], [47, §18.3], [66]), denoted here $\Psi_b^m(X; L_1, L_2)$ for any bundles L_1, L_2, which captures appropriate uniformity of the kernels up to the boundary. The assumption (In.3) implies that the Dirac operator is in the corresponding space of differential operators

$$\eth_E^+ \in \mathrm{Diff}_b^1(X; {}^+S \otimes E, {}^-S \otimes E) \subset \Psi_b^1(X; {}^+S \otimes E, {}^-S \otimes E);$$

it is elliptic. It follows from constructions essentially the same as in the standard case that

$$\begin{matrix} \exp(-t\eth_E^-\eth_E^+) \in \Psi_b^{-\infty}(X; {}^+S \otimes E) \\ \exp(-t\eth_E^+\eth_E^-) \in \Psi_b^{-\infty}(X; {}^-S \otimes E) \end{matrix} \quad \text{in } t > 0. \tag{In.18}$$

As already noted, these conditions do *not* mean that the operators are trace class. Despite this there is an *extension* of the trace functional to a linear functional

$$\text{b-Tr}_\nu : \Psi_b^{-\infty}(X; L) \longrightarrow \mathbb{C}.$$

This extension depends on ν, a trivialization of the normal bundle to ∂X, and is defined simply by regularization of (In.12), as in the work of Hadamard [43]. If $x \in \mathcal{C}^\infty(X)$ is a defining function with $dx \cdot \nu = 1$, the b-trace is

$$\text{b-Tr}_\nu(K) = \lim_{\epsilon \downarrow 0} \left[\int\limits_{x > \epsilon} \operatorname{tr} K(x,x) + \log \epsilon \cdot \widetilde{\operatorname{Tr}}(K) \right]. \tag{In.19}$$

The logarithmic term removed in (In.19) is precisely what is needed to regularize the integral and this fixes the coefficient $\widetilde{\operatorname{Tr}}(K)$.

The appearance of the defect in the index formula is directly related to the failure of (In.13) for the b-trace. There is an algebra homomorphism in the calculus

$$\Psi_b^m(X; L_1, L_2) \xrightarrow{(\ )_\partial} \Psi^m(\partial X; L_1, L_2), \tag{In.20}$$

where the image space consists of the pseudodifferential operators on the boundary acting on the restrictions of the bundles to the boundary. This map is defined by restriction:

$$A_\partial u = A\tilde{u}_{\restriction \partial X} \text{ if } \tilde{u} \in \mathcal{C}^\infty(X; L_1),\ \tilde{u}_{\restriction \partial X} = u.$$

The homomorphism (In.20) can be extended by noting that the calculus is invariant under conjugation by complex powers of x (a boundary defining function)

$$\Psi_b^m(X; L) \ni A \longleftrightarrow x^{-i\lambda} A x^{i\lambda} \in \Psi_b^m(X; L).$$

Then

$$I(A, \lambda) = \left(x^{-i\lambda} A x^{i\lambda}\right)_\partial \in \Psi^m(\partial X; L) \tag{In.21}$$

is an entire analytic family of pseudodifferential operators, the *indicial family* of A. Moreover

$$K \in \Psi_b^{-\infty}(X; L) \text{ is trace class } \iff I(K, \lambda) \equiv 0$$

and

$$I(K, \lambda) \equiv 0 \Longrightarrow \text{b-Tr}_\nu(K) = \text{Tr}(K).$$

The coefficient of the singular term in (In.19) is actually given by

$$\widetilde{\text{Tr}}(K) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Tr}\left(I(K, \lambda)\right) d\lambda.$$

The fundamental formula for the b-trace is:

$$\text{b-Tr}_\nu([A, B]) = \frac{i}{2\pi} \int \text{Tr}\big(\partial_\lambda I(A, \lambda) \circ I(B, \lambda)\big)\, d\lambda. \tag{In.22}$$

The integral on the right converges absolutely. This formula follows directly from (In.21), (In.19) and the definition of the b-calculus, i.e. it is elementary.

5. The proof.

First some bald facts, which need to be interpreted slightly but are 'true' enough. In terms of (In.21):

$$\text{(In.23)}\qquad \begin{cases} I(\eth_E^+,\lambda) = M_-^{-1}(i\lambda + \eth_{0,E})M_+ \\ I(\eth_E^-,\lambda) = M_+^{-1}(-i\lambda + \eth_{0,E})M_- \\ I(\exp(-t\eth_E^-\eth_E^+),\lambda) = M_+^{-1}\exp(-t(\lambda^2+\eth_{0,E}^2))M_+. \end{cases}$$

Here $M_\pm$ are isomorphisms of ${}^{\pm}S$ restricted to ∂X and S_0, the spinor bundle over ∂X, with $M_+^{-1}M_-$ being Clifford multiplication by idx/x. The identity (In.14) still holds, so now taking b-traces it follows from (In.22) that

$$\text{(In.24)}\qquad \begin{aligned} &\frac{d}{dt}\,\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big) \\ &= \frac{1}{2\pi i}\int \mathrm{Tr}\left[\partial_\lambda I(\eth_E^-,\lambda)\circ I(\eth_E^+\exp(-t\eth_E^-\eth_E^+),\lambda)\right]d\lambda. \end{aligned}$$

Using the fact that I is a homomorphism and (In.23), the right side of (In.24) can be rewritten

$$\text{(In.25)}\qquad -\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Tr}\left[(i\lambda+\eth_{0,E})\exp(-t(\lambda^2+\eth_0^2))\right]d\lambda.$$

The λ integral can be carried out, replacing the integrand by its even part and changing variable to $t^{\frac12}\lambda$, to give

$$\text{(In.26)}\qquad -\frac{1}{2\sqrt{\pi}}t^{-\frac12}\,\mathrm{Tr}\big(\eth_{0,E}\exp(-t\eth_{0,E}^2)\big).$$

As $t\to\infty$ (In.11) still holds in case $\partial X\neq\emptyset$, with Tr replaced by b-Tr_ν (the limit is independent of ν because the limiting operator is trace class). Similarly Getzler's scaling argument carries over to this setting to give (In.17), uniformly in x. Finally a similar scaling argument applies to (In.26) (as shown by Bismut and Freed [18], [19]). This allows (In.24) to be integrated over $(0,\infty)$ giving the limiting formula:

$$\text{(In.27)}\qquad \begin{aligned} &\lim_{t\to\infty}\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^-\eth_E^-)\big) \\ &\qquad - \lim_{t\to 0}\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big) \\ &= -\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-\frac12}\,\mathrm{Tr}\left(\eth_{0,E}\exp(-t\eth_{0,E}^2)\right)dt. \end{aligned}$$

This then gives (In.6) provided the eta invariant is *defined* to be

$$\eta(\eth_{0,E}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}\big(\eth_{0,E} \exp(-t\eth_{0,E}^2)\big)\, dt, \tag{In.28}$$

where absolute convergence follows from the scaling argument. This is one of the 'standard formulæ' for the η-invariant, so the 'proof' is complete.

6. Weighting.

In this setting there is no explicit boundary condition on the Dirac operator $\eth_E^+$. Rather it is, precisely when (In.5) holds, a Fredholm operator on the Sobolev space fixed by the metric. In fact all exact b-metrics are quasi-isometric, so these Sobolev spaces are intrinsic to the compact manifold with boundary and are denoted $H_b^m(X;L)$, for sections of a vector bundle L.

To prove the APS theorem it is illuminating to embed the index problem in a one-parameter family. Namely the Sobolev spaces extend to weighted spaces, $x^s H_b^m(X;L)$, for $s \in \mathbb{R}$, where $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary. Then

$$\begin{gathered} \eth_E^+ : x^s H_b^1(X; {}^+S \otimes E) \longrightarrow x^s H_b^0(X; {}^-S \otimes E) \\ \text{is Fredholm} \iff -s \text{ is not an eigenvalue of } \eth_{0,E}. \end{gathered} \tag{In.29}$$

The eigenvalues of $\eth_{0,E}$ form a discrete subset $\operatorname{spec}(\eth_{0,E}) \subset \mathbb{R}$, unbounded above and below, so $\operatorname{ind}_s(\eth_E^+)$ is defined for $-s \in \mathbb{R} \setminus \operatorname{spec}(\eth_{0,E})$. It is convenient to extend the definition of the index, even to the case that the operator is not Fredholm by setting

$$\widetilde{\operatorname{ind}}_s(\eth_E^+) = \lim_{\epsilon \downarrow 0} \frac{1}{2} \left[\operatorname{ind}_{s-\epsilon}(\eth_E^+) + \operatorname{ind}_{s+\epsilon}(\eth_E^+) \right]. \tag{In.30}$$

The parameter, s, can be absorbed into the operator by observing that the weighting factor, x^s, can be treated as 'rescaling' (in the sense of Chapter 8) of the coefficient bundle E to a bundle $E(s)$.

Thus the conjugated operator, $\eth_E^+(s) = x^{-s}\eth_E^+ x^s$, is again the positive part of a (twisted) Dirac operator, however the new total Dirac operator is not self-adjoint; its negative part is $\eth_E^-(s) = x^{-s}\eth_E^- x^s$. All of the discussion above applies, provided $\eth_E^-$ is replaced throughout by $(\eth_E^+)^*$, except for the local index theorem, which no longer holds. The Atiyah-Singer integrand can still be defined as

$$\begin{gathered} \operatorname{AS}(s) = \text{The constant term as } t \downarrow 0 \text{ in} \\ \operatorname{tr}\big(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\big)(x,x) \end{gathered} \tag{In.31}$$

and similarly the modified eta invariant[2] is given by the regularized integral

$$\eta_s(\eth_{0,E}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}\left([\eth_{0,E} + s] \exp(-t[\eth_{0,E} + s]^2) \right) dt, \ \ s \in \mathbb{R}. \tag{In.32}$$

The meaning here is the same as in (In.31): as $t \downarrow 0$ the integral

$$\frac{1}{\sqrt{\pi}} \int_\epsilon^\infty t^{-\frac{1}{2}} \operatorname{Tr}\left([\eth_{0,E} + s] \exp(-t[\eth_{0,E} + s]^2) \right) dt$$

has an expansion in powers of ϵ and $\eta_s(\eth_{0,E})$ is the coefficient of the constant term. The APS theorem then can be written in weighted form

$$\widetilde{\operatorname{ind}}_s(\eth_E^+) = \int_X \operatorname{AS}(s) - \frac{1}{2}\eta_s(\eth_{0,E}) \quad \forall\ s \in \mathbb{R}. \tag{In.33}$$

The proof outlined above only applies directly in the Fredholm case, when $s \notin -\operatorname{spec}(\eth_{0,E})$. However only the discussion as $t \to \infty$ needs to be modified to give (In.33) in general.

In fact $\widetilde{\operatorname{ind}}_s(\eth_E^+)$ is locally constant on the open set of Fredholm values of s. Its jump at a point $s \in -\operatorname{spec}(\eth_{0,E})$ can be computed using the relative index theorem discussed in §6.2 or from the direct analysis of $\eta_s(\eth_{0,E})$ in §8.14:

$$\widetilde{\operatorname{ind}}_s(\eth_{0,E}) = \begin{cases} \lim\limits_{r\downarrow s} \operatorname{ind}_r(\eth_{0,E}) + \frac{1}{2}\dim\operatorname{null}(\eth_{0,E} + s) \\ \lim\limits_{r\uparrow s} \operatorname{ind}_r(\eth_{0,E}) - \frac{1}{2}\dim\operatorname{null}(\eth_{0,E} + s). \end{cases}$$

This allows the value of $\widetilde{\operatorname{ind}}_0(\eth_E^+)$ to be computed explicitly as

$$\widetilde{\operatorname{ind}}_0(\eth_E^+) = \dim\operatorname{null}(\eth_E^+) - \dim\operatorname{null}_-(\eth_E^-) + \frac{1}{2}\dim\operatorname{null}(\eth_{0,E}).$$

Here $\operatorname{null}(\eth_E^+)$ is the null space of $\eth_E^+$ on $L^2(X; {}^+S \otimes E)$ and

$$\operatorname{null}_-(\eth_E^-) = \bigcap_{s<0} \left\{ u \in x^s L^2(X; {}^-S \otimes E); \eth_E^- u = 0 \right\}$$

[2] This is not to be confused with the eta function of $\eth_{0,E}$ which is discussed in Chapter 9.

is the 'extended L^2 null space.' This gives the familiar form of the index theorem in general, from [8], even when (In.5) is not valid:

$$\dim \operatorname{null}(\eth_E^+) - \dim \operatorname{null}_-(\eth_E^-) = \int_X \mathrm{AS} - \frac{1}{2}\left[\eta(\eth_{0,E}) + \dim \operatorname{null}(\eth_{0,E})\right].$$

The integer on the left, which is just the index on $x^s L^2(X; {}^+S \otimes E)$ for small $s > 0$, is sometimes called the extended L^2 index.

The absence of the local index theorem in the weighted case means that the form of the integrand, $\mathrm{AS}(s)$, in (In.33) cannot be so easily computed. However it is a polynomial in s. The removal of the non-constant terms allows the general formula to be recast as

$$\text{(In.34).}\qquad \widetilde{\operatorname{ind}}_s(\eth_E^+) = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}) - \widetilde{N}(\eth_{0,E}, s).$$

Here $\widetilde{N}(\eth_{0,E}, 0) = 0$ and for $s \neq 0$

$$\text{(In.35)}\quad \operatorname{sgn}(s) \times \widetilde{N}(\eth_{0,E}, s) = \text{Number of eigenvalues of } -\eth_{0,E} \text{ in } [0, s],$$

where eigenvalues are counted with their multiplicity and an eigenvalue at an endpoint of $[0, s]$ is counted with half its multiplicity.

7. Outline.

As already noted the remaining chapters are intended to place the proof outlined above on a firm basis and in context. First, in Chapter 1, the one-dimensional case, or rather analogue, of the theorem is discussed, although it is not proved. This discussion is not used later but serves to indicate the different ways of viewing a cylindrical end and introduces the power law behaviour of solutions, and fundamental solutions, which underlies the later analysis. Chapter 2 consists of a brief introduction to Riemannian geometry, the Levi-Civita connection and Riemann curvature tensor, presented in order that the extension to b-metrics should be straightforward. Again the point of view taken is that these metrics correspond in the category of compact manifolds with boundary to Riemann metrics in the boundaryless case. For example, they are fibre metrics on a vector bundle, the b-tangent bundle, which is *not* quite the ordinary tangent bundle but is a perfectly satisfactory replacement for it. The notion of a b-differential operator is introduced, as is the notion of ellipticity in this setting. In Chapter 3 the discussion is extended to examine the Clifford algebra and spin structures. It is shown that the Dirac operator associated to an exact b-spin structure is an elliptic b-differential operator.

The analytic part of the investigation begins, very geometrically, in Chapter 4, with a discussion of the b-stretched product of manifolds with boundary. This is the replacement for the ordinary product that it is convenient to use in the inversion of elliptic b-differential operators and it leads directly to the definition and basic properties of the small b-calculus. In particular the normal homomorphism underlying (In.23) and the product formula are then derived. As noted in the discussion of the composition formula, there is a more elegant, and general, approach using somewhat more differential-geometric machinery (see [63]). Some parts of this approach are introduced in the exercises but it is eschewed here in favour of a more elementary treatment. An effort is made to emphasize the structural properties of the b-calculus. One important feature is the b-trace functional. The commutator identity for this functional, (In.22), plays an important rôle in the proof in that it replaces otherwise cumbersome manipulations of the heat kernel on the cylindrical end, as carried out in [8] and in other versions of the theorem such as [85].

The small calculus of b-pseudodifferential operators reduces to the ordinary calculus of pseudodifferential operators when the compact manifold has no boundary. Philosophically there are two main uses for the ordinary calculus. It is used as an investigative tool (in microlocal analysis) and also to invert elliptic operators. The fact that the same space of operators serves both purposes, when $\partial X = \emptyset$, is somewhat fortuitous. For a manifold with boundary this is no longer the case and to invert elliptic b-differential operators it is necessary to enlarge the calculus. For this purpose, both the 'calculus with bounds' and the 'full calculus' are introduced in Chapter 5. Here the additional boundary terms which appear in the (generalized) inverse are described. The full calculus is applied to the examination of the mapping, and especially Fredholm, properties of elliptic b-differential operators. In Chapter 6 the calculus is further used to establish the relative index theorem and to describe the holomorphy properties of the resolvent family of a self-adjoint operator of second order. The boundary behaviour of the resolvent is also related to scattering theory. As an application of the relative index theorem, using an idea of Gromov and Shubin (see [40]), the Riemann-Roch theorem for surfaces is deduced. This chapter also contains a Hodge theoretic discussion (of course from the point of view of b-metrics) of the cohomology of a compact manifold with boundary.

In Chapter 7 the heat kernel of a second order operator is described. This is done in a manner consistent with the treatment of the b-calculus, i.e. using a blown-up 'heat space' to define the class of admissible kernels. This approach is very similar to the calculi described by Beals and Greiner in [14], by Taylor in [89] and more recently in [31]. Here the geometric structure is made explicit and this has the important consequence that the melding of the heat and b-calculi, to yield a detailed description of the heat kernel of a b-differential operator (such as $\eth_E^-\eth_E^+$) is straightforward. The discussion of the resolvent in Chapter 6 is used to analyze the long-time behaviour of the heat kernel.

Getzler's rescaling argument is formalized in Chapter 8 in the notion of the rescaling, at a boundary hypersurface, of a vector bundle. The weighting of the Sobolev spaces in (In.29) and the b-tangent bundle are both examples of this general procedure. The local index theorem then follows directly from this rescaling, the fundamental observation of Berezin and Patodi on the structure of the supertrace functional on the spin bundle, Lichnerowicz' formula for the difference between the Dirac and connection Laplacians and a generalization of Mehler's formula for the heat kernel of the harmonic oscillator, found by Getzler.

Finally in Chapter 9 the proof of the APS theorem outlined above is reviewed and completed by annotation with references to the intervening material. In fact the theorem is actually proved in the wider context of the Dirac operators on Hermitian Clifford modules (with graded unitary Clifford b-connections) on manifolds with exact b-metrics. The application to the signature formula given in [8] is then explained. It is also shown how the application of the b-calculus allows many of the standard analytic-geometric objects, such as the zeta function, the eta invariant and the Ray-Singer analytic torsion to be transferred to the b-category.

Chapter 1. Ordinary differential operators

The basic analytic tool developed below to carry out the proof of the APS theorem is the calculus of b-pseudodifferential operators. This allows the mapping, especially Fredholm, and spectral properties of $\eth_E^+$ to be readily understood. As motivation for the analytic part of the discussion the one-dimensional case will first be considered, although the result is not proved in detail. This case is 'easy' for many reasons, not least because the dimension is odd, which means there is no interior contribution to the index, and the boundary dimension is zero, so the boundary operator is a matrix, i.e. has finite rank. However, from an analytic point of view the one-dimensional cases serves as quite a good guide to the general case.

1.1. Operators and coordinates.

In one dimension there is *no* spin structure to be concerned about. The only connected one-dimensional compact manifold with non-trivial boundary is the interval $X = [0, 1]$. In particular all bundles are trivial. Of course one should bear in mind that the boundary has two components. So consider a first-order linear differential operator acting on k functions

$$P = A(x)\frac{d}{dx} + B(x), \quad A, B \in \mathcal{C}^\infty([0, 1]; M_{\mathbb{C}}(k)), \tag{1.1}$$

where $M_{\mathbb{C}}(k) \simeq \mathbb{C}^{k^2}$ is the algebra of $k \times k$ complex-valued matrices.

The operator should be *elliptic* in the interior, so $\det(A(x)) \neq 0$ for $x \in (0, 1)$. It should also be of 'b' type at the end-points. This means that P should have regular-singular points at 0 and 1 :

$$A(x) = x(1-x)E(x), \quad E \in \mathcal{C}^\infty([0, 1]; M_{\mathbb{C}}(k)),\ \det E \neq 0 \text{ on } [0, 1]. \tag{1.2}$$

To analyze the *index* of P consider the adjoint, P^*, and set

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim \operatorname{null}(P^*) \tag{1.3}$$

or find some space on which P is *Fredholm*:

a) $P : H_1 \longrightarrow H_2$ continuous, H_1, H_2 Hilbert spaces
b) $\operatorname{null}(P) \subset H_1$ finite dimensional
c) $\operatorname{range}(P) \subset H_2$ closed
d) $\operatorname{range}(P)^\perp \subset H_2$ finite dimensional

and then set

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim(\operatorname{range}(P)^\perp). \tag{1.4}$$

If P is Fredholm these two definitions of the index are the same.

What is a reasonable space on which P can be expected to be Fredholm? Consider the simple case

$$P_c = x(1-x)\frac{d}{dx} + c, \quad c \in \mathbb{C}. \tag{1.5}$$

There are two transformations of the independent variable which yield even simpler operators. First

$$t = \frac{x}{1-x} : [0,\infty) \ni t \longmapsto x = \frac{t}{1+t} \in [0,1) \tag{1.6}$$

and then $s = \log t = \log x - \log(1-x)$, $(-\infty,\infty) \ni s \longmapsto t = e^s \in (0,\infty)$. Notice that

$$\frac{dt}{dx} = \frac{1}{(1-x)^2} = (1+t)^2, \ \frac{dt}{ds} = e^s = t,$$

so

$$x(1-x)\frac{d}{dx} = \frac{x}{1-x}\frac{d}{dt} = t\frac{d}{dt}$$
$$x(1-x)\frac{d}{dx} = \frac{d}{ds}.$$

Thus P_c in (1.5) becomes

$$P_c = t\frac{d}{dt} + c, \ \mathbb{R}^+\text{-invariant on } (0,\infty)$$
$$P_c = \frac{d}{ds} + c, \ \text{translation-invariant on } \mathbb{R}.$$

Certainly then P_c is easy to analyze. Acting on any reasonable class of functions, distributions or even hyperfunctions, P_c has at most a one-dimensional null space, given in the three coordinates systems by

$$P_c u = 0 \Longrightarrow u = \begin{cases} ax^{-c}(1-x)^c \\ at^{-c} & a \in \mathbb{C}. \\ ae^{-cs} \end{cases} \tag{1.7}$$

Then the only question is whether or not this solution is in the domain of P_c.

Suppose P_c^* is taken to be the adjoint with respect to Lebesgue measure, $|ds|$, on $\mathbb{R}$

$$P_c^* = -\frac{d}{ds} + \bar{c}.$$

Thus (1.7) applies to $P_c^* = -P_{-\bar{c}}$. It is reasonable to take the *domain* of P_c to be

$$H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}); \frac{d}{ds}u \in L^2(\mathbb{R})\}, \tag{1.8}$$

the standard Sobolev space.

EXERCISE 1.1. Check that if the domain of P_c^* is defined by

$$\operatorname{Dom}(P_c^*) = \{u \in L^2(\mathbb{R}); H^1(\mathbb{R}) \ni v \longmapsto \int u\overline{P_c v}ds \\ \text{extends by continuity to } L^2(\mathbb{R})\}$$

then $\operatorname{Dom}(P_c^*) = H^1(\mathbb{R})$.

1.2. Index.

Now notice that

$$\exp(-cs) \notin L^2(\mathbb{R}) \qquad \forall\ c \in \mathbb{C}, \tag{1.9}$$

since the exponential is *always* too large in one direction or the other (or both if $c \in i\mathbb{R}$). Thus, with domain (1.8) and definition (1.3), it is always the case that

$$\operatorname{ind}(P_c) = 0, \tag{1.10}$$

which is not too interesting!

The constant in P_c can be changed by conjugating by an exponential

$$e^{-as}(\frac{d}{ds} + c)e^{as}u = \big(\frac{d}{ds} + (a + c)\big)u. \tag{1.11}$$

Since the function e^{as} is real when a is, the adjoint changes to

$$e^{as}(-\frac{d}{ds} + \bar{c})e^{-as} = \big(-\frac{d}{ds} + (\bar{c} + a)\big).$$

This corresponds to replacing the Sobolev space (1.8) by the exponentially weighted space

$$e^{as}H^1(\mathbb{R}) = \{u \in L^2_{\text{loc}}(\mathbb{R}); e^{-as}u \in H^1(\mathbb{R})\},$$

where $L^2_{\text{loc}}(\mathbb{R})$ is the space of locally square-integrable functions on $\mathbb{R}$. Certainly (1.10) still holds on these spaces, but somehow not quite for the 'same' reason in that the part of infinity which causes (1.9) may have changed. This suggest that a less trivial result may follow by looking at spaces which are *weighted* differently at the *two* infinities, i.e. boundary points.

Recall that $H^1_{\text{loc}}(0,1)$ is the space of locally square-integrable functions on $(0,1)$ with first derivative, in the distributional sense, also given by a locally square-integrable function.

DEFINITION 1.2. For $\alpha, \beta \in \mathbb{R}$ set

$$x^{\alpha}(1-x)^{\beta}H^1_b([0,1])$$
$$= \Big\{ u \in H^1_{\mathrm{loc}}((0,1));\ \int_0^1 \Big| x^{-\alpha}(1-x)^{-\beta}u \Big|^2 \frac{dx}{x(1-x)} < \infty,$$
$$\int_0^1 \Big| x^{-\alpha}(1-x)^{-\beta}\big(x(1-x)\frac{du}{dx}\big)\Big|^2 \frac{dx}{x(1-x)} < \infty \Big\}.$$

On passing to the variable s on $\mathbb{R}$, these conditions can be written in terms of $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ as the requirements

$$\begin{aligned} &\int_{-\infty}^0 \Big| e^{-\alpha s} w\Big|^2 ds,\ \int_0^{\infty} \Big| e^{\beta s} w\Big|^2 ds < \infty \\ &\int_{-\infty}^0 \Big| e^{-\alpha s}\frac{dw}{ds}\Big|^2 ds,\ \int_0^{\infty} \Big| e^{\beta s}\frac{dw}{ds}\Big|^2 ds < \infty. \end{aligned} \tag{1.12}$$

Thus in terms of the variable s these spaces are exponentially weighted at infinity, with different weights.

EXERCISE 1.3. Check that in terms of the variable t the spaces in Definition 1.2 become

$$x^{\alpha}(1-x)^{\beta}H^1_b([0,1]) \longleftrightarrow$$
$$\Big\{ v \in H^1_{\mathrm{loc}}((0,\infty));\ \int_0^{\infty} \Big| (\frac{t}{1+t})^{-\alpha}(1+t)^{\beta} v\Big|^2 \frac{dt}{t} < \infty,$$
$$\int_0^{\infty} \Big| (\frac{t}{1+t})^{-\alpha}(1+t)^{\beta}\big(t\frac{d}{dt}v\big)\Big|^2 \frac{dt}{t} < \infty \Big\}.$$

PROPOSITION 1.4. *The operator* P_c *in* (1.5) *is Fredholm as an operator*

$$P_c:\ x^{\alpha}(1-x)^{\beta}H^1_b([0,1]) \longrightarrow x^{\alpha}(1-x)^{\beta}L^2_b([0,1]) \tag{1.13}$$

if and only if

$$\alpha \neq -\operatorname{Re} c,\ \beta \neq \operatorname{Re} c; \tag{1.14}$$

its index is

$$\operatorname{ind}(P_c) = \begin{cases} 1 & \alpha < -\operatorname{Re} c,\ \beta < \operatorname{Re} c \\ 0 & \alpha > -\operatorname{Re} c,\ \beta < \operatorname{Re} c \text{ or } \alpha < -\operatorname{Re} c, \beta > \operatorname{Re} c \\ -1 & \alpha > -\operatorname{Re} c,\ \beta > \operatorname{Re} c. \end{cases} \tag{1.15}$$

PROOF: Certainly P_c is always a continuous linear map (1.13) as follows directly from the definition of these spaces. To study the Fredholm properties, it is enough to *invert* P and then see how the inverse is affected by the weighting. Using (1.11) the constant c can be removed since

$$(\frac{x}{1-x})^{-c} : x^{\alpha}(1-x)^{\beta}H_b^1([0,1]) \longleftrightarrow x^{a-\mathrm{Re}\, c}(1-x)^{b+\mathrm{Re}\, c}H_b^1([0,1])$$

for any $c \in \mathbb{C}$ and

$$(\frac{x}{1-x})^{c}P_c(\frac{x}{1-x})^{-c} = x(1-x)\frac{d}{dx}. \tag{1.16}$$

This reduces the problem to the special case $c = 0$.

Now it is convenient to work in the translation-invariant picture, where $P = d/ds$. An inverse is given by integration, say from 0 :

$$Q'g(s) = \int_0^s g(r)dr \Longrightarrow \frac{d}{ds}Q'g(s) = g. \tag{1.17}$$

Suppose $f \in x^{\alpha}(1-x)^{\beta}L_b^2([0,1])$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R})$ is its expression in terms of the coordinate s. From (1.12) it follows that

$$\beta < 0 \Longrightarrow \int_0^{\infty} \left| e^{s\beta}Q'g(s)\right|^2 ds < \infty$$

$$\alpha < 0 \Longrightarrow \int_{-\infty}^{0} \left| e^{-s\alpha}Q'g(s)\right|^2 ds < \infty.$$

Thus if Qf is $Q'g$ expressed in terms of the coordinate x,

$$\alpha, \beta < 0 \Longrightarrow Q : x^{\alpha}(1-x)^{\beta}L_b^2([0,1]) \longrightarrow x^{\alpha}(1-x)^{\beta}H_b^1([0,1]).$$

In this case P_c is surjective, so certainly Fredholm. From the definition, (1.4), of the index and the fact that the null space is spanned by

$$1 \in x^{\alpha}(1-x)^{\beta}H_b^1([0,1]) \text{ if } \alpha < 0,\ \beta < 0,$$

the validity of the first case in (1.15) follows.

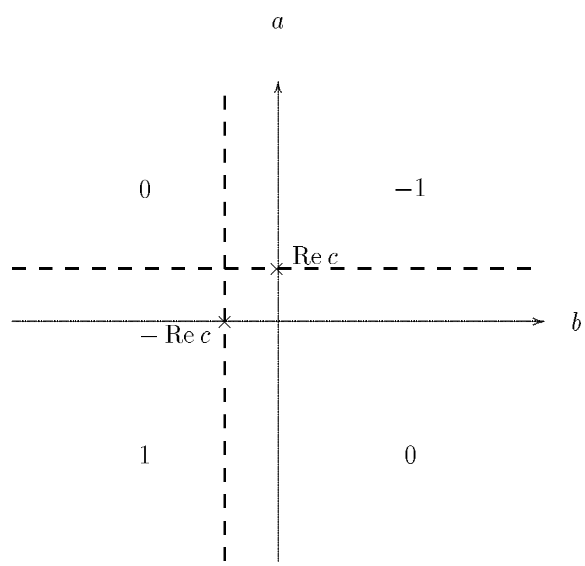

Figure 1. The index of P_c.

Suppose $\beta > 0$, then (1.17) does not give a solution correctly weighted at infinity, unless $\int_0^\infty g(s)ds = 0$. However taking instead

$$Q'g = \int_\infty^s g(r)dr \Longrightarrow \frac{d}{ds}Q'g = g, \tag{1.18}$$

since if $f \in x^\alpha(1-x)^\beta L_b^2$ then g *is* integrable near $s = +\infty$. In this case

$$\alpha < 0,\ \beta > 0 \Longrightarrow P \text{ is surjective, } \operatorname{ind}(P) = 0,$$

since 1 is no longer in $x^\alpha(1-x)^\beta L_b^2$. The same argument applies if $\alpha > 0$, $\beta < 0$ by replacing x by $1-x$, and hence s by $-s$.

Finally if $\alpha > 0$, $\beta > 0$ then (1.18) is still correctly behaved near infinity but

$$Qf \in x^\alpha(1-x)^\beta L_b^2([0,1]),\ \alpha, \beta > 0 \text{ iff } \int_{-\infty}^\infty f(x)\frac{dx}{x(1-x)} = 0.$$

Certainly the constant solution is not in the domain so $\operatorname{ind}(P) = -1$.

This completes the proof of the proposition except for the part of (1.14) which states that P_c is not Fredholm as an operator (1.13) if

$$\alpha = -\operatorname{Re} c \text{ or } \beta = \operatorname{Re} c. \tag{1.19}$$

This is left as an exercise:

Exercise 1.5. Show that if $\alpha = \operatorname{Re} c$ or $\beta = -\operatorname{Re} c$ then P in (1.13) is not Fredholm because the range is *not* closed. [Hint: Find a sequence of functions in $L^2(\mathbb{R})$ of the form du_k/ds, $u_k \in H^1(\mathbb{R})$ such that $du_k/ds \longrightarrow 0$ in L^2 but $||u_k||_{L^2} \longrightarrow \infty$. Use this to show that there exists $f \in L^2(\mathbb{R})$ which is in the closure of the range but $f \neq du/ds$, for any $u \in H^1(\mathbb{R})$.]

1.3. General statement.

Proposition 1.4 can be interpreted informally as saying that the operator, P_c, is Fredholm *unless* there is an element of the null space (on distributions) which is *almost* in the domain, but is *not* in the domain. Notice also that in Figure 1 the index increases by 1 (the dimension of this null space) every time one of the lines in (1.19) is crossed downward. These two ideas will reappear in the higher dimensional setting below.

Now consider the extension of this result to the general case, (1.1) subject to (1.2). This is the prototype for the Dirac operator.

Theorem 1.6. *The operator P in (1.1), subject to (1.2), is always a continuous linear operator*

$$P: \ x^\alpha(1-x)^\beta H^1_b([0,1];\mathbb{C}^k) \longrightarrow x^\alpha(1-x)^\beta L^2_b([0,1];\mathbb{C}^k), \ \alpha,\beta\in\mathbb{R},$$

which is Fredholm if and only if

$$\begin{aligned} &\alpha \neq \operatorname{Re}\lambda \textit{ for any eigenvalue } \lambda \textit{ of } -E(0)^{-1}B(0) \\ &\beta \neq \operatorname{Re}\lambda \textit{ for any eigenvalue } \lambda \textit{ of } E(1)^{-1}B(1) \end{aligned} \tag{1.20}$$

and then its index is

$$\operatorname{ind}(P) = -\frac{1}{2}(\eta^0_\alpha + \eta^1_\beta), \tag{1.21}$$

where if G^i_λ are, for $i = 0, 1$, the eigenspaces with eigenvalue λ of the matrices $(-1)^i(E(i))^{-1}B(i)$,

$$\eta^i_r = \sum_{\operatorname{Re}\lambda > -r} \dim G^i_\lambda - \sum_{\operatorname{Re}\lambda < -r} \dim G^i_\lambda. \tag{1.22}$$

EXERCISE 1.7. Try to prove this result. It is not too hard, using standard results on solutions of ordinary differential equations, as in [28]. It is also illuminating to follow the lines of the proof outlined in the Introduction in this case.

Consider how the formula (1.21) reduces to (In.6). First note that the Atiyah-Singer integrand, AS, vanishes identically because the dimension is odd. It is worth noting the relationship between (1.22) and (In.28). From (In.23) it is reasonable to expect that

$$\eth_0 \longleftrightarrow E(0)^{-1}B(0).$$

To make this correspondence more exact, suppose that $\eth_0 = E(0)^{-1}B(0)$ is a self-adjoint matrix and $a = 0$. Then

$$\operatorname{tr} \eth_0 \exp(-t\eth_0^2) = \sum_{\text{eigenvalues}} \lambda e^{-t\lambda^2}, \tag{1.23}$$

where the eigenvalues are repeated with their multiplicity. By the assumption (In.5), or equivalently the condition (1.20), 0 should not be an eigenvalue. Then, inserting (1.23) into (In.28) gives

$$\begin{aligned}\eta(\eth_0) &= \frac{1}{\sqrt{\pi}} \sum_{\text{eigenvalues}} \operatorname{sgn}(\lambda) \int_0^\infty e^{-t|\lambda|^2} (t|\lambda|^2)^{-\frac{1}{2}} d(t|\lambda|^2) \\ &= \sum_{\text{eigenvalues}} \operatorname{sgn}(\lambda),\end{aligned}$$

since

$$\int_0^\infty e^{-s} \frac{ds}{s^{\frac{1}{2}}} = \sqrt{\pi}.$$

Thus $\eta(\eth_0) = \eta_0^0$ in terms of (1.22). This shows the relationship between (In.6) and (1.21). It also suggests that the eta invariant measures the *spectral asymmetry* of the operator, i.e. the difference between the number of positive and the number of negative eigenvalues.

EXERCISE 1.8. Check the relationship between (1.21) and (In.33).

1.4. Kernels.

To finish this look at the one-dimensional case, consider again the trivial case (1.5). For $\alpha << 0$, $b >> 0$ the solution operator to P_c is obtained

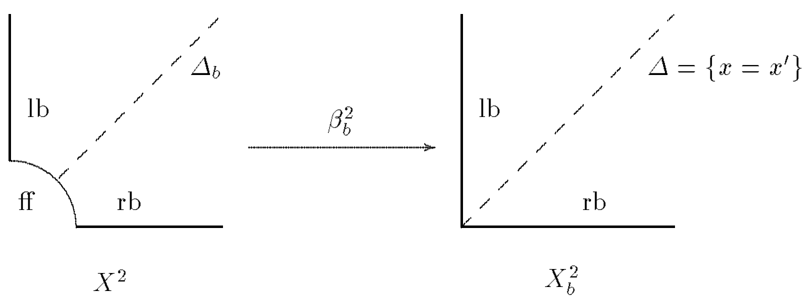

Figure 2. Blow-up of X^2, $X = [0, 1]$.

using the conjugation (1.16), the change of coordinates (1.6) and the integration formula (1.18). Consider this inverse in terms of the compact, x-representation. Then it can be written

$$Qf(x) = \int_1^x f(x') \frac{dx'}{x'(1-x')}.$$

Undoing (1.16) gives the inverse to P_c as

$$Q_c f(x) = \int_1^x \left(\frac{x}{1-x}\right)^c \left(\frac{x'}{1-x'}\right)^{-c} f(x') \frac{dx'}{x'(1-x')}.$$

This can be written as an integral operator

$$Q_c f(x) = \int_0^1 K(x, x') f(x') \frac{dx'}{x'(1-x')},$$

where the Schwartz kernel is

$$K_c(x, x') = -\frac{x^c}{(1-x)^c} \frac{(1-x')^c}{(x')^c} H(x' - x), \tag{1.24}$$

$H(t)$ being the Heaviside function.

Consider the structure of K_c. There are, in principle, five singular terms, with singularities at $x = 0, x = 1, x' = 0, x' = 1$ and $x = x'$ (except that those at $x' = 0$ and $x = 1$ happen to vanish). The singularities are simple power type, except that at the two corners $x = x' = 0$ and $x = x' = 1$

two singularities coincide and things look nasty. It is exactly such kernels that will be analyzed below. To do so, it is convenient to introduce *polar coordinates* around $x = x' = 0$ and $x = x' = 1$.

Actually, to carry out this process of *blowing up,* it is simpler to use the singular coordinates (near $x = x' = 0$)

$$r = x + x', \ \tau = \frac{x - x'}{x + x'}. \tag{1.25}$$

Then $x = \frac{1}{2}r(1+\tau)$, $x' = \frac{1}{2}r(1-\tau)$. Inserting this into (1.24) gives

$$K_c(x, x') = \frac{(1+\tau)^c}{(1-\tau)^c} \times H(-\tau) \times \mathcal{C}^\infty \text{ near } x = x' = 0. \tag{1.26}$$

Notice what is accomplished by this maneuver. The kernel now has singularities at two separated surfaces, $\tau = 0$ and $\tau = -1$. This trick is the basis of the b-calculus.

Exercise 1.9. Write out the relationship between the coordinates r, τ in (1.25) and polar coordinates ρ, θ where $x = \rho\cos\theta$, $x' = \rho\sin\theta$ and $\theta \in [0, \frac{1}{4}\pi]$. Check that the map $\rho, \theta \longrightarrow (r, \tau)$ is a diffeomorphism from $[0, \infty) \times [0, \frac{1}{4}\pi]$ onto $[0, \infty) \times [-1, 1]$.

Exercise 1.10. Find formulæ similar to (1.26) for the Schwartz kernel of a generalized inverse to P_c in the other cases in (1.15).

Chapter 2. Exact b-geometry

Much of this chapter is geometric propaganda. It is intended to convince the reader that there is a 'category' of b-Riemann manifolds in which one can work systematically. This b-geometry can also be thought of as the geometry of manifolds with asymptotically cylindrical ends. There are other geometries which are similar to b-geometry (see [55], [54], [56], [32], [31] and [63] for a general discussion).

Following the definition and discussion of the most basic elements of (exact) b-geometry, the Levi-Civita connection is described *ab initio*. The notion of a b-connection is introduced and its relation to that of an ordinary connection is explained. Finally a brief description of characteristic classes is given. For the reader familiar with differential geometry, the main sections to peruse are §§2.2, 2.3, 2.4, 2.13, 2.16 and 2.17.

2.1. Manifolds.

It is assumed below that the reader is familiar with elementary global differential geometry, i.e. the concept of a manifold. However at various points later, the less familiar notion of a manifold with corners is encountered so, for the sake of clarity, definitions are given here. These have been selected for terseness rather than simplicity or accessibility!

A *topological manifold* of dimension N is a paracompact Hausdorff (connected unless otherwise noted) topological space, X, with the property that each point $p \in X$ is contained in an open set $O \subset X$ which is homeomorphic to $\mathbb{B}^N = \{x \in \mathbb{R}^N; |x| < 1\}$. These open sets, with their maps to $\mathbb{B}^N$, are called coordinate patches.

The algebra of all continuous functions, real-valued unless otherwise stated, is denoted $\mathcal{C}^0(X)$. A subalgebra $\mathcal{F} \subset \mathcal{C}^0(X)$ is said to be a $\mathcal{C}^\infty$ *subalgebra* if for any real-valued $g \in \mathcal{C}^\infty(\mathbb{R}^k)$, for any k, and any elements $f_1, \dots, f_k \in \mathcal{F}$ the continuous function $g(f_1, \dots, f_k) \in \mathcal{F}$. The subalgebra is said to be *local* if it contains each element $g \in \mathcal{C}^0(X)$ which has the property that for every set O_α in some covering of X by open sets there exists $g_\alpha \in \mathcal{F}$ with $g = g_\alpha$ on O_α.

A *manifold* (meaning here always an infinitely differentiable, shortened to $\mathcal{C}^\infty$, manifold) is a topological manifold with a real, local, $\mathcal{C}^\infty$ subalgebra $\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X)$ specified with the following property: X has a covering by open sets O_α, $\alpha \in A$, for each of which there are N elements $f_1^\alpha, \dots, f_N^\alpha \in \mathcal{C}^\infty(X)$ with $F^\alpha = (f_1^\alpha, \dots, f_N^\alpha)$ restricted to O_α making it a coordinate patch and $f \in \mathcal{C}^\infty(X)$ if and only if for each $\alpha \in A$ there exists $g_\alpha \in \mathcal{C}^\infty(\mathbb{B}^N)$ such that $f = g_\alpha \circ F^\alpha$ on O_α.

EXERCISE 2.1. Show that this definition is equivalent to the standard one involving covering by compatible infinitely differentiable coordinate systems. [Hint: Show that the O_α in the previous paragraph form such a covering. To do so first check that if $g \in \mathcal{C}^\infty_c(\mathbb{B}^N)$ is a $\mathcal{C}^\infty$ function vanishing outside a compact subset of $\mathbb{B}^N$ then for fixed α,

$$f = \begin{cases} g \circ F^\alpha & \text{on } O_\alpha \\ 0 & \text{on } X \setminus O_\alpha \end{cases}$$

is an element of $\mathcal{C}^\infty(X)$ – this argument uses both the $\mathcal{C}^\infty$ algebra and locality properties of $\mathcal{C}^\infty(X)$. Use this in turn to show that the coordinate patches are $\mathcal{C}^\infty$ compatible since, under the transition maps, $\mathcal{C}^\infty$ functions of compact support pull back to be $\mathcal{C}^\infty$.]

Open subsets of manifolds are naturally manifolds. A map $f : X \longrightarrow Y$ between manifolds is smooth, i. e. $\mathcal{C}^\infty$, if $f^*g = g \circ f \in \mathcal{C}^\infty(X)$ whenever $g \in \mathcal{C}^\infty(Y)$. A map is a *diffeomorphism* if it is $\mathcal{C}^\infty$ and has a $\mathcal{C}^\infty$, two-sided, inverse. A finite set of real-valued functions $f_1, \dots, f_k \in \mathcal{C}^\infty(X)$, is *independent* at p if there exist $g_1, \dots, g_{N-k}$, where $N = \dim X$, such that $F = (f_1, \dots, f_k, g_1, \dots, g_{N-k})$ restricts to some neighbourhood of p to a diffeomorphism onto an open subset of $\mathbb{R}^N$.

In the sequel we are most involved with manifolds with boundary. A *topological manifold with boundary* is defined exactly as for a topological manifold, except that each point is only required to have an open neighbourhood homeomorphic to either $\mathbb{B}^N$ or to $\mathbb{B}^N_+ = \{x \in \mathbb{R}^N; |x| < 1, x_1 \geq 0\}$. Since these spaces are not themselves homeomorphic, the two cases are distinct and the subset of X consisting of the points with neighbourhoods homeomorphic to $\mathbb{B}^N_+$ constitutes the boundary, ∂X.

EXERCISE 2.2. Show that the interior, $X \setminus \partial X$, and the boundary of a topological manifold with boundary are both topological manifolds.

Infinitely differentiable manifolds with boundary can be defined quite analogously to the boundaryless case. However we give instead a more extrinsic definition of manifolds with corners and then specialize. A *manifold with corners* (always by implication $\mathcal{C}^\infty$) is a topological manifold with boundary with a local $\mathcal{C}^\infty$ subalgebra $\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X)$ specified with the following property: there is a map into a manifold $\widetilde{X}$, $\iota : X \longrightarrow \widetilde{X}$, for which $\mathcal{C}^\infty(X) = \iota^*\mathcal{C}^\infty(\widetilde{X})$ and a finite collection of functions $\rho_i \in \mathcal{C}^\infty(\widetilde{X})$, $i \in I$, for which $\iota(X) = \left\{y \in \widetilde{X}; \rho_i(y) \geq 0 \ \forall\ i \in I\right\}$ and for each $J \subset I$ the ρ_j, for $j \in J$, are independent at each point $p \in \widetilde{X}$ at which they all vanish. The manifold $\widetilde{X}$ is an extension of X.

This definition forces the boundary of a manifold with corners to be a union of embedded hypersurfaces. A direct local coordinate definition (with coordinate patches modeled on $\{x \in \mathbb{R}^N; |x| < 1, x_i \geq 0, i = 1, \dots, k\}$)

does not automatically ensure this. See [63] for a discussion of this point. Smooth maps are defined as before.

EXERCISE 2.3. Show that the product of two manifolds with corners is a manifold with corners.

A *manifold with boundary* is the special case of a manifold with corners when I can be taken to have one element. The boundary of a manifold with boundary is necessarily a manifold without boundary.

A *(real) vector bundle* of rank p over a manifold with corners, Y, consists of a smooth map $f\colon X \longrightarrow Y$ of manifolds with corners where the fibres $f^{-1}(y) \subset X$ have linear structures (of dimension p) which vary smoothly in the sense that the subspace $\mathcal{F} \subset \mathcal{C}^\infty(X)$ of $\mathcal{C}^\infty$ functions which are affine linear on the fibres generates the $\mathcal{C}^\infty$ structure. This last condition means that $\mathcal{C}^\infty(X)$ is the smallest local $\mathcal{C}^\infty$ subalgebra of $\mathcal{C}^0(X)$ containing $\mathcal{F}$ and furthermore $f^*\mathcal{C}^\infty(Y) \subset \mathcal{F}$ is exactly the space of fibre-constant smooth functions.

EXERCISE 2.4. Not only check that this reduces to your favourite definition in case Y (and hence X) has no boundary, but show that such a vector bundle can always be obtained as the restriction of a vector bundle from an extension of Y to a manifold without boundary.

2.2. The b-tangent bundle.

Let X be a compact $\mathcal{C}^\infty$ manifold with boundary. The differential geometry of X will be developed by straightforward extension from the boundaryless case, which is first recalled succinctly. The space X comes equipped with its algebra of $\mathcal{C}^\infty$ functions, $\mathcal{C}^\infty(X)$. Localizing this leads to the cotangent bundle. Thus, if $p \in X$, let

$$\mathcal{I}_p(X) = \left\{u \in \mathcal{C}^\infty(X); u(p) = 0\right\}$$

be the ideal of functions vanishing at p and define the cotangent space at p by $T_p^*X = \mathcal{I}_p/\mathcal{I}_p^2$ where $\mathcal{I}_p^2$ is the linear span of products of pairs of elements of $\mathcal{I}_p$. Clearly T_p^*X has a natural linear structure inherited from $\mathcal{I}_p$. In local coordinates $z^1, \dots, z^N$ based at p any element of $\mathcal{I}_p$ is of the form $\sum_{j=1}^N a_j z^j$, modulo $\mathcal{I}_p^2$. The constants a_j therefore give linear coordinates in T_p^*X. The dual space is the tangent space at p, $T_pX = (T_p^*X)^*$. Both the cotangent and the tangent spaces combine to give $\mathcal{C}^\infty$ vector bundles over X, including in the case of a manifold with boundary:

$$TX = \bigsqcup_{p \in X} T_pX, \quad T^*X = \bigsqcup_{p \in X} T_p^*X.$$

Any $\mathcal{C}^\infty$ function on X defines a section, $df : X \longrightarrow T^*X$, of T^*X where $df(p) = [f - f(p)] \in T_p^*X$. The space of *all* $\mathcal{C}^\infty$ sections of T^*X is just the span

$$\mathcal{C}^\infty(X; T^*X) = \operatorname*{span}_{\mathcal{C}^\infty(X)} \left\{df; f \in \mathcal{C}^\infty(X)\right\}.$$

Over the boundary of X there is a natural subbundle of TX, namely

$$T\partial X \hookrightarrow T_{\partial X}X = \bigsqcup_{p \in \partial X} T_pX. \tag{2.1}$$

This is defined by noting that the conormal space to the boundary

$$N_p^*\partial X = \{df(p); f \in \mathcal{C}^\infty(X), f_{\restriction \partial X} = 0\} \subset T_p^*X, p \in \partial X$$

is a line in T_p^*X such that $0 \longrightarrow N^*\partial X \longrightarrow T_{\partial X}^*X \longrightarrow T^*\partial X \longrightarrow 0$ is a short exact sequence, where the projection is given by pull-back to the boundary. Then (2.1) follows by duality, with $T_p\partial X$ the annihilator of $N_p^*\partial X$.

The space of all $\mathcal{C}^\infty$ sections of TX is the Lie algebra of vector fields; it acts on $\mathcal{C}^\infty(X)$:

$$\begin{gathered}\mathcal{C}^\infty(X; TX) = \mathcal{V}(X) \ni V,\ V : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X)\\ Vf(p) = df(p)\big(V(p)\big).\end{gathered}$$

The Lie bracket is fixed by

$$[V, W]f = VWf - WVf \quad \forall\ f \in \mathcal{C}^\infty(X).$$

Now the inclusion (2.1) allows one to define the subspace of vector fields tangent to the boundary

$$\mathcal{V}_b(X) = \left\{V \in \mathcal{V}(X); V_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; T\partial X)\right\}. \tag{2.2}$$

It is a Lie subalgebra of $\mathcal{V}(X)$. The *fundamental point* leading to *b-geometry* is that $\mathcal{V}_b(X)$ is itself the space of *all* $\mathcal{C}^\infty$ sections of a vector bundle:

$$\mathcal{V}_b(X) = \mathcal{C}^\infty(X; {}^bTX). \tag{2.3}$$

The bundle ${}^bTX$, defined so (2.3) holds, is called the *b-tangent bundle.*

LEMMA 2.5. *For each $p \in X$ (including boundary points) define the vector space*

$$ {}^bT_pX = \mathcal{V}_b(X)/\mathcal{I}_p \cdot \mathcal{V}_b(X). \tag{2.4}$$

Then there is a unique vector bundle structure on

$$ {}^bTX = \bigsqcup_{p \in X} {}^bT_pX$$

as a bundle over X such that (2.3) holds in the sense that, under the natural vector bundle map ${}^bTX \longrightarrow TX$, $\mathcal{V}_b(X)$ pulls back to $\mathcal{C}^\infty(X; {}^bTX)$.

PROOF: By definition

$$\mathcal{I}_p \cdot \mathcal{V}_b(X) = \{V \in \mathcal{V}_b(X); V = \sum_{\text{finite}} f_j V_j, V_j \in \mathcal{V}_b(X), f_j \in \mathcal{I}_p(X)\}.$$

This is certainly a linear space for each $p \in X$. The first thing to check is that the dimension of the quotient, (2.4), is constant. Of course over the interior, essentially by definition,

$${}^bT_pX \simeq T_pX = \mathcal{V}(X)/\mathcal{I}_p \cdot \mathcal{V}(X), \quad p \in X \backslash \partial X.$$

Near any boundary point local coordinates $x, y_1, \dots, y_n$, where $\dim X = n+1$, can always be introduced which are adapted to the boundary, in the sense that

$$x \geq 0, \quad \partial X = \{x = 0\} \text{ near } p$$

and which all vanish at p. Then a $\mathcal{C}^\infty$ vector field can be written in terms of the local derivations

$$\mathcal{V}(X) \ni V \Longrightarrow V_{\restriction O} = a\frac{\partial}{\partial x} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y_j},$$

where $O \subset X$ is the coordinate patch and the coefficients are elements of $\mathcal{C}^\infty(O)$. From (2.2) it follows that

$$\begin{aligned} V \in \mathcal{V}_b(X) &\Longrightarrow a_{\restriction \partial X} = 0 \text{ i.e. } a = x\alpha, \text{ with } \alpha \in \mathcal{C}^\infty(X), \\ &\Longrightarrow V = \alpha x \frac{\partial}{\partial x} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y_j}. \end{aligned} \tag{2.5}$$

This indeed is the main content of the lemma, that $x\partial/\partial x$ and the $\partial/\partial y_j$, for $j = 1, \dots, n$ form a local basis for ${}^bTX$ in any adapted coordinates (which will just be called coordinates from now on). So ${}^bT_pX$ always has dimension equal to $\dim X$, even for $p \in \partial X$. It remains to check that the transition matrix between representations (2.5), for different coordinates, is $\mathcal{C}^\infty$. If $x', y'_1, \dots, y'_n$ are new coordinates then

$$\begin{aligned} &x' = xA(x, y) \text{ with } A(0,0) > 0 \text{ and} \\ &y'_j = Y_j(x, y) \text{ with } \det \frac{\partial Y_j}{\partial y_k}(0,0) \neq 0. \end{aligned}$$

Thus

$$\begin{aligned} x\frac{\partial}{\partial x} &= \big(1 + \frac{x}{A}\frac{\partial A}{\partial x}\big) x' \frac{\partial}{\partial x'} + \frac{x'}{A} \sum_{j=1}^{n} \frac{\partial Y_j}{\partial x} \frac{\partial}{\partial y'_j} \text{ and} \\ \frac{\partial}{\partial y_k} &= \frac{1}{A}\frac{\partial A}{\partial y_k} x' \frac{\partial}{\partial x'} + \sum_{j=1}^{n} \frac{\partial Y_j}{\partial y_k} \frac{\partial}{\partial y'_j}. \end{aligned} \tag{2.6}$$

This shows that the transition matrix is $\mathcal{C}^\infty$, so ${}^bTX$ is a well-defined vector bundle over X and (2.3) holds.

A more general version of this construction of ${}^bTX$, from TX, is described in Chapter 8 as the rescaling of vector bundles. For a discussion of ${}^bTX$ in case X is a manifold with corners see [66] and [63].

EXERCISE 2.6. Show that there is a natural vector bundle map from the cotangent bundle of X, T^*X, to the dual bundle to ${}^bTX$, denoted ${}^bT^*X$, and that this map is an isomorphism over the interior of X. Near any boundary point show that ${}^bT^*X$ has a coordinate basis which pulls back under this bundle map to

$$\frac{dx}{x}, dy_j,\ j = 1, \dots, n. \tag{2.7}$$

2.3. Exact b-metrics.

Now, by b-geometry is meant the analogue of Riemannian geometry for the b-tangent bundle, ${}^bTX$. Thus a b-metric on a compact manifold with boundary is simply a metric on the fibres of ${}^bTX$. Over the interior of X,

$${}^bT_{X\setminus\partial X}X \simeq T_{X\setminus\partial X}X,$$

so a b-metric is a Riemann metric on the interior of X with special uniformity properties at the boundary. Indeed, using Exercise 2.6, in local coordinates at the boundary a b-metric can be written in the form

$$g = a_{00}(\frac{dx}{x})^2 + 2\sum_{j=1}^n a_{0j}\frac{dx}{x}dy^j + \sum_{j,k=1}^n a_{jk}dy^j\,dy^k, \tag{2.8}$$

where the coefficients are $\mathcal{C}^\infty$ and the form is positive definite:

$$a_{00}\lambda^2 + 2\sum_{j=1}^n a_{0j}\eta_j\lambda + \sum_{j,k=1}^n a_{jk}\eta_j\eta_k \ge \epsilon(|\lambda|^2 + |y|^2)\ \forall\ \lambda \in \mathbb{R}, \eta \in \mathbb{R}^n.$$

In practice it is convenient to further restrict the class of metrics. First note some of the intrinsic structure of ${}^bTX$. From (2.6) the element $x\partial/\partial x$ is, at a boundary point, completely invariant:

$$x\frac{\partial}{\partial x} \in T_pX \text{ is well-defined at } p \in \partial X. \tag{2.9}$$

Hence its span ${}^bN\partial X \subset {}^bT_{\partial X}X$ is a canonically trivial line subbundle.

EXERCISE 2.7. Show that there is a short exact sequence of vector bundles

$$0 \longrightarrow {}^bN\partial X \hookrightarrow {}^bT_{\partial X}X \longrightarrow T\partial X \longrightarrow 0 \tag{2.10}$$

where the projection is given by restriction to the boundary.

Now (2.9) shows the condition $(a_{00})_{\restriction\partial X} = 1$ on (2.8) to be meaningful independent of coordinates. This restriction will be strengthened further:

DEFINITION 2.8. An *exact b-metric* on a compact manifold with boundary is a b-metric such that for some boundary defining function $x \in \mathcal{C}^\infty(X)$

$$g = (\frac{dx}{x})^2 + g',\ g' \in \mathcal{C}^\infty(X; T^*X \otimes T^*X). \tag{2.11}$$

That is, for some choice of x, $a_{00} = 1 + O(x^2)$ and $a_{0j} = O(x)$ in (2.8).

EXERCISE 2.9. (An intrinsic characterization of exact b-metrics.) From one point of view (2.11) is rather forced since it just *demands* the exactness of g. It is possible to do a little better. First notice that one can just require that, for some boundary defining function x,

$$(a_{00})_{\restriction\partial X} = 1,\ (a_{0j})_{\restriction\partial X} = 0. \tag{2.12}$$

This means that there may be an additional singular term, $\gamma x\big(\frac{dx}{x}\big)^2$ in (2.11). However setting $x' = x + \frac{1}{2}\gamma x^2$ eliminates this term, without creating any new singular terms. Thus it is only necessary to characterize (2.12). The first condition

$$g = (\frac{dx}{x})^2 \text{ on } {}^bN\partial X \tag{2.13}$$

has already been discussed.

Consider the subbundle which is the g-orthocomplement of ${}^bN\partial X$:

$$\big({}^bN\partial X\big)^\perp \subset {}^bT_{\partial X}X. \tag{2.14}$$

Going back to (2.4) observe that $\mathcal{C}^\infty\big(\partial X; {}^bT_{\partial X}X\big) = \mathcal{V}_b(X)/x\mathcal{V}_b(X)$. Moreover, as already noted, $\mathcal{V}_b(X)$ is a *Lie algebra* and furthermore $x\mathcal{V}_b(X) \subset \mathcal{V}_b(X)$ is an *ideal.* Thus the quotient, $\mathcal{C}^\infty(\partial X; {}^bT_{\partial X}X)$, is also a Lie algebra. From (2.14) the condition

$$\mathcal{C}^\infty(\partial X; ({}^bN\partial X)^\perp) \subset \mathcal{C}^\infty(\partial X; {}^bT_{\partial X}X) \text{ is a Lie subalgebra} \tag{2.15}$$

can be imposed. A metric satisfying (2.13), (2.15) might well be called a *closed* b-metric. Why? Check that these conditions do not *quite* ensure

(2.12), because such coordinates need only exist locally. Observe however that, given (2.15),

$$\begin{array}{c}({}^bN\partial X)^\perp\\ \downarrow\\ \partial X\end{array}$$

becomes a bundle with a flat connection. Show that g is *exact* if and only if the connection is a product connection, i.e. has trivial holonomy.

An exact b-metric fixes a section of $N^*\partial X$, namely dx, up to a global constant multiple on each component of ∂X by demanding that (2.11) hold. Indeed if x' is another such boundary defining function then $(dx'/x')^2 - (dx/x)^2$ must be a $\mathcal{C}^\infty$ quadratic 1-form. Writing $x' = \phi x$, with $\phi > 0$ and $\mathcal{C}^\infty$, this difference is $2d\phi dx/x\phi + (d\phi/\phi)^2$ so it follows that $d\phi = 0$ on ∂X. Thus ϕ restricts to ∂X to be locally constant and positive. Such trivializations are useful at various points below and the first coordinate in any coordinate system at the boundary will be taken to correspond to one.

2.4. Differential operators.

Next elementary differential geometry will be developed in this b-context, starting with forms and exterior differentiation. Let ${}^b\Lambda^kX = \Lambda^k\left({}^bT^*X\right)$ be the k^{th} exterior power of ${}^bT^*X$, i.e. the totally antisymmetric part of the k-fold tensor product. As always there is a canonical isomorphism

$$ {}^b\Lambda^k_{X\setminus\partial X} \simeq \Lambda^k_{X\setminus\partial X}. \tag{2.16}$$

The de Rham complex on any manifold (including a manifold with boundary) is

$$0 \to \mathcal{C}^\infty(X) \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^1) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^N) \to 0, \tag{2.17}$$

where $N = \dim X$. The two main features to note are

$$d \in \mathrm{Diff}^1(X;\Lambda^k,\Lambda^{k+1}),\ d^2 = 0. \tag{2.18}$$

Recall what the first of these means, that d is a first order differential operator. The simplest definition is just to say that $P \in \mathrm{Diff}^k(X;V,W)$ if in any local trivializations of V and W over an open set P is given by a matrix of differential operators of order at most k. For the form bundles local trivializations are always induced from local coordinates and

$$\begin{gathered} u = \sum_{|\alpha|=k} u_\alpha dz^\alpha \Longrightarrow \\ du = \sum_{|\alpha|=k} du_\alpha \wedge dz^\alpha = \sum_{|\alpha|=k}\sum_{j=1}^N \frac{\partial u_\alpha}{\partial z^j} dz^j \wedge dz^\alpha. \end{gathered} \tag{2.19}$$

Thus d is a first order differential operator, as stated in (2.18).

The fact that the order of a differential operator is well defined depends ultimately on the fact that $\mathcal{V}(X)=\mathcal{C}^\infty(X;TX)$ is a Lie algebra. Thus for $V=W=\mathbb{C}$ a differential operator of order k acting on functions is just the sum of up to k-fold products of vector fields:

$$\operatorname{Diff}^k(X)=\operatorname*{span}_{0\le j\le k}\mathcal{V}(X)^j,\ \ \mathcal{V}(X)^0=\mathcal{C}^\infty(X).$$

Using the corresponding Lie algebra in the b setting, $\mathcal{V}_b(X)$, the same definition leads to b-differential operators.

DEFINITION 2.10. For any manifold with boundary, the space $\operatorname{Diff}^k_b(X)$ of b-differential operators of order k consists of those linear maps $P:\mathcal{C}^\infty(X)\longrightarrow\mathcal{C}^\infty(X)$ given by a finite sum of up to k-fold products of elements of $\mathcal{V}_b(X)$ (and $\mathcal{C}^\infty(X)$)

$$P=\sum_{l,k'(l)\le k}a_{(l)}\cdot V_{1,l}\dots V_{k',l},\ \ V_{j,l}\in V_b(X),\ \ a_{(l)}\in\mathcal{C}^\infty(X). \tag{2.20}$$

If V,W are vector bundles then $\operatorname{Diff}^k_b(X;V,W)$ consists of the operators $P:\mathcal{C}^\infty(X;V)\longrightarrow\mathcal{C}^\infty(X;W)$ which are local (i.e. decrease supports) and given in any common trivialization of V and W over an open subset of X by a b-differential operator.

The formal point is that the de Rham complex (2.17) over $X\backslash\partial X$ lifts to the 'b' version of the de Rham complex:

$$0\to\mathcal{C}^\infty(X)\xrightarrow{{}^bd}\mathcal{C}^\infty(X;{}^b\Lambda^1)\xrightarrow{{}^bd}\dots\xrightarrow{{}^bd}\mathcal{C}^\infty(X;{}^b\Lambda^{n+1})\to 0, \tag{2.21}$$

where $\dim X=n+1$ and ${}^bd$ is just the lift of d to an operator on the b-form bundles using (2.16). In fact ${}^bd\in\operatorname{Diff}^1_b(X;{}^b\Lambda^k,{}^b\Lambda^{k+1})$ for all k. To see this, start with the formula (2.19) and the basis (2.7) and observe that near the boundary

$$u\in\mathcal{C}^\infty(X;{}^b\Lambda^k)\Longrightarrow u=\sum_{|\alpha|=k}u_\alpha dy^\alpha+\sum_{|\alpha|=k-1}u'_\alpha\frac{dx}{x}\wedge dy^\alpha,$$

with the coefficients $\mathcal{C}^\infty$, and then

$$\begin{aligned}{}^bdu=\sum_{|\alpha|=k}\big(x\frac{\partial u_\alpha}{\partial x}\big)\frac{dx}{x}\wedge dy^\alpha+&\sum_{|\alpha|=k}\sum_{j=1}^n\frac{\partial u_\alpha}{\partial y_j}dy_j\wedge dy^\alpha\\ &-\sum_{|\alpha|=k-1}\sum_{j=1}^n\frac{\partial u'_\alpha}{\partial y_j}\frac{dx}{x}\wedge dy_j\wedge dy^\alpha.\end{aligned}$$

For reasons of sanity-preservation it is better to set ${}^b d = d$ again.

An element of $\mathcal{V}_b(X)$ can be viewed in several ways. In particular it can be viewed as an element of $\mathrm{Diff}^1_b(X)$. It can also be considered as an element of $\mathcal{C}^\infty(X; {}^bTX)$ and hence as a linear function on the fibres of the dual bundle, i.e. as an element of $P^{[k]}({}^bT^*X)$, the space of $\mathcal{C}^\infty$ functions on ${}^bT^*X$ which are homogeneous polynomials of degree k on the fibres, for $k = 1$. Let ${}^b\sigma_1(V) \in P^{[1]}({}^bT^*X)$ denote the image of $iV \in \mathcal{V}_b(X)$ in this sense. The factor of 'i' is inserted so that D_{y_j} and xD_x have real symbols, where $D_{y_j} = -i\partial/\partial y_j$, $xD_x = -ix\partial/\partial x$ corresponding to normalization of the Fourier and Mellin transforms. Since $\mathcal{V}_b(X)$ is a Lie algebra the map defined by taking the leading terms in (2.20):

(2.22)
$$\begin{aligned}\mathrm{Diff}^k_b(X) \ni P \longmapsto {}^b\sigma_k(P) = \\ \sum_{l, k'(l)=k} a_{(l)}\, {}^b\sigma_1(V_{1,l}) \cdots {}^b\sigma_1(V_{k,l}) \in P^{[k]}({}^bT^*X)\end{aligned}$$

is defined independently of the choice of presentation (2.20).

In the interior of X where ${}^bTX$ is identified with TX and ${}^bT^*X$ with T^*X this gives the symbol in the usual sense. In adapted local coordinates at the boundary

$$P \in \mathrm{Diff}^k_b(X) \Longleftrightarrow P = \sum_{j+|\alpha|\le k} p_{j,\alpha}(x,y)(xD_x)^j D_y^\alpha \tag{2.23}$$

and then the symbol becomes, in terms of the basis (2.7),

$${}^b\sigma_k(P) = \sum_{j+|\alpha|=k} p_{j,\alpha}(x,y)\lambda^j\eta^\alpha. \tag{2.24}$$

Here λ and η are the linear functions on ${}^bT^*X$ defined by the basis, i.e. a general element of ${}^bT^*X$ is written

$$\lambda\frac{dx}{x} + \sum_{i=1}^n \eta_i dy_i.$$

An operator $P \in \mathrm{Diff}^k_b(X)$ is said to be elliptic (really b-elliptic) if ${}^b\sigma_k(P) \neq 0$ on ${}^bT^*X \setminus 0$.

For an operator $P \in \mathrm{Diff}^k_b(X; V, W)$, between vector bundles, the coefficients in (2.24) become local homomorphisms from V to W. Thus the symbol is a well-defined element of the space of homomorphisms from the lift of V to ${}^bT^*X$ to the lift of W and is a homogeneous polynomial of degree k on the fibres:

$${}^b\sigma_k(P) \in P^{[k]}({}^bT^*X; V, W),\ P \in \mathrm{Diff}^k_b(X; V, W). \tag{2.25}$$

Such an operator is elliptic if ${}^b\sigma_k(P)$ is invertible on ${}^bT^*X \setminus 0$.

EXERCISE 2.11. Show that the symbol of $d \in \mathrm{Diff}^1_b(X; {}^b\Lambda^k, {}^b\Lambda^{k+1})$ at $\zeta \in {}^bT^*X$ is

$$ {}^b\sigma_1(d)(\zeta) = i\zeta \wedge . \tag{2.26} $$

2.5. Levi-Civita Connection.

Having dealt with the most elementary parts of differential geometry, consider next the fundamental notions in Riemannian geometry, especially the Levi-Civita connection and Riemann curvature tensor. As above this will be developed with an eye to the generalization to the case of a b-metric. If X is a $\mathcal{C}^\infty$ manifold, then a Riemann metric on X is just a positive-definite fibre metric on the tangent bundle. Since it is non-degenerate, such a metric, g, fixes an isomorphism

$$ TX \xrightarrow{G} T^*X,\ G(v)(w) = g_p(v,w)\ \forall\ v,w \in T_pX\ \forall\ p \in X. $$

This means that the metric can be transferred to the dual bundle, T^*X.

An obvious question to ask is the extent to which the metric can be simplified by choice of local coordinates. The metric g is said to be *flat* near p if it reduces to the Euclidean metric in some local coordinates. How close to flat is a metric? A metric is always flat up to second-order terms in Taylor series around any point. This is the origin of the Levi-Civita connection.

Certainly for any point $p \in X$ there are local coordinates based at p (meaning in which p is mapped to the origin), $z^1, z^2, \dots, z^N$, such that

$$ g_p = (dz^1)^2 + \cdots + (dz^N)^2 \text{ on } T_pX. \tag{2.27} $$

In fact any basis of T^*_pX is given by the differentials at p of some coordinate system, so one can just as well look at the space of orthonormal bases of T^*_pX with respect to g_p :

$$ F_p = \left\{ \phi = (\phi^1, \dots, \phi^N); \phi^j \in T^*_pX \text{ and } g_p = \sum_{j=1}^N (\phi^j)^2 \right\}. $$

Notice that if $\phi = (\phi^1, \dots, \phi^N)$ is one element of F_p and $O \in \mathrm{O}(N)$ is an orthogonal matrix, i.e. a real $N \times N$ matrix with inverse O^t, then

$$ \psi = (\psi^1, \dots, \psi^N) \in F_p,\ \psi^j = \sum_{k=1}^N O_{kj}\phi^k. \tag{2.28} $$

Every element of F_p arises in this way from a fixed element and O can be recovered from ψ and ϕ. The action of $\mathrm{O}(N)$ on F_p given by (2.28)

$$A : F_p \times \mathrm{O}(N) \longrightarrow F_p,\ (\phi, O) \longmapsto \phi O = \psi \tag{2.29}$$

is simple and transitive. The action is on the right:

$$A(A(\phi, O), O') = A(\phi, OO'),\ \text{i.e. } (\phi O)O' = \phi(OO').$$

As p varies locally in X the metric is reduced to the Euclidean metric by a smoothly varying basis of T^*X. Thus the space

$$F = \bigsqcup_{p\in X} F_p \xrightarrow{\pi} X$$

is itself a $\mathcal{C}^\infty$ manifold such that the basis is a smooth section and the $\mathrm{O}(N)$ action, given by (2.29) on each fibre, is $\mathcal{C}^\infty$. In fact F is just the orthonormal coframe bundle of the Riemann manifold X with metric g; it is a principal $\mathrm{O}(N)$-bundle. The map (2.29) will be written as

$$A_O : F \longrightarrow F \quad \forall\ O \in \mathrm{O}(N).$$

It is also useful to consider the closely related map $L_\phi \colon F_p \longleftrightarrow \mathrm{O}(N)$, fixed for each $p \in X$ and each $\phi \in F_p$ by

$$L_\phi(\psi) = O \text{ if } A_O\phi = \psi. \tag{2.30}$$

Going back to the question of simplifying the metric, choose any coordinates based at p in which (2.27) holds. Thus, near p, the metric is of the form

$$g = (dz^1)^2 + \cdots + (dz^N)^2 + \sum_{i,j,k=1}^{N} g_{ij,k}dz^i dz^j z^k + O(|z|^2), \tag{2.31}$$

where $g_{ij,k} = \partial g_{ij}/\partial z^k$ at p is a constant matrix satisfying

$$g_{ij,k} = g_{ji,k} \quad \forall\ i, j, k = 1, \ldots, N. \tag{2.32}$$

The coordinates can be modified, without changing the basis at p, by adding quadratic terms:

$$w^j = z^j + \frac{1}{2}\sum_{l,k=1}^{N} \gamma^j_{lk} z^l z^k,\ j = 1, \ldots, N, \tag{2.33}$$

where the coefficients are arbitrary except for the symmetry condition

$$\gamma^j_{lk} = \gamma^j_{kl} \quad \forall\ j,k,l. \tag{2.34}$$

The inverse of this transformation is, near p, of the form:

$$z^j = w^j - \frac{1}{2}\sum_{lk=1}^{N} \gamma^j_{lk} w^l w^k + O(|w|^3).$$

Inserting this into (2.31) the metric becomes

$$g = (dw^1)^2 + \cdots + (dw^N)^2 + O(|w|^2) \tag{2.35}$$

if and only if the linear terms in z cancel, i.e.

$$\sum_{i,j,k=1}^{N} g_{ij,k}\, dz^i dz^j z^k = \sum_{j,l,k=1}^{N} \gamma^j_{lk}(dz^j z^l dz^k + dz^j dz^l z^k).$$

This is a symmetric form in dz, so, using (2.34), the condition can be written in terms of the coefficients as

$$\gamma^j_{lk} + \gamma^l_{jk} = g_{jl,k} \quad \forall\ j,k,l. \tag{2.36}$$

LEMMA 2.12. *The linear map* $\gamma^j_{lk} \longmapsto g_{ij,k}$ *given by* (2.36) *is an isomorphism of matrices subject to* (2.34) *onto matrices subject to* (2.32).

PROOF: The inverse of (2.36) can be computed explicitly as

$$\gamma^j_{lk} = \frac{1}{2}\left\{g_{jk,l} + g_{jl,k} - g_{lk,j}\right\}. \tag{2.37}$$

This means that once the basis $\phi \in F_p$ is chosen there is a *unique* way of choosing the coordinates, up to cubic terms, so that $\phi = (dw^1, \ldots, dw^n)$ at p and (2.35) holds. This information is elegantly encoded in the concept of a *connection* on F. If $w^1, \ldots, w^N$ are the coordinates constructed so that (2.35) holds then the coordinate basis

$$dw^1, \ldots, dw^N$$

can be orthonormalized, near p, by the Gram-Schmidt procedure and this only changes it by quadratic terms. Thus there is a local orthonormal coframe $\phi^1, \ldots, \phi^N$ of the form

$$\phi^j = dw^j + \sum_{k=1}^{N} a_{jk} dw^k,\ a_{jk} = O(|w|^2). \tag{2.38}$$

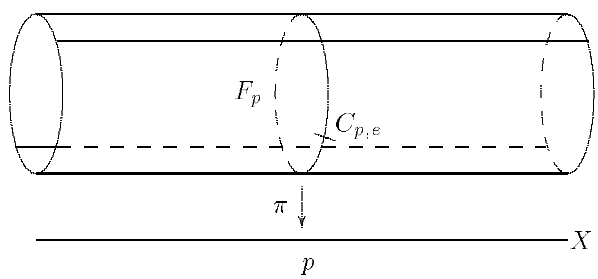

Figure 3. The orthonormal frame bundle and connection.

If the $\phi^j(p)$ are a given orthonormal basis of T_p^*X then the ϕ^j are determined by (2.38) up to quadratic terms at p. This defines an N-plane in the tangent space to F at each point (p, ϕ) :

$$C_{p,\phi} \subset T_{(p,\phi)}F,\ \ \phi \in F_p. \tag{2.39}$$

The uniqueness of the planes $C_{p,\phi}$ shows that the differential of the $\mathrm{O}(N)$-action intertwines the spaces

$$(A_O)_*(C_{p,\phi}) = C_{p,\phi O} \quad \forall\ O \in \mathrm{O}(N) \tag{2.40}$$

and clearly

$$C_{p,\phi} \oplus T_\phi F_p = T_{(p,\phi)}F \quad \forall\ p, \phi \tag{2.41}$$

since $C_{p,\phi}$ is the tangent plane to a local section of F. These are precisely the conditions defining a connection on F. That is:

DEFINITION 2.13. A connection on a principal $\mathrm{O}(N)$-bundle is a smooth N-dimensional distribution, i.e. assignment of an N-plane $C_{p,\phi}$ at each point (p, ϕ) of the bundle, satisfying (2.40) and (2.41).

Thus the discussion above just leads to the Levi-Civita connection:

LEMMA 2.14. *The prescription* (2.39) *fixes a connection (the Levi-Civita connection) on the orthonormal coframe bundle of any Riemann manifold.*

An orthonormal coframe ϕ^l is said to be covariant constant at p if the tangent space at $(p, \phi(p))$ to the section of F defined by it is $C_{p,\phi(p)}$.

2.6. Riemann curvature tensor.

Cartan's reformulation of this notion of a connection is based on the observation that, for each $\phi \in F_p$, the identification L_ϕ in (2.30) maps ϕ to the identity in $\mathrm{O}(N)$, so if the tangent space to $\mathrm{O}(N)$ at Id is taken as the Lie algebra, $\mathfrak{so}(N)$, then

$$(L_\phi)_* : T_\phi F_p \longleftrightarrow \mathfrak{so}(N).$$

Therefore there is a well-defined 1-form with values in $\mathfrak{so}(n)$

$$\omega \in \mathcal{C}^\infty(F; \Lambda^1 F \otimes \mathfrak{so}(N)), \tag{2.42}$$

which is to say N^2 1-forms

$$\omega_{ij} \in \mathcal{C}^\infty(F; \Lambda^1 F) \text{ s.t. } \omega_{ij} + \omega_{ji} = 0 \quad \forall\ i, j = 1, \dots, N,$$

fixed by the condition that at each point of F

$$\omega(v) = (L_\phi)_* v,\ \ v \in T_\phi F_p,\ \ \phi \in F_p \tag{2.43}$$

and

$$\omega_{\restriction C_{p,\phi}} = 0. \tag{2.44}$$

That this fixes the connection form, ω, uniquely follows from (2.41) and then from (2.40) if follows that

$$(A_O)^* \omega_{ij} = \sum_{p,q} O_{ip} O_{jq} \omega_{pq} \quad \forall\ O \in \mathrm{O}(N) \tag{2.45}$$

gives the transformation law under the $\mathrm{O}(N)$-action. Conversely if a 1-form (2.42) satisfying (2.43) and (2.45) is given and the planes $C_{p,\phi} \subset T_{p,\phi} F$ are defined to be the null spaces of ω, so (2.44) holds, then the $C_{p,\phi}$ define a connection on F.

There is another way to look at the connection forms ω_{ij}, more in keeping with the method of moving frames. There are N natural evaluation maps $\Xi_i \colon F \longrightarrow T^* X$, just sending an orthonormal coframe to its ith element. The cotangent bundle carries a *tautological* form, α, given at any point $(p, \beta) \in T^* X$ by the pull back of β under the projection from $T^* X$ to X. Thus F has N such tautological forms, $\alpha_i = \Xi_i^* \alpha$, $i = 1, \dots, N$. These forms vanish when restricted to any fibre but are everywhere independent. The *first structure equation*[3] of the connection is:

$$d\alpha_i + \sum_{p=1}^{N} \omega_{ip} \wedge \alpha_p = 0. \tag{2.46}$$

Notice that this actually determines the ω_{ij}.

To prove (2.46) at a general point $(p, \phi) \in F$ consider an orthonormal frame as in (2.38), near p, taking the value ϕ at p. This gives an identification over a neighbourhood U of p :

$$\begin{gathered} F_{\restriction U} \equiv U \times \mathrm{O}(N) \\ (p', \psi) \longleftrightarrow (p', O), \ \psi^j = \sum_k O_{kj} \phi^k \end{gathered} \tag{2.47}$$

as in (2.28). Then the tautological forms are just $\alpha_i = \sum_l O_{li} \phi^l$. Moreover, $d\phi^l$ vanishes at p so $d\alpha_i = \sum_l dO_{li} \wedge \phi^l$ at $p \times \mathrm{O}(N)$. Inverting (2.47) and noting that the covariant constancy of the basis means that

$$\omega_{ij} = \sum_q O_{qi} dO_{qj} \text{ at } p \times \mathrm{O}(N) = F_p;$$

it follows that

$$d\alpha_i = \sum_l dO_{li} \wedge \sum_q O_{lq} \alpha_q = \sum_q \omega_{qi} \wedge \alpha_q.$$

This gives (2.46) at (p, ϕ) and hence in general.

The *second structure equation* for the connection is given by computing the exterior differential of the connection form.

PROPOSITION 2.15. *The Levi-Civita connection form, ω, defined by* (2.43) *and* (2.44), *satisfies*

$$d\omega_{ij} + \sum_q \omega_{iq} \wedge \omega_{qj} = Q_{ij} \tag{2.48}$$

[3] The convention used here for the exterior product is fixed by demanding

$$\xi_1 \wedge \cdots \wedge \xi_N(v_1, \dots, v_N) = \frac{1}{N!} \det\big(\xi_j(v_k)\big) \quad \forall\, \xi_j \in T_p^* X,\ v_j \in T_p X.$$

where $Q_{ij} \in \mathcal{C}^\infty(F; \Lambda^2 F)$ *annihilates* $T_\phi F_p$ *at each point, i.e.*

$$Q(v,w) = 0 \text{ if } v \in T_{(p,\phi)}F,\ w \in T_\phi F_p. \tag{2.49}$$

PROOF: Any tangent vector $v \in T_\phi F$ can be extended to a vector field, V, near F_p. If $v \in T_\phi F_p$ the vector field V can be chosen to be vertical, i.e. tangent to the leaves of F, near p and to have

$$(L_\psi)_* V = v' = (L_\phi)_* v \ \forall\ \psi \in F_\phi,$$

so constant in $\mathfrak{so}(N)$. This means that V generates a 1-parameter subgroup of the $\mathrm{O}(N)$-action.

Suppose first that $v, w \in T_\phi F_p$ are both vertical. Then, by Cartan's formula for the exterior derivative,

$$d\omega(V,W) = \frac{1}{2}\left[V\omega(W) - W\omega(V) - \omega([V,W])\right]. \tag{2.50}$$

Taking V, W to be generators of 1-parameter subgroups of the $\mathrm{O}(N)$-action as above makes $\omega(V)$ and $\omega(W)$ constant, so the first two terms on the right in (2.50) vanish. Evaluated at (p,ϕ), using (2.43), the third term in (2.50) becomes

$$\begin{aligned}\omega([V,W]) =& (L_\phi)_*([V,W]) \\ =& [(L_\phi)_* V, (L_\phi)_* W] \\ =& \omega(V)\omega(W) - \omega(W)\omega(V)\end{aligned}$$

since the commutator in $\mathfrak{so}(N)$ is just matrix commutation. This gives (2.49) when both vectors are vertical.

Suppose next that $v \in C_{p,\phi}$. Then, using (2.40), the extension can be chosen to be $\mathrm{O}(N)$-invariant and must take values in $C_{p,\psi}$ for all $\psi \in F_p$. Thus $\omega(V) = 0$ on F_p. With W a generator of the $\mathrm{O}(N)$-action the first two terms in (2.50) again vanish. Now $[W,V]$ is the Lie derivative of V along the 1-parameter group generated by W, so vanishes on F_p and hence the third term in (2.50) also vanishes at ϕ. This completes the proof of (2.49).

From (2.49) the form Q at $(p,\phi) \in F$ is determined by its restriction to $C_{p,\phi}$. By (2.41) projection to X gives an isomorphism of T_pX and $C_{p,\phi}$, so one can define

$$\begin{gathered}Q_{ij}^{p,\phi} \in \Lambda_p^2 X \text{ by } Q_{ij}^{p,\phi}(v,w) = Q_{ij}(v',w'), \\ v', w' \in C_{p,\phi},\ \pi_*(v') = v, \pi_*(w') = w.\end{gathered}$$

Then consider the 4-cotensor at p :

$$\begin{gathered}R(p; v, w, \tilde{v}, \tilde{w}) = \sum_{i,j=1}^{n} Q_{ij}^{p,\phi}(v,w)\phi^i(\tilde{v})\phi^j(\tilde{w}), \\ v, w, \tilde{v}, \tilde{w} \in T_pX,\ \phi = (\phi^1, \ldots, \phi^N) \in F_p.\end{gathered} \tag{2.51}$$

The notation here indicates that

$$R(p) \in \Lambda^2_p X \otimes \Lambda^2_p X$$

is independent of the choice of $\phi \in F_p$ involved in (2.51). This indeed is a consequence of the transformation law for Q in (2.48) which follows from (2.45):

$$(A_O)^* Q_{ij} = \sum_{p,q} O_{ip} O_{jq} Q_{pq} \tag{2.52}$$

and the transformation law (2.28) for the ϕ^i. Thus

$$R \in \mathcal{C}^\infty(X; \Lambda^2 X \otimes \Lambda^2 X)$$

is well defined by (2.51). It is the *Riemann curvature tensor.* Notice that, as a consequence of Proposition 2.15, R also determines Q.

THEOREM 2.16. *If the Riemann curvature tensor vanishes near a point p then the metric is flat nearby in the sense that there are local coordinates based at p in terms of which*

$$g = (dz^1)^2 + \cdots + (dz^N)^2 \text{ near } p.$$

PROOF (BRIEF): By Proposition 2.15 and the remarks above the vanishing of R is equivalent to the vanishing of Q in a neighbourhood of F_p in F and hence to

$$d\omega = -\omega \wedge \omega \text{ near } F_p.$$

This in turn means that the subspaces $C_{p,\phi} \subset T_{(p,\phi)}F$ form a foliation near F_p, i.e. the space of vector fields tangent to them at each point is closed under commutation. So, by Frobenius' theorem, through each point $\phi \in F_p$ there is, at least locally, a smooth submanifold, X_ϕ, such that $T_{(p',\psi)}X_e = C_{p',\psi}$ for each $(p',\psi) \in X_\phi$. The transversality condition (2.41) means that $\pi : X_\phi \longrightarrow X$ is a local diffeomorphism, so X_ϕ is a section of F passing through (p,ϕ), i.e. an orthonormal basis $\phi^i(p')$ for g at each point p' near p. The tangency of $C_{p',\psi}$ to X_ϕ means that each $\phi^i(p')$ is actually a closed 1-form, so locally exact and this gives the coordinates in which the metric is flat.

So this is the elementary theory of the Levi-Civita connection and Riemann curvature of a Riemann manifold. Before considering the changes needed to handle b-metrics, recall the identification of the tensor bundles as *associated bundles* to the coframe bundle and the extension of the connection to these bundles in the guise of *covariant differentiation.*

2.7. Associated bundles.

Suppose that E is a vector space with an $\mathrm{O}(N)$-action, $L\colon\ \mathrm{O}(N) \ni O \longmapsto L_O \in \mathrm{Gl}(E)$. For example consider the standard action on $\mathbb{R}^N$:

$$L_O(x_1,\dots,x_N) = (y_1,\dots,y_N),\ y_j = \sum_{k=1}^{N} O_{jk}x_k. \tag{2.53}$$

Given any such linear action consider the space $F \times E$ with the action:

$$F \times E \in (p,\phi,u) \longrightarrow (p, A_O\phi, L_O^{-1}u) \in F \times E. \tag{2.54}$$

The quotient is a vector bundle over X, which can be denoted temporarily as E_L :

$$\begin{aligned}(E_L)_p = \big\{[(p,\phi,u)];(p,\psi,u') \in [(p,\phi,u)] \text{ iff } \exists\ O \in \mathrm{O}(N)\\ \text{with } \psi = A_O\phi = \phi O,\ u' = L_O^{-1}u\big\}.\end{aligned}$$

The linear structure on $(E_L)_p$ comes from the linearity of the action on E, the $\mathcal{C}^\infty$ structure projects from $F \times E$.

Any such associated bundle has a connection induced from F. With $C_{p,\phi}$ the N-planes defined above consider

$$C^L_{p,\phi,u} = C_{p,\phi} \times E \subset T_{(p,\phi,u)}(F \times E)\ \forall\ (p,\phi,u) \in F \times E,$$

where the identification of the tangent space of a vector space with the vector space itself is used. Now from (2.40) and the fact that L_O acts on, i.e. preserves, E the differential of the action (2.54) maps $C^L_{p,\phi,u}$ onto $C^L_{p,\phi O,L_O^{-1}u}$. Thus the spaces $C^L_{p,\phi,u}$ project to well-defined N-planes in $T_{(p,v)}E_L$, which can again be denoted

$$C_{p,v} \subset T_{(p,v)}E_L,\ v \in (E_L)_p,\ p \in X.$$

The transversality condition (2.41) certainly persists so

$$C_{p,v} \oplus T_v(E_L)_p = T_{(p,v)}E_L \quad \forall\ (p,v) \in E_L. \tag{2.55}$$

EXERCISE 2.17. Check carefully that if the representation of $\mathrm{O}(N)$ is taken to be (2.53) then the associated bundle is canonically isomorphic to T^*X where the isomorphism is given by mapping (p,ϕ,u) to

$$\sum_j u_j\phi^j \in T^*_pX.$$

2.8. Covariant differentiation.

It is more usual to rewrite the information (2.55) in terms of a differential operator. A section, $\mu \in \mathcal{C}^\infty(X; E_L)$, is said to be covariant constant at p if its graph

$$\mathrm{Gr}(\mu) = \{(p', \mu(p')) \in E_L; p' \in X\} \text{ has tangent plane } C_{p,\mu(p)} \text{ at } (p, \mu(p)).$$

More generally any section, at any point, can be compared to a covariant constant section. Since

$$T_{(p,\mu(p))}\,\mathrm{Gr}(\mu) \oplus T_{\mu(p)}(E_L)_p = T_{(p,\mu(p))}E_L,$$

there is a well-defined projection $\nabla\mu(p) \colon C_{p,\mu(p)} \longrightarrow T_{\mu(p)}(E_L)_p$. Since the projection to the base identifies $C_{p,\mu(p)}$ with T_pX and the tangent space to the fibre can be identified with the fibre this means that $\nabla\mu(p)$ is a linear map $T_pX \longrightarrow (E_L)_p$ fixed by μ. Equivalently it can be thought of it as an element

$$\nabla\mu(p) \in T_p^*X \otimes (E_L)_p.$$

Clearly this construction varies smoothly with p so defines the covariant derivative of any smooth section of any associated bundle:

$$\nabla : \mathcal{C}^\infty(X; E_L) \longrightarrow \mathcal{C}^\infty(X; T^*X \otimes E_L). \tag{2.56}$$

LEMMA 2.18. *The map* (2.56) *is a first order linear differential operator.*

PROOF: The tangent space to the section only depends on the first derivatives of the section in local coordinates, so certainly $\nabla\mu(p)$ only depends on the first derivatives, and the value, of μ at p. To see that the dependence is linear observe that, by definition, if ϕ is any section of F near p then a local section μ, of E_L, determines and is determined by a map f from a neighbourhood of p in X into E by $\mu(p') \longleftrightarrow [(\phi(p'), f(p'))]$. If ϕ is covariant-constant at p, i.e. has graph tangent to $C_{p,\phi(p)}$, then the covariant derivative of μ is just

$$\nabla\mu(p) = df(p) \in T_p^*X \otimes (E_L)_p, \tag{2.57}$$

where $E \ni e \longrightarrow [(\phi(p), e)]$ identifies E and $(E_L)_p$. The linearity is then clear from (2.57).

Directly from the definition, the zero section of the vector bundle is covariant constant, which is to say that at the zero section $C_{p,0} = T_pX$ is the tangent plane to the zero section. Note that the tangent space to the vector bundle has a natural decomposition $T_{p,0}E = T_pX \oplus E_p$. If μ is any smooth section and $f \in \mathcal{C}^\infty(X)$ vanishes at p then the graph of the section $f\mu$ has tangent space $\{df(p) = 0\} \oplus \operatorname{span}(\mu)$ at p. It follows that the covariant derivative of such a section, at p, is just $df(p) \otimes \mu(p)$. More generally

$$\nabla(f\mu) = df \otimes \mu + f\nabla\mu \quad \forall\ f \in \mathcal{C}^\infty(X),\ \mu \in \mathcal{C}^\infty(X; E). \tag{2.58}$$

Indeed this follows, at a general point p, by writing $f\mu = f(p)\mu + (f - f(p))\mu$.

DEFINITION 2.19. A connection on a vector bundle, E, is a first order linear differential operator

$$\nabla : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; T^*X \otimes E)$$

satisfying (2.58).

The Levi-Civita connection on T^*X is certainly a connection (an affine connection) in this sense.

EXERCISE 2.20. Show that for any connection on a vector bundle over X and each $e \in E_p$, $p \in X$, there is a section $\mu \in \mathcal{C}^\infty(X; E)$ with $\mu(p) = e$ which is covariant constant at p, i.e. $\nabla\mu(p) = 0$. [Hint: choose any section through e at p then add to it $\sum_i f_i\mu_i$, where the μ_i are a local basis of sections and the $f_i \in \mathcal{I}_p$. Use (2.58) to choose the f_i so that the covariant derivative vanishes at p.] Use this to show that a connection is equivalent to a $\mathcal{C}^\infty$ distribution of tangent planes $C_{(p,e)} \subset T_{(p,e)}E$ which project isomorphically onto T_pX and satisfy a linearity condition which can be written

$$C_{(p,e)} + C_{(p,e')} = C_{(p,e+e')}\ \forall\ e, e' \in E_p,\ p \in X.$$

Thus if E has a connection and $p \in X$ one can always find a local basis of sections of E, e_j, which are covariant constant at p. Any other such basis is necessarily of the form

$$\tilde{e}_j = \sum_k a_{jk}e_k + \sum_{jk} f_{jk}e_k,\ a_{jk} \text{ constant and } f_{jk} \in \mathcal{I}_p^2. \tag{2.59}$$

If ∇' and ∇'' are both connections on E and $\phi \in \mathcal{C}^\infty(X)$ then $\nabla = \phi\nabla' + (1 - \phi)\nabla''$ is also a connection. Of course it is only necessary for ∇' to be defined in a neighbourhood of the support of ϕ and similarly for ∇''. This allows one to show that any vector bundle has a connection. For a trivial bundle, $E = X \times \mathbb{R}^n$, a connection is defined by d acting on the coefficients. Since any bundle is locally trivial, this means that a connection exists in an open neighbourhood of each point. If O_j is a covering of X by such neighbourhoods with ∇_j a connection on E over O_j and ϕ_j is a partition of unity subordinate to the cover then $\nabla = \sum_j \phi_j \nabla_j$ is a connection on E.

The covariant derivative can always be evaluated on a vector field (in the factor of T^*X in $T^*X \otimes E$). Thus if $V \in \mathcal{V}(X)$

$$\nabla_V : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;E),\ \nabla_V \mu = \nabla\mu(V,\cdot)\ \forall\ \mu \in \mathcal{C}^\infty(X;E)$$

is always well defined.

LEMMA 2.21. *Every $\mathcal{C}^\infty$ bundle has a connection. If the bundle has a fibre metric then the connection can be chosen to be orthogonal in the sense that*

$$\langle \nabla_V \mu, \mu' \rangle + \langle \mu, \nabla_V \mu' \rangle = d\langle \mu, \mu' \rangle \quad \forall\ \mu, \mu' \in \mathcal{C}^\infty(X;E) \tag{2.60}$$

for all $\mathcal{C}^\infty$ vector fields V.

PROOF: Using the same superposition argument it suffices to construct an orthogonal connection locally, assuming the bundle to be trivial. Then one can simply demand that some orthonormal basis be covariant constant.

If E has a connection then the dual bundle E' has a natural connection given by demanding that a section e' of E' be covariant constant at p if and only if its pairing with any section of E which is covariant constant at p has vanishing differential at p. That is, in terms of the pairing

$$d\langle e, e' \rangle = \langle \nabla e', e \rangle + \langle e', \nabla e \rangle. \tag{2.61}$$

Indeed if e_j is a covariant constant basis at p and e'_j is the dual basis of E' at p then e'_j must be covariant constant. This fixes the connection on E' and it is independent of the choice of basis since another choice, (2.59), clearly gives the same covariant constant sections at p.

Similarly the tensor product of two bundles E and F with connections has a unique connection for which $e \otimes f$ is covariant constant at p if e and f are covariant constant at p as sections of E and F. It is determined by the identity

$$\nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes (\nabla f). \tag{2.62}$$

EXERCISE 2.22. Prove these assertions carefully, in particular showing smoothness of the induced connections.

EXERCISE 2.23. Show that if E and F are bundles associated to a principal $\mathrm{O}(N)$-bundle then $E \otimes F$ is also an associated bundle, corresponding to the tensor product representation. Check that the tensor product of the connections induced on E and F, as in (2.62), is the induced connection on $E \otimes F$.

The Levi-Civita connection therefore defines covariant differentiation acting on tensors on X. Actually this connection arises in two ways, directly because the tensor bundles are associated to the orthonormal coframe bundle and also through the tensor product construction from the connections on the tangent and cotangent bundles; by Exercise 2.23 these are the same. Various properties of the Levi-Civita connection are important later. First since g, the metric, can be considered as a section of $T^*X \otimes T^*X$ (a symmetric one to be sure) its covariant derivative is defined. In fact $\nabla g \equiv 0$. Indeed this follows from (2.57) and the original definition of the connection, since if g is represented in terms of an orthonormal basis which is covariant constant at p it is given, by definition, by the Euclidean form up to second order.

EXERCISE 2.24. For any form bundle, antisymmetrization gives a projection $A_k \colon T^*X \otimes \Lambda^k X \longrightarrow \Lambda^{k+1}X$. Show that composition with covariant differentiation gives, for the Levi-Civita connection,

$$A_{k+1} \circ \nabla = d \colon \mathcal{C}^\infty(X;\Lambda^k) \longrightarrow \mathcal{C}^\infty(X;\Lambda^{k+1})$$

for every k.

EXERCISE 2.25. Using Exercise 2.24 make sense of, and prove, the following formula for any (local) orthonormal coframe ϕ^i with dual frame v_i :

$$d\mu = \sum_{j=1}^{n} \phi^i \wedge \nabla_{v_i}\mu, \tag{2.63}$$

where μ is a $\mathcal{C}^\infty$ k-form.

2.9. Christoffel symbols.

The construction above can be reinterpreted in terms of the full coframe bundle $\widetilde{F}$. Thus

$$\widetilde{F}_p = \left\{\phi = \{\phi^1, \dots, \phi^N\} \text{ is a basis of } T_p^*X\right\}.$$

The N-plane $C_{p,\phi}$ can be defined at all points of $\widetilde{F}$, using (2.40), i.e. a general frame is covariant constant at p if and only if it is of the form

$\sum_j h^i_j \phi^j$, where ϕ^j is an orthonormal frame which is covariant constant at p and h^i_j is an invertible matrix with $dh^j_i = 0$ at p. In a general coordinate system, not assumed to give an orthonormal basis for $T^*_p X$, the formula generalizing (2.37) for the coefficients γ^i_{jk} to make the coordinate basis defined by (2.33) covariant constant at p is

$$\gamma^i_{jk} = \frac{1}{2}\sum_{p=1}^{N} g^{ip}\left\{g_{pj,k} + g_{pk,j} - g_{jk,p}\right\}, \tag{2.64}$$

where g^{ij} is the inverse matrix to g_{ij}. Then from (2.37) it is easy to deduce the local coordinate expression for the covariant derivative of a 1-form. Namely by writing

$$\alpha = \sum_{i=1}^{N} \alpha_i dz^i = \sum_{i=1}^{N} \alpha_i (dz^i + \sum_{j,k=1}^{N} \gamma^i_{jk} dz^j z^k) - \sum_{i,j,k=1}^{N} \alpha_i \gamma^i_{jk} dz^j z^k \tag{2.65}$$

and using (2.57) the familiar expression

$$\nabla\alpha = \sum_{i,j=1}^{N} \frac{\partial \alpha_i}{\partial z^j} dz^j \otimes dz^i - \sum_{i,j,k=1}^{N} \alpha_i \gamma^i_{jk} dz^j \otimes dz^k \tag{2.66}$$

results. The γ^i_{jk} are called the Christoffel symbols and are defined in any local coordinate system by the identity

$$\nabla_{\frac{\partial}{\partial z^i}} dz^j = -\sum_{k=1}^{N} \gamma^j_{ik} dz^k. \tag{2.67}$$

They are not the coefficients of a tensor.

EXERCISE 2.26. Show that the formula for covariant differentiation of a p-form is determined by

$$\nabla_{\frac{\partial}{\partial x^i}} dx^I = -\sum_{l=1}^{p}\sum_{k=1}^{N} \gamma^{I_l}_{ik} dx^{I_1} \wedge \dots dx^{I_{l-1}} \wedge dx^k \wedge dx^{I_{l+1}} \wedge \dots \wedge dx^{I_p}.$$

The identity (2.58) for covariant differentiation means that if V and W are $\mathcal{C}^\infty$ vector fields on X then for any connection on a vector bundle

$$[\nabla_V, \nabla_W](f\mu) - \nabla_{[V,W]}(f\mu) = f\left([\nabla_V, \nabla_W]\mu - \nabla_{[V,W]}\mu\right)$$
$$\forall\ f \in \mathcal{C}^\infty(X), \mu \in \mathcal{C}^\infty(X; E).$$

Thus

$$K_E(V,W)\mu = \left([\nabla_V, \nabla_W] - \nabla_{[V,W]}\right)\mu \tag{2.68}$$

is actually a differential operator of order 0, that is $K_E(V,W)$ is a section of the homomorphism bundle of E. It is called the curvature operator.

For the Levi-Civita connection acting on 1-forms, observe from (2.66) and (2.67) that the curvature operator evaluated on a coordinate basis is

$$\begin{aligned} &K_{\Lambda^1}(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})dz^p \\ &= -\sum_q \left(\frac{\partial}{\partial z^j}\gamma^p_{kq} - \frac{\partial}{\partial z^k}\gamma^p_{jq}\right)dz^q + \sum_r\sum_q \left(\gamma^q_{jr}\gamma^p_{kq} - \gamma^q_{kr}\gamma^p_{jq}\right)dz^r. \end{aligned} \tag{2.69}$$

If the coordinates are chosen so that (2.35) holds then the coefficients γ^i_{jk} vanish at p and

$$K_{\Lambda^1}(p;\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})dz^p = \sum_q \left(\frac{\partial}{\partial z^k}\gamma^p_{jq} - \frac{\partial}{\partial z^j}\gamma^p_{kq}\right)dz^q.$$

LEMMA 2.27. *The curvature operator on T^*X of the Levi-Civita connection is*

$$K_{\Lambda^1}(p;\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k})dz^l = \sum_{s,q=1}^N g^{ls}R_{jkqs}dz^q, \tag{2.70}$$

where R is the Riemann curvature tensor.

PROOF: Consider the local section of $\Phi\colon u \longrightarrow \widetilde{F}$ given by the orthonormalized coordinate differentials, ϕ^j in (2.38), i.e. $\Phi(x) = (\phi^1, \dots, \phi^N)$. Let $\widetilde{\gamma}_{ij} = \Phi^*\omega_{ij}$ be the pull-back of the connection forms under this section, so $\widetilde{\gamma}_{ij} = \sum\limits_{k=1}^N \widetilde{\gamma}^i_{jk}\phi^k$. Recalling that the ϕ^j are equal to the dz^j up to quadratic terms at p, the pull-back of the structure equation (2.48) then shows that

$$d\widetilde{\gamma}_{ij} = \sum_{pq} R_{ijpq}dz^p \wedge dz^q.$$

Since the pull-back of the form α_j is just ϕ^j it follows from (2.46) that (2.70) holds in this case since $g(p)$ is the Euclidean metric. As both sides of (2.70) are tensors it must hold in general.

EXERCISE 2.28. Find the curvature operator of the Levi-Civita connection on the tangent bundle in terms of the Riemann curvature tensor.

Combining (2.69) and (2.70) gives a formula for the Riemann curvature tensor, at any point p, computed in coordinates in which (2.35) holds, namely:

$$R_{ijkl} = \frac{1}{2}\left(-\frac{\partial}{\partial z^j}\frac{\partial}{\partial z^l}g_{ki} + \frac{\partial}{\partial z^j}\frac{\partial}{\partial z^k}g_{il} + \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^l}g_{kj} - \frac{\partial}{\partial z^i}\frac{\partial}{\partial z^k}g_{jl}\right).$$

This shows that the curvature tensor has the following symmetries.

LEMMA 2.29. *The Riemann curvature tensor satisfies the identities:*

$$\begin{aligned} R_{ijkl} &= -R_{jikl} \\ R_{ijkl} &= R_{klij} \\ R_{ijkl} + R_{iklj} + R_{iljk} &= 0 \quad \text{(1st Bianchi identity).} \end{aligned}$$

Since they are coordinate-invariant they hold in any coordinates.

2.10. Warped products.

For later application the connection and curvature for a simple warped product will be computed.

Thus let M be a Riemann manifold with metric h and let $\phi \in \mathcal{C}^\infty(\mathbb{S}^1)$ be a real-valued function on the circle. On $X = \mathbb{S}^1 \times M$ consider the metric

$$g = d\theta^2 + e^{2\phi(\theta)}h, \tag{2.71}$$

where θ is the standard variable on $\mathbb{S}^1$. Let ∇ be the covariant derivative, acting on 1-forms, on M. We shall also let ∇ denote the covariant derivative on the product when $\phi \equiv 0$ since then, with respect to the decomposition

$$T^*X = T^*\mathbb{S}^1 \oplus T^*M,$$

the covariant derivative with respect to $\partial/\partial\theta$ just acts on the coefficients and the covariant derivative with respect to a vector field on M acts through the connection of h.

LEMMA 2.30. *The Levi-Civita connection, $\widetilde{\nabla}$, of the metric g in* (2.71) *is given in terms of the product connection through*

$$\begin{aligned} \widetilde{\nabla}_{\partial_\theta} d\theta = 0,\ \widetilde{\nabla}_{\partial_\theta}\beta &= -\frac{\partial\phi}{\partial\theta}\beta \\ \widetilde{\nabla}_V d\theta &= \frac{\partial\phi}{\partial\theta}e^{2\phi}\mu, \\ \widetilde{\nabla}_V\beta &= \nabla_V\beta - \frac{\partial\phi}{\partial\theta}\beta(V)d\theta \end{aligned} \tag{2.72}$$

for any $\mathcal{C}^\infty$ 1-form β on M and $\mathcal{C}^\infty$ vector field V on M, with μ the dual form to V, i.e. $\mu = h(V, \cdot)$.

PROOF: From (2.64) the Christoffel symbols $\widetilde{\gamma}$ for g can be computed for coordinates $z^0 = \theta$ and z^j, $j = 1, \dots, n = \dim M$ in M in terms of the Christoffel symbols, γ, of h :

$$
\begin{aligned}
&\widetilde{\gamma}^i_{jk} = \gamma^i_{jk},\ \widetilde{\gamma}^0_{jk} = -\frac{\partial\phi}{\partial\theta}e^{2\phi}h_{jk} \\
&\widetilde{\gamma}^i_{0k} = \widetilde{\gamma}^i_{k0} = \frac{\partial\phi}{\partial\theta}\delta^i_k,\ \widetilde{\gamma}^0_{0k} = \widetilde{\gamma}^0_{k0} = \widetilde{\gamma}^0_{00} = 0.
\end{aligned} \tag{2.73}
$$

From these formulæ, and (2.67), (2.72) follows directly.

LEMMA 2.31. *The curvature matrix $\widetilde{R}$ of the metric* (2.71) *is given in terms of the curvature of M through*

$$
\begin{aligned}
&\widetilde{R}_{kj*0} = 0,\ \widetilde{R}_{0j0k} = h_{jk}\big(\frac{\partial^2\phi}{\partial\theta^2} - (\frac{\partial\phi}{\partial\theta})^2\big) \\
&\widetilde{R}_{kjpq} = e^{2\phi}R_{kjpq} + (\frac{\partial\phi}{\partial\theta})^2 e^{4\phi}\left(h_{kp}h_{jq} - h_{kq}h_{jq}\right)
\end{aligned} \tag{2.74}
$$

where $j, k, p, q = 1, \dots, n = \dim M$.

PROOF: This follows by assiduous use of (2.70) and Lemma 2.30.

2.11. Curvature formulæ.

If E' is the dual bundle to E then the transpose of a linear map defines an isomorphism

$$\hom(E) \equiv \hom(E'),\ A \longmapsto A^t.$$

Applying (2.61), repeatedly, shows that the curvature of the connection on the dual bundle satisfies

$$
\begin{gathered}
\langle K_{E'}(V,W)\mu', \mu\rangle + \langle\mu', K_E(V,W)\mu\rangle = 0 \\
\forall\ \mu \in \mathcal{C}^\infty(X;E), \mu' \in \mathcal{C}^\infty(X;E').
\end{gathered}
$$

It follows easily that the curvature operator satisfies

$$K_{E'} = -K_E^t \tag{2.75}$$

(which provides a solution to Exercise 2.28). Similarly if E and F are two vector bundles with connections then (2.62) fixes a connection on $E \otimes F$. The homomorphism bundle of the tensor product is

$$\hom(E \otimes F) \equiv \hom(E) \otimes \hom(F).$$

In terms of this isomorphism a similar computation shows the curvature operator to be

$$K_{E\otimes F} = K_E \otimes \operatorname{Id} + \operatorname{Id} \otimes K_F. \tag{2.76}$$

As a special case the homomorphism bundle of E itself acquires a connection through the identity

$$\nabla(Au) = (\nabla A)u + A(\nabla u). \tag{2.77}$$

This is the same as the connection induced by the identification $\hom(E) = E \otimes E'$. As an operator on $\hom(E)$, the curvature is therefore given by

$$K_{\hom(E)}(V,W) = K_E(V,W) \cdot - \cdot K_E(V,W), \tag{2.78}$$

for any vector fields V, W, where the dots indicate that the two terms act by composition on the left and right respectively, i.e.

$$\begin{aligned} K_{\hom(E)}(V,W)A =& K_E(V,W)A - AK_E(V,W) \\ =& [K_E(V,W), A] \quad \forall\ A \in \hom(E). \end{aligned}$$

It is also important later to consider the 'big' homomorphism bundle, $\operatorname{Hom}(E)$, since this arises whenever operators on sections of E are considered. If E is a bundle over X this is the bundle over X^2 with fibre $\hom(E_{x'}, E_x)$ at (x, x'). This is sometimes called the exterior tensor product:

$$\operatorname{Hom}(E) = E \boxtimes E' \text{ over } X^2, \tag{2.79}$$

meaning that E should be pulled back to X^2 from the left factor of X and E' from the right, i.e.

$$\operatorname{Hom}(E) = \pi_L^* E \otimes \pi_E^* E' \text{ over } X^2, \tag{2.80}$$

where π_L and π_R are the two projections from X^2 to X onto the left and right factors.

Recall that the pull-back of a bundle E over Y under a smooth map $f\colon X \longrightarrow Y$ is always a bundle over X with fibre at x just $E_{f(x)}$. A section of E, $\mu \in \mathcal{C}^\infty(Y; E)$, defines a section of f^*E, namely $f^*\mu = \mu \circ f$. In fact every section of f^*E is locally a sum of products of $\mathcal{C}^\infty$ functions on X and these pulled-back sections:

$$\mathcal{C}^\infty(X; f^*E) = \mathcal{C}^\infty(X) \cdot f^*\mathcal{C}^\infty(Y; E). \tag{2.81}$$

This allows a connection on E to be pulled back to a connection on f^*E simply by defining

$$\begin{aligned} \nabla_{f^*E}\mu &= \sum_j (dh_j \otimes f^*\mu_j + h_j f^*(\nabla_E \mu_j)) \\ \text{if } \mu &= \sum_j h_j f^*\mu_j,\ h_j \in \mathcal{C}^\infty(X),\ \mu_j \in \mathcal{C}^\infty(Y; E). \end{aligned} \tag{2.82}$$

Thus sections covariant constant at $y \in Y$ pull back to be covariant constant at each $x \in f^{-1}(y)$.

LEMMA 2.32. *If $f\colon X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map and E is a bundle over Y with connection and curvature operator K_E then the action of the curvature operator of the pulled-back connection on f^*E is*

$$\begin{gathered} K_{f^*E}(V,W)\mu = K_E(f_*V, f_*W)\mu \in E_{f(x)} = (f^*E)_x \\ \forall\ V,W \in T_xX,\ \mu \in (f^*E)_x, \end{gathered} \tag{2.83}$$

i.e. if K_E is regarded as a section of $\Lambda^2 X \otimes \hom(E)$ then $K_{f^*E} = f^*K_E$.

PROOF: It suffices to compute the curvature operator acting on covariant constant sections at a given point $x \in X$. In fact it is enough to consider sections $f^*\mu'$ with $\mu' \in \mathcal{C}^\infty(Y;E)$ covariant constant at $f(x)$. The second term in (2.68) is therefore zero at x and the first evaluates at x, using (2.82), to $f^*([\nabla_v, \nabla_w]\mu')$, where v, w are any $\mathcal{C}^\infty$ vector fields on Y with $v = f_*V$ and $w = f_*W$ at x. This gives (2.83).

The decomposition (2.80) of $\mathrm{Hom}(E)$ shows that it has a natural connection. Applying Lemma 2.32, (2.76) and (2.75) shows that the curvature operator for this connection is

$$K_{\mathrm{Hom}(E)}(V,W) = K_E\big((\pi_L)_*V, (\pi_L)_*W\big)\cdot - \cdot K_E\big((\pi_R)_*V, (\pi_R)_*W\big); \tag{2.84}$$

again the dots indicate that the terms act as operators on the left and right.

EXERCISE 2.33. Check that the connection on $\mathrm{Hom}(E)$ is consistent with that on $\hom(E)$ in the following sense. Let $\iota\colon \Delta \hookrightarrow X^2$ be the embedding of the diagonal, the set of points (x,x). Projection onto either factor of X gives the same diffeomorphism $\Delta \equiv X$. Thus pulling back $\mathrm{Hom}(E)$ under i gives a bundle over X. Show that this bundle is canonically isomorphic to $\hom(E)$ and that the pulled-back connection is the one defined above.

2.12. Orientation.

As noted already, the coframe bundle, a principal bundle with structure group $\mathrm{Gl}(N,\mathbb{R})$, of any compact manifold has a reduction to a principal $\mathrm{O}(N)$-bundle; namely the orthonormal coframe bundle, F, for any Riemann metric. In fact they were defined above in the other order. Each fibre, F_x, has two components. The manifold is orientable if and only if F itself has two components (assuming X to be connected). In this case either of the components gives a reduction of the structure group to $\mathrm{SO}(N)$. Thus an orientation, ${}_oF \subset F$, is simply a subbundle of the orthonormal frame bundle which is a principal $\mathrm{SO}(N)$-bundle, this subbundle is then the bundle of oriented orthonormal frames.

If a Riemann manifold is orientable and is given an orientation then the Riemann density becomes a form of maximal degree. Thus if ϕ^j, $j = 1,\dots,N$, is an oriented orthonormal frame in F_p then

$$\upsilon = \phi^1 \wedge \cdots \wedge \phi^N \in \Lambda^N_p(X)$$

is the volume form at p. It is independent of the choice of oriented orthonormal frame. The choice of orientation also fixes the Hodge (star) isomorphism:

$$\star\colon \Lambda^j_p(X) \longleftrightarrow \Lambda^{N-j}_p(X),$$

for each $p \in X$ and each j. If ϕ^j is an oriented orthonormal frame then

$$\star(\phi^{i_1} \wedge \cdots \wedge \phi^{i_j}) = \sigma \phi^{i_{j+1}} \wedge \cdots \wedge \phi^{i_N} \tag{2.85}$$

where σ is the sign of the permutation $(1, \dots, N) \longleftrightarrow (i_1, \dots, i_N)$. The forms on the left give a basis for $\Lambda^j_p(X)$, so $\star$ is defined by extending it linearly. The spaces $\Lambda^j(X)$ carry fibre metrics derived from the metric, on T^*X, by demanding that $\phi^1 \wedge \cdots \wedge \phi^j$ should have length 1 for any orthonormal basis of T^*_pX. In terms of this inner product on the form bundles, the operator (2.85) satisfies

$$\langle \star w, v \rangle \upsilon = w \wedge v. \tag{2.86}$$

In fact this shows that it is uniquely determined, independent of choice of oriented frame. It is therefore a $\mathcal{C}^\infty$ bundle map.

Moreover it follows from (2.85) that

$$\star^2 = (-1)^{j(N-j)} \text{ on } \Lambda^j(X). \tag{2.87}$$

The identity (2.86) makes $\star$ useful in analyzing the adjoint, δ, of d. The adjoint is defined by requiring

$$\int_X \langle \delta w, v \rangle \upsilon = \int_X \langle w, dv \rangle \upsilon, \ \forall\ w \in \mathcal{C}^\infty(X; \Lambda^j),\ \ v \in \mathcal{C}^\infty(X; \Lambda^{j-1}).$$

Using Stokes' theorem and (2.86) twice gives, for any $\mathcal{C}^\infty$ k form u and $N-k-1$ form v,

$$\int_X \langle \star dw, v \rangle \upsilon = \int_X dw \wedge v = (-1)^{k+1} \int_X w \wedge dv = (-1)^{k+1} \int_X \langle \star w, dv \rangle \upsilon.$$

Then using (2.87) it follows readily that

$$\delta = (-1)^{Nj+N+1} \star d \star \ \text{ on } \mathcal{C}^\infty(X; \Lambda^j). \tag{2.88}$$

Whilst the Hodge star operator is real it is useful to consider, in the even dimensional case, the complex operator

$$\tau = i^{j(j-1)+k} \star\colon \Lambda^j(X) \longrightarrow \Lambda^{2k-j}(X),\ \dim X = 2k. \tag{2.89}$$

By direct computation one then finds:

LEMMA 2.34. *If* $\dim X = 2k$ *is even then* τ *is an involution,* $\tau^2 = \mathrm{Id}$, *and*

$$\tau(d+\delta) + (d+\delta)\tau = 0.$$

In the odd-dimensional case there is a similar operator,

$$\tau' = i^{j(j-1)+k} \star R_\Lambda : \Lambda^j(X) \longrightarrow \Lambda^{2k-1-j}(X), \ \dim X = 2k-1,$$

where R_Λ is the parity involution

$$R_\Lambda = \begin{cases} 1 & \text{on } \Lambda^{\mathrm{evn}} \\ -1 & \text{on } \Lambda^{\mathrm{odd}}. \end{cases} \tag{2.90}$$

2.13. b-connections.

The connection on $\widetilde{F}$ can be recovered from (2.66) by noting that a frame is covariant constant at a point p if and only if each element of it is covariant constant. Then (2.67) can be used to extend the discussion to b-metrics. If g is a general b-metric on a manifold with boundary the problem is to determine what happens to the $C_{p,\phi}$ as p approaches the boundary. In the interior, but near the boundary, local coordinates

$$z^0 = \log x, z^j = y_j, \ j \geq 1, \tag{2.91}$$

can be used, where $x, y_1, \dots, y_n$ is a coordinate system up to the boundary and the dimension is now taken to be $N = n+1$. By definition the metric is smooth (in x, y) and non-degenerate with respect to the induced basis of ${}^bT^*X$,

$$g = \sum_{i,j=0}^{n} g_{ij}(x,y) dz^i dz^j, \ dz^0 = \frac{dx}{x}, \ dz^j = dy_j, \ j \geq 1. \tag{2.92}$$

Using the formulae (2.64) for the Christoffel symbols, noting that $\partial_{z^0} = x\partial/\partial x$ while $\partial_{z^j} = \partial_{y_j}$ for $j \geq 1$ and (2.66) for the covariant derivative gives:

LEMMA 2.35. *The covariant derivative for a general b-metric on a manifold with boundary, X, extends from the interior to a first order differential operator:*

$$\begin{gathered} {}^b\nabla : \mathcal{C}^\infty(X; {}^b\Lambda^1) \longrightarrow \mathcal{C}^\infty(X; {}^bT^*X \otimes {}^b\Lambda^1), \\ {}^b\nabla \in \mathrm{Diff}^1_b(X; {}^b\Lambda^1, {}^bT^*X \otimes {}^b\Lambda^1) \\ {}^b\nabla(f\mu) = f{}^b\nabla\mu + df \otimes \mu \ \forall \ f \in \mathcal{C}^\infty(X), \mu \in \mathcal{C}^\infty(X; {}^bT^*X). \end{gathered} \tag{2.93}$$

In the last equation in (2.93), df should be interpreted as a b-form, really ${}^b df$.

PROOF: Applying (2.67) to the coordinates (2.91) gives

$$ {}^b\nabla dz^j = -\sum_{ik} \gamma^j_{ik} dz^i \otimes dz^k, $$

where the Christoffel symbols are given by (2.64). Since g_{ij} in (2.92) is non-degenerate, $g_{ij,0} = x\partial g_{ij}/\partial x$, and $g_{ij,l} = \partial g_{ij}/\partial y_l$ for $l > 0$, the γ^j_{ik} are indeed $\mathcal{C}^\infty$ down to $x = 0$. Thus the ${}^b\nabla dz^j$ are smooth sections of ${}^bT^*X \otimes {}^b\Lambda^1$ over the coordinate patch. The third part of (2.93) then follows by continuity and in turn leads to the other two parts.

Reversing the arguments above shows that an affine b-connection in the sense of (2.93) fixes the condition that a general b-frame, i.e. smoothly varying basis of ${}^bT^*X$, should be covariant constant at a point. Let

$$ {}^bF \subset {}^b\widetilde{F} $$

be the orthonormal coframe bundle for the b-metric as a subbundle (a principal $\mathrm{O}(n+1)$-bundle) of the general b-coframe bundle. Thus ${}^b\widetilde{F}_p$ consists of all the bases of ${}^bT^*X$ and ${}^bF_p$ all the orthonormal bases. Both are manifolds with boundary and ${}^bF$ is compact if X is compact. It follows that the $(n+1)$-planes $C_{p,e}$ extend smoothly from the interior to define

$$ \begin{aligned} {}^b\widetilde{C}_{p,\phi} &\subset {}^bT_{(p,\phi)}{}^b\widetilde{F} \quad \forall\ (p,\phi) \in {}^b\widetilde{F} \\ {}^bC_{p,\phi} &\subset {}^bT_{(p,\phi)}{}^bF \quad \forall\ (p,\phi) \in {}^bF. \end{aligned} $$

This is the more sophisticated formulation of the metric, i.e. Levi-Civita, b-connection, where (2.40) (and its generalization to $\mathrm{Gl}(n+1,\mathbb{R})$) and the analogue of (2.41):

$$ {}^bC_{p,\phi} \oplus T_\phi {}^bF_p = {}^bT_{(p,e)}{}^bF $$

continue to hold by continuity.

EXERCISE 2.36. Show that the structure equation as in (2.48) holds with $Q \in \mathcal{C}^\infty({}^bF; {}^b\Lambda^2 F \otimes \mathfrak{so}(n+1))$ and that in consequence the Riemann curvature tensor becomes

$$ R \in \mathcal{C}^\infty(X; {}^b\Lambda^2 X \otimes {}^b\Lambda^2 X). $$

Of course the symmetry properties in Lemma 2.29 continue to hold by continuity.

Now what happens if the metric is assumed to be an exact b-metric? Taking local coordinates (2.91) in which (2.11) holds, the metric tensor satisfies

$$\begin{aligned} g_{00},\ g^{00} =& 1 + O(x^2) \\ g_{0i},\ g^{0i} =& O(x),\ i = 1, \dots, n. \end{aligned}$$

Then for the Christoffel symbols, with respect to the coordinates z^i introduced above, (2.64) implies that

$$\gamma^i_{j0} = O(x),\ i, j = 0, \dots, n.$$

This means that (2.66) can be rewritten as

(2.94)

$$\begin{aligned} \nabla\alpha = \sum_{i=0}^{n} \partial_x \alpha_i dz^i \otimes dx + \sum_{i=0}^{n}\sum_{j=1}^{n} \partial_{z^j} \alpha_i dz^i \otimes dz^j \\ - \sum_{i,j=0}^{n} \alpha_i (\frac{1}{x}) \gamma^i_{j0} dz^j dx - \sum_{i,j=0}^{n}\sum_{k=1}^{n} \alpha_i \gamma^i_{jk} dz^j dy^k \end{aligned}$$

and so proves:

PROPOSITION 2.37. *For an exact b-metric the Levi-Civita connection extends by continuity to a connection on the b-frame bundle, and equivalently to a covariant derivative*

(2.95)

$$\nabla : \mathcal{C}^\infty(X; {}^b\Lambda^1) \longrightarrow \mathcal{C}^\infty(X; T^*X \otimes {}^b\Lambda^1),\ \nabla \in \operatorname{Diff}^1(X; {}^b\Lambda^1, T^*X \otimes {}^b\Lambda^1).$$

EXERCISE 2.38. Does the Levi-Civita connection of an exact b-metric define a b-connection on T^*X? Does it define a connection on T^*X?

Consider in a little more detail exactly what happens at the boundary in the case of an exact b-metric. The b-1-form dx/x is well-defined at ∂X as the section of length one of the orthocomplement in ${}^bT^*_{\partial X}X$ to $T^*\partial X$. If $F(\partial X)$ is the orthonormal frame bundle for the metric h induced on the boundary by the exact b-metric (2.11) then extending an orthonormal basis by adding dx/x to it gives the embedding

$$F(\partial X) \hookrightarrow {}^bF_{\partial X}(X).$$

This means that the structure group of ${}^bF(X)$, the orthonormal frame bundle, reduces to $\mathrm{O}(n)$ at ∂X. It is important to see the properties of the curvature of the b-metric which follow.

First note that the form dx/x, always the one appearing in (2.11) for the exact b-metric, satisfies

$$ {}^b\nabla\left(\frac{dx}{x}\right) = 0 \text{ in } {}^bT^*X \otimes {}^bT^*X \text{ at } \partial X. $$

Indeed this follows from (2.94). Furthermore, from (2.95),

$$ {}^b\nabla_{x\frac{\partial}{\partial x}}\alpha = 0 \text{ in } {}^bT^*X \otimes {}^bT^*X \text{ at } \partial X \quad \forall\ \alpha \in \mathcal{C}^\infty(X; {}^bT^*X). $$

Combining these observations with the commutation properties of vector fields tangent to the boundary easily gives:

PROPOSITION 2.39. *The Riemann curvature tensor of an exact b-metric is a $\mathcal{C}^\infty$ tensor in the usual sense:*

$$ R \in \mathcal{C}^\infty(X; \Lambda^2 \otimes \Lambda^2) $$

and has the property that its restriction to the boundary (in all arguments) is the Riemann curvature tensor of the induced metric on the boundary.

In this sense exact b-metrics correspond to truly cylindrical ends. The problem of the extension of all the discussion above, and the APS theorem itself, to the case of non-exact b-metrics is quite analogous to the recent discussion by Grubb [42] and Gilkey [38] of the extension of the more standard 'elliptic' formulation of the APS theorem to the case where the metric is not a product near the boundary.

PROOF: The commutator of any vector field tangent to the boundary with a vector field vanishing on the boundary also vanishes on the boundary, so the curvature operator on 1-forms vanishes at ∂X if either argument is $x\partial/\partial x$. This, and the symmetry properties, show that the curvature tensor is smooth in the ordinary sense, since any term containing dx/x vanishes at ∂X. It follows immediately from the definition as the curvature operator on 1-forms that the pull-back to the boundary of the Riemann tensor of the b-metric is the Riemann tensor of the boundary metric.

In general a b-connection on a vector bundle E is given by a first order differential operator ${}^b\nabla \in \mathrm{Diff}_b^1(X; E, {}^bT^*X \otimes E)$ which is a connection in the interior. It therefore satisfies (2.58) and so by continuity

$$ {}^b\nabla(f\mu) = df \otimes \mu + f({}^b\nabla\mu),\ f \in \mathcal{C}^\infty(X),\ \mu \in \mathcal{C}^\infty(X; E), \tag{2.96} $$

where df is, as in (2.93), interpreted as a section of ${}^bT^*X$. As for the Levi-Civita connection of an exact b-metric, one way to get such a b-connection is just to start with an ordinary connection and restrict its action to tangent vector fields. To characterize such connections amongst general b-connections, consider the dual sequence to (2.10):

$$0 \longrightarrow T^*\partial X \longrightarrow {}^bT^*_{\partial X}X \longrightarrow {}^bN^*\partial X \longrightarrow 0.$$

Here ${}^bN^*\partial X$ is a canonically trivial bundle over ∂X, with dx/x a section independent of the choice of defining function x. Projection onto this line therefore defines a differential operator, namely ${}^b\nabla_{x\partial/\partial x}$ evaluated at ∂X. The identity (2.96) shows that this is actually of order zero:

$$ {}^b\nabla_{x\partial/\partial x} \in \hom(E_{\partial X}). \tag{2.97}$$

Proposition 2.40. *A b-connection on a bundle E over a compact manifold with boundary is (induced by) a connection if and only if the homomorphism ${}^b\nabla_{x\partial/\partial x}$ of $E_{\partial X}$ vanishes identically.*

Proof: Certainly if ${}^b\nabla$ is induced by a connection ∇ then ${}^b\nabla_{x\partial/\partial x} = x\nabla_N$, where for some choice of defining function x the $\mathcal{C}^\infty$ vector field N satisfies $Nx = 1$ at ∂X. The homomorphism (2.97) then vanishes. Conversely if this homomorphism vanishes and N is such a normal vector field, any $\mathcal{C}^\infty$ vector field on X can be written in the form

$$W = V + fN,\ f \in \mathcal{C}^\infty(X),\ V \in \mathcal{V}_b(X).$$

The covariant derivative of a section e of E can then be defined by

$$\nabla_W e = {}^b\nabla_V + \frac{f}{x}({}^b\nabla_{xN}).$$

This defines a connection on E which induces the b-connection ${}^b\nabla$.

Note that if the homomorphism ${}^b\nabla_{x\partial/\partial x}$ is non-zero then the b-connection does not induce a connection on the boundary.

Exercise 2.41. Show that a section of ${}^bT^*X$ over X with the property that its restriction to ∂X projects to the zero section of ${}^bN^*\partial X$ is just a $\mathcal{C}^\infty$ section of T^*X. Use (2.96) to show that the difference between two b-connections is given by the tensor product with a $\mathcal{C}^\infty$ section of ${}^bT^*X$. Combine these two observations to give an alternative proof of Proposition 2.40.

Even when the homomorphism (2.97) for a b-connection is non-zero, it is still possible for the curvature operator to be a smooth form in the ordinary sense, rather than just a smooth b-form. Notice that ${}^b\nabla_V u$, for $V \in \mathcal{C}^\infty(\partial X; {}^bTX)$ and $u \in \mathcal{C}^\infty(\partial X; E)$, is a well-defined element of $\mathcal{C}^\infty(\partial X; E)$. From the discussion in §2.11 it follows that ${}^b\nabla_V F$ is similarly well defined if $F \in \mathcal{C}^\infty(\partial X; \hom(E))$.

LEMMA 2.42. *If X is a manifold with boundary and E is a vector bundle over X with b-connection then the curvature operator $K_E \in {}^b\Lambda^2 \otimes \hom(E)$ is a smooth form, i.e. $K_E \in \Lambda^2 \otimes \hom(E)$ if and only if the homomorphism* (2.97) *is covariant constant on the boundary in the sense that*

$$ {}^b\nabla_V({}^b\nabla_{x\partial/\partial x}) = 0 \ \forall \ V \in {}^bT_{\partial X}X. \tag{2.98}$$

PROOF: The b-connection on $\hom(E)$ satisfies (2.77), so if $A = {}^b\nabla_{x\partial/\partial x}$

$$({}^b\nabla_{x\partial/\partial x}A)u = {}^b\nabla_{x\partial/\partial x}(Au) - A\,{}^b\nabla_{x\partial/\partial x}u. \tag{2.99}$$

Thus ${}^b\nabla_{x\partial/\partial x}A = [A, A] = 0$. So (2.98) only depends on $V \in \mathcal{V}(\partial X)$. In fact (2.99) just corresponds to $K_E(x\partial/\partial x, x\partial/\partial x) = 0$, by antisymmetry. More generally if $V \in {}^bT_{\partial X}X$ then

$$\begin{aligned} K_E(x\partial/\partial x, V)u &= {}^b\nabla_{x\partial/\partial x}{}^b\nabla_V u - {}^b\nabla_V{}^b\nabla_{x\partial/\partial x}u - {}^b\nabla_{[x\partial/\partial x, V]}u \\ &= -\,({}^b\nabla_V A)u \end{aligned}$$

since $[x\partial/\partial x, V] = 0$ as a section of ${}^bT_{\partial X}X$. Thus (2.98) implies that, at ∂X, $K_E(x\partial/\partial x, \cdot) = 0$ and hence $K_E \in \Lambda^2 \otimes \hom(E)$ and conversely.

2.14. Characteristic classes.

No attempt is made to give a full treatment of characteristic classes here. However a brief discussion of the Weil homomorphism is included so that the integrands in (In.1) and (In.6) can be understood. See in particular [27] and [20]. Suppose that

$$h\colon \mathfrak{so}(N) \longrightarrow \mathbb{C} \tag{2.100}$$

is a function on $\mathfrak{so}(N)$ which is invariant under the adjoint action of $\mathrm{SO}(N)$, i.e.

$$h(GAG^{-1}) = h(A) \quad \forall \ A \in \mathfrak{so}(N), \ G \in \mathrm{SO}(N) \tag{2.101}$$

(or if X is not orientable under $\mathrm{O}(N)$.) The assumption that h is invariant, i.e. (2.101), means that the derivative satisfies

$$h'(A, [A, B]) = 0 \ \forall \ A, B \in \mathfrak{so}(N). \tag{2.102}$$

where $h'(A, B) = dh(A + tB)/dt$ at $t = 0$.

Consider the curvature form Q in (2.48) for X a compact manifold without boundary. It is a 2-form on F which takes values in $\mathfrak{so}(N)$. Alternatively it can be considered as an element of $\mathfrak{so}(N)$, the space of antisymmetric real 2×2 matrices, with entries which are 2-forms on F. The important point to note is that the exterior multiplication of 2-forms is commutative. Thus if h is a polynomial there is no ambiguity in evaluating $h(Q)$ as a form on F. In fact the wedge product of more than $N/2$ 2-forms vanishes, so even if h is a smooth function, $h(Q)$ is well defined as a 2-form on F by replacing h by at least the first $N/2$ terms in its Taylor series at 0. The point of the assumption, (2.101), of invariance of h is that (2.52) shows that

$$(A_O)^* h(Q) = h(OQO^t) = h(Q)$$

is actually invariant under the action of $\mathrm{SO}(N)$. A form on F which is invariant under the $\mathrm{SO}(N)$ action and vanishes if any one of its arguments is tangent to the fibre of $\pi \colon F \longrightarrow X$ is the lift of a uniquely defined form on the base.

EXERCISE 2.43. Show that if $\phi \colon U \longrightarrow F$ is a section over some open set of X and $h \in \mathcal{C}^\infty(F; \Lambda^* F)$ is invariant, $A_O^* h = h$ for all $O \in \mathrm{O}(N)$, and $h(v, \cdot, \cdots, \cdot) = 0$ if $v \in T_{\phi(p)} F_p$ then $\gamma = \phi^* h$ is a smooth form on U which is independent of the choice of ϕ and is such that $h = \pi^* \gamma$.

Thus there is a well-defined form $h(R)$ on X such that

$$h(Q) = \pi^*(h(R)),$$

$\pi \colon F \longrightarrow X$ being the bundle map. This form on the base represents the characteristic class associated to the invariant polynomial h and to the metric.

EXERCISE 2.44. Show that the notation here is a reasonable one, in that $h(R)$ expressed with respect to any orthonormal coframe ϕ^j is just h evaluated on $\sum\limits_{pq} R_{ijpq} \phi^p \wedge \phi^q$.

The structure equation (2.48) has the important consequence:

LEMMA 2.45. *The form $h(R)$ defined by an invariant polynomial* (2.100) *is always closed.*

PROOF: Since wedge product of 2-forms commutes with all forms, in particular 3-forms, the exterior derivative of $h(Q)$ can be written unambiguously as

$$dh(Q) = h'(Q, dQ), \tag{2.103}$$

where the derivative of h, $h'(Q, A)$, is a linear functional in $A \in \mathfrak{so}(N)$. From (2.48)

$$dQ_{ij} = \sum_q \left[d\omega_{iq} \wedge \omega_{qj} - \omega_{iq} \wedge d\omega_{qj} \right] = \sum_q \left[Q_{iq} \wedge \omega_{qj} - \omega_{iq} \wedge Q_{qj} \right].$$

since the terms with three factors of ω cancel. This is just the commutator in the Lie algebra $\mathfrak{so}(N)$ so

$$dQ = [Q, \omega] \quad \text{(2nd Bianchi identity)}. \tag{2.104}$$

Inserting (2.104) in (2.103) and using (2.102) shows that $h(R)$ is indeed a closed form.

The classic case of this construction gives the Pontriagin forms. These arise by taking the invariant polynomial

$$h(A) = \det(\mathrm{Id} + \frac{A}{2\pi});$$

the invariance being a consequence of the multiplicativity of the determinant. The kth Pontrjagin form is the term of degree $4k$ in $h(R)$; it is easy to see that the terms of degree $4k+2$ all vanish.

The de Rham theorem allows $h(R)$ to be interpreted as a cohomology class on X. It is important to note that this class is independent of the choice of metric. Indeed any two metrics, g^0 and g^1, can be connected by a smooth one-parameter family of metrics g^t. The construction of the connection is clearly smooth in the parameter, so the forms ω^t and Q^t depend smoothly on t. Differentiating the structure equation gives:

$$\frac{dQ^t_{ij}}{dt} = d\frac{d\omega^t_{ij}}{dt} + \sum_q \left[\frac{d\omega^t_{iq}}{dt} \wedge \omega^t_{qj} + \omega^t_{iq} \wedge \frac{d\omega^t_{qj}}{dt} \right].$$

Consider the one-parameter family of closed forms $h(Q^t)$ on F:

$$\begin{aligned} \frac{dh(Q^t)}{dt} &= h'(Q^t; \frac{dQ^t}{dt}) \\ &= h'(Q^t; d\frac{d\omega^t}{dt}) + h'(Q^t; \frac{d\omega^t}{dt} \wedge \omega^t + \omega^t \wedge \frac{d\omega^t}{dt}). \end{aligned}$$

The first term here can be written $d\,[h'(Q^t; d\omega^t/dt)]$ since Q^t is closed. The second term is $\mathrm{O}(N)$-invariant and only ω^t is non-zero on vertical vectors, so evaluated on $v \in T_\phi F_p$, with $A = \omega^t(v)$,

$$h'(Q^t; \frac{d\omega^t}{dt} \wedge \omega^t + \omega^t \wedge \frac{d\omega^t}{dt})(v, \cdot) = h'(Q^t; [\frac{d\omega^t}{dt}, A])(\cdot) = 0$$

by (2.102). Since it also vanishes with ω^t, i.e. on the connection planes, this second term vanishes identically and hence

$$h(Q^1) - h(Q^0) = dT, \; T = \int_0^1 h'(Q^t, \frac{d\omega^t}{dt})dt. \tag{2.105}$$

This transgression formula shows that the de Rham class of $h(R)$, i.e. its equivalence class up to exact forms, is independent of the choice of metric used to define it. It therefore corresponds to a cohomology class.

The nilpotence of the exterior action of a 2-form means that to define $h(R)$ it suffices for the polynomial h to be obtained as the Taylor series of a function which is just defined near the origin in $\mathfrak{so}(N)$, provided it is invariant there. Such examples arise directly in the computation of the index form. In particular the $\widehat{A}$-genus:

$$\widehat{A}(X) = \det{}^{\frac{1}{2}}\left(\frac{R/4\pi i}{\sinh(R/4\pi i)}\right) \tag{2.106}$$

is a well-defined form, depending only on the metric and orientation. Here $\det^{\frac{1}{2}}$ is the positive square root of the determinant of a matrix. It is defined near the identity and is invariant under conjugation by $\mathrm{O}(N)$. Its argument is a formal power series with first term the identity matrix, so the square root of the determinant is a well-defined invariant formal power series.

EXERCISE 2.46. Show that $\widehat{A}(X)$ can be expressed in terms of the Pontrjagin forms.

Similarly the L-genus of Hirzebruch is

$$L(X) = \det{}^{\frac{1}{2}}\left(\frac{R/4\pi i}{\tanh(R/4\pi i)}\right). \tag{2.107}$$

It arises in the signature formula (see §9.3).

2.15. Hermitian bundles.

So far we have been dealing mainly with real bundles. In practice many of the examples below are complex. A connection on a complex bundle is automatically assumed to be complex, i.e. $\mathbb{C}$-linear. The existence of such a connection follows as in the discussion leading to Lemma 2.21.

A complex bundle is Hermitian if it carries a sesqui-linear inner product. A connection on the bundle is unitary (or Hermitian) if it satisfies (2.61) with respect to this pairing, i.e.

$$V\langle e, f\rangle = \langle \nabla_V e, f\rangle + \langle e, \nabla_V f\rangle \tag{2.108}$$

for all real vector fields V.

EXERCISE 2.47. Show that the proof of Lemma 2.21 also implies that any Hermitian bundle has a unitary connection.

For Hermitian bundles with unitary connections there is a similar construction of characteristic classes as in the real case, leading to the Chern classes. The curvature operator

$$K_E(V,W) = [\nabla_V, \nabla_W] - \nabla_{[V,W]} \in \mathcal{C}^\infty(X; \hom(E))$$

takes values in the subbundle, $\mathfrak{u}(E) \subset \hom(E)$, formed by the Lie algebras of the unitary groups of the Hermitian structures. An invariant polynomial

$$h\colon \mathfrak{u}(m) \longrightarrow \mathbb{C},\ \ h(GAG^*) = h(A)\ \forall\ A \in \mathfrak{u}(m), G \in \mathrm{U}(m),$$

where m is the fibre dimension of E, fixes a bundle map $\mathcal{C}^\infty(X; \mathfrak{u}(E)) \longrightarrow \mathcal{C}^\infty(X)$. Regarding K_E as a 2-form with values in $\mathfrak{u}(E)$ it follows, as in the discussion above, that $h(K_E)$ is a well-defined closed form on X. Necessarily it has only terms of even degree.

For example the Chern forms, $\mathrm{cc}_j(E)$, of the bundle are the terms of degree $2j$ in the total Chern form

$$\mathrm{cc}(E) = \det\left(\mathrm{Id} + i\frac{K_E}{2\pi}\right).$$

There is a similar transgression formula showing that $\mathrm{cc}_j(E)$ is well defined as a cohomology class independent of the choice of Hermitian structure and unitary connection on E. The Chern character of the bundle is defined as the characteristic class

$$\mathrm{Ch}(E) = \mathrm{tr}\left[\exp\left(\frac{iK_E}{2\pi}\right)\right]. \tag{2.109}$$

The term which is homogeneous of degree k in $A \in \mathfrak{u}(2k)$ in the trace of the exponential

$$\mathrm{Pf}(A) = \frac{1}{k!}\frac{d^k}{dr^k}\,\mathrm{tr}\exp(rA) \tag{2.110}$$

is an invariant polynomial called the Pfaffian. It occurs in the Gauss-Bonnett formula (see §9.2).

The combination of (2.106) and (2.109) means that the Atiyah-Singer integrand in (In.2) is now defined. The integral in (In.1) picks out the term of maximal form degree, $\dim X$, and integrates it over the oriented manifold X.

2.16. de Rham cohomology.

Before briefly discussing the extension of the construction of characteristic classes to the 'b' setting, first consider the various forms of the de Rham theorem on a compact manifold with boundary. The standard complex, (2.17), yields the 'absolute' de Rham spaces:

$$\mathcal{H}^k_{\mathrm{dR,abs}}(X) = \left\{u \in \mathcal{C}^\infty(X;\Lambda^k); du = 0\right\} / d\mathcal{C}^\infty(X;\Lambda^{k-1}). \tag{2.111}$$

On the other hand the operator d also defines maps

$$d\colon \dot{\mathcal{C}}^\infty(X;\Lambda^k) \longrightarrow \dot{\mathcal{C}}^\infty(X;\Lambda^{k+1}),$$

where for any vector bundle E over a manifold with boundary (or corners) $\dot{\mathcal{C}}^\infty(X;E) \subset \mathcal{C}^\infty(X;E)$ is the subspace of section vanishing to all orders at the boundary. Thus $u \in \dot{\mathcal{C}}^\infty(X;E)$ if it is the restriction to X of some section $u' \in \mathcal{C}^\infty(\widetilde{X};\widetilde{E})$ which vanishes on $\widetilde{X} \setminus X$, for some extensions of X and E. The relative de Rham spaces are

$$\mathcal{H}^k_{\mathrm{dR,rel}}(X) = \left\{u \in \dot{\mathcal{C}}^\infty(X;\Lambda^k); du = 0\right\} / d\dot{\mathcal{C}}^\infty(X;\Lambda^{k-1}). \tag{2.112}$$

THEOREM 2.48. *The absolute and relative de Rham cohomology spaces of a compact manifold with boundary are canonically isomorphic to the (singular) cohomology spaces of the interior, with arbitrary and compact supports respectively.*

There is also a natural b-de Rham space. Namely one can take the cohomology of the complex (2.21):

$$\mathcal{H}^k_{\mathrm{dR,b}}(X) = \left\{u \in \mathcal{C}^\infty(X;{}^b\Lambda^k); {}^b du = 0\right\} / {}^b d\mathcal{C}^\infty(X;{}^b\Lambda^{k-1}), \tag{2.113}$$

where the notation ${}^b d$ is only briefly resuscitated for emphasis.

PROPOSITION 2.49. ([4]) *The b-de Rham spaces of a compact manifold decompose canonically:*

$$\mathcal{H}^k_{\mathrm{dR,b}}(X) = \mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X) \oplus \mathcal{H}^k_{\mathrm{dR,abs}}(X). \tag{2.114}$$

PROOF (BRIEF): There is an obvious map at the level of forms from $\mathcal{H}^k_{\mathrm{dR,abs}}(X)$ into $\mathcal{H}^k_{\mathrm{dR,b}}(X)$ since closed smooth sections of $\Lambda^k X$ are naturally closed smooth sections of ${}^b\Lambda^k X$. At the level of forms the map from $\mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X)$ is given in terms of a collar decomposition of X near ∂X, a

[4] Unpublished joint result with R.R. Mazzeo.

boundary coordinate x and $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with $\phi(0) = 1$ and $\phi(x)$ supported in the collar by

$$\mathcal{C}^\infty(\partial X; \Lambda^{k-1}) \ni \alpha \longmapsto \phi(x)\frac{dx}{x} \wedge \alpha \in \mathcal{C}^\infty(X; {}^b\Lambda^k). \tag{2.115}$$

At the level of cohomology these maps give (2.114). The inverse to (2.115) is given by contraction with $x\partial/\partial x \in {}^bT_{\partial X}X$:

$$\mathcal{H}^k_{\mathrm{dR,b}}(X) \ni \beta \longmapsto \beta(x\frac{\partial}{\partial x}, \cdot) \in \mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X).$$

2.17. b-characteristic classes.

Consider the alterations which must be made to the treatment of characteristic classes in §2.14 and §2.15 in the case of a b-metric or a b-connection.

The formulæ in §2.14 are local in nature and so continue to hold when interpreted as involving b-forms. In particular, with R the curvature tensor of a b-metric and h an invariant polynomial, $h(R)$ is a well-defined smooth closed b-form on X, which is now a compact manifold with boundary. Applying Proposition 2.49, such a form defines both an absolute cohomology class on X and a class on ∂X. If the metric is an exact b-metric then R is a smooth form and the second term vanishes, so $h(R)$ defines an absolute cohomology class.

On the other hand observe what happens in the transgression formula (2.105). In the case of a general b-metric the form T is a smooth b-form on F but, because of the presence of $d\omega^t/dt$, is not necessarily a smooth form. If the deformation is through exact b-metrics then the connection form is always smooth, so T will in fact be smooth. In any case it follows that the characteristic classes give well-defined absolute cohomology classes on X.

Thus the first term in (In.6) is defined as the integral over X, a compact manifold with boundary, of the volume form component of AS, given by (In.2). As distinct from the boundaryless case, this does not have direct topological significance, since it does not correspond to a pairing in cohomology (as absolute cohomology classes pair with relative homology classes). In particular under a deformation of the metric the two terms in (In.6) need not be separately preserved. The variation of the first (and hence of the second) can be obtained from the transgression formula, (2.105), and Stokes' theorem. This is discussed in more detail in §8.15.

Chapter 3. Spin structures

The twisted Dirac operators involved in the APS theorem, as stated in the Introduction, are defined in terms of a spin structure on the manifold. This is now discussed, as is the more general notion of a Dirac operator associated to a Clifford module with connection. Needless to say these concepts are extended to the case of compact manifolds with boundary equipped with b-metrics.

3.1. Euclidean Dirac operator.

Dirac, wanting to quantize the electron, looked for a first-order differential operator with square the d'Alembertian, i.e. the wave operator. This is just the Laplacian for a metric of signature $+,-,-,-$ and one can as well do the same thing for the ordinary Laplacian. The question is: Does there exist a matrix-valued constant coefficient operator

$$\eth^+ = \sum_{j=1}^{n} \gamma_j D_j, \quad D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \tag{3.1}$$

such that

$$(\eth^+)^* \cdot \eth^+ = \sum_{j=1}^{n} D_j^2 = \Delta? \tag{3.2}$$

Here the adjoint is the usual matrix adjoint and the operator adjoint with respect to Lebesgue measure, i.e. $D_j^* = D_j$. Clearly (3.2) holds if and only if the matrices satisfy

$$\gamma_j^*\gamma_j = \mathrm{Id},\ \gamma_j^*\gamma_k + \gamma_k^*\gamma_j = 0,\ j \neq k,\ j,k = 1,\dots,n. \tag{3.3}$$

In one and two dimensions it is easy to solve (3.3). If $n = 1$, taking $\gamma_1 = i$ gives $\eth^+ = d/dx$. If $n = 2$, then $\gamma_1 = i$, $\gamma_2 = -1$ are 1×1 matrix solutions. So for $n = 2$ the "obvious" solution (leading to complex analysis and hence a good part of nineteenth century mathematics) is

$$\eth^+ = \bar\partial = (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$
$$\Longrightarrow (\eth^+)^* = -\partial = -(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}),\ \Delta = -\partial\bar\partial = \eth^{+*}\eth^+.$$

Following Dirac, (3.3) can be solved with 2×2 matrices for $n = 4$. Taking $\gamma_1 = i\,\mathrm{Id}$, as seems reasonable, the remaining conditions become

$$\gamma_j^* = \gamma_j,\ \gamma_j^2 = \mathrm{Id},\ \gamma_j\gamma_k + \gamma_k\gamma_j = 0, \quad j \neq k,\ j,k \geq 2. \tag{3.4}$$

Thus the remaining matrices are self-adjoint anticommuting involutions. It is then easy to guess a solution, for example

$$\gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \gamma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \gamma_4 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = i\gamma_2\gamma_3. \tag{3.5}$$

EXERCISE 3.1. Check that these *Pauli matrices* do solve the problem (3.1), (3.2) for $n = 4$.

As will be shown below there is a solution of (3.1), (3.2) for $n = 2k$ amongst $2^{k-1} \times 2^{k-1}$ matrices. Of course having found a solution for any one n, it works for smaller values, simply by dropping some of the matrices. Note that the original idea was to find $\eth$ so that $\eth^2 = \Delta \cdot \mathrm{Id}$. If $\gamma_1 = i\,\mathrm{Id}$ and the γ_j for $j > 1$ satisfy (3.4) then $\gamma_1 = -i\,\mathrm{Id}$ and γ_j also satisfy (3.3). So setting $\eth^- = \eth^*_+$, the operator

$$\eth = \begin{bmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{bmatrix} \text{ satisfies } \eth^2 = \Delta \cdot \mathrm{Id}, \ \eth^* = \eth.$$

It will therefore follow that for $n = 2k, 2k+1$ there is a self-adjoint solution of (3.1) and (3.2) amongst operators with values in $2^k \times 2^k$ matrices (and no smaller).

The construction of solutions to (3.1), (3.2) in general will be through the Clifford algebra. To see how this arises, suppose $\eth$ is a $p \times p$ matrix-valued operator satisfying $\eth^2 = \Delta \cdot \mathrm{Id}$. The symbol of $\eth$ is a matrix depending linearly on $\xi \in \mathbb{R}^n$:

$$\sigma(\eth) = \sum_{i=1}^{n} \gamma_i \xi_i.$$

If $M_{\mathbb{C}}(p)$ is the algebra of complex $p \times p$ matrices then $\sigma(\eth)$ is a linear map

$$\mathbb{R}^n \ni \xi \longmapsto \sigma(\eth)(\xi) \in M_{\mathbb{C}}(p), \tag{3.6}$$

such that $\sigma(\eth)(\xi) \cdot \sigma(\eth)(\xi) = |\xi|^2\,\mathrm{Id}$.

This must be true for all ξ, so replacing ξ by $\xi + \eta$ and expanding gives

$$\sigma(\eth)(\xi) \cdot \sigma(\eth)(\eta) + \sigma(\eth)(\eta) \cdot \sigma(\eth)(\xi) = 2\langle \xi, \eta \rangle\,\mathrm{Id}. \tag{3.7}$$

This identity, derived for a linear map (3.6), makes perfectly good sense for a linear map

$$c : V \ni \xi \longmapsto \mathrm{cl}(\xi) \in A \tag{3.8}$$

if V is a (finite dimensional) vector space with an inner product $\langle , \rangle$ and A is an associative algebra with identity.

Any linear map, (3.8), extends to a map from the j-fold tensor product

$$\mathrm{Ten}_j(V) = V \overset{j \text{ factors}}{\otimes \cdots \otimes} V \longrightarrow A,$$
$$\sum a_{i_1 \dots i_j} e_{i_1} \otimes \cdots \otimes e_{i_j} \longmapsto \sum a_{i_1 \dots i_j} \mathrm{cl}(e_{i_1}) \dots \mathrm{cl}(e_{i_1}).$$

Thus such a map extends to the full tensor algebra

$$\hat{c}\colon \mathrm{Ten}(V) = \bigoplus_{j=0}^{\infty} \mathrm{Ten}_j(V) \longrightarrow A. \tag{3.9}$$

The identity (3.7) means that under (3.9) the ideal

$$\mathcal{I}_{\mathrm{Cl}}(V) = \Big\{ \sum_{\text{finite}} u_\alpha \big(\xi_\alpha \otimes \eta_\alpha + \eta_\alpha \otimes \xi_\alpha - 2\langle \xi_\alpha, \eta_\alpha \rangle \,\mathrm{Id}\big) v_\alpha; \quad \xi_\alpha, \eta_\alpha \in V, u_\alpha, v_\alpha \in \mathrm{Ten}(V) \Big\} \tag{3.10}$$

should be mapped to zero. Thus the quotient algebra

$$\mathrm{Cl}(V) = \mathrm{Ten}(V) \big/ \mathcal{I}_{\mathrm{Cl}}(V) \tag{3.11}$$

has the *universal property* that any linear map (3.8) satisfying[5]

$$\mathrm{cl}(\xi) \cdot \mathrm{cl}(\eta) + \mathrm{cl}(\eta) \cdot \mathrm{cl}(\xi) = 2\langle \xi, \eta \rangle \,\mathrm{Id} \tag{3.12}$$

extends to an algebra homomorphism

$$\tilde{c}\colon \mathrm{Cl}(V) \longrightarrow A.$$

3.2. Clifford algebra.

This makes it natural to study the *Clifford algebra* of V (with inner product $\langle , \rangle$) defined by (3.10), (3.11). The Clifford algebra is *filtered* by the order gradation from the tensor algebra:

$$\mathrm{Cl}^{(k)}(V) = \left\{ u \in \mathrm{Cl}(V); \exists\, v \in \bigoplus_{j=0}^{k} \mathrm{Ten}_j(V),\ u = [v] \right\}. \tag{3.13}$$

Thus the *order* of an element of $\mathrm{Cl}(V)$ is the minimum order of a representative in the tensor algebra. Note that the relation, (3.12), equates the

[5] Many authors use a different sign convention, so that the square of an element of V of unit length is $-\mathrm{Id}$. This corresponds to replacing the matrices γ_j by $i\gamma_j$.

symmetric part of an element of order 2 with an element of order 0. Thus the quotients are naturally identified:

$$\mathrm{Cl}^{(k)}(V)/\mathrm{Cl}^{(k-1)}(V) \equiv \Lambda^k(V),\ A_k\colon\ \mathrm{Cl}^{(k)}(V) \longrightarrow \Lambda^k(V),\ \forall\ k. \tag{3.14}$$

The map here is just the usual total antisymmetrization map, which means for example that if $c_1, c_2 \in V$ then

$$A_2(c_1c_2) = A_2(\frac{1}{2}(c_1\cdot c_2 + c_2\cdot c_1) + \frac{1}{2}[c_1, c_2]) = c_1 \wedge c_2.$$

The null space is the ideal generated by $\xi\otimes\eta + \eta\otimes\xi$, $\xi, \eta \in V$.

Let $e_1, \dots, e_n$ be an orthonormal basis of V. As a consequence of (3.14) the Clifford algebra $\mathrm{Cl}(V)$ has a basis given by the products in strictly increasing order:

$$e_{i_1}\cdot e_{i_2}\cdots e_{i_j},\ i_1 < i_2 < \cdots < i_j,\ j = 0, \dots, n = \dim V.$$

In particular

$$\dim_{\mathbb{R}} \mathrm{Cl}(V) = 2^{\dim V}. \tag{3.15}$$

Thus every element of $\mathrm{Cl}(V)$ has order at most $n = \dim V$:

$$\mathbb{R}\cdot\mathrm{Id} = \mathrm{Cl}^{(0)}(V) \subset \mathrm{Cl}^{(1)}(V) \subset \cdots \subset \mathrm{Cl}^{(n)}(V) = \mathrm{Cl}(V).$$

From (3.15) the dimension of $\mathrm{Cl}(V)$ is equal to that of the exterior algebra

$$\Lambda^* V = \bigoplus_{k=0}^{\dim V} \Lambda^k V.$$

In fact the exterior algebra arises as the Clifford algebra when the inner product is taken to vanish identically. From the form of the relation (3.12)

$$\mathrm{Cl}^{(1)}(V) \equiv V \oplus (\mathbb{R}\cdot\mathrm{Id}) = V \oplus \mathrm{Cl}^{(0)}(V). \tag{3.16}$$

There is another close relationship between $\mathrm{Cl}(V)$ and $\Lambda^* V$, namely an action of $\mathrm{Cl}(V)$ on $\Lambda^* V$:

$$c\colon\ \mathrm{Cl}(V) \longrightarrow \hom(\Lambda^* V). \tag{3.17}$$

In (3.16), the identity is mapped under (3.17) to the identity whereas

$$\mathrm{cl}(\xi) = i\,[\mathrm{ext}(\xi) - \mathrm{int}(\xi)] : \Lambda^* V \longrightarrow \Lambda^* V. \tag{3.18}$$

The *contraction* or inner product with ξ, $\operatorname{int}(\xi)$, is defined by evaluation on the vector dual to ξ,

$$v_\xi \in V^*, \quad v_\xi(w) = \langle \xi, w\rangle \quad \forall\ w \in V.$$

Then $\alpha \in \Lambda^k V$ can be identified as a totally antisymmetric multilinear map and if

$$\begin{aligned} \alpha\colon V^* \otimes \cdots \otimes V^* &\longrightarrow \mathbb{R}, \\ \operatorname{int}(\xi)\alpha = k!\alpha(v_\xi, \dots, .) &\in \Lambda^{k-1}V. \end{aligned} \tag{3.19}$$

The exterior product $\operatorname{ext}(\xi)$ is just wedge product (on the left) with ξ :

$$\operatorname{ext}(\xi)\alpha = \xi \wedge \alpha.$$

EXERCISE 3.2. Show, using (2.70), that the action of the curvature operator of the Levi-Civita connection on the exterior algebra can be written in terms of the Riemann curvature tensor as

$$\sum_{ij} dz^i \wedge dz^j \sum_{kl} R_{ijlk} \operatorname{ext}(dz^k) \operatorname{int}(dz^l). \tag{3.20}$$

Clifford multiplication, (3.18), extends to an algebra homomorphism (3.17) since, essentially by definition, it extends to the tensor algebra, as a map into the algebra $\hom(\Lambda^* V)$ and from (3.18), (3.19)

$$\operatorname{cl}(\xi) \cdot \operatorname{cl}(\xi) = \operatorname{ext}(\xi) \operatorname{int}(\xi) + \operatorname{int}(\xi) \operatorname{ext}(\xi) = |\xi|^2.$$

Thus c projects to (3.17).

So far the coefficients in the Clifford algebra have been real, i.e. the tensor product in (3.9) is over $\mathbb{R}$. Even when V itself is real, it is useful to consider the *complexified* Clifford algebra

$$\mathbb{C}\mathrm{l}(V) = \mathbb{C} \otimes_{\mathbb{R}} \operatorname{Cl}(V). \tag{3.21}$$

This just arises by taking complex coefficients in the definition of the tensor algebra and the ideal. Since the coefficient matrices, γ_i, are expected to be complex it is reasonable to consider (3.21). In any case the structure of the complexified algebra is simpler.

3.3. Periodicity.

For each integer $k \geq 1$ let $\operatorname{Cl}(k)$ be the Clifford algebra generated by $\mathbb{R}^k$ with the Euclidean inner product, so $\mathbb{C}\mathrm{l}(k)$ is its complexification. Similarly, $M_{\mathbb{C}}(r)$ denotes the complexification of $M(r)$, i.e. the algebra of complex $r \times r$ matrices. The fundamental *periodicity* result is:

Lemma 3.3. *For any $k \in \mathbb{N}$*

$$\mathbb{C}l(k+2) \simeq \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2). \tag{3.22}$$

Proof: Let $\xi_1, \dots, \xi_{k+2}$ be the natural basis of $\mathbb{R}^{k+2}$. It certainly suffices to show that there is a linear map

$$\mathbb{R}^{k+2} \longrightarrow \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2)$$

which extends to give the isomorphism (3.22). Since $\xi_1, \dots, \xi_k$ can be taken as the canonical basis of $\mathbb{R}^k$, it makes sense to define

$$\xi_j \longmapsto \xi_j \otimes \gamma_2, \quad j = 1, \dots, k \tag{3.23}$$

and for the last two generators to take the other Pauli matrices:

$$\begin{aligned} \xi_{k+1} &\longmapsto \mathrm{Id} \otimes \gamma_3 \\ \xi_{k+2} &\longmapsto \mathrm{Id} \otimes \gamma_4, \end{aligned}$$

with the γ_i fixed by (3.5). Of course $\mathrm{Id} \in \mathbb{C}l(k+2)$ is to be sent to $\mathrm{Id} \in \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2)$. Since it is defined on the generators this gives an algebra homomorphism $\mathrm{Ten}_{\mathbb{C}}(\mathbb{R}^{k+2}) \longrightarrow \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2)$. To check that it descends to $\mathbb{C}l(k+2)$ consider the generating relations. Certainly from (3.23)

$$\xi_j \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \xi_l \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \xi_j \cdot \xi_l \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad j, l = 1, \dots, k.$$

The remaining relations follow from the properties of the Pauli matrices in (3.4). This gives an algebra homomorphism

$$\mathbb{C}l(k+2) \longrightarrow \mathbb{C}l(k) \otimes M_{\mathbb{C}}(2). \tag{3.24}$$

To see that this map is surjective, notice that off-diagonal elements can be obtained as follows

$$\begin{aligned} \alpha_e \cdot \xi_{k+1} &\longmapsto \alpha_e \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \alpha_o \cdot \xi_{k+1} &\longmapsto \alpha_o \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ \alpha_e \cdot \xi_{k+2} &\longmapsto \alpha_e \otimes \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \\ \alpha_o \cdot \xi_{k+2} &\longmapsto \alpha_o \otimes \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \end{aligned} \tag{3.25}$$

where α_e is a sum of multiples of products of even numbers of the elements $\xi_1, \dots, \xi_k$ and similarly α_o is a sum of multiples of products of odd numbers of these basis elements. Similarly the diagonal elements can be obtained by noting that

$$
\begin{aligned}
\alpha_e &\longmapsto \alpha_e \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\alpha_o &\longmapsto \alpha_o \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\alpha_e \cdot \xi_{k+1} \cdot \xi_{k+2} &\longmapsto \alpha_e \otimes \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
\alpha_o \cdot \xi_{k+1} \cdot \xi_{k+2} &\longmapsto \alpha_o \otimes \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}.
\end{aligned}
\tag{3.26}
$$

This also shows the injectivity of the map (3.24), proving the lemma.

As a direct consequence of this isomorphism

$$\mathbb{Cl}(2k) \simeq M_{\mathbb{C}}(2^k) \tag{3.27}$$

$$\mathbb{Cl}(2k+1) \simeq M_{\mathbb{C}}(2^k) \oplus M_{\mathbb{C}}(2^k). \tag{3.28}$$

Indeed the block decomposition of matrices shows that

$$M_{\mathbb{C}}(r) \otimes M_{\mathbb{C}}(2) = M_{\mathbb{C}}(2r) \quad \forall\ r.$$

So to get (3.27) and (3.28) it suffices to observe that

$$
\begin{aligned}
\mathbb{Cl}(2) &\simeq M_{\mathbb{C}}(2) \\
\mathbb{Cl}(1) &\simeq \mathbb{C} \oplus \mathbb{C}.
\end{aligned}
$$

The first of these follows from the proof of Lemma 3.3. The second is given by the explicit isomorphism

$$\mathbb{Cl}(1) = \operatorname*{span}_{\mathbb{C}} \{1, \xi\} \longmapsto \operatorname*{span}_{\mathbb{C}} \left\{ \frac{1+\xi}{2}, \frac{1-\xi}{2} \right\}.$$

The proof of Lemma 3.3 also shows the importance of considering the $\mathbb{Z}_2$-*grading* of $\mathbb{Cl}(k)$ (or $\mathrm{Cl}(k)$)

$$\mathbb{Cl}(k) = \mathbb{Cl}^+(k) \oplus \mathbb{Cl}^-(k), \tag{3.29}$$

where $\mathbb{C}\mathrm{l}^+(k)$ and $\mathbb{C}\mathrm{l}^-(k)$ are respectively the sums of multiples of products of even and odd numbers of the generators. More generally

$$\begin{aligned}\mathrm{Cl}^+(V) &= \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k}(V) \Big/ \left(\mathcal{I}_{\mathrm{Cl}}(V) \cap \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k}(V)\right)\\ \mathrm{Cl}^-(V) &= \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k+1}(V) \Big/ \left(\mathcal{I}_{\mathrm{Cl}}(V) \cap \bigoplus_{k=0}^{\infty} \mathrm{Ten}_{2k+1}(V)\right),\end{aligned}$$

where (3.29) follows from the fact that $\mathcal{I}_{\mathrm{Cl}}$ is graded, i.e. it is the sum of its even and odd parts, since the relation in (3.10) respects parity. It is particularly important that (3.29) is a grading of the algebra:

$$\begin{aligned}\mathrm{Cl}^+(V) \cdot \mathrm{Cl}^+(V) &\subset \mathrm{Cl}^+(V)\\ \mathrm{Cl}^-(V) \cdot \mathrm{Cl}^-(V) &\subset \mathrm{Cl}^+(V)\\ \mathrm{Cl}^+(V) \cdot \mathrm{Cl}^-(V) &\subset \mathrm{Cl}^-(V)\\ \mathrm{Cl}^-(V) \cdot \mathrm{Cl}^+(V) &\subset \mathrm{Cl}^-(V).\end{aligned}$$

The $\mathbb{Z}_2$-grading defines, and is determined by, an involution, i.e. a linear map

$$R_{\mathrm{Cl}}\colon \mathrm{Cl}(V) \longleftrightarrow \mathrm{Cl}(V)$$

given by the obvious sign reversal:

$$R_{\mathrm{Cl}} \restriction \mathrm{Cl}^+(V) = \mathrm{Id},\ R_{\mathrm{Cl}} \restriction \mathrm{Cl}^-(V) = -\mathrm{Id}\,.$$

Next note how the relationship between the two cases (3.27), (3.28) can be understood directly. Suppose that V is a real vector space with a Euclidean inner product. On $V \times \mathbb{R}$ take the obvious extension of this

$$\langle (v,r),(v',r')\rangle = \langle v,v'\rangle + rr'.$$

Then the map, at the level of the tensor algebra

$$\begin{aligned}\mathrm{Ten}_{\mathbb{C},2k}(V) \ni \alpha &\longmapsto \alpha \in \mathrm{Ten}_{\mathbb{C},2k}(V\times\mathbb{R})\\ \mathrm{Ten}_{\mathbb{C},2k+1}(V) \ni \alpha &\longmapsto i\alpha\otimes\tau \in \mathrm{Ten}_{\mathbb{C},2k+2}(V\times\mathbb{R}),\end{aligned} \tag{3.30}$$

where $\tau = (0,1) \in V\times\mathbb{R}$, projects to an isomorphism

$$\mathbb{C}\mathrm{l}(V) \longleftrightarrow \mathbb{C}\mathrm{l}^+(V\times\mathbb{R}). \tag{3.31}$$

Exercise 3.4. Prove that (3.31) is an isomorphism of algebras.

3.4. Clifford bundle.

So far a general solution to (3.1)–(3.3) has not been obtained. Before doing so, consider how to transfer the notions above to a general Riemann manifold; eventually this will lead to the notion of a (generalized) Dirac operator. Certainly if X is a Riemann manifold then T^*_xX is, for each $x \in X$, a vector space with Euclidean inner product. So $\mathrm{Cl}_x(X) = \mathrm{Cl}(T^*_xX)$, $\mathbb{C}\mathrm{l}_x(X) = \mathbb{C}\mathrm{l}(T^*_xX)$ fixes the real and complex Clifford algebra at each point. These spaces combine into bundles of algebras over X

$$\mathrm{Cl}(X) = \bigsqcup_{x\in X} \mathrm{Cl}_x(X), \quad \mathbb{C}\mathrm{l}(X) = \bigsqcup_{x\in X} \mathbb{C}\mathrm{l}_x(X). \tag{3.32}$$

If $O \in \mathrm{O}(V)$ is an orthogonal transformation on V then O extends to $\mathrm{Ten}(V)$ as the sum of the tensor actions. The ideal $\mathcal{I}_{\mathrm{Cl}} \subset \mathrm{Ten}(V)$ is invariant under this action, since $\langle Ov, Ov' \rangle = \langle v, v' \rangle$ by definition. Thus $\mathrm{Cl}(V)$ has a natural $\mathrm{O}(V)$-action consistent both with the inclusion $V \subset \mathrm{Cl}(V)$ and the algebra structure, i.e.

$$O\alpha \cdot O\beta = O(\alpha \cdot \beta).$$

In this way $\mathrm{Cl}(X)$ is identified as a bundle associated to the orthonormal frame bundle:

$$\mathrm{Cl}(X) \simeq \big(F \times \mathrm{Cl}(N)\big)\big/\,\mathrm{O}(N), \ \dim X = N. \tag{3.33}$$

As discussed in Chapter 2 this means that $\mathrm{Cl}(X)$ has a natural Levi-Civita connection,

$$\nabla \in \mathrm{Diff}^1(X; \mathrm{Cl}(X), T^*X \otimes \mathrm{Cl}(X)).$$

Notice also that Clifford multiplication, defined by (3.18), is consistent with the $\mathrm{O}(V)$ action on Λ^*V, i.e.

$$\mathrm{cl}(O\xi) \cdot O\alpha = O(\mathrm{cl}(\xi) \cdot \alpha).$$

This means that the same formula, (3.18), extends to define Clifford multiplication on the manifold:

$$\mathcal{C}^\infty(X; \mathbb{C}\mathrm{l}(X)) \times \mathcal{C}^\infty(X; \Lambda^*X) \ni (\mu, \alpha) \longmapsto \mathrm{cl}(\mu) \cdot \alpha \in \mathcal{C}^\infty(X; \Lambda^*X).$$

This is a tensorial operation.

3.5. Clifford modules.

The exterior algebra, $\Lambda^* X$, is an example of a Clifford module over X. In general a vector bundle $E \longrightarrow X$ is a Clifford module[6] if there is a pointwise action, i.e. linear map

$$\mathrm{cl}\colon \mathrm{Cl}_x(X) \longrightarrow \hom(E_x) \tag{3.34}$$

such that the resulting tensorial operator is $\mathcal{C}^\infty$, i.e.

$$\mathcal{C}^\infty(X; \mathrm{Cl}(X)) \times \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; E),\ (\alpha, e) \longmapsto \mathrm{cl}(\alpha)e.$$

Usually the bundle E will be a complex bundle and the action that of the complexified Clifford algebra. By convention the Clifford action is assumed to be non-trivial.

By (3.27), in the $2k$-dimensional case, the complexified Clifford algebra is isomorphic to a full matrix algebra. It follows that for any (non-trivial) Clifford action (3.34) the subspace of invariant elements, $E'_x \subset E_x$, is of dimension $\dim E_x - 2^k$. Choosing a complementary space E''_x on which the algebra acts (simply by taking the span of the orbit of an element outside E'_x) gives the decomposition

$$E_x = E''_x \oplus E'_x \Longrightarrow \hom(E_x) = \mathbb{C}\mathrm{l}_x \otimes \hom'_{\mathbb{C}\mathrm{l}}(E_x), \tag{3.35}$$

where $\hom'_{\mathbb{C}\mathrm{l}}(E_x) \subset \hom(E_x)$ is the space of linear operators commuting with the Clifford action. Whilst the decomposition of the space is not unique, that of the homomorphism bundle is. It follows that the corresponding decomposition of the bundle $\hom(E)$ is smooth.

Suppose that E has a connection ∇. The obvious compatibility condition to demand for ∇ to be a 'Clifford connection,' is

$$\nabla_V(\mathrm{cl}(\alpha)e) = \mathrm{cl}(\nabla_V \alpha)e + \mathrm{cl}(\alpha)\nabla_V e,\ \forall\ V \in \mathcal{V}(X), \tag{3.36}$$

where $\nabla_V \alpha$ is the Levi-Civita connection on $\mathrm{Cl}(X)$.

EXERCISE 3.5. Show that any Clifford module has a Clifford connection. [Hint: See Lemma 3.24.]

LEMMA 3.6. *If $\dim X$ is even then the homomorphism bundle of any Clifford module over X decomposes smoothly as the tensor product*

$$\hom(E) = \mathbb{C}\mathrm{l}(X) \otimes \hom'_{\mathbb{C}\mathrm{l}}(E). \tag{3.37}$$

If E has a Clifford connection then the curvature operator on E also decomposes as

$$K_{\hom(E)} = K_{\mathbb{C}\mathrm{l}} \otimes \mathrm{Id} + \mathrm{Id} \otimes K_{\hom'_{\mathbb{C}\mathrm{l}}(E)}. \tag{3.38}$$

PROOF: The identity (3.37) has already been proved. Applying (2.76) to it gives (3.38).

[6] One should perhaps call these bundles of Clifford modules, but that is too clumsy.

DEFINITION 3.7. If $E \longrightarrow X$ is a Clifford module over a Riemann manifold X the generalized Dirac operator associated with a connection on E is

$$\eth_E = -i\tilde{c} \cdot \nabla \in \mathrm{Diff}^1(X; E), \tag{3.39}$$

given by composing the connection with $-i\tilde{c}$ where

$$\tilde{c} \colon \mathcal{C}^\infty(X; T^*X \otimes E) \longrightarrow \mathcal{C}^\infty(X; E),\ e \otimes \xi \longmapsto \mathrm{cl}(\xi) \cdot e. \tag{3.40}$$

Notice that by taking a local orthonormal coframe, ϕ^j, and the dual frame with respect to the metric, v_j, for TX the definition (3.39) becomes

$$\eth_E e = \sum_j \mathrm{cl}(\phi^j) \nabla_{iv_j}. \tag{3.41}$$

EXERCISE 3.8. Check directly that (3.41) is independent of the choice of orthonormal basis.

The pure imaginary factor is included in the definition so that $\eth_E$ is at least a symbolic square root of the Laplacian. More generally, if E is an Hermitian bundle, i.e. has a smoothly varying positive-definite sesquilinear inner product on the fibres, then a Clifford action is said to be Hermitian (or unitary) if

$$\mathrm{cl}(\xi)^* = \mathrm{cl}(\xi) \quad \forall\, \xi \in T^*X.$$

PROPOSITION 3.9. *If E is a Clifford module over X then the generalized Dirac operator satisfies*

$$\sigma_1(\eth_E)(\xi) = \mathrm{cl}(\xi) \quad \forall\, \xi \in T^*X. \tag{3.42}$$

If the bundle and Clifford action are Hermitian then

$$\sigma_2(\eth_E^* \eth_E)(\xi) = |\xi|^2 \quad \forall\, \xi \in T^*X \tag{3.43}$$

and if the connection is unitary and a Clifford connection then $\eth_E$ is self-adjoint.

PROOF: The symbol map is a homomorphism so the symbol of $\eth_E$ is given by the composite of the symbol of the connection, which is the tensor product with $i\xi$, and $-i\tilde{c}$, i.e. at $\xi \in T_x^*X$

$$E_x \ni e \longmapsto e \otimes \xi \longmapsto \mathrm{cl}(\xi) \cdot e$$

which gives (3.42). If E is Hermitian then the adjoint of $\eth_E$, using the Riemann density on X, is well defined and has symbol given by the adjoint of the symbol, i.e. $\mathrm{cl}(\xi)^*$. If the Clifford action is Hermitian this reduces to $\mathrm{cl}(\xi)$ and hence (3.43) follows from the multiplicativity of the symbol map and the definition of the Clifford algebra.

The inner product on E induces a natural inner product on $T^*X \otimes E$ given by the usual tensor product formula

$$\langle \xi \otimes e, \eta \otimes f \rangle = \langle \xi, \eta \rangle \langle e, f \rangle.$$

This has the property that if $\xi \in \mathcal{C}^\infty(X; T^*X)$ is real and $V \in \mathcal{V}(X)$ is the dual vector field then

$$\langle \nabla e, \xi \otimes f \rangle = \langle \nabla_V e, f \rangle.$$

If the connection on E is unitary, i.e. (2.108) holds, then it follows that the adjoint of the connection is

$$\nabla^*(\xi \otimes e) = -\nabla_V e + (\operatorname{div} V) e. \tag{3.44}$$

Here $\operatorname{div} V$ is the metric divergence of V defined as the difference of the adjoint of $V \in \mathrm{Diff}^1(X)$ and $-V$. Similarly if the Clifford action is Hermitian then the adjoint of (3.40) satisfies

$$\langle \tilde{c}^* e, \xi \otimes f \rangle = \langle e, \mathrm{cl}(\xi) \cdot f \rangle = \langle \mathrm{cl}(\xi) \cdot e, f \rangle.$$

Thus $\tilde{c}^*(e)(v_j) = \mathrm{cl}(\phi^j) e$ if ϕ^j and v_j are dual local orthonormal bases of T^*X and TX respectively, i.e.

$$\tilde{c}^*(e) = \sum_i \phi^j \otimes \mathrm{cl}(\phi^j) e.$$

Combining this with (3.44), and using (3.36), this shows that with respect to such a basis

$$\eth_E^* e = i \sum_j \nabla_{v_j}^* [\mathrm{cl}(\phi^j) \cdot e]$$

$$= -i \sum_j \mathrm{cl}(\phi^j) \nabla_{v_j} \cdot e + i \sum_j \operatorname{div}(v_j)\, \mathrm{cl}(\phi^j) \cdot e + ic \left(\sum_j \nabla_{v_j} \phi^j \right) e.$$

The first sum on the right is just $\eth_E e$. The remaining part vanishes since it must be independent of the choice of orthonormal basis and the basis can be chosen covariant constant at any given point, in which case both $\operatorname{div}(v_j)$ and $\nabla_{v_j}\phi^j$ vanish at that point. Thus $\eth_E$ is self-adjoint as claimed.

Particularly in the even-dimensional case it is natural to consider Clifford modules which are graded. Thus the bundle has a decomposition

$$E = E^+ \oplus E^-$$

such that

$$\mathbb{Cl}_x^+ : E_x^\pm \longrightarrow E_x^\pm,\ \mathbb{Cl}_x^- : E_x^\mp \longrightarrow E_x^\pm\ \forall\ x \in X.$$

From the formula (3.41) and Proposition 3.9 one easily deduces:

COROLLARY. *If X is an even-dimensional Riemann manifold and E is a graded Hermitian Clifford module with a connection which is unitary, Clifford and preserves the grading then the Dirac operator is graded in the sense that*

$$\eth_E = \begin{pmatrix} 0 & \eth_E^- \\ \eth_E^+ & 0 \end{pmatrix},\ (\eth_E^+)^* = \eth_E^-.$$

3.6. Clifford bundle of ${}^bTX$.

The definitions above extend directly to the case of a b-metric. Thus (3.32) and (3.33) serve just as well to define the Clifford bundle, over any manifold with boundary, associated to a b-metric, now of course with ${}^b\mathrm{Cl}_x(X) = \mathrm{Cl}({}^bT_x^*X)$. The Levi-Civita connection of the b-metric extends to a b-connection on ${}^b\mathrm{Cl}(X)$. If the metric is an exact b-metric then Proposition 2.37 shows that the b-connection is actually a connection on ${}^bT^*X$, and hence on $\mathrm{Cl}({}^bT^*X)$. Simply by continuity the consistency condition demanded of a b-Clifford connection should be

$${}^b\nabla(\mathrm{cl}(\alpha)\cdot e) = \mathrm{cl}({}^b\nabla\alpha)\cdot e + \mathrm{cl}(\alpha)\cdot {}^b\nabla e.$$

Then the proof of Proposition 3.9 applies, essentially *verbatim,* to give the b-analogue:

PROPOSITION 3.10. *If E is a Clifford module for an exact b-metric on a compact manifold with boundary, X, the generalized Dirac operator ${}^b\eth_E \in \mathrm{Diff}_b^1(X;E)$ satisfies ${}^b\sigma_1({}^b\eth_E)(\xi) = \mathrm{cl}(\xi)$ for $\xi \in {}^bT^*X$ and if the bundle and Clifford action are Hermitian then ${}^b\sigma_2({}^b\eth_E^*\,{}^b\eth_E) = |\xi|^2$ for $\xi \in {}^bT^*X$, with ${}^b\eth_E$ self-adjoint for a unitary Clifford connection.*

EXERCISE 3.11. Go through the proof of this Proposition carefully to make sure that you understand why this generalization to the b-metrics is *immediate!*

Certainly the Corollary to Proposition 3.9 also extends by continuity to the case of a generalized Dirac operator associated to a graded Hermitian Clifford module over an even-dimensional exact b-manifold when the connection on the module is graded, Clifford and unitary. It is not necessary to assume that the connection is a true connection, only that it is a b-connection. This is especially convenient because it allows the weighting discussed in §6 to be incorporated in the general case. The assumption that the metric is an exact b-metric and that the connection is Clifford means that the part of the connection on the Clifford factor in (3.37) will always be a true connection.

3.7. Spin group.

Naturally the question arises as to the existence of Clifford modules. One way these arise is through the notion of a spin structure and the associated spinor bundles. This leads to the definition of 'the' Dirac operator and more generally the twisted Dirac operators which appear in the APS theorem of the introduction. They were introduced in this degree of generality by Atiyah and Singer [13].

The first thing to consider is the relationship between the Clifford algebra and the orthogonal group of a Euclidean vector space. Recall that there is a natural embedding $V \hookrightarrow \mathrm{Cl}(V)$. Consider the elements of V of unit length as elements of $\mathrm{Cl}(V)$. The anticommutation relation

$$\xi \cdot \eta + \eta \cdot \xi = 2\langle \xi, \eta\rangle \,\mathrm{Id}, \quad \xi, \eta \in V$$

can then be rewritten

$$\xi \cdot \eta - \big(-\eta + 2\langle \xi, \eta\rangle \xi\big) \cdot \xi = 0, \ |\xi| = 1, \tag{3.45}$$

since $\xi^2 = \mathrm{Id}$. Notice that

$$\hat{\eta} = \eta - 2\langle \xi, \eta\rangle \xi = R_\xi \eta$$

is the reflection of η in the plane orthogonal to ξ. Thus for any $\xi \in V$, $|\xi| = 1$

$$R_\xi v = v - 2\langle \xi, v\rangle \xi \quad \forall\ v \in V \tag{3.46}$$

is an element of $\mathrm{O}(V)$.

The anticommutation relations amongst these orthogonal involutions are:

$$R_\xi \cdot R_\eta = R_{\hat{\eta}} \cdot R_\xi \ , \ \hat{\eta} = R_\xi \eta, \ |\xi| = |\eta| = 1 \tag{3.47}$$

as follows directly from (3.46). Since these are the same relations as (3.45) (noting that $R_\xi = R_{-\xi}$) and are the only relations amongst these involutions, it follows that the map

$$\{\eta \in V \subset \mathrm{Cl}(V);\ |\eta| = 1\} \longrightarrow \mathrm{O}(V) \tag{3.48}$$

can be extended multiplicatively. That is, setting

$$\mathrm{Pin}(V) = \{a \in \mathrm{Cl}(V); a = \eta_1 \cdot \ldots \cdot \eta_p \text{ for some } p, \eta_i \in V,\ |\eta_i| = 1\}$$

leads to

Proposition 3.12. *The map generated by* (3.48) *is a group homomorphism*

$$Z\colon \operatorname{Pin}(V) \twoheadrightarrow \mathrm{O}(V) \tag{3.49}$$

which is a 2-1 *covering of the orthogonal group, with kernel* $\pm \mathrm{Id} \in \mathrm{Cl}(V)$.

Proof: First observe that $\operatorname{Pin}(V)$ *is* a group, since it has an associative product and the inverse of $\eta_1 \dots \eta_k$ is just $\eta_k \dots \eta_1$. Consider (3.48) another way. Namely the orthogonal transformation R_η can be written

$$V \ni v \longmapsto R_\eta v = -\eta \cdot v \cdot \eta, \quad |\eta| = 1, \ \eta \in V$$

in terms of Clifford multiplication. Indeed if $v = a\eta + b\eta^\perp$, $\langle \eta, \eta^\perp \rangle = 0$, then

$$\eta \cdot v = a - b\eta^\perp \cdot \eta \Longrightarrow \eta \cdot v \cdot \eta = a\eta - b\eta^\perp = -R_\eta v. \tag{3.50}$$

This shows directly that (3.49), given by (3.46), is a group homomorphism.

It remains to show that the map Z is surjective and to find its kernel. Any orthogonal transformation can be written as a product of reflections, in fact a product of at most $\dim V$ reflections.

Exercise 3.13. Recall how to prove this. If $O \in \mathrm{O}(V)$ then O is unitary (being real $O^* = O^t = O^{-1}$) so has a spectral decomposition

$$O = \sum_j \lambda_j P_j,$$

where the P_j are complex projections onto the eigenspaces associated to the λ_j. The λ_j's all have modulus 1 and, since O is real, come in complex-conjugate pairs, or are real, and similarly for the P_j's. This decomposes V into a sum of one and two dimensional orthogonal spaces

$$V = \bigoplus_j V_j$$

which O preserves. This reduces the problem to the one and two dimensional cases, where it is easy enough.

If $O = R_{\eta_1} \cdot \dots \cdot R_{\eta_k}$, with $|\eta_i| = 1$, $\eta_i \in V$, then $O = Z\big(\pm\eta_1 \dots \eta_k\big)$ as is clear from (3.50). This shows the surjectivity.

Clearly $Z(\pm \mathrm{Id}) = \mathrm{Id}$. Conversely suppose $Z(a) = \mathrm{Id}$, $a \in \operatorname{Pin}(V)$. From (3.50) this means that if $a = \eta_1 \dots \eta_k \in \mathrm{Cl}^{(k)}(V)$

$$\eta_1 \cdot \dots \cdot \eta_k v = (-1)^k v \eta_1 \dots \eta_k \quad \forall\ v \in V.$$

Thus a must satisfy

$$a \cdot v = (-1)^k v \cdot a \quad a \in \mathrm{Cl}^k(V). \tag{3.51}$$

Now decompose a into its unique presentation as a sum over the basis of the subspace of $\mathrm{Cl}^{(k)}(V)$ of the same parity as k, given by products $e_{i_1} \dots e_{i_l}$ $i_1 < i_2 < \cdots < i_l$, $l \equiv k$ modulo 2, where the $e_i \in V$ form an orthonormal basis:

$$a = \sum_{l \equiv k \mod 2} c_I e_{i_1} \dots e_{i_l}.$$

Applying (3.51) for each j, $v = e_j$, shows that $c_I = 0$ if $i_r = j$ for some r. Thus $a = c_0 \,\mathrm{Id}$ and then (3.51) can hold if and only if $c_0 = \pm 1$, $k = 0$. This proves the proposition.

The orthogonal group has two components:

$$\mathrm{O}(V) = \mathrm{SO}(V) \sqcup \mathrm{O}^-(V), \ \dim V \geq 2,$$

where

$$O \in \mathrm{SO}(V) \Longleftrightarrow O \in \mathrm{O}(V) \text{ and } \det(O) = 1.$$

Since a reflection has determinant -1, $O \in \mathrm{SO}(V)$ if and only if it is a product of an *even* number of reflections. Thus, in (3.47),

$$\mathrm{Spin}(V) \stackrel{\text{def}}{=} Z^{-1}(\mathrm{SO}(V)) = \mathrm{Pin}(V) \cap \mathrm{Cl}^+(V).$$

LEMMA 3.14. *If* $\dim V \geq 3$, *the group* $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}^+(V)$ *is the unique simply-connected double cover of* $\mathrm{SO}(V)$.

PROOF: Since $\mathrm{SO}(V)$ is connected and Z is a double cover, $\mathrm{Spin}(V)$ can have at most two components.

EXERCISE 3.15. Show that $\mathrm{SO}(V)$ is connected. Choose an orthonormal basis of V so that the problem is reduced to $V = \mathbb{R}^N$, $\mathrm{SO}(V) = \mathrm{SO}(N)$. Given $O \in \mathrm{SO}(N)$ let $f^j = O(e^j)$ be the image of the standard orthonormal basis. It is easy to check that $\mathrm{SO}(2)$ is connected. Using a rotation in the first two variables it can be arranged that $f^2 \perp e^1$. Similarly a rotation in the first and third variables allows one to arrange $f^3 \perp e^1$. Thus O is connected in $\mathrm{SO}(N)$ to an element O' such that the corresponding basis $(f')^j \perp e^1$, $j \geq 2$. This means $f^1 = \pm e^1$, so by a further rotation in the first two variables it can be assumed that $f^1 = e^1$. Now proceed by induction.

To show that $\mathrm{Spin}(V)$ is connected it therefore suffices to check that -1 is connected to 1. If $\xi \perp \eta$, with $|\xi| = |\eta| = 1$, then

$$\xi\eta\xi\eta = -\xi\xi\eta\eta = -1.$$

Since, for $\dim V \geq 2$, the sphere is connected, a path ξ_t in V with $|\xi_t| = 1$, $\xi_0 = \eta$, $\xi_1 = \xi$, can be chosen and then $\xi_t\eta\xi_t\eta$ connects 1 to -1. In fact a similar argument shows that if $\dim V \geq 3$ then $\mathrm{Spin}(V)$ is simply connected. In this case the unit sphere in V is simply connected. Any curve in $\mathrm{Spin}(V)$ can be approximated closely by a curve of the form $\xi_1(t) \cdot \xi_2(t) \cdot \ldots \cdot \xi_N(t)$, where $N = \dim V$ or $\dim V - 1$, as $\dim V$ is even or odd. Each of these $\xi_j(t)$ are curves on the unit sphere so can be contracted to a fixed element. Since $\mathrm{SO}(N)$ is connected this shows that the double cover

$$Z\colon \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V) \tag{3.52}$$

is non-trivial, and in fact yields the well-known result that the fundamental group of $\mathrm{SO}(V)$ is $\mathbb{Z}_2$, if $\dim V \geq 3$.

The spin group of the Euclidean vector space V has been defined as a submanifold of the vector space $\mathrm{Cl}^+(V)$, so the Lie algebra must be a linear subspace. In fact

Lemma 3.16. *The Lie algebra of* $\mathrm{Spin}(V) \subset \mathrm{Cl}^+(V)$ *is the linear span of antisymmetric products:*

$$\mathfrak{sn}(V) = \operatorname{span}\{\xi\xi' - \xi'\xi; \xi, \xi' \in V\} \subset \mathrm{Cl}^+(V). \tag{3.53}$$

Proof: Let $\xi \in V$ be a unit vector and choose $\xi' \perp \xi$ with $|\xi'| = 1$. Then setting

$$\xi_s = (1 - s^2)^{\frac{1}{2}}\xi + s\xi' \in V,\ |\xi_s| = 1 \Longrightarrow c(s) = \xi\xi_s \in \mathrm{Spin}(V). \tag{3.54}$$

Moreover $\xi_0 = \xi$ so $c(0) = \mathrm{Id}$. Thus the tangent vector at $s = 0$ to the curve $c(s)$ is an element of $\mathfrak{sn}(V)$, it is just

$$c'(0) = \xi\xi' = \frac{1}{2}(\xi\xi' - \xi'\xi) \text{ since } \xi \perp \xi'.$$

Taking linear combinations, it follows that the subset of the space on the right in (3.53) with $\xi \perp \xi'$ is contained in $\mathfrak{sn}(V)$. Since $\xi\xi = 1$, $\mathfrak{sn}(V)$ contains the vector space in (3.53). The dimensions being the same, this proves the lemma.

The double covering (3.52) gives a canonical identification of the Lie algebras, $\mathfrak{so}(N)$ of $\mathrm{SO}(N)$ and $\mathfrak{sn}(N)$ of $\mathrm{Spin}(N)$. Since the former is also naturally identified with the linear space of antisymmetric real matrices, (3.53) gives a natural map

$$\begin{aligned}\mathfrak{so}(n) = \{A_{ij}; A_{ij} + A_{ji} = 0,\ i,j = 1,\dots,n\} \longrightarrow& \\ \operatorname{span}\{\xi\xi' - \xi'\xi; \xi, \xi' \in V\} = \mathfrak{sn}(n) \subset& \ \mathrm{Cl}^+(n).\end{aligned} \tag{3.55}$$

Lemma 3.17. *The natural identification* (3.55) *is*

$$A_{ij} \longmapsto \frac{1}{4}\sum_{i,j=1}^{n} A_{ij}\phi^i\phi^j. \tag{3.56}$$

Proof: Consider the curve (3.54) in $\mathrm{Spin}(N)$ with $\xi = \phi^i$ and $\xi' = \phi^j$. It has tangent vector $\frac{1}{2}(\phi^i\phi^j - \phi^j\phi^i)$ at 0. By (3.46) and (3.47) its image as an orthogonal transformation is

$$v \longmapsto v - 2\langle v, e_s\rangle e_s - 2\langle v, e^i\rangle e^i + 4\langle v, e_s\rangle\langle e_s, e^i\rangle e^i,\ e_s = (1-s^2)^{\frac{1}{2}}\phi^i + s\phi^j.$$

The tangent vector at $s = 0$ to this curve in $\mathrm{SO}(N)$ is just

$$v \longmapsto -2v_i\phi^j + 2v_j\phi^i.$$

This is the correspondence (3.56).

3.8. Spin representations.

Next the embedding $\mathrm{Spin}(V) \hookrightarrow \mathrm{Cl}(V)$ will be used to find the spin representations, which are representations of $\mathrm{Spin}(V)$ but *not* representations of $\mathrm{SO}(V)$. In fact (3.26) already gives a representation of $\mathbb{C}\mathrm{l}(V)$. Thus, if $\dim V = 2k$, there is an identification $\mathrm{Cl}(V) \simeq \mathrm{Cl}(2k)$ and then

$$c\colon\ \mathbb{C}\mathrm{l}(2k) \longleftrightarrow M_{\mathbb{C}}(2^k). \tag{3.57}$$

This gives a representation

$$\mathrm{Pin}(V) \longrightarrow GL(2^k, \mathbb{C}), \quad \dim V = 2k. \tag{3.58}$$

Moreover (3.58) must be irreducible, since the $\mathbb{C}$-linear span of $\mathrm{Pin}(V)$ is $\mathbb{C}\mathrm{l}(V)$ which certainly has no invariant subspace (by (3.57).)

So consider what happens when (3.58) is restricted to the connected component of the identity, i.e. $\mathrm{Spin}(V)$.

LEMMA 3.18. *If* $\dim V = 2k$, *the restriction of* (3.58) *to the spin subgroup:*

$$\mathrm{SR}\colon \operatorname{Spin}(V) \longrightarrow U(2^k) \tag{3.59}$$

is a unitary representation which is the direct sum of two irreducible representations

$$\mathrm{SR}^{\pm}\colon \operatorname{Spin}(V) \longrightarrow GL({}^{\pm}S), \quad \mathbb{C}^{2^k} = S = {}^{+}S \oplus {}^{-}S. \tag{3.60}$$

PROOF: From (3.31) it follows that $\mathbb{C}l^{+}(2k) \cong \mathbb{C}l(2k-1)$. Then (3.28) shows that this is isomorphic to two copies of $M_{\mathbb{C}}(2^{k-1})$. The restriction of the representation (3.58) to $\mathrm{Cl}^{+}(2k)$ therefore decomposes into two representations, where $\dim {}^{+}S = \dim {}^{-}S = 2^{k-1}$. Since the $\mathbb{C}$-linear span of $\operatorname{Spin}(2k)$ is $\mathbb{C}l^{+}(2k)$, it follows that these two representations are irreducible and (3.60) holds.

Notice that the combined representation is in fact unitary. This can be seen directly from the discussion leading to (3.28). In $\mathbb{C}l(2k)$ there is a natural involution, defined as the identity on V and 1 and extended to be conjugate-linear and product reversing:

$$\alpha^{*} = \sum_{I} c_I e_{i_1} \cdots e_{i_j} \Longrightarrow \alpha^{*} = \sum_{I} \bar{c}_I e_{i_j} \cdots e_{i_1}.$$

From (3.27) and (3.28) it follows that the isomorphism (3.24) is a $*$-isomorphism, with the usual adjoint operation on $M_{\mathbb{C}}(2)$. Since the elements of $\operatorname{Spin}(V)$ are unitary, in the sense that $S^{*} = S^{-1}$ if $S \in \operatorname{Spin}(V)$, the representation (3.59) is unitary.

The odd-dimensional case is even simpler.

LEMMA 3.19. *If* $V = \mathbb{R}^{2k+1}$ *the restriction of the representation*

$$\mathbb{C}l^{+}(V) \longleftrightarrow \mathbb{C}l(2k) \longleftrightarrow \hom(\mathbb{C}^{2^k}) \tag{3.61}$$

to $\operatorname{Spin}(V)$ *is an irreducible unitary representation.*

PROOF: The first identification in (3.61), given by (3.31), is a $*$-isomorphism, so restriction to $\operatorname{Spin}(V)$ gives a unitary representation. Since $\operatorname{Spin}(V)$ spans $\mathbb{C}l^{+}(V)$, which is to say $M_{\mathbb{C}}(2^k)$, over $\mathbb{C}$ the representation is necessarily irreducible.

Alternatively the spin representation in the odd dimensional case, $V = \mathbb{R}^{2k+1}$, can be obtained from (3.31) as the identification

$$\mathbb{C}l(V) \longleftrightarrow \mathbb{C}l^{+}(2k+2),$$

allowing either of the spin representations of $\operatorname{Spin}(2k+2)$ to be restricted to $\operatorname{Spin}(V)$ where they give equivalent irreducible representations, also equivalent to that above. This is discussed in more detail in §3.13 below.

These two lemmas fix the spin representations of $\operatorname{Spin}(\mathbb{R}^n)$ and solve the problem posed at the beginning of this chapter. Namely if $n = 2k$ the $2^{k-1} \times 2^{k-1}$ matrices

$$\gamma_j = i\,\mathrm{SR}^+(\phi^j \phi^1),\ j = 1, \dots, 2k$$

defined by the representation satisfy (3.4), with $\gamma_1 = i\,\mathrm{Id}$. This gives $\eth^+$ defined by (3.1) satisfying (3.2). In fact, just taking the γ_i to be $\mathrm{SR}^+(\phi^i)$ gives a solution of (3.3). As the notation indicates the representation of the Clifford algebra in (3.57) is generally written as Clifford multiplication:

$$\mathrm{cl}(\alpha) \in \hom(S),\ \xi \in \mathbb{C}\mathrm{l}(2k).$$

Then the full Dirac operator can be written

$$\eth = \sum_{j=1}^{2k} \mathrm{cl}(\phi^j) D_j.$$

EXERCISE 3.20. Check that $\eth^2 = \Delta\,\mathrm{Id}$.

3.9. Spin structures.

To transfer this construction to a Riemann manifold requires the introduction of the notion of a spin structure. As discussed in §2.12 an orientation on X corresponds to a reduction of the structure group of the coframe bundle to $\mathrm{SO}(N)$. Similarly one can ask when the bundle of oriented orthonormal frames can be extended to a principal $\operatorname{Spin}(N)$-bundle.

DEFINITION 3.21. A principal $\operatorname{Spin}(N)$-bundle which is a double cover $s\colon {}_sF \longrightarrow {}_oF$ is called a *spin structure* if the $\operatorname{Spin}(N)$ action is consistent with the $\mathrm{SO}(N)$ action, i.e. if $A \in \operatorname{Spin}(N)$ and $\pi(A) \in \mathrm{SO}(N)$ is its image then the diagram

$$\begin{array}{ccc} {}_sF_x & \xrightarrow{A} & {}_sF_x \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ {}_oF_x & \xrightarrow[\pi(A)]{} & {}_oF_x \end{array} \tag{3.62}$$

commutes.

EXERCISE 3.22. Recall, for example from [67], the definition of the Stiefel-Whitney classes $w_i \in H^i(X, \mathbb{Z}_2)$ of (the tangent bundle of) an oriented manifold. Show that the existence of a spin structure on X is equivalent to the vanishing of w_2.

The fact that the covering map, $s\colon {}_sF \longrightarrow {}_oF$, is discrete means that the Levi-Civita connection on F, which certainly restricts to ${}_oF$ when the manifold is oriented, lifts to a unique, compatible, connection on ${}_sF$. The compatibility condition is just the requirement that the image of the connection on ${}_sF$ be that on ${}_oF$ under s_*.

The existence of a spin structure on X allows the spin bundles to be defined as bundles associated to the principal $\mathrm{Spin}(N)$-bundle ${}_sF$. Thus the fibre at any point $x \in X$ is just

$$S_x = ({}_sF_x \times S(N)) \,/\, \mathrm{SR},\ N = \dim X, \tag{3.63}$$

where $\mathrm{SR}\colon \mathrm{Spin}(N) \longrightarrow \hom(S(N))$ is the spin representation on $S(N)$. That is,

$$S = ({}_sF \times S(N)) \,/\, \mathrm{SR}$$

is a complex vector bundle over X of dimension 2^k where $n = 2k$ or $n = 2k+1$. In the even dimensional case the splitting of the spin representation gives a global splitting of the bundle:

$$S = {}^+S \oplus {}^-S,\ \dim X = 2k. \tag{3.64}$$

Since the spin representations are unitary the spin bundles are Hermitian, i.e have a natural positive definite Hermitian form on their fibres.

The Levi-Civita connection on ${}_sF(X)$ induces a connection on the spin bundles, and this connection is unitary. It is also clear that the spin bundles are Clifford modules. Certainly the spin group $\mathrm{Spin}(T^*_xX)$ acts on each fibre S_x and by definition the spin representation extends to a representation of $\mathrm{Pin}(T^*_xX)$ and to a representation of $\mathbb{C}l(T^*_xX)$. Indeed this is how the spin representations were defined in the first place. It is important to notice for later reference that, again directly from the definition, in the even-dimensional case the action of $\mathbb{C}l(T^*_xX)$ on S_x actually gives all linear transformations.

LEMMA 3.23. *If X is an even-dimensional spin manifold then*

$$\mathbb{C}l(T^*_xX) \longleftrightarrow \hom(S_x) \quad \forall\ x \in X \tag{3.65}$$

gives an isomorphism of the Clifford bundle and the homomorphism bundle of the spinor bundle.

This is a special case of Lemma 3.6.

3.10. Clifford connections.

The spin bundle of a manifold with spin structure is an Hermitian Clifford module. To see that the Levi-Civita connection on it is a unitary Clifford connection it remains only to show that (3.36) holds. This follows directly from the definitions, since if $\alpha \in \mathcal{C}^\infty(X; \Lambda^1)$ and $s \in \mathcal{C}^\infty(X; S)$ are both covariant constant at a point p then so is $\mathrm{cl}(\alpha)s$. Using these examples we can now easily prove:

LEMMA 3.24. *Any Clifford module over an even-dimensional manifold has a Clifford connection and any Hermitian Clifford module has a unitary Clifford connection.*

PROOF: Suppose first that X is even dimensional. Lemma 3.6 and (3.65) can be used to decompose the given Clifford bundle as a tensor product $S \otimes G$ locally near any point, with S the spin bundle for a spin structure near that point. This shows that there is always a Clifford connection locally; superposition using a partition of unity gives a globally defined Clifford connection. If the Clifford bundle is Hermitian this also allows the connection to be chosen unitary.

EXERCISE 3.25. Extend Lemma 3.24 to the odd-dimensional case.

For later computations it is also useful to have an explicit formula for covariant differentiation of sections of the spin bundle, analogous to (2.66) for 1-forms and its extension to higher forms in Exercise 2.26. Let ϕ^i be an oriented local orthonormal coframe, i.e. a local orthonormal basis of T^*X which defines a local section of ${}_oF$, and let v_i be the dual orthonormal basis of TX near some point p. Then define, by analogy with (2.67), the Christoffel symbols for this frame by

$$\nabla_{v_i}\phi^j = -\sum_{k=1}^N \gamma^j_{ik}\phi^k. \tag{3.66}$$

Suppose f^i is a local section of ${}_sF$ which covers ϕ^i; there are two possible choices by (3.62). Then a section $s \in \mathcal{C}^\infty(X; S)$ of S, the spinor bundle, is locally reduced to map a s_f into $S(N)$, the spinor space for $\mathbb{R}^N$ by definition in (3.63).

LEMMA 3.26. *With this notation for a section s of the spinor bundle*

$$(\nabla_{v_i}s)_f = v_i s_f(x) - \frac{1}{4}\Gamma_i(x)s,\ \Gamma_i(x) = \sum_{k,j=1}^N \gamma^k_{ij}\phi^j\phi^k, \tag{3.67}$$

where $\Gamma_i(x) \in \mathrm{Cl}_x$ acts by Clifford multiplication on the fibres of S.

Proof: It suffices to prove (3.67) at each point p. If the basis ϕ^i happens to be covariant constant at p then the Christoffel symbols vanish at that point, the spin frame f^i is also covariant constant and so (3.67) holds at p. In general a frame ψ^i equal to ϕ^i at p and covariant constant at p can always be found. Indeed

$$\psi^i = \sum_{j=1}^{n} O_{ij}(x)\phi^i, \; O_{ij} = \delta_{ij} + A_{ij}(x),$$

where $A_{ij}(p) = 0$. The associated frame of ${}_sF$

$$g^i = \sum_{j=1}^{n} O_{ij}(x) f^j \text{ is covariant constant at } p.$$

Thus if $s_{f'}$ is the map into $S(N)$ representing s with respect to this frame then

$$(\nabla_{v_i} s)_{f'} = v_i s_{f'}.$$

Since $O_{ij}(p) = \delta_{ij}$

$$\begin{aligned}(\nabla_{v_i} a)_f(p) =& (\nabla_{v_i} s)_{f'}(p) = v_i s_{f'}(p) \\ =& v_i\left(\mathrm{SR}(O(x)) s_f\right)(p) = v_i s_f + \frac{1}{4} G^i s_f .\end{aligned}$$

Here Lemma 3.17 is used to write the action of the Lie algebra, so

$$G = \sum_{pq} G_{pq} \phi^p \phi^q$$

is acting by Clifford multiplication, where $G^i_{pq} = v_i A_{pq}$. To find the G^i_{ij} in terms of the Christoffel symbols, simply use the same computation for the connection on T^*X :

$$\nabla_{v_i} \xi^j(p) = \nabla_{v_i}(\sum_{k=1}^{n} O_{jk}\psi^k(p) = \sum_{k=1}^{n} (v_i O_{jk})\phi^k(p) = \sum_k v_i A_{jk}\phi^k .$$

Comparing this with (3.66) gives the formula (3.67).

As a direct consequence of this computation we can deduce a formula for the curvature of the Levi-Civita connection on the spin bundle:

LEMMA 3.27. *For the Levi-Civita connection on the spin bundle on a spin manifold*

$$K_S(v,w) = -\frac{1}{4}\operatorname{cl} R(v,w). \tag{3.68}$$

3.11. Twisted Dirac operators.

Definition 3.7 fixes the Dirac operator $\eth \in \mathrm{Diff}^1(X;S)$ whenever X is a spin manifold, i.e. an oriented Riemann manifold with spin structure. This operator is often considered to be *the* Dirac operator. More generally suppose that E is any other $\mathcal{C}^\infty$ Hermitian bundle over X with a unitary connection. Then on the bundle $S \otimes E$ there is both a unitary connection and a Clifford action (acting purely on S). Again using Definition 3.7 this gives the 'twisted Dirac operator' $\eth_E \in \mathrm{Diff}^1(X; S\otimes E)$.

PROPOSITION 3.28. *The twisted Dirac operator on a compact spin manifold (without boundary), X, for an Hermitian coefficient bundle, E, with unitary connection is a formally self-adjoint operator which, if the dimension is even, decomposes into a sum:*

$$\eth_E = \begin{pmatrix} 0 & \eth_E^- \\ \eth_E^+ & 0 \end{pmatrix},\ \dim X = 2k, \tag{3.69}$$

where

$$\eth_E^\pm \in \mathrm{Diff}^1(X; {}^\pm S \otimes E, {}^\mp S \otimes E).$$

PROOF: The self-adjointness is a special case of Proposition 3.9. If $\dim X$ is even then the connection restricts to ${}^\pm S$ since both are associated bundles to the spin bundle. The connection is therefore graded and the Corollary to Proposition 3.9 applies.

3.12. Spin structure for a b-metric.

Finally having defined the notion of a twisted Dirac operator on a spin manifold consider the case of a b-spin manifold. Thus let X be a compact manifold with a b-metric. Again the orientability of X is equivalent to the existence of a subbundle

$${}^b_oF \subset {}^bF, \text{ with structure group } \mathrm{SO}(N).$$

Further consider the existence of a b-spin structure, meaning a principal $\mathrm{Spin}(N)$ bundle which double covers ${}^b_oF$ and is consistent with it, as in (3.62).

EXERCISE 3.29. Show that a b-spin structure exists if and only if a spin structure exists in the ordinary sense.

The Levi-Civita connection, a b-connection, lifts to a connection on the spin frame bundle which will be denoted ${}^b_sF$. Of course over the interior of X these notions just reduce to those discussed above. Thus the Dirac operator, or more generally the twisted Dirac operator for a coefficient bundle E with b-connection, is defined as a differential operator over the interior of X. More precisely

LEMMA 3.30. *The twisted Dirac operator $\eth_E$ for a bundle with b-connection over a b-spin manifold is an element of* $\mathrm{Diff}^1_b(X; S \otimes E)$.

PROOF: As in Definition 3.7, $i{}^b\eth_E$ is just the composite of the b-connection and Clifford multiplication:

$$
\begin{gathered}
{}^b\nabla\colon \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X; {}^bT^*X \otimes E),\ {}^b\nabla \in \mathrm{Diff}^1_b(X; E, {}^bT^*X \otimes E)\\
\tilde{c}\colon \mathcal{C}^\infty(X; {}^bT^*X \otimes E) \longrightarrow \mathcal{C}^\infty(X;E),\ e \otimes \xi \longmapsto \mathrm{cl}(\xi)\cdot e.
\end{gathered}
$$

EXERCISE 3.31. Show that for any Hermitian bundle, E, with unitary b-connection over a b-spin manifold X (compact with boundary) the twisted Dirac operator defines a formally self-adjoint element ${}^b\eth_E \in \mathrm{Diff}^1_b(X; S\otimes E)$ with a splitting (3.69) if the dimension is even.

The general case considered in Chapter 9 is covered by the following consequence of this discussion:

LEMMA 3.32. *Let E be an Hermitian Clifford bundle over a compact manifold with boundary with an exact b-metric. Suppose further that E has a b-connection which is Hermitian and Clifford. Then the Dirac operator in the sense of Definition 3.7 is a self-adjoint element of* $\mathrm{Diff}^1_b(X;E)$. *If $E = E^+ \oplus E^-$ is graded, the manifold is even-dimensional and the connection preserves the $\mathbb{Z}_2$-grading then the Dirac operator is of the form* (3.69).

3.13. Boundary behaviour.

It is of fundamental importance to examine the behaviour of the Dirac operator at the boundary.

First note that there is no completely natural way that a general b-connection induces a connection on the boundary. This corresponds to $T\partial X$ being a quotient of ${}^bT_{\partial X}X$ rather than a subbundle. However an exact b-metric (see Exercise 2.9) gives an embedding (as the orthocomplement to $x\partial_x$) $T\partial X \longrightarrow {}^bT_{\partial X}X$ which induces a Lie algebra homomorphism on the spaces of sections, so ${}^bT^*_{\partial X}X \longrightarrow T^*\partial X$ can be taken as orthogonal projection. If $V \in \mathcal{C}^\infty(\partial X; T\partial X)$ let $\widetilde{V} \in \mathcal{C}^\infty(\partial X; {}^bTX)$ be the image of V. Then if ${}^b\nabla$ is a b-connection on some bundle E set

$$\nabla_V e = {}^b\nabla_{\widetilde{V}} e,\ e \in \mathcal{C}^\infty(\partial X; E).$$

This defines the induced connection on ∂X, corresponding to the choice of an exact b-metric.

EXERCISE 3.33. Check that if ${}^b\nabla$ is actually a true connection this induced connection on the boundary is the usual one. More generally show that a different choice of exact b-metric changes the induced connection by a homomorphism (*cf.* (2.97))

$$d\psi \otimes {}^b\nabla_{x\partial/\partial x},\ \psi \in \mathcal{C}^\infty(\partial X).$$

As in (3.41) the action of $\eth_E$ can be written in terms of any local orthonormal frame, v_j and dual coframe ϕ^j :

$$\eth_E u = -i \sum_j \operatorname{cl}(\phi^j)^b\nabla_{v_j} u. \tag{3.70}$$

Recall that over the boundary the spin bundle in the even dimensional case splits as the direct sum of two copies of the induced spin bundle. In fact using the identification (3.30) of the Clifford algebra over the boundary with the positive part of the Clifford algebra of the manifold, restricted to the boundary

$$\operatorname{Cl}(\partial X) \longrightarrow \operatorname{Cl}^+_{\partial X} X,\ \phi^j \longmapsto i\phi^j \cdot \frac{dx}{x}, \tag{3.71}$$

the spin bundle over the boundary, S_0, can be identified with ${}^+S_{\restriction \partial X}$. Denote this identification

$$M_+ : {}^+S_{\restriction \partial X} \longrightarrow S_0.$$

Then the other half of the spin bundle can be identified with S_0 through Clifford multiplication:

$$M_-^{-1} = -\operatorname{cl}(i\frac{dx}{x})M_+^{-1} : S_0 \longleftrightarrow {}^-S_{\restriction \partial X}.$$

Thus $M_-^{-1}M_+ = -i\operatorname{cl}(\frac{dx}{x})$. As an operator on ${}^+S_{\restriction \partial X}$ one then finds

$$M_+\eth_0 M_+^{-1} = \sum_{j=1}^n i\operatorname{cl}(\phi^j)\operatorname{cl}(\frac{dx}{x})\nabla_{iv_j},\ N = n+1.$$

LEMMA 3.34. *With these identifications, the restriction to the boundary of the twisted Dirac operator $\eth_E^-$ for an even-dimensional exact b-spin manifold and coefficient bundle with (true) connection is given in terms of the twisted Dirac operator for the induced spin structure and connection on the boundary by*

$$(\eth_E u)_{\restriction \partial X} = \begin{pmatrix} 0 & M_+^{-1}\eth_{0,E}M_- \\ M_-^{-1}\eth_{0,E}M_+ & 0 \end{pmatrix} (u_{\restriction \partial X}) \tag{3.72}$$

for all $u \in \mathcal{C}^\infty(X; S \otimes E)$.

PROOF: As usual the orthonormal basis should be chosen with first element reducing to $x\partial_x$ at the boundary. The assumption that the connection on E is a true connection means that ${}^b\nabla_{x\partial_x}$ vanishes at ∂X and then (3.72) follows directly from (3.70) and (3.71).

In the case of a general b-connection on E formula (3.72) becomes

$$(\eth_E u)_{\restriction \partial X} = \begin{pmatrix} 0 & M_+^{-1}\eth_{0,E}M_- \\ M_-^{-1}\eth_{0,E}M_+ & 0 \end{pmatrix} (u_{\restriction \partial X}) . \tag{3.73}$$

Indeed the same approach works in the case of a graded Hermitian module. Define

$$E_0 = E^+_{\restriction \partial X},$$

with Clifford action given by (3.71). Then the b-connection induces a connection on E_0 and:

COROLLARY. *The same result,* (3.73), *holds for the case of the generalized Dirac operators in Lemma 3.32 provided* $\eth_{0,E}$ *is the generalized Dirac operator induced on the boundary Clifford module* E_0.

EXERCISE 3.35. Check the details of the proof of this corollary.

Later this result will be extended somewhat, once the indicial homomorphism has been defined, to arrive at the first two parts of (In.23).

3.14. Dirac operators of warped products.

We proceed to compute the form of the Dirac operator on a warped product, as considered in §2.10.

Consider the product case. Thus suppose that M is an odd-dimensional spin manifold, with metric h and spinor bundle S_0, and consider the even-dimensional manifold $X = \mathbb{S}^1 \times M$ with the product metric $d\theta^2 + h$. The discussion in the preceding two sections applies with minor changes. So X has a spin structure the spinor bundle of which can be identified with $S_0 \oplus S_0$, where $M_+ : {}^+S \equiv S_0$ is regarded as an equality and $M_- : {}^-S \longleftrightarrow S_0$ is given by $M_- = -iM_+ \operatorname{cl}(d\theta)$. Clifford multiplication, cl_0, on S_0 and $S = {}^+S \oplus {}^-S$ are related as in (3.71), i.e.

$$\operatorname{cl}_0(\eta) = -i \operatorname{cl}(d\theta) \operatorname{cl}(\eta), \tag{3.74}$$

where in the right $\eta \in T^*M$ is regarded as a 1-form on X. The Dirac operator on X then becomes

$$\eth = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \frac{1}{i}\frac{\partial}{\partial\theta} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eth_0. \tag{3.75}$$

To generalize this formula to the metric (2.71) we first need to identify the spin bundle over X with the bundle in the product case. To define the spin structure for the metric g, the orthonormal coframe bundle of g can be identified with that in the product case. If the η_i for $i = 1, \dots, 2k-1$ are a local orthonormal frame for h on M then $d\theta, \eta_i$ is a local orthonormal frame for X with the product metric and $d\theta, e^{\phi}\eta_i$ an orthonormal frame for g. This gives a local identification of the orthonormal coframe bundles; since it is independent of the choice of the η_i it extends globally. This identification extends to define a spin structure for g by identifying the spin (structure) bundle of it with the product case. Since the spinor bundles are associated bundles of these principal bundles they too are identified. Thus the Dirac operator for the metric g acts on the same bundle as (3.75), so the two operators can be compared.

To find the form of the Dirac operator we simply compute expressions for Clifford multiplication $\widetilde{\mathrm{cl}}$ and covariant differentiation, $\widetilde{\nabla}$, for the metric g, on the bundle $S = S_0 \oplus S_0$. Since Clifford multiplication is defined algebraically

$$\begin{aligned} \widetilde{\mathrm{cl}}(\eta) &= \mathrm{cl}(e^{-\phi}\eta), \ \ \eta \in \mathcal{C}^{\infty}(M; T^*M), \\ \widetilde{\mathrm{cl}}(d\theta) &= \mathrm{cl}(d\theta). \end{aligned} \tag{3.76}$$

Now consider a point $p \in M$ and fix local coordinates z^j near p for which the differentials are orthonormal at p and covariant constant there, all with respect to the metric h. Then combining (3.67) and (2.73)

$$\begin{aligned} \widetilde{\nabla}_{\partial_\theta} s &= \frac{\partial s}{\partial \theta} - \frac{2k-1}{4}\frac{\partial \phi}{\partial \theta} e^{-2\phi} s \\ \widetilde{\nabla}_j s &= \nabla_j s + \frac{1}{4}\frac{\partial \phi}{\partial \theta} e^{\phi}\, \mathrm{cl}(dz^j)\, \mathrm{cl}(d\theta). \end{aligned} \tag{3.77}$$

Inserting these formulæ into the definition of the Dirac operator gives:

LEMMA 3.36. *With the identification above of the spinor bundle for the metric (2.71) on $X = \mathbb{S}^1 \times M$ with the spinor bundle in case $\phi \equiv 0$ the Dirac operator becomes*

$$\widetilde{\eth} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \left\{ \frac{1}{i}\frac{\partial}{\partial \theta} + F(\theta) \right\} + e^{-\phi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eth_0, \tag{3.78}$$

where $F(\theta)$ is a function of θ.

This formula is used in Chapter 8 in the proof of the absolute convergence of the integral defining the eta invariant.

EXERCISE 3.37. Show that

$$F(\theta) = \frac{i(2k-1)}{4}\frac{\partial \phi}{\partial \theta}\left(e^{-2\phi} - 1\right). \tag{3.79}$$

Chapter 4. Small b-calculus

The next major topic is the calculus of b-pseudodifferential operators, since this allows the analytic properties of elliptic b-differential operators to be examined. The treatment starts very geometrically. In particular Figure 2 of Chapter 1 will be explained. Recall from (1.24) and (1.26) that the kernels of the inverses of operators such as ${}^b\eth_+^*\,{}^b\eth_+ + 1$ can be expected to be simplest when expressed in terms of the singular coordinates (1.25). In the one-dimensional case this is rather a minor issue. In the higher-dimensional examples of interest here it is more significant, as there will be a countably infinite superposition of terms like (1.26), with different constants c. To handle these systematically the properties of the coordinate change (1.25) will be examined with some care.

4.1. Inward-pointing spherical normal bundle.

Let X be a compact manifold with boundary. The first goal is to define X_b^2, obtained from X^2 by the introduction of singular coordinates, so that

$$\frac{x - x'}{x + x'} \in \mathcal{C}^\infty(X_b^2),$$

with x and x' are defining functions for the boundary of X lifted from the respective factors, x from the left and x' from the right. The space will come equipped with a surjective smooth *blow-down* map

$$\beta_b : X_b^2 \longrightarrow X^2. \tag{4.1}$$

To construct X_b^2 as a set (i.e. independent of the choice of coordinates) assume that ∂X is connected and consider

$$B = \partial X \times \partial X = \{x = x' = 0\} \subset X^2. \tag{4.2}$$

This is the manifold which should be replaced by a bigger set in which some information concerning the direction of approach to B is included. Directions are associated to curves, so consider all $\mathcal{C}^\infty$ curves with only their endpoint on B and which are not tangent to B at that point:

$$\chi\colon [0,1] \longrightarrow X^2,\ \chi(t) \in B \Longleftrightarrow t = 0,\ \chi'(0) \notin T_{\chi(0)}B. \tag{4.3}$$

This space, although not linear, has a natural topology as a subspace of the space of all $\mathcal{C}^\infty$ curves in X^2, corresponding to uniform convergence of all derivatives on compact subsets of coordinate patches.

The equivalence relation placed on these curves is:

$$\chi_1 \sim \chi_2 \Longleftrightarrow \chi_1(0) = \chi_2(0) = p,\ \chi_1'(0) \equiv a\chi_2'(0) \pmod{T_pB},\ a \in \mathbb{R}^+.$$

This identifies two curves if they correspond to the same end point, in B, and have derivatives at that point which are multiples, up to a term tangent to B. Thus the set of equivalence classes is the inward-pointing unit (or really spherical) normal bundle to B :

$$S_+N_p(B) = \{\chi \text{ in } (4.3), \chi(0) = p\}/\sim \tag{4.4}$$

and then

$$S_+N(B) = \bigsqcup_{p\in B} S_+N_p(B)$$

gives a bundle over B with fibre diffeomorphic to $[-1,1]$. This will be proven directly by constructing coordinates on $S_+N(B)$.

LEMMA 4.1. *If $x \in \mathcal{C}^\infty(X)$ is a defining function for ∂X and $\tilde{x} = \pi_L^* x$, $\tilde{x}' = \pi_R^* x$ denote its lifts to X^2 from the left and right factors then for any curve (4.3) the limit*

$$\tau([\chi]) = \lim_{t\downarrow 0} \chi^* \left(\frac{\tilde{x} - \tilde{x}'}{\tilde{x} + \tilde{x}'} \right) \tag{4.5}$$

exists and depends only on $[\chi] \in S_+N(B)$. This function gives an identification

$$S_+N(B) \ni [\chi] \longmapsto (p = \chi(0), \tau([\chi])) \in B \times [-1,1]. \tag{4.6}$$

Another choice of x determines, through (4.6), a diffeomorphism of the form

$$B \times [-1,1] \ni (p,\tau) \longmapsto \left(p, \tau' = \frac{A(p) + \tau}{1 + A(p)\tau}\right) \in B \times [-1,1], \tag{4.7}$$

where $A \in \mathcal{C}^\infty(B)$ is given by

$$A(p) = \frac{a(y) - a(y')}{a(y) + a(y')},\ p = (y,y') \in \partial X \times \partial X,\ 0 < a \in \mathcal{C}^\infty(\partial X). \tag{4.8}$$

PROOF: If $\tilde{x}$ and $\tilde{x}'$ are the elements in $\mathcal{C}^\infty(X^2)$ indicated then at each point $p \in B$, $d\tilde{x}$ and $d\tilde{x}'$ are linear coordinates in the two dimensional space $N_pB = T_pX^2/T_pB$. By definition, in (4.4),

$$S_+N_p(B) = \{0 \neq v \in N_pB;\ d\tilde{x}(v) \geq 0, d\tilde{x}'(v) \geq 0\}/\mathbb{R}^+.$$

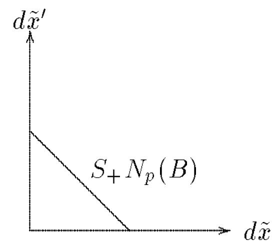

Figure 4. The spherical normal fibre.

Thus there is an identification

$$S_+N_p(B) \simeq \{v \in N_pB;\ d\tilde{x}(v) \geq 0,\ d\tilde{x}'(v) \geq 0, d\tilde{x}(v) + d\tilde{x}'(v) = 1\}.$$

With this identification the function (4.5) is just $\tau(v) = d\tilde{x}(v) - d\tilde{x}'(v) \in [-1, 1]$. Under change of the defining function from x to $\bar{x} = a(y)x + O(x^2)$, $0 < a \in \mathcal{C}^\infty(\partial X)$, the new function (4.5) is

$$\tau' = \lim_{t \downarrow 0} \chi^* \left(\frac{\tilde{\bar{x}} - \tilde{\bar{x}}'}{\tilde{\bar{x}} + \tilde{\bar{x}}} \right) = \frac{a(y)(1+\tau) - a(y')(1-\tau)}{a(y)(1+\tau) + a(y')(1-\tau)}.$$

This gives (4.7) and (4.8).

Thus not only does $S_+N(B)$ have a $\mathcal{C}^\infty$ structure, given by (4.6), but the fibres have a *projective* structure. Indeed if $\alpha \in [-1, 1]$ then

$$M_\alpha \colon S_+N(B) \longrightarrow S_+N(B),\ M_\alpha \tau = \frac{\alpha + \tau}{1 + \alpha\tau} \tag{4.9}$$

is independent of the choice of projective coordinate τ, since

$$\tau' = \frac{A + \tau}{1 + A\tau'} \Longrightarrow \frac{\alpha + \tau'}{1 + \alpha\tau'} = \frac{A + M_\alpha \tau}{1 + AM_\alpha \tau}.$$

Using the coordinate

$$s = \frac{1+\tau}{1-\tau} \in (0, \infty) \ \text{ on } \ S_+N(B) \backslash \partial(S_+N(B)) \tag{4.10}$$

the action (4.9) becomes

$$s \longmapsto \beta s, \quad \beta = \frac{1-\alpha}{1+\alpha} \in (0, \infty).$$

That is, the interior of each fibre of $S_+N(B)$ has a natural $(0, \infty)$-action.

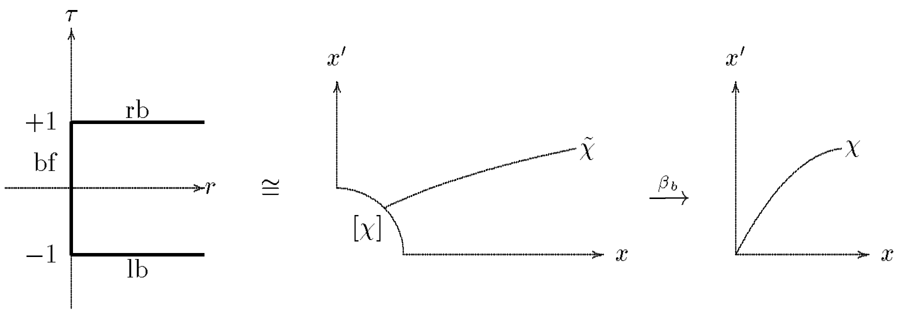

Figure 5. $\beta_b\colon X^2_b \longrightarrow X^2$.

4.2. The b-stretched product.

If ∂X is connected the b-stretched product is obtained by replacing B by $S_+N(B)$ in X^2, i.e.

$$X^2_b = S_+N(B) \sqcup (X^2 \backslash B).$$

This is certainly well-defined as a set and (4.1) arises by letting β_b act as the identity on $X^2\backslash B$ and taking it to be the projection from $S_+N(B)$ to B, $\beta_b([\chi]) = \chi(0) \in B$. It remains to show that X^2_b has a natural $\mathcal{C}^\infty$ structure with respect to which β_b is $\mathcal{C}^\infty$. Fix the topology on X^2_b by noting that each curve χ in (4.3) defines a map

$$\tilde{\chi}\colon [0,1] \longrightarrow X^2_b,\ \tilde{\chi}(t) = \chi(t),\ t > 0,\ \tilde{\chi}(0) = [\chi].$$

A subset $U \subset X^2_b$ is open if it meets $X^2 \setminus B$ and $S_+N(B)$ in open sets and provided that whenever χ is a curve, as in (4.3), such that $\tilde{\chi}(0) \in U$ there exists $\epsilon > 0$ and a neighbourhood of χ amongst such curves for which $\tilde{\mu}(t) \in U$ for all $0 \le t < \epsilon$ and for all curves μ in the neighbourhood. It would actually be enough to take the $\mathcal{C}^1$ topology on curves to get the same topology on X^2_b. Notice in particular that the function $\tau = (\tilde{x} - \tilde{x}')/(\tilde{x} + \tilde{x}')$ on $X^2\backslash B$, extended to $S_+N(B)$ by (4.5), is in $\mathcal{C}^0(X^2_b)$. Then define $\mathcal{C}^\infty(X^2_b) \subset \mathcal{C}^0(X^2_b)$ as consisting of those functions which can be expressed locally near each point of $S_+N(B)$ in the form $F(\tau, \tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')$, with F a $\mathcal{C}^\infty$ function, i.e. $\mathcal{C}^\infty(X^2_b)$ is generated locally by τ and $\beta_b^*\mathcal{C}^\infty(X^2)$. As in the discussion above, under a change of defining function to $\bar{x} = a(x,y)x$, $0 < a \in \mathcal{C}^\infty(X)$, τ is transformed to

$$\bar{\tau} = \frac{A(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') + \tau}{1 + A(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tau}$$

where
$$A(\tilde{x},\tilde{y},\tilde{x}',\tilde{y}') = \frac{a(\tilde{x},\tilde{y}) - a(\tilde{x}',\tilde{y}')}{a(\tilde{x},\tilde{y}) + a(\tilde{x}',\tilde{y}')}.$$
Thus $\bar{\tau} \in \mathcal{C}^\infty(X_b^2)$ and the definition is indeed independent of the choice of x. As illustrated in Figure 5 there is a neighbourhood of $S_+N(B)$ in X_b^2 of the form $[-1,1] \times [0,\epsilon) \times \partial X \times \partial X$. In fact for any boundary defining function $x \in \mathcal{C}^\infty(X)$ the set $G = \{\tilde{x} + \tilde{x}' < \epsilon\}$ is, for small $\epsilon > 0$, such a neighbourhood. The functions τ and $\tilde{x} + \tilde{x}'$ on G give the desired decomposition. It follows that X_b^2 is a compact manifold with corners (just as is X^2) with the blow-down map, β_b in (4.1), $\mathcal{C}^\infty$. The construction of X_b^2 from X^2 by blowing up B in (4.2) is a special case of the general process of (real) blow-up of a submanifold. In the notation of [63] or [31]
$$X_b^2 = [X^2; B]$$
but the general construction is not needed here.

So far the construction has been carried out under the assumption that ∂X consists of one component manifold. If the boundary has several components:

$$\partial X = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_r \tag{4.11}$$

then set

$$B = \bigsqcup_{j=1}^{r} (Y_j \times Y_j) \subset \partial X \times \partial X. \tag{4.12}$$

The b-stretched product X_b^2 is defined by blowing up B, i.e. by blowing up each of the components of B in (4.12).

EXERCISE 4.2. Make sure that you understand that 'blow-up' is just the introduction of polar coordinates around the submanifold. In particular check that $r = (\tilde{x}^2 + (\tilde{x}')^2)^{\frac{1}{2}}$ and $\theta = \arctan(\tilde{x}/\tilde{x}')$ are $\mathcal{C}^\infty$ functions on X_b^2. How would you express the naturality of the $\mathcal{C}^\infty$ structure on X_b^2 from this point of view, i.e. what would you have to do to prove it?

4.3. Submanifolds of X_b^2.

It is important to gain a clear understanding of the geometry of X_b^2, which will be the carrier for the Schwartz kernels of the operators in the b-calculus. The boundary hypersurfaces of X_b^2 will be denoted lb, rb and bf, where

$$\mathrm{bf}(X_b^2) = S_+N(B) \tag{4.13}$$

is the 'new' or 'front face' replacing B. The left and right boundary faces, lb and rb, are just the 'old' boundary faces:

$$\mathrm{lb}(X_b^2) = \mathrm{cl}\big(\beta_b^{-1}(\partial X \times \overset{\circ}{X})\big),\ \mathrm{rb}(X_b^2) = \mathrm{cl}\big(\beta_b^{-1}(\overset{\circ}{X} \times \partial X)\big) \text{ in } X_b^2. \tag{4.14}$$

Notice that $\partial X \times \overset{\circ}{X} = (\partial X \times X) \setminus B$. The lift is defined in this way, as the closure in X_b^2 of the part of the submanifold (in this case $\mathrm{lb}(X^2) = \partial X \times X$) outside B, rather than just the preimage because

$$\beta_b^{-1}(\mathrm{lb}(X^2)) = \mathrm{bf}(X_b^2) \cup \mathrm{lb}(X_b^2)$$

and the front face is not to be thought of as part of the left boundary. Indeed the main idea is to separate the left and right boundaries. Observe that (provided ∂X is connected)

$$\mathrm{lb}(X_b^2) \cap \mathrm{rb}(X_b^2) = \emptyset.$$

There is another submanifold of primary importance in the analysis of differential operators, namely the diagonal

$$\Delta = \{(p,p) \in X^2;\ p \in X\} \simeq X.$$

The *lifted* or b-diagonal is defined, as in (4.14), by setting

$$\Delta_b = \mathrm{cl}(\beta_b^{-1}(\Delta \backslash B)) \text{ in } X_b^2. \tag{4.15}$$

LEMMA 4.3. *The lifted diagonal is an embedded, closed submanifold of X_b^2 which is diffeomorphic to X under β_b and which meets the boundary of X_b^2 only in $\mathrm{bf}(X_b^2)$, which it intersects transversally.*

PROOF: Of course in $X^2 \backslash B$ nothing has been changed, so it is only necessary to consider a neighbourhood of $\mathrm{bf}(X_b^2)$. From the definition of $\tau = (\tilde{x} - \tilde{x}')/(\tilde{x} + \tilde{x}')$ in $X^2 \backslash B$ it follows, by continuity, that $\tau \equiv 0$ on Δ_b. Similarly $\tilde{y} = \tilde{y}'$ on Δ_b for any local coordinates in X. This gives $\dim X$ independent defining functions and shows that Δ_b is an embedded submanifold. The local coordinates x, y lift to local coordinates $\tilde{x}, \tilde{y}$ on Δ_b, so $\beta_b \colon \Delta_b \longleftrightarrow X$ is a diffeomorphism. Notice that

$$\mathrm{bf}(X_b^2) = \{r = 0\},\ r = \tilde{x} + \tilde{x}' \in \mathcal{C}^\infty(X_b^2).$$

Since $dr = 2d\tilde{x}$ on Δ_b is non-vanishing,

$$\mathrm{bf}(X_b^2) \pitchfork \Delta_b = \\ \{[\chi] \in S_+N(B); \tau([\chi]) = 0,\ p = \chi(0) \in \Delta_{\partial X} \subset \partial X \times \partial X\}.$$

This also completes the explanation of Figure 2 in Chapter 2.

4.4. Lifting vector fields.

What has been accomplished by the introduction of X_b^2? Later it will be seen to lead to a simplification of the Schwartz kernels of b-differential operators and this simple characterization makes the definition of the b-pseudodifferential operators reasonably obvious. The main step in doing this is to see what happens to the elements of $\mathcal{V}_b(X)$ when they are lifted to X^2, to act just on the left factor, and then are further lifted to X_b^2. In general one cannot lift a vector field under a $\mathcal{C}^\infty$ map, however:

PROPOSITION 4.4. *Each element of $\mathcal{V}_b(X)$, as a $\mathcal{C}^\infty$ vector field on the left factor of X in X^2, lifts to a $\mathcal{C}^\infty$ vector field on X_b^2.*

PROOF: As already observed, near each point of $\mathrm{bf}(X_b^2)$ the coordinates

$$\tau = \frac{x - x'}{x + x'}, r = x + x', y, y' \tag{4.16}$$

can be used, where (x, y) are coordinates in the left factor and (x', y') are coordinates in the right factor, with $x = x' \in \mathcal{C}^\infty(X)$. From now on the rather heavy-handed tilde notation used above to distinguish between a function on X and the lift to one of the factors of X^2 will be dropped. A general element of $\mathcal{V}_b(X)$ is of the form

$$V = \alpha_0 x\frac{\partial}{\partial x} + \sum_{j\geq 1} \alpha_j \frac{\partial}{\partial y_j},$$

with the coefficients, α_l, $\mathcal{C}^\infty$ functions in local coordinates. Since the coefficients certainly lift to be $\mathcal{C}^\infty$ and the $\partial/\partial y_j$ lift to $\partial/\partial y_j$ only the lift of $x\partial/\partial x$ needs to be examined. From (4.16),

$$x\frac{\partial}{\partial x} = x\frac{\partial}{\partial r} + \frac{2xx'}{(x+x')^2}\frac{\partial}{\partial \tau} = \frac{1}{2}(1+\tau)r\frac{\partial}{\partial r} + \frac{1}{2}(1-\tau^2)\frac{\partial}{\partial \tau} \tag{4.17}$$

which is $\mathcal{C}^\infty$, proving the proposition.

Notice what has happened in (4.17). The vector field $x\partial/\partial x$ vanishes at $\mathrm{lb}(X^2)$, i.e. $\{x = 0\}$. When lifted to X_b^2, near $\mathrm{bf}(X_b^2)$, it again vanishes at $\mathrm{lb}(X_b^2)$, but not on $\mathrm{bf}\setminus\mathrm{lb}$. Indeed the lifts of the $\partial/\partial y_j$ and $x\partial/\partial x$ are independent at each point of Δ_b and no non-trivial linear combination of them is tangent to Δ_b. This can be reexpressed in the form:

LEMMA 4.5. *The lift to X_b^2 of $\mathcal{V}_b(X)$ from the left factor of X^2 is a Lie subalgebra of $\mathcal{V}_b(X_b^2)$ which is transversal to Δ_b.*

At each point of Δ_b the normal space is

$$N_p(\Delta_b) = T_pX_b^2/T_p\Delta_b,\ p \in \Delta_b. \tag{4.18}$$

Thus the values at p of the elements of $\mathcal{V}_b(X)$ define a subspace of $T_pX_b^2$ of dimension equal to that of X. By Lemma 4.5 this space is transversal to Δ_b and can be identified with ${}^bT_qX$, where $\beta_b(p) = (q,q)$. Thus

$$T_pX_b^2 = T_p\Delta_b \oplus {}^bT_qX,\ \beta_b(p) = (q,q),\ p \in \Delta_b. \tag{4.19}$$

Combined with (4.18) this gives:

LEMMA 4.6. *The normal bundle to Δ_b in X_b^2 is naturally isomorphic to ${}^bTX$ and dually*

$$N^*\Delta_b \cong {}^bT^*X. \tag{4.20}$$

This turns out to be important in the discussion of the symbol mapping.

EXERCISE 4.7. Prove (4.20) directly, in the spirit of b-geometry (see [63] for more details). First show that the composite map $\pi_{b,L}^2 = \pi_L^2 \cdot \beta_b : X_b^2 \longrightarrow X$, where $\pi_L^2 : X^2 \longrightarrow X$ is projection onto the left factor, is a b-map, meaning that if ρ is a defining function for the one boundary hypersurface of X then

$$(\pi_{b,L}^2)^*\rho = \rho_{\mathrm{bf}}^{e(\mathrm{bf})}\rho_{\mathrm{rb}}^{e(\mathrm{rb})}\rho_{\mathrm{lb}}^{e(\mathrm{lb})}, \tag{4.21}$$

where the ρ_F's are defining functions for the various boundary hypersurfaces of X_b^2. In this case $e(\mathrm{bf}) = 1$, $e(\mathrm{rb}) = 0$ and $e(\mathrm{lb}) = 1$. The reason for considering such a condition as (4.21) is that it implies that under pull-back smooth b-forms become smooth b-forms. In this case

$$(\pi_{b,L}^2)^*\frac{d\rho}{\rho} = \frac{d\rho_{\mathrm{bf}}}{\rho_{\mathrm{bf}}} + \frac{d\rho_{\mathrm{lb}}}{\rho_{\mathrm{lb}}}. \tag{4.22}$$

Use this to see (4.20).

Having gone this far it is probably worthwhile to introduce the most important notion described in [63] for maps between manifolds with corners, namely the notion of a b-fibration. This will not be used below (but makes for a much cleaner proof of the composition formula, etc.) For b-maps the differential extends, by continuity from the interior of the manifold, to a b-differential

$$(\pi_{b,L}^2)_* : {}^bT_pX_b^2 \longrightarrow {}^bT_qX,\ q = \pi_{b,L}^2(p). \tag{4.23}$$

Indeed this is just the dual statement to the pull-back on ${}^bT^*X^2$ which is expressed by (4.22). There are two properties of (4.23) which are not possessed by general b-maps. First

(4.24) The b-differential is surjective,

which is easy enough to check. A map with this property is called a b-submersion. So check that $\pi^2_{b,L}$ is a b-submersion. In addition there is another, independent, condition satisfied by $\pi^2_{b,L}$. Recall that at each point of $q \in X^2_b$ (or any manifold with corners) the b-normal space to the boundary is a well-defined subspace of ${}^bT_qX^2_b$, just given by the values at that point of the vector fields which vanish at the boundary in the ordinary sense (i.e. is spanned by $x\partial/\partial x$ at a boundary point of X.) It is easy to see that the b-differential of a b-map must always map the b-normal space at any point into the b-normal space at the image. The map is said to be b-normal if

(4.25) The b-differential is surjective as a map between b-normal spaces.

Check that $\pi^2_{b,L}$ has this property too. A b-map with the two properties (4.24) and (4.25), i.e. a b-submersion which is b-normal, is called a b-fibration in [63]. Thus $\pi^2_{b,L}$ is a b-fibration. Why should you care? Part of the answer to this can be found in Chapter 5 (or [62], better, in [63] of course). Roughly speaking a b-fibration has properties analogous to that of a fibration, especially as regards the push-forward of distributions.

Check that the blow-down map β_b is a b-submersion which is not b-normal, hence not a b-fibration. Give an example of a b-map which is not a b-submersion but is b-normal.

So far only the lift of $\mathcal{V}_b(X)$ from the left factor has been examined. To avoid accusations of 'handism' (oddhandedness??) simply consider the natural involution on X^2 :

$$I\colon X^2 \ni (p,p') \longmapsto (p',p) \in X^2.$$

This fixes each point of B and (therefore) lifts to a diffeomorphism, I_b, of X^2_b. Indeed, in the local coordinates (τ, r, y, y') near $\mathrm{bf}(X^2_b)$

$$I_b\colon X^2_b \longrightarrow X^2_b,\ I_b^2 = \mathrm{Id},\ I_b^*\tau = -\tau, I_b^*r = r, I_b^*y = y'.$$

Of course I_b interchanges lb and rb and shows that the discussion of the lift from the left extends immediately to the lift from the right.

EXERCISE 4.8. Show that the same isomorphism, (4.20), arises by using the decomposition analogous to (4.19) with $\mathcal{V}_b(X)$ the lift from the right factor.

EXERCISE 4.9. Justify the 'therefore' in parenthesis above by checking that under the blow-down map of a submanifold (or just for β_b) any $\mathcal{C}^\infty$ vector field tangent to the submanifold blown-up (in this case B) lifts to be $\mathcal{C}^\infty$ on the blown-up manifold (see also [63]).

The simultaneous lift from left and right is also something to consider. If V and W are vector bundles over X let $V \boxtimes W$ be the exterior tensor product, namely the bundle over X^2 with fibres $V_p \otimes W_q$ over $(p, q) \in X^2$. Then it is clear that

$$ {}^bTX \boxtimes {}^bTX \cong {}^bT(X^2) $$

is a vector bundle on X^2 with the property analogous to that for which ${}^bTX$ was defined in the first place:

$$ \begin{aligned} \mathcal{V}_b(X^2) &= \{V \in \mathcal{C}^\infty(X^2; TX); V \text{ is tangent to } \mathrm{rb} \cup \mathrm{lb}\} \\ &\cong \mathcal{C}^\infty(X^2; {}^bTX^2). \end{aligned} $$

Another way of seeing this is to note that the sum of the lifts from left and right of $\mathcal{V}_b(X)$ spans $\mathcal{V}_b(X^2)$ over $\mathcal{C}^\infty(X^2_b)$.

LEMMA 4.10. *The lift to X^2_b of $\mathcal{V}_b(X^2)$ spans*

$$ \mathcal{V}_b(X^2_b) = \{V \in \mathcal{C}^\infty(X^2_b; TX^2_b); V \textit{ is tangent to } \mathrm{lb} \cup \mathrm{rb} \cup \mathrm{bf}\} $$

over $\mathcal{C}^\infty(X^2_b)$ and therefore

$$ \mathcal{V}_b(X^2_b) = \mathcal{C}^\infty(X^2_b; {}^bTX^2_b) \textit{ with } {}^bTX^2_b \cong \beta_b^*({}^bTX^2). \tag{4.26} $$

PROOF: Away from $\mathrm{bf}(X^2_b)$ there is nothing to show because β_b is a diffeomorphism there. Consider local coordinates x, y in the left factor and x', y' in the right factor of X and corresponding coordinates (4.16) near the front face of X^2_b. Then the tangential vector fields $\partial/\partial y_j$ and $\partial/\partial y'_j$ lift to the same vector fields; the lift of $x\partial/\partial x$ is given by (4.17) and similarly

$$ x'\frac{\partial}{\partial x'} = x'\frac{\partial}{\partial r} - \frac{2xx'}{(x+x')^2}\frac{\partial}{\partial \tau} = \frac{1}{2}(1-\tau)r\frac{\partial}{\partial r} - \frac{1}{2}(1-\tau^2)\frac{\partial}{\partial \tau}. \tag{4.27} $$

The sum of (4.27) and (4.17) is $r\partial/\partial r$ so the span of the lifted vector fields is the same as that of

$$ r\frac{\partial}{\partial r},\ (1-\tau^2)\frac{\partial}{\partial \tau},\ \frac{\partial}{\partial y_j} \text{ and } \frac{\partial}{\partial y'_j}. $$

Since these vector fields clearly span $\mathcal{V}_b(X^2_b)$, locally, this proves the lemma.

EXERCISE 4.11. Extend Lemma 4.5 to show that elements of $\mathrm{Diff}^m_b(X^2)$ lift from X^2 to X^2_b under β_b, defining a map

$$\beta_b^*\colon \mathrm{Diff}^m_b(X^2) \longrightarrow \mathrm{Diff}^m_b(X^2_b). \tag{4.28}$$

Show further that the range of this map spans $\mathrm{Diff}^m_b(X^2_b)$ over $\mathcal{C}^\infty(X^2_b)$.

4.5. Densities.

The b-stretched product just discussed is the starting point for the description of the b-calculus on a compact manifold with boundary, i.e. the calculus of b-pseudodifferential operators. This calculus is designed to include the generalized inverses of elliptic b-differential operators and not too much more. The term *calculus* is used to indicate that the b-pseudodifferential operators do *not* form an algebra. However, when composition of two such operators is possible the composite is in the calculus. The first step is to define the *small* calculus of b-pseudodifferential operators which *is* an algebra.

First a few words about densities, and in particular b-densities. As a general principle, from now on almost all analytic discussion will be carried out for operators acting on b-half-densities, in order to reduce the bookkeeping overhead associated with bundles. The extension to the general case is then mainly a matter of notation. Recall that if V is a vector space of dimension n then the space of s-densities on V is, for $s \in \mathbb{R}$,

$$\Omega^s V = \{u\colon \Lambda^n V^* \setminus \{0\} \longrightarrow \mathbb{R}; u(t\alpha) = |t|^s u(\alpha) \quad \forall\ \alpha \in \Lambda^n V^*, t \neq 0\}.$$

Thus $\mu \in \Omega^s V$ is fixed by its value at any one $0 \neq \alpha \in \Lambda^n V^*$ and so $\Omega^s V$ is always a one-dimensional vector space with an orientation, fixed by $u(\alpha) > 0$. Directly from the definition there are canonical isomorphisms:

$$\begin{gathered} \Omega^s V \otimes \Omega^t V \equiv \Omega^{s+t} V \quad \forall\ s, t \in \mathbb{R} \\ \Omega^0 V \equiv \mathbb{R} \\ \Longrightarrow \Omega^{-s} V \equiv (\Omega^s V)^*. \end{gathered} \tag{4.29}$$

Similarly if V and W are any finite dimensional vector spaces there is a canonical isomorphism

$$\Omega^s(V \oplus W) \equiv \Omega^s(V) \otimes \Omega^s(W)\ \forall\ s \in \mathbb{R}; \tag{4.30}$$

this just arises from $\Lambda^{n+m}(V^* \oplus W^*) \equiv \Lambda^n V^* \otimes \Lambda^m W^*$ if $\dim V = n$ and $\dim W = m$.

If X is a compact manifold (with or without boundary) then the spaces

$$\Omega^s_x = \Omega^s(T^*_x X)$$

form $\mathcal{C}^\infty$ bundles over X. Note that, being oriented line bundles, the density bundles are always trivial. Only for $s = 0$ is Ω^s canonically trivial.

EXERCISE 4.12. Show that

$$\Omega^s X = \bigsqcup_{x \in X} \Omega^s_x$$

is a bundle associated to the coframe bundle of X.

Most importantly 1-densities, usually just called densities with Ω^1 denoted simply as Ω, can be integrated. The *integral* is then a well-defined linear functional

$$\int_X : \mathcal{C}^\infty(X;\Omega) \longrightarrow \mathbb{R}. \tag{4.31}$$

EXERCISE 4.13. Local coordinates in X induce a basis of $T^*_x X$, hence the basis element $\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}$ in $\Lambda^n(T_x X)$. Show that

$$|dx|^s(a\partial_{x_1} \wedge \cdots \wedge \partial_{x_n}) = |a|^s, \ s \in \mathbb{R}$$

is a local basis element of $\Omega^s X$. Use this to show that (4.31) can be defined as the Riemann integral in local coordinates. Thus if $f \in \mathcal{C}^\infty(X;\Omega)$ has support in a coordinate patch, so $f = g(x)|dx|$ in terms of the coordinate trivialization with g a $\mathcal{C}^\infty$ function, then

$$\int_X f = \int_{\mathbb{R}^n} g(x)dx. \tag{4.32}$$

Check that (4.32) is independent of the coordinate system containing the support of f and hence that there is a unique functional (4.31) extending (4.32) by linearity.

The most important fractional densities here are the half-densities. From (4.29) there is a product

$$\mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}) \cdot \mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(X;\Omega). \tag{4.33}$$

This extends to complex sections (i.e. sections of the complexified bundles) and hence gives a sesquilinear pairing:

$$\langle u, v\rangle = \int_X u\overline{v}, \ u, v \in \mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}). \tag{4.34}$$

If $\mathcal{C}^\infty(X;\Omega^{\frac{1}{2}})$ is completed with respect to this pairing the result is the Hilbert space of square-integrable half-densities, $L^2(X;\Omega^{\frac{1}{2}})$. For example the Hilbert-Schmidt operators (which will be useful later) can be easily described. Thus:

$$K \in L^2(X^2;\Omega^{\frac12}) \Longrightarrow A_K\colon L^2(X;\Omega^{\frac12}) \longrightarrow L^2(X;\Omega^{\frac12}), \quad A_K u(x) = \int_X K(x,y)u(y). \tag{4.35}$$

Here if $\phi \in L^2(X;\Omega^{\frac12})$

$$\langle A_K u, \phi\rangle = \int_{X^2} K(x,y)\phi(x)u(y) = \langle K, \phi \boxtimes u\rangle$$

using the (fairly obvious) form of Fubini's theorem

$$L^2(X;\Omega^{\frac12}) \boxtimes L^2(X;\Omega^{\frac12}) \subset L^2(X^2;\Omega^{\frac12}).$$

This can be generalized to operators on distributions. The pairing (4.33) invites the definition of the space of *distributional* half-densities as consisting of the continuous linear maps

$$u\colon \mathcal{C}^\infty(X;\Omega^{\frac12}) \longrightarrow \mathbb{C} \text{ continuous and linear } \Longleftrightarrow u \in \mathcal{C}^{-\infty}(X;\Omega^{\frac12}). \tag{4.36}$$

Here continuity is with respect to the topology of uniform convergence of all derivatives of the half-density on compact subsets of coordinate patches. Then the pairing (4.33) gives an inclusion

$$\mathcal{C}^\infty(X;\Omega^{\frac12}) \hookrightarrow \mathcal{C}^{-\infty}(X;\Omega^{\frac12})$$

and in fact the range is dense in the weak topology. For a general vector bundle, E, over X define

$$\mathcal{C}^{-\infty}(X;E) = \left(\mathcal{C}^\infty(X;E' \otimes \Omega)\right)',$$

where E' is the dual bundle. This is consistent with (4.36) since $(\Omega^{\frac12})' \equiv \Omega^{-\frac12}$ from (4.29) so $(\Omega^{\frac12})' \otimes \Omega \equiv \Omega^{\frac12}$.

The general case of a continuous linear operator

$$A\colon \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^{-\infty}(Y;F), \tag{4.37}$$

where E is a $\mathcal{C}^\infty$ vector bundle over X and F is a $\mathcal{C}^\infty$ vector bundle over Y, is covered by Schwartz' kernel theorem:

THEOREM 4.14. *(Schwartz) The continuous linear maps* (4.37) *are in* 1-1 *correspondence with the distributions*

$$K \in \mathcal{C}^{-\infty}\left(Y \times X; F \boxtimes [E' \otimes \Omega(X)]\right),$$

where $A \longleftrightarrow K$ *if*

$$\langle Au, \phi \rangle = \langle K, \phi \boxtimes u \rangle \quad \forall\ u \in \mathcal{C}^{\infty}(X; E),\ \phi \in \mathcal{C}^{\infty}(Y; F').$$

Quite straightforward proofs of this result are now available (using the Fourier transform) see [46].

For $E = F = \Omega^{\frac{1}{2}}$ this simplifies as in (4.35), i.e.

$$\begin{aligned} &\left\{A \colon \mathcal{C}^{\infty}(X; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}(Y; \Omega^{\frac{1}{2}});\ \text{continuous and linear}\right\} \\ &\qquad\qquad \longleftrightarrow \left\{K \in \mathcal{C}^{-\infty}(Y \times X; \Omega^{\frac{1}{2}})\right\}. \end{aligned} \tag{4.38}$$

The fact that only half-density bundles appear throughout here is one reason for working with them.

4.6. The space of pseudodifferential operators.

Now recall, for later generalization, the definition of the space of pseudodifferential operators on X, $\partial X = \emptyset$. These spaces were defined by Hörmander in [44] and by Kohn and Nirenberg in [49] although closely related singular integral operators had been used before. These operators can be defined in terms of their Schwartz kernels, using (4.38):

$$\Psi^m(X; \Omega^{\frac{1}{2}}) \longleftrightarrow I^m(X^2, \Delta; \Omega^{\frac{1}{2}}). \tag{4.39}$$

On the right is the space of conormal distributional sections of $\Omega^{\frac{1}{2}}$ of order m associated to $\Delta \subset X^2$. In fact $I^m(Z, Y; E) \subset \mathcal{C}^{-\infty}(Z; E)$ is defined for any closed embedded submanifold $Y \subset Z$, any $m \in \mathbb{R}$ and any vector bundle E over Z (see [45]).

DEFINITION 4.15. The space $I^m(Z, Y; E) \subset \mathcal{C}^{-\infty}(Z; E)$ consists of the distributional sections of E satisfying

$$K_{\restriction Z \setminus Y} \in \mathcal{C}^{\infty}(Z \setminus Y; E) \tag{4.40}$$

and such that in any local coordinates $x_1, \dots, x_p$ in Z with respect to which $Y = \{x_1 = \cdots = x_q = 0\}$ and on the coordinate patch, O, over which E is trivial,

$$K_{\restriction O} = (2\pi)^{-q} \int\limits_{\mathbb{R}^q} e^{ix' \cdot \xi} a(x'', \xi) d\xi, \tag{4.41}$$

where $x' = (x_1, \dots, x_q)$, $x'' = (x_{q+1}, \dots, x_p)$ and $a = (a_1, \dots, a_r)$, taking values in $\mathbb{C}^r$ where $E_{\restriction O} \equiv O \times \mathbb{C}^r$, satisfies

$$
\begin{aligned}
&|D^\alpha_{x''} D^\beta_\xi a_i(x'', \xi)| \le C_{\alpha,\beta,K}(1 + |\xi|)^{m'-|\beta|}, \\
&\qquad x'' \in K \subset\subset O, \xi \in \mathbb{R}^q, i = 1, \dots, r;
\end{aligned}
\tag{4.42}
$$

here $m' = m - q/4 + n/2$, $n = \dim Z$.

The estimates (4.42) are the *symbol estimates* on the amplitude a, introduced in this form by Hörmander [44]. It is important that the combination (4.41), (4.42) is coordinate invariant, see [47]. Thus, given (4.40), if (4.41) holds in a covering by coordinate patches it holds in any other coordinate patch.

There are more refined classes of operators than those with symbols just satisfying the estimates (4.42). Namely the class of one-step polyhomogeneous operators corresponds to symbols having complete asymptotic expansion with integral step:

$$
a_i \sim \sum_{k=0}^{\infty} a_{ik}(x'', \xi)|\xi|^{m'-k}. \tag{4.43}
$$

Here the coefficient functions a_{ik} are $\mathcal{C}^\infty$ in $\xi \neq 0$ and homogeneous of degree zero and the meaning of (4.43) is that for each N the difference

$$
a_{i,(N)}(x'', \xi) = a_i(x'', \xi) - \sum_{k<N} a_{ik}(x'', \xi)|\xi|^{m'-k}
$$

satisfies the estimates (4.42) in $|\xi| \ge 1$ with m' replaced by $m' - N$. Again the existence of such an expansion is a condition independent of coordinates.

If $I^m_{\text{os}}(Z, Y; E) \subset I^m(Z, Y; E)$, the subspace of one-step polyhomogeneous conormal distributions, is used in (4.39) the result is what is often called the 'classical' algebra of pseudodifferential operators on a compact manifold:

$$
\Psi^m_{\text{os}}(X; \Omega^{\frac12}) \longleftrightarrow I^m_{\text{os}}(X^2, \Delta; \Omega^{\frac12}).
$$

The algebra of pseudodifferential operators can be considered as the microlocalization of the algebra of differential operators. Thus for $k \in \mathbb{N}$

$$
\mathrm{Diff}^k(X; \Omega^{\frac12}) \subset \Psi^k_{\text{os}}(X; \Omega^{\frac12}).
$$

The kernels of differential operators are the smooth Dirac half-densities supported by the diagonal (as discussed in §4.8). The passage from smooth Dirac sections to conormal sections can be viewed as the process of microlocalization. It is the space $\Psi^*_{\text{os}}(X; \Omega^{\frac12})$ which is to be generalized to compact manifolds with boundary, by this same process of microlocalization.

4.7. Distributions.

To carry out the generalization, first briefly consider distributions on these spaces. For a fuller treatment see [63]. As before the discussion is initially restricted to half-densities, but in this case it is preferable to consider b-half-densities. Thus for any $s \in \mathbb{R}$ the oriented vector spaces ${}^b\Omega^s_x X = \Omega^s({}^bT^*_x X)$ combine to form the $\mathcal{C}^\infty$ bundles

$$
{}^b\Omega^s X = \bigsqcup_{x \in X} {}^b\Omega^s_x X.
$$

There is a very simple relationship between these bundles and the bundles of s-densities in the ordinary sense, which are also well-defined over a manifold with boundary. Namely:

$$
\nu \in \mathcal{C}^\infty(X; {}^b\Omega^s) \Longleftrightarrow x^s \nu \in \mathcal{C}^\infty(X; \Omega^s).
$$

In particular it follows from this that if

$$
\dot{\mathcal{C}}^\infty(X; E) = \{u \in \mathcal{C}^\infty(X; E); u \text{ vanishes to all orders at } \partial X\} \tag{4.44}
$$

for any vector bundle E then

$$
\dot{\mathcal{C}}^\infty(X; {}^b\Omega^s) \equiv \dot{\mathcal{C}}^\infty(X; \Omega^s). \tag{4.45}
$$

Using (4.45) the integral (4.31) can be transferred to $\dot{\mathcal{C}}^\infty(X; {}^b\Omega)$. The product map (4.33) also extends to

$$
\mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \times \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty(X; {}^b\Omega)
$$

since if one of the factors vanishes to infinite order at the boundary so does the product. Then the pairing (4.34) extends to

$$
\mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \times \dot{\mathcal{C}}^\infty(X; \Omega^{\frac{1}{2}}) \ni (\phi, \psi) \longmapsto \int_X \phi\overline{\psi}. \tag{4.46}
$$

It turns out to be rather important to note that this bilinear form does not extend directly to the whole of the product of $\mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}})$ with itself, since the integral in (4.31) does not extend directly to all of $\mathcal{C}^\infty(X; {}^b\Omega)$. Certainly

$$
\mathcal{C}^\infty(X; \Omega) = x\mathcal{C}^\infty(X; {}^b\Omega) \xrightarrow{\int_X} \mathbb{R}
$$

but in general if $u \in \mathcal{C}^\infty(X; {}^b\Omega)$ the local integral diverges logarithmically near the boundary. This is actually the reason that the trace functional is not defined on the operators in (In.18) and why the b-trace functional is introduced below.

Part of the reason that the b-densities are convenient to deal with is that there is a natural restriction map. Recall that there is a canonically trivial b-normal line subbundle ${}^bN\partial X \subset {}^bT_{\partial X}X$, spanned in local coordinates by $x\partial/\partial x$, and the quotient is naturally

$$ {}^bT_{\partial X}X/{}^bN\partial X \equiv T\partial X. $$

EXERCISE 4.16. Check that (4.30) is also valid when $V \subset U$ is a subspace in the sense that there is a natural isomorphism

$$ \Omega^s(U) \equiv \Omega^s(V) \otimes \Omega^s(U/V). $$

Thus for any $s \in \mathbb{R}$

$$ {}^b\Omega^s_{\partial X}X \equiv \Omega^s\partial X \otimes \Omega^s({}^bN\partial X). $$

Since the b-normal bundle is canonically trivial, by (2.9), so is its density bundle, hence there is a natural restriction map:

$$ \mathcal{C}^\infty(X; {}^b\Omega^s) \xrightarrow{\restriction_{\partial X}} \mathcal{C}^\infty(\partial X; \Omega^s). \tag{4.47} $$

EXERCISE 4.17. Show that in local coordinates $x, y_1, \dots, y_n$ near a boundary point the map (4.47) is just

$$ a(x,y)|\frac{dx}{x}dy|^s \longmapsto a(0,y)|dy|^s. \tag{4.48} $$

The subspace $\dot{\mathcal{C}}^\infty(X; E) \subset \mathcal{C}^\infty(X; E)$ given by (4.44) is closed. As a result there are two natural spaces of distributional half-densities:

$$ \begin{aligned} \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12}) &= \left(\dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12})\right)' \text{ (extendible distributions)} \\ \dot{\mathcal{C}}^{-\infty}(X; {}^b\Omega^{\frac12}) &= \left(\mathcal{C}^\infty(X; {}^b\Omega^{\frac12})\right)' \text{ (supported distributions).} \end{aligned} \tag{4.49} $$

The second space includes distributions supported on the boundary, the second does not. For the most part, only the spaces of extendible distributions occur below. The pairing (4.46) gives an identification of $\mathcal{C}^\infty(X; {}^b\Omega^{\frac12})$

as a subspace of the space of extendible distributions, as in the boundaryless case:

$$\begin{gathered}\mathcal{C}^\infty(X;{}^b\Omega^{\frac12}) \ni \nu \longmapsto \nu' \in \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}),\\ \nu'(\phi) = \int\limits_X \nu\phi,\ \ \phi \in \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12}).\end{gathered}$$

As usual there is rarely any point in distinguishing between a $\mathcal{C}^\infty$ density and the distribution it represents.

The corresponding construction for manifolds with corners is also needed to a limited extent. To avoid a detailed discussion it is enough to note that they behave locally as products of manifolds with boundary. Then spaces such as $\dot{\mathcal{C}}^\infty(X;E)$ can be defined by reference to any local product decomposition. For more details see [63]. For a general vector bundle E over a compact manifold with corners, X, define

$$\mathcal{C}^{-\infty}(X;E) = \left(\dot{\mathcal{C}}^\infty(X;E'\otimes{}^b\Omega)\right)'.$$

This reduces to the first case of (4.49) when $E = {}^b\Omega^{\frac12}$. The Schwartz kernel theorem extends readily to manifolds with boundary in terms of these spaces:

THEOREM 4.18. *(Schwartz) There is a* 1-1 *correspondence between continuous linear maps*

$$\dot{\mathcal{C}}^\infty(X;E) \longrightarrow \mathcal{C}^{-\infty}(Y;F), \tag{4.50}$$

where X and Y are compact manifolds with boundary and E and F are vector bundles over them, and the space $\mathcal{C}^{-\infty}(Y\times X; F\boxtimes[E^*\otimes{}^b\Omega])$; the correspondence is given by

$$\begin{gathered}A \longrightarrow K \Longleftrightarrow \langle A\phi,\psi\rangle = \langle K,\psi\boxtimes\phi\rangle\\ \forall\ \phi\in\dot{\mathcal{C}}^\infty(X;E), \psi\in\dot{\mathcal{C}}^\infty(Y;F^*\otimes{}^b\Omega).\end{gathered} \tag{4.51}$$

Actually this result is not really used below. Rather it is of philosophical importance, since it shows that questions about operators can always be reduced to questions about distributions. It is worth noting the general approach to understanding operators that underlies this discussion. As seen in Chapter 1, the kernels of operators related to b-differential operators tend to have rather complicated singularities at the corner of the manifold $X^2 = X\times X$. The stretched product X^2_b has been introduced, since the kernels are simplified when lifted to it. Observe that the blow-down map has the property

$$(\beta^2_b)^* : \dot{\mathcal{C}}^\infty(X^2;{}^b\Omega^{\frac12}) \longleftrightarrow \dot{\mathcal{C}}^\infty(X^2_b;{}^b\Omega^{\frac12}). \tag{4.52}$$

EXERCISE 4.19. Check (4.52) in local coordinates.

Therefore, by duality,

$$(4.53)\qquad (\beta_b^2)_* : \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac12}) \longleftrightarrow \mathcal{C}^{-\infty}(X^2; {}^b\Omega^{\frac12}).$$

Thus, nothing is gained by blowing up the corner! Of course that is not the point. The idea is that, by (4.53) and its extension to sections of other bundles, one can just as well examine the kernels of operators (4.50) on X_b^2 as on X^2. There are 'more' $\mathcal{C}^\infty$ functions on X_b^2, so more kernels will be considered as admissible than on X^2.

A useful combination of (4.53) and Theorem 4.18 in the special case that $X = Y$ is:

LEMMA 4.20. *If E and F are $\mathcal{C}^\infty$ vector bundles over a compact manifold with boundary X then the continuous linear operators*

$$\dot{\mathcal{C}}^\infty(X; E) \longrightarrow \mathcal{C}^{-\infty}(X; F)$$

are in 1-1 correspondence with the elements of the space of distributional sections $\mathcal{C}^{-\infty}(X_b^2; (\beta_b^2)^*(F \boxtimes [E' \otimes {}^b\Omega]))$, where

$$(4.54)\qquad A \longleftrightarrow \kappa \Longleftrightarrow \langle A\phi, \psi\rangle = \langle \kappa, (\beta_b^2)^*[\psi \boxtimes \phi]\rangle.$$

In particular if $E = F = {}^b\Omega^{\frac12}$, which will be assumed until §4.16, then $F \boxtimes [E' \otimes {}^b\Omega] \equiv {}^b\Omega^{\frac12}(X^2)$. Moreover it follows from the discussion above that there is a natural isomorphism (*cf.* (4.26))

$$(4.55)\qquad (\beta_b^2)^*[{}^b\Omega^{\frac12}(X^2)] \equiv {}^b\Omega^{\frac12}(X_b^2).$$

Thus, finally, operators on distributional densities can be identified with the lifts of their kernels to X_b^2 :

(4.56)
$$\left\{A_K : \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12})\right\} \longleftrightarrow \left\{K_A \in \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac12})\right\}.$$

4.8. Kernels of b-differential operators.

Consider (4.56) applied to the identity operator on b-half-densities. The action of the identity is given in local coordinates by

$$\mathrm{Id}(\phi \left|\frac{dx}{x}dy\right|^{\frac12}) = \int \delta(x-x')\delta(y-y')\phi(x',y')dx'dy' \left|\frac{dx}{x}dy\right|^{\frac12}.$$

Thus the kernel in the sense of (4.51) is

$$(4.57)\qquad K_{\mathrm{Id}} = x'\delta(x-x')\delta(y-y')\left|\frac{dx'}{x'}dy'\frac{dx}{x}dy\right|^{\frac12}.$$

By a smooth Dirac section of order (at most) k of ${}^b\Omega^{\frac{1}{2}}$, with respect to Δ_b, is meant $\kappa \in \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac{1}{2}})$ which has support contained in Δ_b and in local coordinates takes the form

$$\sum_{p+|\alpha|\in k} a_{p,\alpha}(x,y) D_s^p \delta(s-1) D_y^\alpha \delta(y-y')$$

where the coefficients are $\mathcal{C}^\infty$. One expression of the utility of X_b^2 is

LEMMA 4.21. *Under the isomorphism* (4.56) *the space* $\mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ *is mapped isomorphically onto the space of all smooth Dirac sections of order* k, *with respect to* Δ_b.

PROOF: Lifting to X_b^2, where only a neighbourhood of Δ_b needs to be considered, the coordinates x' and $x/x' = s$ can be used. Thus (4.57) becomes

$$\delta(s-1)\delta(y-y') \left| \frac{ds}{s} dy \frac{dx'}{x'} dy' \right|^{\frac{1}{2}}.$$

Writing $P = P \circ \mathrm{Id}$ for a general element of $\mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ and using (4.28) it follows that the lift is a smooth Dirac section and conversely that all such sections arise in this way.

Thus the degeneracy of the kernels on X^2 is removed by the lift to X_b^2. The space of kernels is then much the same as in the boundaryless case and the process of microlocalization can be applied directly.

4.9. The small space of b-pseudodifferential operators.

After all this preparation the (small) space of b-pseudodifferential operators can now be defined.

DEFINITION 4.22. The (small) space, $\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$, of b-pseudodifferential operators of order m, acting on b-half-densities, consists of those continuous linear operators which under (4.56) correspond to conormal sections of order m associated to the lifted diagonal and vanishing to all orders at $\mathrm{lb} \cup \mathrm{rb}$:

$$\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}}) \longleftrightarrow \left\{ \kappa \in I^m(X_b^2, \Delta_b; {}^b\Omega^{\frac{1}{2}}); \kappa \equiv 0 \text{ at } \mathrm{lb} \cup \mathrm{rb} \right\}. \tag{4.58}$$

Here $\equiv$ means equality of Taylor series at the indicated set.

Often (4.58) will be treated as an equality. Of course the precise meaning of the space on the right in (4.58) still needs to be explained. As noted in Lemma 4.1, the lifted diagonal is an embedded $\mathcal{C}^\infty$ submanifold, of dimension $\dim X$, which meets the boundary of X_b^2 only at the front face,

$\mathrm{bf}(X_b^2)$. Thus the vanishing conditions in (4.58) make sense since all elements of $I^m(X_b^2,\Delta_b;{}^b\Omega^{\frac12})$ are $\mathcal{C}^\infty$ in a neighbourhood of $\mathrm{lb}\cup\mathrm{rb}$, this being disjoint from Δ_b.

So consider this definition in a little more detail. The meaning of (4.58) is that the kernel, κ, must have the following properties:

$$\begin{gathered}\kappa_{|X_b^2\setminus\Delta_b}\in\mathcal{C}^\infty\left(X_b^2\setminus\Delta_b;{}^b\Omega^{\frac12}\right)\\ \kappa\equiv 0 \text{ at } \mathrm{lb}\sqcup\mathrm{rb}\end{gathered}\tag{4.59}$$

$$\kappa(z,z')=(2\pi)^{-n-1}\int e^{i(z-z')\zeta}a(z',\zeta)d\zeta|dzdz'|^{\frac12}\text{ near }\Delta_b\setminus\mathrm{bf}\tag{4.60}$$

$$\begin{aligned}\kappa(r,\tau,y,y')=(2\pi)^{-n-1}&\int e^{i\tau\lambda+i(y-y')\cdot\eta}b(r,y',\lambda,\eta)d\lambda d\eta\\ &\times\left|\frac{dr}{r}d\tau dydy'\right|^{\frac12}\text{ near }\Delta_b\cap\mathrm{bf}\,.\end{aligned}\tag{4.61}$$

In (4.60) $z_0,\dots,z_n$ and $z_0',\dots,z_n'$ are the same coordinates in the two factors of $X^2\simeq X_b^2$ near $\Delta_b\setminus\mathrm{bf}\simeq\Delta\setminus(\partial X)^2$ and in (4.61) if x,y,x',y' are the local coordinates with the usual conventions then $\tau=(x-x')/(x+x')$, $r=x+x'$. The amplitudes a and b are also required to be symbols of order m, as in (4.42). If the symbols have full asymptotic expansion as in (4.43) then the 'classical' or one-step polyhomogeneous class results:

$$\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\longleftrightarrow\left\{\kappa\in I^m_{\mathrm{os}}(X_b^2,\Delta_b;{}^b\Omega^{\frac12});\kappa\equiv 0\text{ at }\mathrm{lb}\sqcup\mathrm{rb}\right\}.\tag{4.62}$$

For the most part the various properties of these conormal spaces (including coordinate-invariance) will be taken for granted, since that is precisely what is covered in a standard treatment of pseudodifferential operators (on manifolds without boundary). A little enlightenment may be gained from the exercises starting at Exercise 4.24 below. A safer alternative for the uninitiated is to consult [47, Chapter 18] or one of many reasonable introductions to pseudodifferential operators (for example [83], [88]). Another possibility is [65].

4.10. Symbol map.

One particular property of conormal spaces is that they have symbol mappings delineating their order filtrations. Let $S^{[m]}({}^bT^*X)$ denote the space of $\mathcal{C}^\infty$ functions, on ${}^bTX\setminus 0$, which are homogeneous of degree m. Summarizing the invariance discussion in the exercises leads to:

PROPOSITION 4.23. *The local symbols in* (4.60), (4.61) *together fix the symbol map, giving a short exact sequence*

$$0 \longrightarrow \Psi_{b,\mathrm{os}}^{m-1}(X;{}^b\Omega^{\frac{1}{2}}) \hookrightarrow \Psi_{b,\mathrm{os}}^{m}(X;{}^b\Omega^{\frac{1}{2}}) \xrightarrow{{}^b\sigma_m} S^{[m]}({}^bT^*X) \longrightarrow 0. \tag{4.63}$$

EXERCISE 4.24. Check that if κ is defined by (4.60) or (4.61), where a or b satisfies the estimates of (4.42), then it satisfies (4.59). [Hint: These integrals are inverse Fourier transforms. Show that if $m < -q - N$ for the order in (4.42) then κ is $\mathcal{C}^N$. Then use the standard identities for the Fourier transform of $\tau\kappa$ and $(y-y')_j\kappa$ to show that if $|\tau|^2 + |y-y'|^2 \neq 0$ then κ is in $\mathcal{C}^N$ for each N.]

EXERCISE 4.25. Use Exercise 4.24 to show that if $U \subset X_b^2$ is a preassigned neighbourhood of Δ_b and κ satisfies (4.59) – (4.61) for some covering of Δ_b by coordinate patches then $\kappa = \kappa_1 + \kappa_2$, where $\kappa_1 \in \mathcal{C}^\infty(X_b^2;{}^b\Omega^{\frac{1}{2}})$ satisfies (4.59) and κ_2 has support in U and still satisfies (4.59) – (4.61) with respect to the same coordinate covering.

EXERCISE 4.26. Use Exercise 4.25 to show that κ satisfies (4.59) – (4.61) for a particular covering by coordinate patches if and only if $\kappa = \kappa' + \kappa''$, where κ'' is $\mathcal{C}^\infty$ and κ' is a sum of terms, each supported in one coordinate patch and satisfying (4.60) or (4.61) in those coordinates.

EXERCISE 4.27. For distributions supported in the coordinate patch check directly that (4.48) and (4.61) are invariant under changes of coordinates on X_b^2 which are linear in $z-z'$ or $y-y'$ and τ but arbitrary in z' or r and y'. Show that under such transformations the symbol a or b projects to a well-defined element of $S^{[m]}({}^bT^*X)$, i.e. the leading part is unchanged.

EXERCISE 4.28. Finally (this is the trickiest part) show that under coordinate transformations which induce a trivial transformation on $N\Delta_b$ the form of (4.60) or (4.61) is preserved and the symbol is fixed in $S^{[m]}({}^bT^*X)$. Now check that you understand where (4.63) comes from.

The utility of the algebra of pseudodifferential operators on a compact manifold without boundary is closely linked to the symbol homomorphism. This is a map into a commutative algebra (really a *family* of maps into a *family* of algebras) which determines the leading part of an operator. Using this symbol map one can readily construct parametrices for elliptic operators. For the (small) calculus of b-pseudodifferential operators there are *two* such "normal homomorphisms." The first, in (4.63), is a direct extension of the standard symbol map and has similar properties. The second, indicial, homomorphism discussed below, takes values in a non-commutative algebra and is therefore somewhat different. Both homomorphisms are involved in the construction of parametrices.

4.11. Elementary mapping properties.

By (4.53) b-pseudodifferential operators do indeed correspond to linear operators

$$\dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}), \tag{4.64}$$

but they have much better properties than this. To exploit the properties of the kernel it is useful to have a representation of its action, as in (4.54), which uses the lift of the kernel to X^2_b. The operator can be written out in terms of its kernel on X^2

$$A\phi(z) = \int\limits_X K(z,z')\phi(z'). \tag{4.65}$$

Suppose that $\phi \in \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12})$ has support in a small, product, neighbourhood of the boundary where x is a defining function for the boundary. Then (4.65) can be written

$$A\phi(x,y) = \int\limits_0^\epsilon \int\limits_{\partial X} K(x,y,x',y')\phi(x',y')\frac{dx'}{x'}dy'|\frac{dx}{x}dy|^{\frac12},\ x<\epsilon, \tag{4.66}$$

where the density factors have also be written explicitly.

In (4.66) only the properties of the kernel near $x = x' = 0$ are involved. For simplicity assume that

$$\operatorname{supp}(K) \subset [0,\epsilon]\times\partial X\times[0,\epsilon]\times\partial X.$$

Then the coordinate τ from (4.16) can be used to lift from X^2 to X^2_b :

$$A\phi(x,y) = \int\limits_{-1}^1 \int\limits_{\partial X} K(x,y,x\frac{1-\tau}{1+\tau},y')\phi(x\frac{1-\tau}{1+\tau},y')\frac{d\tau}{1-\tau^2}dy'|\frac{dx}{x}dy|^{\frac12}.$$

Since the lift of the kernel to X^2_b is by definition

$$\kappa(r,\tau,y,y')|\frac{d\tau}{1-\tau^2}drdydy'|^{\frac12} = K(x,y,x\frac{1-\tau}{1+\tau},y')|\frac{dx}{x}dy\frac{dx'}{x'}dy'|^{\frac12},$$

this can be written as

$$A\phi(x,y) = \int\limits_{-1}^1 \int\limits_{\partial X} \kappa(\frac{2x}{1+\tau},\tau,y,y')\phi(x\frac{1-\tau}{1+\tau},y')\frac{d\tau}{1-\tau^2}dy'|\frac{dx}{x}dy|^{\frac12}. \tag{4.67}$$

In the form (4.67) it can be seen directly how the properties of the kernel demanded in (4.58) will be reflected in the properties of the operators. However it is usually clearer to look at the kernel as a function of the improper variables s, x or t, x', where s is given by (4.10):

$$s = \frac{1+\tau}{1-\tau} = \frac{x}{x'} \text{ and } t = \frac{1}{s}.$$

The coordinates s, x fail near rb and lb $\cap$ bf since $s = \infty$ at rb and at lb $\cap$ bf $x = s\rho_{\mathrm{lb}}$, where ρ_{lb} is a defining function for $\mathrm{lb}(X_b^2)$, has vanishing differential. On the other hand t, x' fail as coordinates at lb and rb $\cap$ bf . (Of course in all cases local coordinates y, y' in the two factors of ∂X are also needed.) However, by definition in (4.58) the kernel κ is $\mathcal{C}^\infty$ and vanishes rapidly at $s = 0$ and $s = \infty$, which are just lb and rb . Thus s can be used in place of τ in (4.67) without having to worry about convergence problems. This leads to the two useful representations:

$$A\phi(x,y) = \int_0^\infty \int_{\partial X} \kappa'(x,s,y,y')\phi(x/s,y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac{1}{2}}, \tag{4.68}$$

$$A\phi(x,y) = \int_0^\infty \int_{\partial X} \kappa''(x/s,s,y,y')\phi(x/s,y')\frac{ds}{s}dy'|\frac{dx}{x}dy|^{\frac{1}{2}}, \tag{4.69}$$

where κ' stands for the kernel expressed in terms of x, s, y, y', κ'' for the kernel expressed in terms of x', s, y, y' and $\phi \in \dot{\mathcal{C}}_c^\infty([0,\epsilon) \times \partial X; {}^b\Omega^{\frac{1}{2}})$. These representations will be used immediately to improve (4.64):

PROPOSITION 4.29. *Each $A \in \Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ defines a linear operator*

$$A : \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}). \tag{4.70}$$

PROOF: Divide the kernel of the operator into three pieces, using a $\mathcal{C}^\infty$ partition of unity on X^2. Namely

$$A = A_1 + A_2 + A_3, \tag{4.71}$$

where A_3 has kernel supported in $x \geq \epsilon$, A_2 has kernel supported in $x \leq 2\epsilon$, $x' > x \geq \epsilon$, and A_1 has kernel supported in $x, x' \leq 2\epsilon$.

Of the pieces, A_2 is the most innocuous, since the support of its kernel does not touch the diagonal in X^2 and hence it is $\mathcal{C}^\infty$ on X^2 and vanishes to infinite order at both boundaries. It is convenient to have a name for such operators:

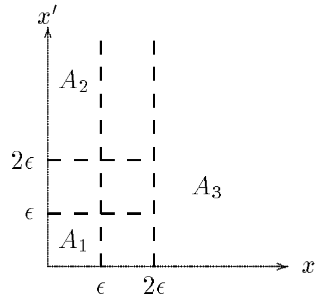

Figure 6. Decomposition in (4.69).

DEFINITION 4.30. The *maximally residual operators,* in $\Psi^{-\infty,\emptyset}(X;{}^b\Omega^{\frac{1}{2}})$, are those operators with kernels in $\dot{\mathcal{C}}^\infty(X^2;{}^b\Omega^{\frac{1}{2}}) = \dot{\mathcal{C}}^\infty(X^2_b;{}^b\Omega^{\frac{1}{2}})$.

The notation here is a precursor to that for the full calculus in Chapter 5. The $-\infty$ refers to the absence of any singularity at the diagonal, while the $\emptyset$ means that the kernel vanishes to infinite order at all boundaries. The absence of the, otherwise ubiquitous, subscript 'b' is also no accident; it refers to the fact that these kernels are characterized directly on X^2.

Essentially from the definition of the action of an operator in terms of its Schwartz kernel and the spaces involved

$$A \in \Psi^{-\infty,\emptyset}(X;{}^b\Omega^{\frac{1}{2}}) \Longleftrightarrow A\colon \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac{1}{2}}). \tag{4.72}$$

EXERCISE 4.31. Make sure you understand why the mapping property (4.72) is a necessary and sufficient condition for an operator to be maximally residual.

Certainly then A_2 satisfies (4.70). Slightly more seriously consider A_3. Removing another maximally residual term the kernel of A_3 can be supposed to have support contained strictly in the interior. From the definition A_3 is then simply a pseudodifferential operator, in the usual sense, on the interior $\mathring{X}$ of X with kernel having compact support in $(\mathring{X})^2$. One of the standard mapping properties of pseudodifferential operators is that they preserve $\mathcal{C}^\infty$ and taking into account the support property this means

$$A_3\colon \mathcal{C}^\infty(X;{}^b\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac{1}{2}}). \tag{4.73}$$

Recall where this mapping property comes from. It is really a result on the regularity of push-forwards of conormal distributions. More prosaically it follows from (and is essentially equivalent to) the fact that if K is a kernel as in (4.60) (with compact support) then

$$\int K(z,z')dz' \in \mathcal{C}^\infty. \tag{4.74}$$

EXERCISE 4.32. Show that (4.74) follows from (4.60), the properties of symbols discussed in the exercises starting at Exercise 4.24 and integration by parts.

So finally consider A_1. Here the representation (4.68) can be used. The kernel κ vanishes to all orders at $s=0$ and rapidly as $s \longrightarrow \infty$, so it follows that the product

$$\begin{gathered}\kappa(x,s,y,y')\phi(x/s,y') \text{ is } \mathcal{C}^\infty \text{ in } |s-1| \neq 0\\ \text{and vanishes rapidly as } s \longrightarrow \infty \text{ and as } s \downarrow 0.\end{gathered} \tag{4.75}$$

Indeed, when $s \neq 1$, κ itself is $\mathcal{C}^\infty$ and if $s>0$ then so is $\phi(x/s,y')$. Since, by assumption, κ vanishes rapidly as $s \downarrow 0$ so does the product; the derivatives of $\phi(x/s,y')$ being of at most polynomial growth as $s \downarrow 0$. As $s \longrightarrow \infty$ both factors vanish rapidly (since x is bounded). This rapid decrease at infinity compensates for the non-compactness of the domain of integration, so the integral of (4.75) in s and y' behaves just as for (4.74). Thus $A_1\phi$ is $\mathcal{C}^\infty$. Since $\phi(x',y')$ is assumed to vanish with all derivatives at $x'=0$, i.e. $\phi(x',y')=(x')^N\phi_N(x',y')$ the same argument can be applied to $\kappa(x,s,y,y')s^{-N}\phi_N(x/s,y')$ and hence it follows that $A_1\phi \in x^N\mathcal{C}^\infty(X;{}^b\Omega^{\frac12})$ for any N. This proves (4.70) and therefore Proposition 29

EXERCISE 4.33. Using a (simple) duality argument show that

$$A \in \Psi_b^m(X;{}^b\Omega^{\frac12}) \Longrightarrow A\colon \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12}) \tag{4.76}$$

is a continuous linear operator.

Essentially the same argument as used to prove Proposition 4.29 also leads to

PROPOSITION 4.34. *Each element $A \in \Psi_b^*(X;{}^b\Omega^{\frac12})$ defines a continuous linear map*

$$A\colon \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{C}^\infty(X;{}^b\Omega^{\frac12}). \tag{4.77}$$

PROOF: Using the same decomposition (4.71) both A_2 and A_3 have the property (4.77) as consequences of (4.72) and (4.73). So again it is enough to consider A_1. Notice that $\dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac12})$ is not dense in $\mathcal{C}^\infty(X;{}^b\Omega^{\frac12})$ in the topology of the latter, but it is dense in the topology of $\mathcal{C}^{-\infty}(X;{}^b\Omega^{\frac12})$. Thus the definition of the map in (4.77) follows from (4.76). Explicitly this convergence follows by taking

$$\phi_n(x,y) = \mu(nx)\phi(x,y),\ \mu \in \mathcal{C}^\infty([0,\infty)), \begin{cases} \mu(t) & = 1,\ t > 1 \\ \mu(t) & = 0,\ t < \frac12. \end{cases}$$

Inserting ϕ_n into (4.68) the integral remains, for $x > 0$, absolutely convergent as $n \longrightarrow \infty$. Thus the remainder of the proof of Proposition 4.29 can be followed since the rapid vanishing of ϕ is only used at the very last step. Thus (4.77) holds.

4.12. Asymptotic completeness.

A companion property to the symbol map in (4.63) is the *asymptotic completeness* of the space. Define

$$\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}) = \bigcap_m \Psi_b^m(X;{}^b\Omega^{\frac12}).$$

Then, under (4.62), there is an identification

$$\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}) \longleftrightarrow \left\{\kappa \in \mathcal{C}^\infty(X_b^2;{}^b\Omega^{\frac12}); \kappa \equiv 0 \text{ at } \mathrm{lb} \sqcup \mathrm{rb}\right\}.$$

EXERCISE 4.35. Show that, for any $m \in \mathbb{R}$,

$$\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}) = \bigcap_j \Psi_{b,\mathrm{os}}^{m-j}(X;{}^b\Omega^{\frac12}).$$

Give an example to make sure you understand that this space is *a lot bigger* than $\Psi^{-\infty,\emptyset}(X;{}^b\Omega^{\frac12})$.

Asymptotic completeness is a general property of conormal spaces. For the one-step polyhomogeneous spaces it becomes:

LEMMA 4.36. *If* $A_j \in \Psi_{b,\mathrm{os}}^{m-j}(X;{}^b\Omega^{\frac12})$, $j = 0, 1, \dots$ *then there exists an asymptotic sum* $A \in \Psi_{b,\mathrm{os}}^m(X;{}^b\Omega^{\frac12})$ *such that*

$$A - \sum_{j=0}^{N-1} A_j \in \Psi_{b,\mathrm{os}}^{m-N}(X;{}^b\Omega^{\frac12})\ \forall\ N. \tag{4.78}$$

The relationship (4.78) is written

$$A \sim \sum_{j=0}^{\infty} A_j$$

and determines A uniquely up to an element of $\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$.

4.13. Small parametrix.

Of course there is one other important property that guided the definition of the small calculus from the beginning. Namely

$$\mathrm{Diff}_b^k(X;{}^b\Omega^{\frac{1}{2}}) \subset \Psi_{b,\mathrm{os}}^k(X;{}^b\Omega^{\frac{1}{2}}) \ \forall\ k \in \mathbb{N}_0. \tag{4.79}$$

This will be used with the following special case of a more general composition formula (Proposition 5.20) proved later:

LEMMA 4.37. *For any $k \in \mathbb{N}$, $m \in \mathbb{R}$*

$$\begin{gathered}\mathrm{Diff}_b^k(X;{}^b\Omega^{\frac{1}{2}}) \circ \Psi_{b,\mathrm{os}}^m(X;{}^b\Omega^{\frac{1}{2}}) \subset \Psi_{b,\mathrm{os}}^{m+k}(X;{}^b\Omega^{\frac{1}{2}})\\ {}^b\sigma_{m+k}(P\circ A) = {}^b\sigma_k(P)\cdot {}^b\sigma_m(A),\end{gathered} \tag{4.80}$$

where ${}^b\sigma_k(P)$ is the symbol defined in (2.23) and ${}^b\sigma_m(A)$ is from (4.63). Since all these spaces are invariant under passage to adjoints, the same result holds for the composite in the other order.

PROOF: To prove (4.79) the first thing to check is that

$$\mathrm{Id} \in \Psi_{b,\mathrm{os}}^0(X;{}^b\Omega^{\frac{1}{2}}). \tag{4.81}$$

This however is a direct consequence of Lemma 4.21.

Now (4.79) can be deduced from (4.81) by trivializing the density bundles and recalling (2.20). Thus it is only necessary to show that if $\nu_b^{\frac{1}{2}}$ is a non-vanishing b-half-density

$$\Psi_{b,\mathrm{os}}^m(X;{}^b\Omega^{\frac{1}{2}}) \ni \kappa \longmapsto \nu_b^{\frac{1}{2}} V \nu_b^{-\frac{1}{2}}\cdot\kappa \in \Psi_{b,\mathrm{os}}^{m+1}(X;{}^b\Omega^{\frac{1}{2}}) \ \forall\ V \in \mathcal{V}_b(X). \tag{4.82}$$

Here V acts on the left. The lift of these vector fields to X_b^2 is described by Proposition 4.4. Since the lifted vector field is $\mathcal{C}^\infty$ on X_b^2 and tangent to the boundary, (4.82) holds, as can be seen directly from the local representations of the kernels in (4.61). From this (4.79) and (4.80) follow.

These elementary properties of the small calculus allow the first parametrix construction to be made. Really a parametrix should be an inverse up to compact errors; this is *not* achieved yet, as discussed below. However it is worth stretching the concept a little and thinking of this as a parametrix construction:

PROPOSITION 4.38. *If $P \in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac{1}{2}})$ is elliptic then there exists $G_s \in \Psi_{b,\mathrm{os}}^{-k}(X;{}^b\Omega^{\frac{1}{2}})$ such that*

$$R_s = \mathrm{Id} - P\circ G_s \in \Psi_b^{-\infty}(X;{}^b\Omega^{\frac{1}{2}}), \tag{4.83}$$

and G_s with this property is unique up to an element of $\Psi_b^{-\infty}(X;{}^b\Omega^{\frac{1}{2}})$.

The subscript 's' is intended to indicate that the element is in the small calculus.

PROOF: This is just *the* standard symbolic construction. The ellipticity of P means that there exists

$$q \in S^{-k}({}^bT^*X) \text{ with } q \cdot {}^b\sigma_k(P) - 1 \in S^{-\infty}({}^bT^*X).$$

Indeed q can be taken to be equal to $1/{}^b\sigma_k(P)$ outside any compact neighbourhood of the zero section of ${}^bT^*X$. Then the surjectivity in (4.63) shows that there exists

$$G_s^0 \in \Psi_{b,\mathrm{os}}^{-k}(X; {}^b\Omega^{\frac{1}{2}}) \text{ with } {}^b\sigma_{-k}(G_s^0) = q.$$

From (4.80) it follows that $P \circ G_s^0 \in \Psi_{b,\mathrm{os}}^0(X; {}^b\Omega^{\frac{1}{2}})$ and

$${}^b\sigma_0(P \circ G_s^0) = 1.$$

Certainly ${}^b\sigma_0(\mathrm{Id}) = 1$, so ${}^b\sigma_0(R_s^1) = 0$ if

$$R_s^1 = \mathrm{Id} - P \circ G_s^0. \tag{4.84}$$

From the exactness in (4.63), $R_s^1 \in \Psi_{b,\mathrm{os}}^{-1}(X; {}^b\Omega^{\frac{1}{2}})$. Moreover G_s^0 is determined up to the addition of an element of $\Psi_{b,\mathrm{os}}^{-k-1}(X; {}^b\Omega^{\frac{1}{2}})$. Now proceed by induction. Suppose that

$$G_s^{(j)} \in \Psi_{b,\mathrm{os}}^{-k-j}(X; {}^b\Omega^{\frac{1}{2}}) \text{ is chosen for } j < l \tag{4.85}$$

so that

$$R_s^l = \mathrm{Id} - P \circ \left(\sum_{j=0}^{l-1} G_s^{(j)} \right) \in \Psi_{b,\mathrm{os}}^{-l}(X; {}^b\Omega^{\frac{1}{2}}). \tag{4.86}$$

Then (4.63) can be used again to find $G_s^{(l)} \in \Psi_{b,\mathrm{os}}^{-k-l}(X; {}^b\Omega^{\frac{1}{2}})$ with

$${}^b\sigma_{-k-l}(G_s^{(l)}) = q \cdot {}^b\sigma_{-l}(R_s^l).$$

This means $R_s^l - P \circ G_s^{(l)} \in \Psi_{b,\mathrm{os}}^{-l-1}(X; {}^b\Omega^{\frac{1}{2}})$, so the inductive hypothesis (4.84), (4.85) is recovered.

Finally take

$$G_s \sim \sum_{j=0}^{\infty} G_s^{(j)} \in \Psi_{b,\mathrm{os}}^{-k}(X; {}^b\Omega^{\frac{1}{2}}).$$

Then from (4.78) and (4.86)

$$R_s = \operatorname{Id} - P \circ G_s \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}});$$

this proves (4.83). The essential uniqueness follows from the fact that if $G'_s \in \Psi_b^{-k}(X; {}^b\Omega^{\frac{1}{2}})$ is another solution then

$$P \circ (G_s - G'_s) \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}).$$

Proceeding inductively shows that $G_s - G'_s \in \Psi_b^{-k-j}(X; {}^b\Omega^{\frac{1}{2}})$ for all j, i.e. $G_s - G'_s \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$.

A solution to

$$R'_s = \operatorname{Id} - \tilde{G}_s \circ P \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}) \tag{4.87}$$

can be constructed by a similar argument. Moreover $\tilde{G}_s$, constructed to satisfy (4.87), is essentially equal to G_s. To show this requires a particular case of Proposition 5.20, where m' or $m = -\infty$. Then,

$$\tilde{G}_s - \tilde{G}_s \circ R_s = \tilde{G}_s \circ P \circ G_s = G_s - R'_s \circ G_s.$$

EXERCISE 4.39. Show, directly from the definition, that $\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ is invariant under passage to adjoints. Use this to show that a left parametrix, as in (4.87), can be obtained from a right parametrix for the adjoint of P.

4.14. Non-compactness.

As already noted, the error terms in (4.83) or (4.87) are not really very 'small'. In particular a general element of $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ is *not* compact on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ (so far it has not even been shown to be bounded, but it is). To see why it is not compact, consider, for simplicity, the one-dimensional case. Then if $u = \phi(x)|dx/x|^{\frac{1}{2}}$ has support near $x = 0$ in $X = [0, 1]$,

$$Au(x) = \int_0^\infty \kappa(x, s)\phi(\frac{x}{s})\frac{ds}{s}\left|\frac{dx}{x}\right|^{\frac{1}{2}}. \tag{4.88}$$

Here $s = x/x'$ and κ can be assumed to have support in $[0, \frac{1}{2}] \times [\frac{1}{2}, 2]$, i.e $0 \le x \le \frac{1}{2}$, $\frac{1}{2} \le s \le 2$.

Now in (4.88) consider a sequence which converges weakly in L^2 :

$$u_t = \phi(\frac{x}{t})\left|\frac{dx}{x}\right|^{\frac{1}{2}} \quad \text{as } t \downarrow 0 \ u_t \rightharpoonup 0.$$

From (4.88) it follows that $Au_t = w_t(\frac{x}{t})|dx/x|^{\frac{1}{2}}$ if

$$w_t(x) = \int_0^\infty \kappa(xt, s)\phi(\frac{x}{s})\frac{ds}{s}.$$

Thus, as $t \downarrow 0$, w_t converges in $\mathcal{C}^\infty$ to

$$w_0(x) = \int_0^\infty \kappa(0, s)\phi(\frac{x}{s})\frac{ds}{s}.$$

It is straightforward to check that if $w_t \longrightarrow w_0$ in $\mathcal{C}^\infty([0, \frac{1}{2}])$ then $w_t(\frac{x}{t}) = Au_t \rightharpoonup 0$ (converges weakly to 0) in $L^2(X; {}^b\Omega^{\frac{1}{2}})$ but *cannot* converge strongly to zero unless $w_0 \equiv 0$.

EXERCISE 4.40. Show that if A is compact on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ then $\kappa(0, s) \equiv 0$.

4.15. Indicial operator.

Thus for an element of $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ the *whole* of $\kappa \restriction \mathrm{bf}(X_b^2)$ appears to be (and is) an obstruction to compactness. This is not completely surprising since, under β_b, $\mathrm{bf}(X_b^2)$ gets mapped into $B \subset \Delta$, so the kernel on X^2 is singular at B if $\kappa \restriction \mathrm{bf} \neq 0$.

This suggests that this *obstruction* be looked at directly. To do so consider the map

$$\Psi_b^m(X; {}^b\Omega^{\frac{1}{2}}) \ni A \longmapsto \kappa_{\restriction \mathrm{bf}(X_b^2)} \in I^{m+\frac{1}{4}}(\mathrm{bf}, \Delta_{b,0}; {}^b\Omega^{\frac{1}{2}}). \tag{4.89}$$

Here $\Delta_{b,0} = \Delta_b \cap \mathrm{bf} = \{r = 0, \tau = 0, y = y'\}$, and ${}^b\Omega^{\frac{1}{2}}$ is the restriction to $\mathrm{bf}(X_b^2)$ of ${}^b\Omega^{\frac{1}{2}}(X_b^2)$ (see (4.47)). In fact there is a canonical identification

$$ {}^b\Omega^{\frac{1}{2}}(\mathrm{bf}) \simeq {}^b\Omega^{\frac{1}{2}}(X_b^2)_{\restriction \mathrm{bf}(X_b^2)}. \tag{4.90}$$

EXERCISE 4.41. If r, τ, y, y' are the usual type of coordinates in X_b^2 near bf, so $\mathrm{bf} = \{r = 0\}$, show that

$$\left|\frac{d\tau}{1-\tau^2}\frac{dr}{r}dydy'\right| \longmapsto \left|\frac{d\tau}{1-\tau^2}dydy'\right|$$

gives the canonical isomorphism

$$ {}^b\Omega(X_b^2)_{\restriction \mathrm{bf}(X_b^2)} \simeq {}^b\Omega(\mathrm{bf}(X_b^2))$$

and hence also gives (4.90).

The apparent increase in order in (4.89), from m to $m + \frac{1}{4}$, is a figment of the convention for the order of a conormal distribution, in particular of its relation to the dimension. This convention, introduced by Hörmander in [45], may seem a little strange but it is very natural in terms of pull-back and push-forward operations.

Recall, from (4.13), exactly what the front face is. A more primitive version of (4.4) can be used to define a compactified normal bundle to ∂X itself (rather than the spherical normal bundle to $B = \partial X \times \partial X$ in X^2). If $p \in \partial X$ then

$$
{}_+N_p(\partial X) = \{v \in T_pX; dx(v) \geq 0\}/T_p\partial X \tag{4.91}
$$

is a closed half-line, forming a bundle over ∂X. Since it is reassuring to keep things, compact consider instead the product $X \times [0,\infty) \supset \partial X \times \{0\}$ and the space

$$
\widetilde{X}_p = [(\{v \in T_pX; dx(v) \geq 0\} \times [0,\infty))\backslash\{0\}]/(T_p\partial X \times \mathbb{R}^+), \tag{4.92}
$$

where the factor of $\mathbb{R}^+$ acts multiplicatively on T_pX.

This is very like (4.91), indeed

$$
{}_+N_p\partial X \ni [v] \longrightarrow [(v,1)] \in \widetilde{X}_p \tag{4.93}
$$

embeds ${}_+N_p\partial X$ in $\widetilde{X}_p$. Again it follows that

$$
\widetilde{X} = \bigsqcup_{p \in \partial X} \widetilde{X}_p
$$

is a compact manifold with boundary, diffeomorphic to $\partial X \times [-1,1]$ with a projective structure on its fibres; it is a natural compactification of ${}_+N\partial X$. It is useful to think of $\widetilde{X}$ as a *model* for X near ∂X. It has a natural $\mathbb{R}^+$-action, induced by the multiplicative action on the fibres at ${}_+N\partial X$. In (4.92) the $\mathbb{R}^+$-action is therefore

$$
[(v,1)] \longrightarrow [(sv,1)] \quad s > 0. \tag{4.94}
$$

The two pieces of the boundary will be denoted

$$
\partial\widetilde{X} \simeq \partial X \times \{-1\} \sqcup \partial X \times \{1\} = (\partial_0 X) \sqcup (\partial_1 X).
$$

The stretching construction described above can be carried out, leading to the model stretched product $\widetilde{X}^2_b = (\widetilde{X})^2_b$.

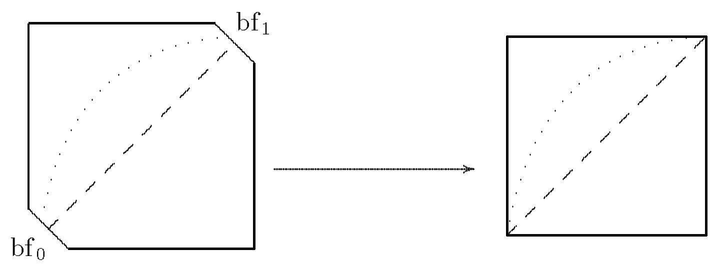

Figure 7. The blow-down map $\beta_b\colon \widetilde{X}^2_b \longrightarrow \widetilde{X}^2$ (with $\mathbb{R}^+$-orbit).

LEMMA 4.42. *The front face of $\widetilde{X}^2_b$ has two pieces,*

$$\mathrm{bf}(\widetilde{X}^2_b) = \mathrm{bf}_0(X^2_b) \sqcup \mathrm{bf}_1(X^2_b), \tag{4.95}$$

which are each canonically isomorphic to $\mathrm{bf}(X^2_b)$.

PROOF: First observe that (4.93) gives a canonical identification of one part of $\mathrm{bf}(\widetilde{X}^2_b)$ with $\mathrm{bf}([{}_+N\partial X]^2_b)$. Thus it is only necessary to show that

$$\mathrm{bf}(X^2_b) \cong \mathrm{bf}([{}_+N\partial X]^2_b).$$

This in turn is clear from (4.13).

As already noted there is a natural $\mathbb{R}^+$-action, M_a, on $\widetilde{X}$, coming from the multiplicative action on the fibres of ${}_+N\partial X$. This is generated by a naturally defined vector field, which will be denoted

$$x\frac{\partial}{\partial x} \in \mathcal{V}_b(\widetilde{X}).$$

The product $\widetilde{X}^2$ has the product action which, being generated by $x\partial/\partial x + x'\partial/\partial x'$, lifts to a $\mathcal{C}^\infty$ $\mathbb{R}^+$-action, M^2_a, on $\widetilde{X}^2_b$. Observe that for an operator the equivalence of the commutation property and $\mathbb{R}^+$-invariance:

$$M^*_a(Au) = A(M^*_a u) \Longleftrightarrow (M^2_a)^*\kappa = \kappa \ \forall\ a \in \mathbb{R}^+. \tag{4.96}$$

Thus $\widetilde{X}^2$ can be pictured as a square, suppressing the extra factor of $(\partial X)^2$. The action of M^2_a has orbits passing from one of the 'diagonal' corners to the other. Then $\widetilde{X}^2_b$ is the square with the two diagonal corners 'cut off,' i.e. blown up. The action of M^2_a is transversal to the front faces produced. Notice that, by assumption, the kernels of elements of the small calculus vanish to infinite order at boundary hypersurfaces other than the two front faces. Let $\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \subset \Psi^m_{b,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12})$ be the subspace of invariant elements in the sense of (4.96).

PROPOSITION 4.43. *Restriction to the first piece of the front face in* (4.95)*, as in* (4.89)*, gives an isomorphism:*

$$\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \longrightarrow \left\{\kappa \in I^{m+\frac14}_{\mathrm{os}}(\mathrm{bf}_0,\Delta^0_b;{}^b\Omega^{\frac12});\kappa\equiv 0 \text{ at } \mathrm{lb}\sqcup\mathrm{rb}\right\}. \tag{4.97}$$

Since the image space in (4.97) is exactly the same as for the restriction map (4.89) (on one-step polyhomogeneous operators) this defines the second 'normal homomorphism' which will be called the indicial homomorphism. Notice that the null space of (4.89) is exactly the space of kernels of the form $\rho_{\mathrm{bf}}\kappa$, where $\kappa\in\Psi^m_b(X;{}^b\Omega^{\frac12})$ and ρ_{bf} is a defining function for $\mathrm{bf}(X^2_b)$. Similarly for one-step polyhomogeneous operators:

PROPOSITION 4.44. *The restriction map* (4.89) *defines, using* (4.97)*, a map*

$$\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \ni A \longmapsto I(A)\in\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \tag{4.98}$$

which gives a short exact sequence

$$\begin{aligned}0\longrightarrow \rho_{\mathrm{bf}}\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \hookrightarrow &\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\\ &\xrightarrow{I}\!\!\!\!\to \Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12})\longrightarrow 0\end{aligned} \tag{4.99}$$

where the first space in (4.99) *is fixed by*

$$\rho^k_{\mathrm{bf}}\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) = \\ \left\{\kappa\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12});\kappa=\rho^k_{\mathrm{bf}}G,\ G\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\right\}.$$

The indicial operator $I(P)$ of $P\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})$ is thus an operator, which is $\mathbb{R}^+$-invariant, on the model space $\widetilde{X}$. As will be shown eventually, the Fredholm properties of P, given that it is elliptic, are captured by $I(P)$. Once again the map I is actually a homomorphism into the (noncommutative) algebra $\Psi^*_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12})$. For the moment the full force of this result is not needed, however it is useful to observe:

LEMMA 4.45. *The passage to indicial operators gives a homomorphism*

$$I\colon\ \mathrm{Diff}^k_b(X;{}^b\Omega^{\frac12})\longrightarrow\!\!\!\!\to \mathrm{Diff}^k_{b,I}(\widetilde{X};{}^b\Omega^{\frac12}) \tag{4.100}$$

such that if $P\in\mathrm{Diff}^k_b(X;{}^b\Omega^{\frac12})$*, and* $A\in\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})$ *then*

$$I(PA)=I(P)\circ I(A). \tag{4.101}$$

Proof: For the identity it is clear that $I(\mathrm{Id}) = \mathrm{Id}$. If x, y, x', y' are coordinates as usual and $s = x/x'$, $r = x + x'$ are projective coordinates near the front face of X_b^2 via the left action lifted to X_b^2

$$x\frac{\partial}{\partial x} \longmapsto s\frac{\partial}{\partial s} + r\frac{\partial}{\partial r},\ \frac{\partial}{\partial y_j} \longrightarrow \frac{\partial}{\partial y_j}.$$

The same is true for operators on $\widetilde{X}$. Clearly then if $A \in \Psi^m_{b,\mathrm{os}}(X; {}^b\Omega^{\frac{1}{2}})$ the indicial operator satisfies

$$I(x\frac{\partial}{\partial x} \circ A) = s\frac{\partial}{\partial s} I(A),\ I(\frac{\partial}{\partial y_j} \circ A) = \frac{\partial}{\partial y_j} I(A).$$

By specializing A to the identity it follows that

$$\begin{aligned} P &= \sum_{j+|\alpha|\le k} p_{j,\alpha}(x,y)(xD_x)^j D_y^\alpha \Longrightarrow \\ I(P) &= \sum_{j+|\alpha|\le k} p_{j,\alpha}(0,y)(sD_s)^j D_y^\alpha, \end{aligned} \tag{4.102}$$

where the coordinates induced by x, y in $\widetilde{X}$ are used. This gives (4.100) and (4.101) follows as well.

Consider for a moment where this map comes from 'algebraically'. By definition $\mathrm{Diff}^*_b(X)$ is generated by $\mathcal{C}^\infty(X)$ and $\mathcal{V}_b(X)$ acting on $\mathcal{C}^\infty$ functions on X. For the 'coefficients' there is a natural map defined by restriction and lifting:

$$\mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty_I(\widetilde{X}),\ \phi \longmapsto \phi_{\restriction \partial X} \longmapsto \pi^*(\phi_{\restriction \partial X}) \in \mathcal{C}^\infty(\widetilde{X}), \tag{4.103}$$

where the subscript 'I' refers to invariance under the $\mathbb{R}^+$-action on $\widetilde{X}$.

There is a similar map for $\mathcal{V}_b(X)$. Since all elements of $\mathcal{V}_b(X)$ are tangent to the boundary the subspace

$$x\mathcal{V}_b(X) = \{V \in \mathcal{V}_b(X); V = xW,\ W \in \mathcal{V}_b(X)\}$$

is an ideal:

$$[\mathcal{V}_b(X), x\mathcal{V}_b(X)] \subset x\mathcal{V}_b(X),\ [V, xW] = xfW + x[V,W],\ f = \frac{Vx}{x}.$$

It follows that the quotient is also a Lie algebra

$$\mathcal{C}^\infty(\partial X; {}^bTX) = \mathcal{V}_b(X)/x\mathcal{V}_b(X).$$

In local coordinates this process is just freezing the coefficients at the boundary. The resulting map is a Lie algebra homomorphism

$$\mathcal{V}_b(X) \xrightarrow{I} \mathcal{C}^\infty(\partial X; {}^bTX). \tag{4.104}$$

The range space here can be reinterpreted:

Lemma 4.46. *There is a natural isomorphism:*

$$\mathcal{C}^\infty(\partial X; {}^bTX) \cong \mathcal{V}_{b,I}(\widetilde{X}), \tag{4.105}$$

where $\widetilde{X}$ is the compactified normal bundle to ∂X and $\mathcal{V}_{b,I}(\widetilde{X}) \subset \mathcal{V}_b(\widetilde{X})$ is the subspace consisting of the $\mathbb{R}^+$-invariant elements for the action (4.94).

Proof: Use of a partition of unity on ∂X reduces the question to a local one, with coordinates x, y. Then both sides of (4.105) are locally spanned over $\mathcal{C}^\infty(\partial X)$ by the basis vector fields $x\partial/\partial x$ and $\partial/\partial y_j$. Moreover this isomorphism is clearly independent of the choice of coordinates.

One can give a more invariant proof by noting that both sides of (4.105) have subalgebras which are commutative and isomorphic to $\mathcal{C}^\infty(\partial X)$. On the left it is just the span of $x\partial/\partial x$, which is a well-defined section of ${}^bT_{\partial X}X$, and on the right it is the subspace tangent to the fibres of $\widetilde{X}$ as a $[-1,1]$-bundle over ∂X. Both quotients are just $\mathcal{C}^\infty(\partial X; T\partial X)$ and the actions giving decompositions as semidirect products are also isomorphic.

Thus (4.104) can be written

$$I : \mathcal{V}_b(X) \longrightarrow \mathcal{V}_{b,I}(\widetilde{X}) \tag{4.106}$$

giving a Lie algebra homomorphism consistent with (4.103). The extension of (4.106) to the enveloping algebras gives the indicial operator in the differential case:

$$I : \mathrm{Diff}_b^k(X) \longrightarrow \mathrm{Diff}_{b,I}^k(\widetilde{X}) \ \forall\ k. \tag{4.107}$$

Clearly (4.102) just gives the local coordinate form of this map.

Exercise 4.47. Make sure that you can visualize these operations on X_b^2 and that you can see why (4.107) is the same map as (4.100) except for the density factor.

4.16. General coefficients.

The calculus of b-pseudodifferential operators has been developed purely for operators acting on b-half-densities. This restriction is completely arbitrary and was made up to this point simply to limit the notational overhead. The extension to the case of operators on sections of general vector bundles involves nothing essentially new and is described here as much to give a summary of the results proved so far for the small calculus (and a few not proved until later) as from real necessity.

Let E and F be vector bundles over a compact manifold with boundary. Rather than repeat the definition by localization, one can take as the space of kernels the space of finite linear combinations of products of kernels in the b-half-density case and smooth homomorphisms. If $\mathrm{Hom}(E,F)$ is the bundle over X^2 with fibres $\hom(E_z,F_{z'})$ at the point $(z,z')\in X^2$ then

$$\begin{gathered}\Psi^m_{b,\mathrm{os}}(X;E,F)=\\ \Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})\otimes_{\mathcal{C}^\infty(X^2_b)}\mathcal{C}^\infty(X^2_b;\beta_b^*\,\mathrm{Hom}(F\otimes{}^b\Omega^{-\frac12},E\otimes{}^b\Omega^{-\frac12})).\end{gathered}\tag{4.108}$$

For the case $E=F={}^b\Omega^{\frac12}$ the second factor here is just $\mathcal{C}^\infty(X^2_b)$, so the definition reduces to the original one in that case. By localization to neighbourhoods where the bundles are trivial it is easy to extend the results above. Thus:

$$\begin{aligned}A\in\Psi^m_{b,\mathrm{os}}(X;E,F)\Longrightarrow&\\ &A\colon\dot{\mathcal{C}}^\infty(X;E)\longrightarrow\dot{\mathcal{C}}^\infty(X;F)\\ &A\colon\mathcal{C}^\infty(X;E)\longrightarrow\mathcal{C}^\infty(X;F)\\ &A\colon\mathcal{C}^{-\infty}(X;E)\longrightarrow\mathcal{C}^{-\infty}(X;F).\end{aligned}$$

Composition of such operators makes sense, and the results in §5.9 extend to give

$$\Psi^m_{b,\mathrm{os}}(X;G,F)\circ\Psi^{m'}_{b,\mathrm{os}}(X;E,G)\subset\Psi^{m+m'}_{b,\mathrm{os}}(X;E,F).$$

The symbol map gives a short exact sequence

$$\begin{aligned}0\longrightarrow\Psi^{m-1}_{b,\mathrm{os}}(X;E,F)\hookrightarrow\Psi^m_{b,\mathrm{os}}(X;E,F)\xrightarrow{{}^b\sigma_m}&\\ S^{[m]}({}^bT^*X;\pi^*\hom(E,F))\longrightarrow 0,&\end{aligned}\tag{4.109}$$

where $\hom(E,F)$ is the bundle over X with fibres $\hom(E_z,F_z)$ at $z\in X$ and $\pi\colon{}^bT^*X\longrightarrow X$ is the projection. Furthermore the symbol map is multiplicative

$$\begin{gathered}{}^b\sigma_{m+m'}(A\circ B)={}^b\sigma_m(A)\cdot{}^b\sigma_{m'}(B),\\ A\in\Psi^m_{b,\mathrm{os}}(X;G,F),\ B\in\Psi^{m'}_{b,\mathrm{os}}(X;E;G).\end{gathered}\tag{4.110}$$

It will be shown below that the indicial homomorphism extends to

$$I\colon\Psi^m_{b,\mathrm{os}}(X;E,F)\longrightarrow\Psi^m_{b,I,\mathrm{os}}(\widetilde{X};E,F),$$

where in the image E and F are really the restrictions of E and F to the boundary of X pulled back to the compactified normal bundle $\widetilde{X}$ by the natural projection from $\widetilde{X}$ to ∂X. The corresponding short exact sequence is

$$0 \longrightarrow \rho\Psi^m_{b,\mathrm{os}}(X;E,F) \hookrightarrow \Psi^m_{b,\mathrm{os}}(X;E,F) \\ \longrightarrow \Psi^m_{b,I,\mathrm{os}}(\widetilde{X};E,F) \longrightarrow 0.$$

This is also multiplicative

$$I(A \circ B) = I(A) \circ I(B), \tag{4.111}$$

as in (4.110). There is a connection between the symbol and indicial homomorphisms:

$$^b\sigma_m(A)_{\restriction {}^bT^*_{\partial X}X} = {}^b\sigma_m(I(A))_{\restriction {}^bT^*_{\partial_0 X}\widetilde{X}}$$

where $\partial_0 X$ is one of the boundary faces of $\widetilde{X}$; this uses the the natural identification of the b-cotangent bundles of X and $\widetilde{X}$ over their boundaries.

EXERCISE 4.48. Show that the definition of the indicial homomorphism for differential operators, in (4.102), does indeed extend to a well-defined map

$$I\colon \operatorname{Diff}^k_b(X;E,F) \longrightarrow \operatorname{Diff}^k_{b,I}(\widetilde{X};E,F)$$

for any vector bundles E, F. Check that it has the composition property (4.111). If E and F are Hermitian and X is equipped with a non-vanishing b-density, ν, then the adjoint of an operator is defined by

$$\int_X \langle P^* f, e\rangle \nu = \int_X \langle f, Pe\rangle \ \forall\ e \in \dot{\mathcal{C}}^\infty(X;E),\ f \in \dot{\mathcal{C}}^\infty(X;F).$$

Show that the boundary value of ν fixes a non-vanishing, $\mathbb{R}^+$-invariant, b-density on $\widetilde{X}$ with respect to which the adjoint of a differential operator satisfies

$$I(P^*) = [I(P)]^*. \tag{4.112}$$

4.17. Examples.

Having defined the indicial operator in general, at least for b-differential operators, it is opportune to compute it for the examples considered so far. Choosing a trivialization of $N\partial X$, i.e. the differential, dx, at ∂X of a boundary defining function, reduces $\widetilde{X}$ to $[-1,1] \times \partial X$. Reverting to the non-compact variable x on $N\partial X$, a general element $Q \in \operatorname{Diff}^m_{b,I}(\widetilde{X};E,F)$ is of the form

$$Q = \sum_{k \le m} Q_k (x\frac{\partial}{\partial x})^k,\ Q_k \in \operatorname{Diff}^{m-k}(\partial X; E_{\restriction \partial X}, F_{\restriction \partial X}).$$

Consider first the exterior differential in (2.21). The choice of trivialization of the normal bundle fixes the section dx/x of ${}^bT^*_{\partial X}X$ which is complementary to $T^*\partial X \subset {}^bT^*_{\partial X}X$ and so gives a decomposition

$$ {}^bT^*_{\partial X}X = \operatorname{span}(\frac{dx}{x}) \oplus T^*\partial X. $$

Taking exterior powers

$$ {}^b\Lambda^k_{\partial X}X \equiv \Lambda^k\partial X \oplus \Lambda^{k-1}\partial X,\ u = u' + \frac{dx}{x} \wedge u''. \tag{4.113} $$

Using this the indicial operator of d becomes a 2×2 matrix (except for $k = 0$ and $k = \dim X$)

$$ I(d) = \begin{pmatrix} d & 0 \\ x\frac{\partial}{\partial x} & -d \end{pmatrix}, $$

where on the right d is the exterior differential on the boundary.

Applying Exercise 4.48 to the adjoint, δ, of d with respect to an exact b-metric it follows easily that

$$ I(\delta) = \begin{pmatrix} \delta & -x\frac{\partial}{\partial x} \\ 0 & -\delta \end{pmatrix}, $$

provided the decomposition (4.113) is taken with respect to the form dx/x defined by the metric. Here δ on the right is the adjoint of δ with respect to the metric induced on the boundary.

For the Laplacian of an exact b-metric in b-forms it follows that

$$ I({}^b\Delta) = \begin{pmatrix} \Delta_0 + (xD_x)^2 & 0 \\ 0 & \Delta_0 + (xD_x)^2 \end{pmatrix}, $$

where Δ_0 is the Laplacian on the boundary.

Next consider the Levi-Civita connection of an exact b-metric, acting on any associated bundle. Here the trivialization should be chosen so that the metric takes the form (2.11) with respect to it. Then

$$ \begin{aligned} I(\nabla) = (x\frac{\partial}{\partial x}, \nabla_0) \colon \mathcal{C}^\infty(\widetilde{X}; E) &\longrightarrow \mathcal{C}^\infty(\widetilde{X}; {}^bT^*_{\partial X}X \otimes E) \\ &= \mathcal{C}^\infty(\widetilde{X}; E) \oplus \mathcal{C}^\infty(\widetilde{X}; T^*\partial X \otimes E), \end{aligned} \tag{4.114} $$

where ∇_0 is the induced connection on the boundary.

EXERCISE 4.49. Prove (4.114).

From (3.71) the indicial operator of the twisted Dirac operator on an even-dimensional exact b-spin manifold can be found. It is an operator

$$I(\eth_E^\pm)\colon \mathcal{C}^\infty(\widetilde{X}; {}^\pm S) \longrightarrow \mathcal{C}^\infty(\widetilde{X}; {}^\mp S).$$

Recalling that over the boundary the spinor bundles are isomorphic to the spinor bundle of the boundary,

$$M_\pm \colon {}^\pm S \longleftrightarrow S_0 \text{ over } \partial X.$$

In terms of this

$$I(\eth_E^\pm) = M_\mp^{-1}\left(\pm x\frac{\partial}{\partial x} + \eth_0\right) M_\pm \tag{4.115}$$

which is just as in (In.23).

EXERCISE 4.50. Prove (4.115).

This formula can be extended to the generalized Dirac operators considered in Lemma 3.32 and the Corollary to Lemma 3.34. Consider the homomorphism (2.97) restricted to the boundary:

$$\nabla_{x\partial/\partial x} \in \hom(E_{\restriction \partial X}). \tag{4.116}$$

By assumption this preserves the grading. Moreover since the connection is assumed to be Clifford, it commutes with Clifford multiplication by $\frac{dx}{x}$ and so induces an isomorphism on the induced Clifford module on the boundary; denote this by τ. Since the extra term in the indicial operator, as opposed to the boundary operator computed in (3.72), arises from $-i\,\mathrm{cl}(\frac{dx}{x})\nabla_{x\partial/\partial x}$ it follows that:

LEMMA 4.51. *For the generalized Dirac operators of Lemma 3.32 the indicial operator is*

$$I({}^b\eth_E) = \begin{pmatrix} 0 & I({}^b\eth_{-,E}) \\ I({}^b\eth_{+,E}) & 0 \end{pmatrix},$$

where in place of (4.115)

$$I({}^b\eth_{\pm,E}) = M_\mp^{-1}\left(\pm x\frac{\partial}{\partial x} + \tau + \eth_0\right) M_\pm.$$

4.18. Trace class operators.

Although it is clear from the discussion in §4.14 that a b-pseudodifferential operator of order $-\infty$ is compact only if its indicial operator vanishes, it is very useful to extend the trace functional to these operators. Recall that for $N \times N$ matrices the trace is the sum of the diagonal entries

$$\operatorname{tr}(A) = \sum_{i=1}^{N} a_{ii}, \quad A = (a_{ij})_{i,j=1}^{N}. \tag{4.117}$$

If A and B are $N \times N$ matrices then the commutator is traceless:

$$\begin{gathered}\operatorname{tr}([A, B]) = 0,\ [A, B] = AB - BA \\ [A, B]_{ij} = \sum_{k=1}^{N} (a_{ik} b_{kj} - b_{ik} a_{kj}),\end{gathered} \tag{4.118}$$

as follows directly from the definition. Thus if T is an invertible matrix then

$$\operatorname{tr}(TAT^{-1}) = \operatorname{tr}(A) \quad \forall\ A, \tag{4.119}$$

since $TAT^{-1} - A = [TA, T^{-1}]$. The trace is therefore well-defined as a linear functional on the space of (continuous) linear operators on any finite dimensional vector space. The Jordan normal form shows that if the eigenvalues of A are summed with their multiplicities

$$m(\lambda) = \lim_{k \to \infty} \dim \operatorname{null}(A - \lambda)^k, \quad \lambda \in \mathbb{C}$$

then

$$\operatorname{tr}(A) = \sum_{\lambda \in \mathbb{C}} m(\lambda)\lambda, \tag{4.120}$$

where the sum is finite for a linear operator on a finite dimensional space. On a Hilbert space, H, the finite rank operators are those with kernels of the form

$$A = \sum_{j=1}^{N} e_j f_j, \quad e_j, f_j \in H. \tag{4.121}$$

Conventionally they act through

$$A\phi = \sum_{j=1}^{N} \langle \phi, f_j \rangle e_j,$$

and hence act on the finite dimensional subspace spanned by the e_j and f_j. The trace, given by (4.120) on this subspace, is then

$$\operatorname{tr}(A) = \sum_{j=1}^{N} \langle e_j, f_j \rangle. \tag{4.122}$$

EXERCISE 4.52. Check that (4.122) holds and is independent of choices.

The trace norm of A, of finite rank, is

$$\|A\|_{\mathrm{Tr}} = \sup \sum_{j=1}^{N} |\langle Af_j, e_j\rangle|, \tag{4.123}$$

with the supremum taken over all pairs of orthonormal bases, $\{e_j\}$ and $\{f_j\}$ of H. Again (4.118) and (4.119) hold for finite rank operators. If H is a Hilbert space the space of trace class operators on H is the closure of the finite rank operators with respect to the norm (4.123).

EXERCISE 4.53. Check that (4.123) does indeed define a norm. Show that, for finite rank operators,

$$|\operatorname{tr}(A)| \le \|A\|_{\mathrm{Tr}}$$

and hence conclude that the functional tr extends by continuity in the trace norm to all trace class operators.

If A has finite rank and B is a bounded operator on H then both AB and BA have finite rank and, from (4.118), $\operatorname{tr}[A, B] = 0$. It can be shown that

$$\|AB\|_{\mathrm{Tr}} \le \|A\|_{\mathrm{Tr}}\|B\|$$

with $\|B\|$ the norm of B as a bounded operator. This and a simple convergence argument lead to the following result:

PROPOSITION 4.54. *The trace class operators on a Hilbert space form an ideal in the bounded operators with* $\operatorname{tr}[A, B] = 0$ *if A is trace class and B bounded and hence* $\operatorname{tr}(TAT^{-1}) = \operatorname{tr}(A)$ *if A is trace class and T is invertible.*

The relevant Hilbert spaces here are Sobolev spaces, in particular spaces of square-integrable functions. For compact manifolds without boundary the following form of Lidskii's theorem gives an alternative way of computing the trace.

PROPOSITION 4.55. *On a compact manifold without boundary, X, the smoothing operators are trace class and*

$$\operatorname{tr}(A) = \int_X K_A(x, x), \quad A \in \Psi^{-\infty}(X; \Omega^{\frac{1}{2}}). \tag{4.124}$$

By definition the kernel, K_A, is a $\mathcal{C}^\infty$ half-density on X^2. Restricted to $\Delta \subset X^2$, the diagonal, the half-density bundle of X^2 is canonically isomorphic to the density bundle on X, so (4.124) makes sense. Thus

$$
\begin{aligned}
\Omega^{\frac{1}{2}}(X^2)_{\restriction\Delta} &\cong \Omega^{\frac{1}{2}}(\Delta) \otimes \Omega^{\frac{1}{2}}_{\text{fibre}}(N\Delta) \\
&\cong \Omega^{\frac{1}{2}}(X) \otimes \Omega^{\frac{1}{2}}_{\text{fibre}}(TX) \\
&\cong \Omega^{\frac{1}{2}}(X) \otimes \Omega^{\frac{1}{2}}(X) \\
&\cong \Omega(X).
\end{aligned}
\tag{4.125}
$$

Here the isomorphisms for densities discussed in Chapter 2 are used successively.

EXERCISE 4.56. Prove (4.124) following Hörmander [46,47] (or otherwise). Starting again with finite rank operators, on $L^2(X;\Omega^{\frac{1}{2}})$, consider the Hilbert-Schmidt norm:

$$
||A||_{\text{HS}} = \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}} = \left(\sum_j \int_X |e_j|^2 \int_X |f_j|^2\right)^{\frac{1}{2}}
\tag{4.126}
$$

in terms of (4.117), (4.121) respectively where, in the latter case it can also be assumed that $e_1, \dots e_N$ and $f_1, \dots f_N$ are (separately) orthogonal families. Relate this to (4.123) by showing that for finite rank operators

$$
||A||_{\text{Tr}} \le ||B||_{\text{HS}} ||C||_{\text{HS}}, \quad A = BC.
$$

Show that defining the Hilbert-Schmidt operators as the closure of the finite rank operators with respect to (4.126) gives an ideal which, for $H = L^2(X;\Omega^{\frac{1}{2}})$, is exactly the space of operators with square integrable *kernels* and

$$
||A||^2_{\text{HS}} = \int_{X^2} |K(x,y)|^2.
$$

Show that if $A = BC$, with B and C Hilbert-Schmidt, then A is trace class, and

$$
\operatorname{tr}(A) = \int_{X^2} K_B(x,y) K_C(y,x).
\tag{4.127}
$$

Check that any $B \in \Psi^{-\frac{n}{2}-\epsilon}(X;\Omega^{\frac{1}{2}})$, $\epsilon > 0$, $n = \dim X$, is Hilbert-Schmidt. From the composition formula for pseudodifferential operators (or otherwise) show that *any* $A \in \Psi^{-n-\epsilon}(X;\Omega^{\frac{1}{2}})$ is trace class and deduce (4.124) from (4.127).

This is a lot of work, but it is important to get to the case of most interest here, that of a compact manifold with boundary. As already observed the direct generalization of Proposition 4.55 to the b-calculus does *not* hold; namely the operators of order $-\infty$ are *not* in general trace class; they are not even compact in general.

Proposition 4.57. *If X is a compact manifold with boundary then an operator $A \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ is trace class on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ if and only if*

$$I(A) = 0 \tag{4.128}$$

and then

$$\operatorname{Tr}(A) = \int_X (K_A)_{\restriction \Delta_b}. \tag{4.129}$$

Similar remarks apply here as in Lidskii's Theorem, Proposition 4.55, but with an important twist. The kernel, written $K_A \in \mathcal{C}^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$, can be restricted to the lifted diagonal as a $\mathcal{C}^\infty$ section of ${}^b\Omega^{\frac{1}{2}}(X_b^2)$. Recall that there is an isomorphism similar to that in (4.125). First, since Δ_b is transversal to the boundary (by Lemma 4.3),

$${}^b\Omega^{\frac{1}{2}}(X_b^2)_{\restriction \Delta_b} \cong {}^b\Omega^{\frac{1}{2}}(X) \otimes \Omega^{\frac{1}{2}}_{\text{fibre}}(N\Delta_b), \tag{4.130}$$

where the fact that $\Delta_b \cong X$ has been used. In Lemma 4.6 the normal bundle to Δ_b is identified with ${}^bTX$ and clearly

$$\Omega^{\frac{1}{2}}_{\text{fibre}}({}^bTX) \cong {}^b\Omega^{\frac{1}{2}}(X),$$

so from (4.130)

$${}^b\Omega^{\frac{1}{2}}(X_b^2)_{\restriction \Delta_b} \cong {}^b\Omega(X).$$

Thus the restriction of the kernel to Δ_b defines a section

$$(K_A)_{\restriction \Delta_b} \in \mathcal{C}^\infty(X; {}^b\Omega). \tag{4.131}$$

As discussed in §4.19 a general smooth section of ${}^b\Omega$ cannot be integrated over X. However the condition (4.128) means that in (4.131) the section vanishes at the boundary. So, given (4.128), $(K_A)_{\restriction \Delta_b} \in \mathcal{C}^\infty(X; \Omega)$ and (4.129) is meaningful.

Proof: The most important part is the sufficiency, that (4.128) implies that $A \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ is trace class. From the boundedness properties of b-pseudodifferential operators, discussed in the next chapter, it follows that

$$A\colon L^2(X;{}^b\Omega^{\frac12}) \longrightarrow H_b^\infty(X;{}^b\Omega^{\frac12})\ \forall\ A\in\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}). \tag{4.132}$$

The vanishing condition (4.128) means precisely that $A = xB$, where B is an element of $\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$. To prove that A is then trace class, it is enough to use the obvious extension of (4.124) to manifolds with boundary. Namely if p is large enough then

$$K\in\dot{\mathcal{C}}^p(X^2;\Omega^{\frac12}) \Longrightarrow K \text{ is trace class and } \operatorname{tr}(K) = \int\limits_X K_{\restriction\Delta}. \tag{4.133}$$

Of course the problem is that the kernel of A is certainly *not* this smooth, just because of (4.128). However there is a trick to make it so.

Given a positive integer p and a boundary defining function $\rho\in\mathcal{C}^\infty(X)$ for ∂X, set

$$\begin{aligned}\mathcal{C}^\infty(X_{p^{-1}}) = \big\{u\colon X\longrightarrow\mathbb{C};\, u = F(u_1,\dots,u_q,\rho^{\frac1p}),\\ u_1,\dots,u_q\in\mathcal{C}^\infty(X),\, F\in\mathcal{C}^\infty(\mathbb{R}^{q+1})\big\}.\end{aligned} \tag{4.134}$$

It is straightforward to check that this space of functions turns X into a $\mathcal{C}^\infty$ manifold, denoted $X_{p^{-1}}$. As a point-set $X_{p^{-1}}$ and X are identical. Differentially however there are 'more' $\mathcal{C}^\infty$ functions on $X_{p^{-1}}$; it is in fact X with the boundary blown up to pth order. Directly from (4.134) the trivial map

$$\beta_p\colon X_{p^{-1}}\longrightarrow X \text{ is } \mathcal{C}^\infty. \tag{4.135}$$

EXERCISE 4.58. Show that, if $p > 1$, (4.135) is *not* an isomorphism. However, show that X and $X_{p^{-1}}$ *are* always isomorphic (but there is no natural isomorphism).

Although (4.135) is not an isomorphism it does have many properties of an isomorphism:

LEMMA 4.59. *Under β_p each non-vanishing section of ${}^b\Omega(X)$ lifts to a non-vanishing $\mathcal{C}^\infty$ section of ${}^b\Omega(X_{p^{-1}})$, also $\mathcal{V}_b(X)$ lifts into $\mathcal{V}_b(X_{p^{-1}})$ and β_p induces an isomorphism*

$$\begin{aligned}\beta_p^*\colon L^2(X;{}^b\Omega^{\frac12}) &\longleftrightarrow L^2(X_{p^{-1}};{}^b\Omega^{\frac12})\\ \beta_p^*\colon H_b^\infty(X;{}^b\Omega^{\frac12}) &\longleftrightarrow H_b^\infty(X_{p^{-1}};{}^b\Omega^{\frac12}).\end{aligned} \tag{4.136}$$

PROOF: If $\rho \in \mathcal{C}^\infty(X)$ is a defining function for ∂X then $\beta_p^* \rho = x^p$, with $x \in \mathcal{C}^\infty(X_{p^{-1}})$ a defining function for $\partial X_{p^{-1}}$. Thus $\beta_p^*(d\rho/\rho) = p dx/x$, from which the statement about sections of ${}^b\Omega$ follows. If $\rho, y_1, \dots, y_n$ are local coordinates near ∂X then $x, y_1, \dots, y_n$ are local coordinates in the preimage, near $\partial X_{p^{-1}}$. Then

$$(\beta_p)_*(x\partial_x) = p\rho\partial_\rho,\ (\beta_p)_*(\partial_{y_j}) = \partial_{y_j}.$$

This shows that $\mathcal{V}_b(X)$ lifts into (and spans) $\mathcal{V}_b(X_{p^{-1}})$. Then (4.136) is immediate.

From (4.136) and the invariance of the trace class, trace class operators are the same on $L^2(X; {}^b\Omega^{\frac{1}{2}})$ and $L^2(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}})$. From (4.132), if $A \in x\Psi^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ then

$$\begin{aligned} A\colon L^2(X; {}^b\Omega^{\frac{1}{2}}) &\longrightarrow \rho H_b^\infty(X; {}^b\Omega^{\frac{1}{2}}) \Longrightarrow \\ A\colon L^2(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}}) &\longrightarrow x^p H_b^\infty(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}}). \end{aligned}$$

In fact it is easy to see that under β_p

$$\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \Psi_b^{-\infty}(X_{p^{-1}}; {}^b\Omega^{\frac{1}{2}}). \tag{4.137}$$

EXERCISE 4.60. Prove (4.137) by showing that the blow-down map β_p, acting on each factor, extends from the interior of $(X_{p^{-1}})_b^2$ to define a $\mathcal{C}^\infty$ map $\beta_p^2\colon (X_{p^{-1}})_b^2 \longrightarrow X_b^2$.

Taking p sufficiently large, (4.133) applies to $A \in \rho\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ as an operator on $X_{p^{-1}}$ so A is trace class and the sufficiency of (4.128) for (4.129) follows. The necessity in Proposition 4.57 will be discussed below.

There are more elementary ways to prove the sufficiency of (4.128). However this blow-up argument seems to be appropriate technology and also β_p is an interesting example of a map which in the b-category is close to an isomorphism.

EXERCISE 4.61. Show that β_p is a b-fibration in the sense of Exercise 4.7 and so is the map β_p^2 of Exercise 4.60.

4.19. The b-integral.

Having observed that general elements of $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ are *not* trace class it is nevertheless important for present purposes to define their traces! To do this the first step is to define an extension of the integral, a b-integral defined on $\mathcal{C}^\infty(X; {}^b\Omega)$. This is a regularization, in the sense of Hadamard, of the generally divergent integral. Recall that restriction to the boundary is a natural map

$$\mathcal{C}^\infty(X;{}^b\Omega) \ni \phi \longmapsto \phi_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X;\Omega)$$
$$\psi \cdot \Big|\frac{dx}{x}dy\Big| \longmapsto \psi_{\restriction \partial X}|dy|.$$

LEMMA 4.62. *Let $\nu \in \mathcal{C}^\infty(\partial X; {}_+N\partial X)$ be a trivialization of the normal bundle of a compact manifold with boundary and suppose $x \in \mathcal{C}^\infty(X)$ is a defining function for ∂X with $dx \cdot \nu = 1$ at ∂X, then*

$$\tag{4.138} {}^\nu\!\!\int_X \phi = \lim_{\epsilon \downarrow 0}\left[\int_{x>\epsilon} \phi + \log\epsilon \cdot \int_{\partial X} \phi_{\restriction \partial X}\right]$$

exists for all $\phi \in \mathcal{C}^\infty(X;{}^b\Omega)$ and depends only on ϕ and ν. If $\nu' = a\nu$, $0 < a \in \mathcal{C}^\infty(\partial X)$, is another trivialization then

$$\tag{4.139} {}^{\nu'}\!\!\int_X \phi - {}^\nu\!\!\int_X \phi = \int_{\partial X} \log a \cdot (\phi_{\restriction \partial X}).$$

PROOF: Choose $\delta > 0$ small so that

$$\{x \le \delta\} \cong \partial X \times [0,\delta]_x.$$

Then, in $x \le \delta$, $\phi = \psi \cdot \frac{dx}{x}\mu$, $\mu \in \mathcal{C}^\infty(\partial X;\Omega)$, $\psi \in \mathcal{C}^\infty(X)$, so

$$\begin{aligned}\int_{x>\epsilon}\phi &= \int_{x\ge\delta}\phi + \int_{\partial X}\int_{x=\epsilon}^{x=\delta}\psi(x,\cdot)\frac{dx}{x}\mu\\ &= \int_{x\ge\delta}\phi + \int_{\partial X}\left(\int_\epsilon^\delta\left[\frac{\psi(x,\cdot)-\psi(0,\cdot)}{x}\right]dx\right)\mu + (\log\frac{\delta}{\epsilon})\int_{\partial X}\phi_{\restriction \partial X}\end{aligned}$$

since $\psi(0,\cdot)\mu = \phi_{\restriction \partial X}$. Clearly the terms on the right, except the last, converge as $\epsilon \downarrow 0$, so the limit in (4.138) does exist.

If $x' = a(x,y)x$ is another defining function then

$$\int_{x'>\epsilon}\phi - \int_{x>\epsilon}\phi = \int_{\partial X}\int_{x=\epsilon}^{xa(x,y)=\epsilon}\psi\frac{dx}{x}\mu \longrightarrow \int_{\partial X}\phi_{\restriction \partial X}\log a(0,y) \text{ as } \epsilon \downarrow 0.$$

This proves (4.139).

EXERCISE 4.63. In §5.1 the indicial family of a b-pseudodifferential operator is introduced. For a b-differential operator it can be defined in terms of the representation (4.102) by replacing xD_x by the variable λ :

$$I_\nu(P,\lambda) = \sum_{j+|\alpha|\le k} p_{j,\alpha}(0,y)\lambda^j D_y^\alpha,$$

where the suffix ν refers to the choice of trivialization of the normal bundle which is involved. Show that with this definition

$$\begin{gathered} {}^{\nu}\!\!\int Pu\cdot\overline{v} - {}^{\nu}\!\!\int u\cdot\overline{P^*v} = \frac{1}{i}\int_{\partial X}\partial_\lambda I_\nu(P,0)u\cdot\overline{v} \\ \forall\ u,v\in\mathcal{C}^\infty(X;{}^b\Omega^{\frac12}),\ P\in\operatorname{Diff}_b^k(X;{}^b\Omega^{\frac12}). \end{gathered} \tag{4.140}$$

4.20. The b-trace functional.

Having defined the regularized integral the b-trace can now be defined:

DEFINITION 4.64. If $\nu\in\mathcal{C}^\infty(\partial X;{}_+N\partial X)$ is a trivialization of the normal bundle then for $A\in\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})$

$$\text{b-Tr}_\nu(A) = {}^{\nu}\!\!\int_X A_{\restriction\Delta_b}.$$

Now, from (4.138), observe that

$${}^{\nu}\!\!\int_X\phi = \int_X\phi \quad \forall\ \phi\in\mathcal{C}^\infty(X;{}^b\Omega),\ \phi_{\restriction\partial X}=0.$$

Thus from Proposition 4.57 it follows that

$$A\in\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}))\text{ trace class }\Longrightarrow \text{b-Tr}_\nu(A)=\operatorname{Tr}(A)\ \forall\ \nu.$$

This means that these b-trace functionals

$$\text{b-Tr}_\nu\colon\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})\longrightarrow\mathbb{C} \tag{4.141}$$

are *extensions* of the usual trace functional in the following sense. In the short exact sequence (4.99) for $m=-\infty$ the trace functional is defined on the first space; the b-trace functional is defined on the middle space and vanishes identically on the space of $\mathbb{R}^+$-invariant operators in which the indicial operator takes values. Thus the diagramme:

$$\begin{array}{ccccc} \rho_{\mathrm{bf}}\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) & \longrightarrow & \Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) & \xrightarrow{\ I\ } & \Psi^m_{b,I,\mathrm{os}}(\widetilde{X};{}^b\Omega^{\frac12}) \\ \Big\downarrow{\scriptstyle\mathrm{Tr}} & & \Big\downarrow{\scriptstyle\text{b-Tr}_\nu} & & \Big\downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} & \longrightarrow & \{0\} \end{array}$$

commutes.

EXERCISE 4.65. Check this last statement that

$$\text{b-Tr}_\nu(A) = 0 \ \forall \ A \in \Psi^{-\infty}_{b,I,\text{os}}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}).$$

The trace of a commutator, for trace class operators, is always zero, so it is reasonable to expect the trace of a commutator to be expressible in terms of the indicial operators. The fundamental identity established in §5.5 shows that the b-trace does not vanish on the commutator subspace and so implies that is not *really* a trace functional.

Chapter 5. Full calculus

In Chapter 4 the small b-pseudodifferential calculus was used to carry out the ('symbolic') construction of a parametrix. This gives an inverse up to errors in the small-residual space, $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$. The fact that these operators are *not* generally compact shows that more needs to be done in order to understand the mapping, especially Fredholm, properties of elliptic b-differential operators. What is required is the inversion of the other symbol, i.e. the indicial operator. The Mellin transform can be applied, so reducing the question to the invertibility properties of the indicial family. For this the symbolic construction is a convenient starting point.

The structure of the inverse of the indicial operator is used as a guide to the extension of the calculus so that it includes a parametrix, up to compact errors. The composition properties of the extended calculus are then analyzed and the structure of generalized inverses is discussed.

The extensive discussion of the polyhomogeneous calculus in the latter part of this chapter leads to a quite detailed description of the kernels of such generalized inverses. Rather little of this is needed for a minimal proof of the APS theorem. In particular nothing from §5.18 onward is required, except for the extension, as in §5.24, of the calculus with bounds to general bundles.

5.1. Mellin transform.

To examine the invertibility properties of $I(P)$, the indicial operator of an elliptic b-differential operator as in (4.102), or just an elliptic element $Q \in \mathrm{Diff}^k_{b,I}(\widetilde{X})$, it is natural to use the *Mellin transform*

$$u_{M,\nu}(\lambda, y) = \int_0^\infty x^{-i\lambda} u(x,y) \frac{dx}{x}, \ u \in \dot{\mathcal{C}}^\infty(\widetilde{X}). \tag{5.1}$$

Here $\nu : \widetilde{X} \longleftrightarrow [-1,1] \times \partial X$ is a (projective) trivialization of $\widetilde{X}$ and x is the corresponding linear coordinate. Thus some defining function $\rho \in \mathcal{C}^\infty(X)$ for ∂X fixes ν and x is just the linear function $d\rho$ on the interior of $\widetilde{X}$. Since

$$u \in \dot{\mathcal{C}}^\infty(\widetilde{X}) \Longleftrightarrow \sup_{\widetilde{X}} |x^p D_x^q T u| < \infty \quad \forall\ p \in \mathbb{Z}, q \in \mathbb{N}_0, T \in \mathrm{Diff}^*(\partial X),$$

the integral in (5.1) certainly converges absolutely. Indeed the Paley-Wiener theorem characterizes the range:

THEOREM 5.1. *The Mellin transform,* (5.1)*, is an isomorphism*

$$\dot{\mathcal{C}}^\infty(\widetilde{X}) \ni u \longleftrightarrow u_{M,\nu} \in \Big\{U : \mathbb{C} \longrightarrow \mathcal{C}^\infty(\partial X) \text{ entire};\\ \sup_{|\operatorname{Im}\lambda|\le p} |(1+|\lambda|)^k \partial_\lambda^j TU| < \infty \ \forall\ k,j,p \in \mathbb{N}_0, T \in \operatorname{Diff}^*(\partial X)\Big\} \tag{5.2}$$

with inverse

$$u(x,y) = \frac{1}{2\pi}\int\limits_{\operatorname{Im}\lambda = r} x^{i\lambda} u_{M,\nu}(\lambda,y)d\lambda \tag{5.3}$$

for any $r \in \mathbb{R}$.

EXERCISE 5.2. Review the proof of Theorem 5.1. First introduce $s = \log x$ as coordinate so that (5.1) becomes

$$u_{M,\nu}(\lambda,y) = \int_{-\infty}^{\infty} e^{-is\lambda} u(e^s,y)ds. \tag{5.4}$$

Show that $u \in \dot{\mathcal{C}}^\infty(\widetilde{X})$ is equivalent to the faster-than-any-exponential decay of all derivatives of $v(s,y) = u(e^s,y)$. Then recover (5.2) and (5.3) by reference to the standard results for the Fourier(-Laplace) transform.

Clearly the Mellin transform satisfies the identities

$$(xD_xu)_{M,\nu} = \lambda u_{M,\nu} \text{ and}\\ (Tu)_{M,\nu} = Tu_{M,\nu},\ T \in \operatorname{Diff}^*(\partial X). \tag{5.5}$$

It follows that if $Q \in \operatorname{Diff}^k_{b,I}(\widetilde{X})$

$$Q = \sum_{j+|\alpha|\le k} q_{j,\alpha}(y)(xD_x)^j D_y^\alpha \Longrightarrow (Qu)_{M,\nu} = \widehat{Q}u_{M,\nu},\\ \text{where } \widehat{Q}(\lambda) = \sum_{j+|\alpha|\le k} q_{j,\alpha}(y)\lambda^j D_y^\alpha. \tag{5.6}$$

If $P \in \operatorname{Diff}^k_b(X)$ is given by (4.102) then the operator

$$I_\nu(P,\lambda) = \sum_{j+|\alpha|\le k} p_{j,\alpha}(0,y)\lambda^j D_y^\alpha = \widehat{I(P)} \tag{5.7}$$

is the *indicial family* of P, already introduced in Exercise 4.63. It is a differential operator on ∂X depending parametrically, and polynomially, on $\lambda \in \mathbb{C}$.

5.2. Inversion of the indicial family.

Before proceeding to describe the basic result on the invertibility of the indicial family, note that $I_\nu(P,\lambda)$ depends a little on the choice of a trivialization, ν, of the normal bundle. Any other (oriented) trivialization, ν', corresponds to multiplication, on the fibres of $N\partial X$, by some $0 < a \in \mathcal{C}^\infty(\partial X)$, i.e. $x' = x/a$. Directly from (5.1) it follows that

$$I_{\nu'}(P,\lambda) = a^{i\lambda} I_\nu(P,\lambda) a^{-i\lambda}$$

is given by conjugation with the complex powers. Thus invertibility results will be independent of the choice of trivialization.

By the *order* of a pole of a meromorphic function, F, will be meant the negative of the most singular power which occurs in the Laurent series expansion around the singular point, i.e.

$$\operatorname{ord}(z) = \min\left\{k \in \mathbb{N}; (\lambda - z)^k F(\lambda) \text{ is holomorphic near } \lambda = z\right\}.$$

Thus

$$F(\lambda) = \sum_{j=1}^{\operatorname{ord}(z)} (\lambda - z)^{-j} A_j + F_0(\lambda) \tag{5.8}$$

with F_0 holomorphic near $\lambda = z$. The residue at z is A_1. On the other hand, for a meromorphic function $F(\lambda)$, with values in a space of linear operators, given as the inverse of a holomorphic family $Q(\lambda)$, the *rank* of a pole is the dimension of the singular range in the sense that

$$\begin{aligned} \operatorname{rank}(z) = \dim \Bigg\{ u = \sum_{j=1}^{\operatorname{ord}(z)} (\lambda - z)^{-j} u_j, \\ u_j \in \mathcal{C}^\infty(\partial X); Q(\lambda)u(\lambda, y) \text{ is holomorphic near } z \Bigg\}. \end{aligned} \tag{5.9}$$

Clearly $\operatorname{ord}(z) \le \operatorname{rank}(z)$. For a function with values in $\mathbb{C}$ the order and rank are always equal.

PROPOSITION 5.3. *If $Q \in \operatorname{Diff}^k_{b,I}(\widetilde{X})$ is elliptic then there is a discrete set*

$$\operatorname{spec}_b(Q) = \left\{\lambda \in \mathbb{C}; \widehat{Q}(\lambda) \textit{ is not invertible on } \mathcal{C}^\infty(\partial X)\right\}$$

and the inverse is a meromorphic family

$$[\mathbb{C} \setminus \operatorname{spec}_b(Q)] \ni \lambda \longmapsto Q(\lambda)^{-1} \in \Psi^{-k}(\partial X)$$

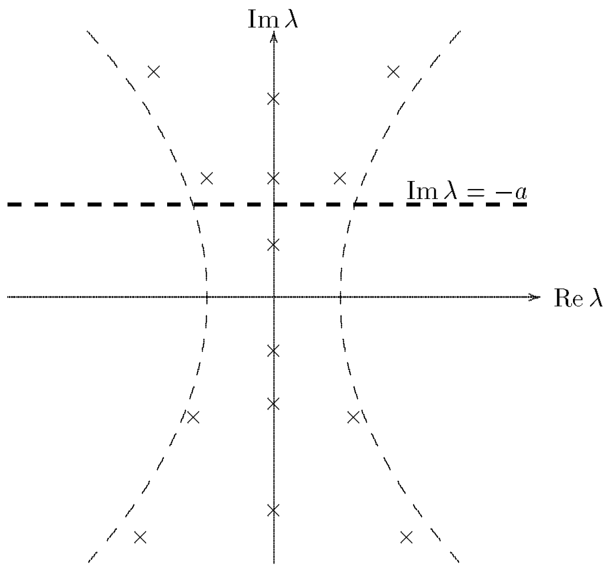

Figure 8. $\operatorname{spec}_b(P),\ a \notin -\operatorname{Im}\operatorname{spec}_b(P)$.

with poles of finite order and rank, with residues finite rank smoothing operators at points of $\operatorname{spec}_b(Q)$ *(the boundary spectrum) and furthermore*

$$\lambda_j \in \operatorname{spec}_b(Q),\ |\lambda_j| \longrightarrow \infty \Longrightarrow |\operatorname{Im}\lambda_j| \longrightarrow \infty.$$

In fact the proof will give slightly more than the statement in that continuity estimates on the inverse of the indicial family arise directly from the discussion. Nevertheless, Proposition 5.3 captures the most important fact, that the indicial family of an elliptic b-differential operator is invertible for almost all values (all except a discrete set) of λ. We shall also have occasion to use a more precise version of spec_b :

$$\text{(5.10)}\quad \operatorname{Spec}_b(Q) = \\ \Big\{(\lambda, k); \lambda \in \operatorname{spec}_b(Q) \text{ and } \widehat{Q}(z)^{-1} \text{ has a pole of order } k+1 \text{ at } \lambda\Big\}.$$

To prove Proposition 5.3 it is useful to extend the notion of the indicial family to the small calculus of b-pseudodifferential operators. To do so (5.7) can first be interpreted in terms of the indicial operator of P, thought of as the restriction to the front face of the Schwartz kernel. Recall that the kernel, in the sense of (4.58), of the identity is given in local coordinates by

$$\kappa_{\mathrm{Id}} = \delta(s-1)\delta(y-y')|\frac{ds}{s}\frac{dr}{r}dydy'|^{\frac{1}{2}}. \tag{5.11}$$

Thus the indicial family arises by taking the Mellin transform in s after restricting to $\mathrm{bf}(X_b^2)$, i.e. $r = x + x' = 0$

$$I_\nu(\mathrm{Id}, \lambda) = \int_0^\infty s^{-i\lambda}\kappa(\mathrm{Id})_{|\mathrm{bf}}\frac{ds}{s} \equiv \mathrm{Id}\,. \tag{5.12}$$

For a general element of $\mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ the kernel at the front face is just $I(P)$ applied to (5.11). From (5.12) it follows that

$$I_\nu(P, \lambda) = \int_0^\infty s^{-i\lambda}\kappa(P)_{|\mathrm{bf}}\frac{ds}{s}. \tag{5.13}$$

Thus the indicial family, with respect to ν, of elements of the small calculus can simply be *defined* by (5.13) and this is consistent with (5.6).

LEMMA 5.4. *The indicial family of $A \in \Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ is an entire family*

$$\mathbb{C} \ni \lambda \longmapsto I_\nu(A, \lambda) \in \Psi^m(\partial X; \Omega^{\frac{1}{2}}) \tag{5.14}$$

which, if $m \le 0$, satisfies the uniform continuity estimates:

$$\begin{gathered} I_\nu(A,\lambda)\colon L^2(\partial X; \Omega^{\frac{1}{2}}) \longrightarrow H^{-m}(\partial X; \Omega^{\frac{1}{2}}), \\ \|I_\nu(A,\lambda)\|_{0,-m} \le C_{A,N},\ |\operatorname{Im}\lambda| \le N \in \mathbb{N} \end{gathered} \tag{5.15}$$

$$\begin{gathered} I_\nu(A,\lambda)\colon L^2(\partial X; \Omega^{\frac{1}{2}}) \longrightarrow L^2(\partial X; \Omega^{\frac{1}{2}}), \\ \|I_\nu(A,\lambda)\|_{0,0} \le C_{A,N}(1+|\lambda|)^m,\ |\operatorname{Im}\lambda| \le N \in \mathbb{N}. \end{gathered} \tag{5.16}$$

EXERCISE 5.5. Write down (and prove) appropriate continuity estimates when $m \ge 0$.

PROOF: This follows from the representation (4.59) – (4.61) of the Schwartz kernel of an element $A \in \Psi^m(X; {}^b\Omega^{\frac{1}{2}})$. In (5.13) only the kernel at $\mathrm{bf}(X_b^2)$ is of interest. Away from the lifted diagonal, it is $\mathcal{C}^\infty$ and vanishes rapidly at the boundary. Thus, directly from Theorem 5.1 it follows that if $A \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ then $I_\nu(A, \lambda)$ is an entire family of smoothing operators in $\Psi^{-\infty}(\partial X; \Omega^{\frac{1}{2}})$ with kernel rapidly decreasing, with all derivatives, as

$|\operatorname{Re}\lambda| \longrightarrow \infty$ in any region where $|\operatorname{Im}\lambda|$ is bounded. Such an operator certainly satisfies (5.15) and (5.16) for any m.

Thus it can be assumed that the kernel of A is supported in any small neighbourhood of the lifted diagonal and, by taking a partition of unity, in a coordinate patch. The representation (4.61) arises by taking as amplitude

$$b(r, y', \lambda, \eta) = \int e^{i\lambda \log s + i\eta\cdot(y-y')} k(r, s, y, y') \frac{ds}{s} dy'$$

since Δ_b is locally defined by $\log s = 0$, $y = y'$. Since k has support in s compact in $[0, \infty)$ it follows that $b(r, y', \lambda, \eta)$ is actually entire in λ and satisfies the symbol estimates (4.42), with $\xi = (\operatorname{Re}\lambda, \eta)$, uniformly in any strip where $|\operatorname{Im}\lambda|$ is bounded. In terms of the representation (4.61) the kernel of the indicial family is just

$$I_\nu(A, \lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(y-y')\cdot\eta} b(0, y', \lambda, \eta) d\eta \tag{5.17}$$

so is indeed an entire family of pseudodifferential operators of order m on ∂X. The continuity estimates on an operator (5.17) only depend on the symbol estimates (and the $\mathcal{C}^\infty$ estimates on the kernel away from the diagonal) see [47]. If $m < 0$ the symbol estimates (4.42) on b :

$$|D^j_\lambda D^\alpha_{y'} D^\beta_\eta b(0, y', \lambda, \eta)| \le C_{j,\alpha,\beta,K}(1 + |\lambda| + |\eta|)^{m-j-|\beta|},\ y' \in K \subset\subset \mathbb{R}^n$$

give the uniform estimates

$$\begin{aligned} |D^\alpha_{y'} D^\beta_\eta b(0, y', \lambda, \eta)| &\le C_{\alpha,\beta,K}(1 + |\eta|)^{m-|\beta|},\ y' \in K \subset\subset \mathbb{R}^n \\ |D^\alpha_{y'} D^\beta_\eta b(0, y', \lambda, \eta)| &\le C_{\alpha,\beta,K}(1 + |\lambda|)^m (1 + |\eta|)^{-|\beta|},\ y' \in K \subset\subset \mathbb{R}^n. \end{aligned}$$

To get the corresponding continuity estimates (5.15) and (5.16), one should treat $I_\nu(A, \lambda)$ as an element of the space $\Psi^m(\partial X, \Omega^{\frac12})$ and then as an element of $\Psi^0(\partial X; \Omega^{\frac12})$.

The indicial operators for the small calculus, given as the restriction of the kernels to the front face of X^2_b, satisfy

$$I(P \circ A) = I(P) \circ I(A),\ P \in \operatorname{Diff}^k_b(X; {}^b\Omega^{\frac12}). \tag{5.18}$$

From the definition, (5.13), of the indicial families it follows that

$$I_\nu(P \circ A, \lambda) = I_\nu(P, \lambda) \circ I_\nu(A, \lambda). \tag{5.19}$$

That is, the indicial family gives an entire family of maps

$$\Psi_b^m(X;{}^b\Omega^{\frac12}) \ni A \longmapsto I_\nu(A,\lambda) \in \Psi^m(\partial X;\Omega^{\frac12}),\ \lambda \in \mathbb{C} \tag{5.20}$$

which are multiplicative over $\mathrm{Diff}_b^*(X;{}^b\Omega^{\frac12})$. The maps (5.20) are actually algebra homomorphisms, a fact which will not be used for the moment but which allows the construction below of parametrices to be extended to elliptic elements of the small calculus.

PROOF OF PROPOSITION 5.3: A parametrix for P in the small calculus was constructed in Chapter 4. Namely $G_s \in \Psi_b^{-k}(X;{}^b\Omega^{\frac12})$ is such that

$$P \circ G_s = \mathrm{Id} - R_s,\ R_s \in \Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}).$$

Now, using (5.19) to pass to the indicial families, with respect to some boundary defining function,

$$I_\nu(P,\lambda) \circ I_\nu(G_s,\lambda) = \mathrm{Id} - I_\nu(R_s,\lambda). \tag{5.21}$$

The remainder term, $I_\nu(R_s,\lambda)$ is, by Lemma 5.62, an entire family of smoothing operators which are rapidly decreasing at real infinity. From (5.16) there is a real function $F : [0,\infty) \longrightarrow [0,\infty)$ such that $\mathrm{Id} - I_\nu(R_s,\lambda)$ is, as an operator on $L^2(\partial X;\Omega^{\frac12})$, invertible for $|\operatorname{Re}\lambda| \ge F(|\operatorname{Im}\lambda|)$. Set

$$(\mathrm{Id} - I_\nu(R_s,\lambda))^{-1} = \mathrm{Id} - S(\lambda),\ |\operatorname{Re}\lambda| \ge F(|\operatorname{Im}\lambda|). \tag{5.22}$$

Then

$$S(\lambda) = I_\nu(R_s,\lambda) + I_\nu(R_s,\lambda)\circ I_\nu(R_s,\lambda) + I_\nu(R_s,\lambda)\circ S(\lambda)\circ I_\nu(R_s,\lambda) \tag{5.23}$$

from which it follows that $S(\lambda)$ is also a smoothing operator depending holomorphically on λ in the region (5.22) and satisfying the same type of rapid decay estimates as $I_\nu(R,\lambda)$ in this region.

5.3. Analytic Fredholm theory.

In fact by analytic Fredholm theory it follows that $S(\lambda)$ extends to a meromorphic function on the whole of $\mathbb{C}$, with values in the smoothing operators. It is perhaps worthwhile to take the time to explain this well-known idea. First fix any $\lambda_0 \in \mathbb{C}$ at random. Since $I_\nu(R_s,\lambda_0)$ is a smoothing operator on a compact manifold the null space, N_0, of $\mathrm{Id} - I_\nu(R_s,\lambda_0)$ acting on $L^2(\partial X;\Omega^{\frac12})$ is finite dimensional and contained in $\mathcal{C}^\infty(\partial X;\Omega^{\frac12})$. Furthermore the range of $\mathrm{Id} - I_\nu(R_s,\lambda_0)$ is closed with a finite dimensional complement of the same dimension as N_0, $N_1 \subset \mathcal{C}^\infty(\partial X; \Omega^{\frac12})$. Thus the operator $I_\nu(R_s, \lambda)$ can be split into a 2×2 matrix:

$$I_\nu(R_s, \lambda) = \begin{bmatrix} R_{00}(\lambda) & R_{01}(\lambda) \\ R_{10}(\lambda) & R_{11}(\lambda) \end{bmatrix}$$

acting on $L^2(\partial X; \Omega^{\frac12}) = N \oplus N^\perp$, where N is a finite dimensional subspace of $\mathcal{C}^\infty(\partial X; \Omega^{\frac12})$, $\mathrm{Id} - R_{11}(\lambda_0)$ is invertible and all the entries are analytic near λ_0. Then solving

$$[\mathrm{Id} - I_\nu(R_s, \lambda)]u = f \in L^2(\partial X; \Omega^{\frac12}),\ f = f_N + f',\ u = u_N + u' \tag{5.24}$$

is equivalent to solving

$$[\mathrm{Id} - R_{00}(\lambda)]u_N + R_{01}(\lambda)u' = f_N,\ u_N + [\mathrm{Id} - R_{11}(\lambda)]u' = f'. \tag{5.25}$$

If λ is close to λ_0 then $\mathrm{Id} - R_{11}(\lambda)$ must still be invertible, so u' can be eliminated from (5.25) and the solvability of (5.24) reduces to that of

$$[\mathrm{Id} - R_{00} + R_{10} \circ (\mathrm{Id} - R_{11})^{-1}]u_N = f_N - R_{10} \circ (\mathrm{Id} - R_{11})^{-1} f'. \tag{5.26}$$

Here all the operators act on N, i.e. are finite matrices, depending holomorphically on λ in $|\lambda - \lambda_0| < \epsilon$. Thus (5.26) is always solvable if and only if the determinant

$$\det[\mathrm{Id} - R_{00} + R_{10} \circ (\mathrm{Id} - R_{11})^{-1}] \neq 0. \tag{5.27}$$

The set of λ for which $\mathrm{Id} - I_\nu(R_s, \lambda)$ is invertible near λ_0 is therefore exactly the set at which the holomorphic function in (5.27) does not vanish. Starting at the boundary of the set in (5.22) it follows that this must be a discrete set, i.e. the determinant in (5.27) does not vanish identically. From this discussion, and (5.23) as before, it follows that $S(\lambda)$ is indeed a meromorphic operator with finite rank residues at the points where it is singular. From (5.21) the inverse of $I_\nu(P, \lambda)$ is also meromorphic since it can be written

$$I_\nu(P, \lambda)^{-1} = I_\nu(G_s, \lambda) \circ (\mathrm{Id} - S(\lambda)).$$

The composition properties of pseudodifferential operators on ∂X show that the residues of the poles of $I_\nu(P, \lambda)^{-1}$ are also finite rank smoothing operators.

Not only does this complete the proof of Proposition 5.3 but it allows the inverse of the indicial family to be written

$$I_\nu(P, \lambda)^{-1} = I_\nu(G_s, \lambda) + R'(\lambda),\ G_s \in \Psi_b^{-k}(X; {}^b\Omega^{\frac12}), \tag{5.28}$$

with $R'(\lambda)$ a meromorphic family of smoothing operators having residues of finite rank and decaying rapidly as $|\operatorname{Re}\lambda| \longrightarrow \infty$ in any region where $|\operatorname{Im}\lambda|$ is bounded.

Exercise 5.6. Extend Proposition 5.3 to give a similar description of the inverse of the indicial family of an elliptic element $Q \in \Psi^m_{b,I,\mathrm{os}}(X; {}^b\Omega^{\frac12})$.

5.4. Conjugation by powers.

There is a direct approach to the indicial family of a b-pseudodifferential operator, as in (5.20), which is mentioned in the Introduction. First observe that $\Psi^m_b(X; {}^b\Omega^{\frac12})$ is invariant under conjugation by complex powers of a boundary defining function.

Proposition 5.7. *If $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary of X and $z \in \mathbb{C}$ then*

$$\Psi^m_b(X; {}^b\Omega^{\frac12}) \ni A \longmapsto x^{-z} A x^z \in \Psi^m_b(X; {}^b\Omega^{\frac12}) \tag{5.29}$$

is an isomorphism

Proof: It is enough to show that the conjugated operator is in the space $\Psi^m_b(X; {}^b\Omega^{\frac12})$ as indicated, since the inverse map is obtained by replacing z by $-z$. The Schwartz kernel of $x^{-z}Ax^z$ is $(\frac{x}{x'})^{-z}K$, where K is the Schwartz kernel of A. Lifting to X^2_b, the kernel in the sense of (4.54) is just

$$(\frac{1-\tau}{1+\tau})^{-z}\kappa, \tag{5.30}$$

where κ is the kernel of A as in Definition 4.22. Since the multiplier here is $\mathcal{C}^\infty$ and non-vanishing in X^2_b, except at $\mathrm{lb} \cup \mathrm{rb}$, where however all derivatives have polynomial growth, the product (5.30) satisfies (4.59) - (4.61) too, i.e. (5.29) holds.

These observations can be used to obtain a result intermediate between Proposition 4.29 and Proposition 4.34:

Corollary. *If $k \in \mathbb{N}$ then each $A \in \Psi^*_b(X; {}^b\Omega^{\frac12})$ defines a continuous linear map*

$$A \colon x^k\mathcal{C}^\infty(X; {}^b\Omega^{\frac12}) \longrightarrow x^k\mathcal{C}^\infty(X; {}^b\Omega^{\frac12}).$$

This in turn allows (5.20) to be defined very simply when $\lambda = 0$. Namely if $\psi \in \mathcal{C}^\infty(\partial X; \Omega^{\frac12})$ then

$$A_\partial\psi = A\phi_{\restriction\partial X} \in \mathcal{C}^\infty(\partial X; \Omega^{\frac12}),\ \phi \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac12}),\ \phi_{\restriction\partial X} = \psi \tag{5.31}$$

is independent of the choice of ϕ. The operator A_∂ defined by (5.31) is equal to $I_\nu(A, 0)$ for any choice of trivialization ν. This follows from the representation (4.68), which was extended to elements of $\mathcal{C}^\infty(X; {}^b\Omega^{\frac12})$ in the proof of Proposition 4.34. Restricting (4.68) to $x = 0$ shows that the kernel of A_∂ is just

$$\int_0^\infty \kappa(0, s, y, y')\frac{ds}{s}.$$

This corresponds to (5.13) for $\lambda = 0$ as claimed.

More generally the indicial family of a b-pseudodifferential operator can also be determined in this way:

PROPOSITION 5.8. *If $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary and $dx \cdot \nu = 1$ then the indicial family, defined by* (5.13) *is also given by*

$$I_\nu(A, \lambda) = [x^{-i\lambda} A x^{i\lambda}]_\partial, \tag{5.32}$$

where the boundary operator is fixed by (5.31).

5.5. Commutator identity for the b-trace.

Now that the indicial family of a b-pseudodifferential operator has been introduced in general, the formula for the b-trace of a commutator mentioned in §4.20 can be deduced.

PROPOSITION 5.9. *If $A, B \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$ then*

$$\text{b-Tr}_\nu([A, B]) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \operatorname{tr}\left(\partial_\lambda I_\nu(A, \lambda) \circ I_\nu(B, \lambda)\right) d\lambda. \tag{5.33}$$

PROOF: It suffices to prove (5.33) for the elements of a subset the span of which is dense in $\Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$. Thus it can be assumed that both A and B have kernels with small supports in X_b^2 and disjoint from $\operatorname{lb}(X_b^2)$ and $\operatorname{rb}(X_b^2)$ (but *not* $\operatorname{bf}(X_b^2)$ since this would not give a dense subset). Consider first the product $A \circ B$. The b-trace is, by definition,

$$\text{b-Tr}_\nu(A \circ B) = \lim_{\epsilon \downarrow 0}\left\{\int_{x>\epsilon} (AB)_{\restriction\Delta} + \log\epsilon \cdot \gamma\right\}$$

for the appropriate constant γ; here the kernels are again denoted as the operators. In $x > \epsilon$ the kernel is given by a convergent integral (since no points near the corner of X^2 are involved):

$$\int_{x>\epsilon} (AB)_{\restriction\Delta} = \int_{x>\epsilon}\left(\int_X A(z, z')B(z', z)\right). \tag{5.34}$$

Let $R\colon X^2 \longleftrightarrow X^2$ be the factor-exchanging isomorphism, $R(z,z') = (z',z)$. Thus R^*A is the kernel of A^t. Then (5.34) can be written

$$\int\limits_{x>\epsilon} (AB)_{\restriction\Delta} = \int\limits_{\{x>\epsilon\}\cap X^2} A \cdot R^*B. \tag{5.35}$$

Here x is the boundary defining function lifted to the left factor. Since $x = 0$ at $\mathrm{bf}(X_b^2)$, the lift of (5.35) to X_b^2 is trivial and

$$\int\limits_{x>\epsilon} (AB)_{\restriction\Delta} = \int\limits_{\{x>\epsilon\}\cap X_b^2} A \cdot R^*B. \tag{5.36}$$

Of course the same discussion applies to $B \circ A$. Since R lifts to an isomorphism of X_b^2 exchanging x and x', the lift of the defining function from the right factor gives

$$\int\limits_{x>\epsilon} (BA)_{\restriction\Delta} = \int\limits_{\{x>\epsilon\}\cap X_b^2} B \cdot R^*A = \int\limits_{\{x'>\epsilon\}\cap X_b^2} A \cdot R^*B. \tag{5.37}$$

The integrands in (5.36) and (5.37) are the same, only the domains differ. To complete the computation introduce local coordinates (as can certainly be done by assuming that the supports of the kernels are small). The supports do not meet $\mathrm{lb}(X_b^2)$ or $\mathrm{rb}(X_b^2)$, so projective coordinates can be used:

$$x, s = \frac{x}{x'}, y, y', A = \alpha \left| \frac{ds}{s}\frac{dx}{x} dy dy' \right|^{\frac12}, B = \beta \left| \frac{ds}{s}\frac{dx}{x} dy dy' \right|^{\frac12}.$$

Then the involution R is $(x,s,y,y') \longleftrightarrow (\frac{x}{s}, 1/s, y', y)$. From (5.36) and (5.37) it follows that

$$\begin{aligned} &\int\limits_{x>\epsilon} (AB - BA)_{\restriction\Delta} \\ &= \int\limits_{\partial X \times \partial X} \int\limits_0^\infty \int\limits_\epsilon^{\epsilon s} \alpha(x,s,y,y')\beta\left(\frac{x}{s}, 1/s, y', y\right) \frac{dx}{x}\frac{ds}{s} dy dy'. \end{aligned}$$

Here α, β have compact support in $[0,\infty)$ in s. Replacing α and β by their values at $x = 0$ gives integrable errors which therefore disappear in the limit $\epsilon \downarrow 0$, so

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_{x>\epsilon}(K_{AB}-K_{BA})_{\restriction\Delta}\\
&=\lim_{\epsilon\downarrow 0}\int_{\partial X\times\partial X}\int_0^\infty\int_\epsilon^{\epsilon s}\alpha(0,s,y,y')\beta(0,\frac{1}{s},y',y)\frac{dx}{x}\frac{ds}{s}dydy'\\
&=\int_{\partial X\times\partial X}\int_0^\infty\log s\cdot\alpha(0,s,y,y')\beta(0,\frac{1}{s},y',y)\frac{ds}{s}dydy'
\end{aligned}
\tag{5.38}
$$

exists, without regularization.

The kernels $\alpha(0,s,y,y')|\frac{ds}{s}dydy'|^{\frac{1}{2}}$ and $\beta(0,s,y,y')|\frac{ds}{s}dydy'|^{\frac{1}{2}}$ are, by definition, those of the indicial operators. Using the inverse of the Mellin transform they can be written in terms of the kernels of the indicial families as

$$
\begin{aligned}
\alpha(0,s,y,y')|\frac{ds}{s}dydy'|^{\frac{1}{2}}&=\frac{1}{2\pi}\int s^{i\lambda}I_\nu(A,\lambda)(y,y')d\lambda|\frac{ds}{s}|^{\frac{1}{2}}\\
\overline{\beta(0,\frac{1}{s},y,y')}|\frac{ds}{s}dydy'|^{\frac{1}{2}}&=\frac{1}{2\pi}\int s^{i\lambda}\overline{I_\nu(B,\lambda)}(y,y')d\lambda|\frac{ds}{s}|^{\frac{1}{2}}.
\end{aligned}
$$

The Plancherel formula for the Mellin (i.e. Fourier) transform combined with integration by parts, shows that

$$
(\log s)\alpha(0,s,y,y')|\frac{ds}{s}dydy'|=\frac{i}{2\pi}\int s^{i\lambda}\partial_\lambda I_\nu(A,\lambda)(y,y')d\lambda|\frac{ds}{s}|^{\frac{1}{2}},
$$

and so allows (5.38) to be rewritten as

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\int_{x>\epsilon}(K_{AB}-K_{BA})_{\restriction\Delta}\\
&=-\frac{1}{2\pi i}\int_{\partial X\times\partial X}\int_{-\infty}^{\infty}\partial_\lambda I_\nu(A,\lambda)(y,y')\cdot I_\nu(B,\lambda)(y',y)d\lambda.
\end{aligned}
\tag{5.39}
$$

The integrals over ∂X just give the trace of the composite operator, so this is precisely (5.33).

In fact it is most important to have (5.33) when one of the operators, say A, is a b-differential operator. This is easily proved directly (see Exercise 5.11) but at this stage it is perhaps more enlightening to deduce it from (5.33).

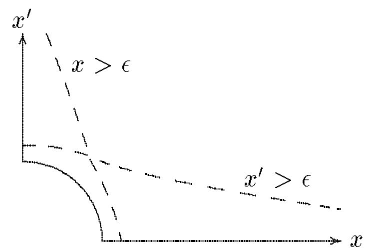

Figure 9. The domains in X_b^2 for (5.34), (5.35).

LEMMA 5.10. *The identity* (5.33) *holds if* $A \in \mathrm{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$ *and* $B \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$.

PROOF: In fact, by continuity, (5.33) extends to allow $A \in \Psi_b^*(X; {}^b\Omega^{\frac{1}{2}})$ if $B \in \Psi_b^{-\infty}(X; {}^b\Omega^{\frac{1}{2}})$.

EXERCISE 5.11. Give a direct proof of Lemma 5.10 by lifting the operator $A \in \mathrm{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$ to an operator $\tilde{A} \in \mathrm{Diff}_b^m(X_b^2; {}^b\Omega^{\frac{1}{2}})$ from the left factor and $\tilde{A}^t \in \mathrm{Diff}_b^m(X_b^2; {}^b\Omega^{\frac{1}{2}})$ from the right factor and observing that the kernel of $[A, B]$ is

$$\tilde{A} \cdot K_B - \tilde{A}^t \cdot K_B \quad \text{on } X_b^2.$$

Restrict this to the (lifted) diagonal, in $x > \epsilon$, and integrate to get the left side of (5.39) in this case. Now (for example working in local coordinates by assuming that B has small support) show that integration by parts reduces the integral over $x > \epsilon$ to one over $x = \epsilon$ which, in the limit as $\epsilon \downarrow 0$, becomes an integral over $\mathrm{bf}(X_b^2)$. With a little manipulation this integral can be brought to the form of the right side of (5.33).

5.6. Invertibility of the indicial operator.

The discussion above of the invertibility properties of the indicial family of an elliptic b-pseudodifferential operator on a compact manifold with boundary can be used to examine the mapping and invertibility properties of the indicial operator itself. This in turn is the main step in obtaining an understanding of the invertibility properties of the original operator.

Recall that the indicial family of $P \in \Psi_b^m(X; {}^b\Omega^{\frac{1}{2}})$ is defined, in (5.13), by taking the Mellin transform of the indicial operator, which is fixed by the kernel of P restricted to the front face of X_b^2. To show the implications of this for $I(P)$, first consider some Sobolev spaces on which it should act.

The Mellin transform (5.1) extends by continuity to an isomorphism

$$L^2\left(\widetilde{X};{}^b\Omega^{\frac12}\right) \overset{()_{M,\nu}}{\longleftrightarrow} L^2\left(\mathbb{R}\times\partial X;|d\lambda|^{\frac12}\cdot\Omega^{\frac12}\right). \tag{5.40}$$

Indeed this is just Plancherel's theorem applied to (5.4). This isomorphism can be used to define the b-Sobolev spaces of any order:

(5.41)

$$\begin{aligned}H_b^m(\widetilde{X};{}^b\Omega^{\frac12}) = \Big\{ & u\in L^2(\widetilde{X};{}^b\Omega^{\frac12});\, u_{M,\nu}\in L^2(\mathbb{R};H^m(\partial X;\Omega^{\frac12})),\\ & (1+|\lambda|^2)^{m/2}u_{M,\nu}\in L^2(\mathbb{R}\times\partial X;|d\lambda|^{\frac12}\Omega^{\frac12})\Big\},\ m\geq 0.\end{aligned}$$

EXERCISE 5.12. Show that the space defined by (5.41) is independent of the section ν used to define the Mellin transform.

In fact this space is actually invariant under arbitrary diffeomorphisms of $\widetilde{X}$ as a compact manifold with boundary and so can be transferred to X itself. For present purposes it suffices to see this when $m\in\mathbb{Z}$. When $m\in\mathbb{N}$, (5.41) can be directly formulated on an arbitrary compact manifold with boundary as

(5.42)

$$\begin{aligned}H_b^m(X;{}^b\Omega^{\frac12}) = \{ & u\in L^2(X;{}^b\Omega^{\frac12});\\ & \operatorname{Diff}_b^m(X;{}^b\Omega^{\frac12})u\subset L^2(X;{}^b\Omega^{\frac12})\},\ m\in\mathbb{N},\end{aligned}$$

and the coordinate-invariance follows from that of $L^2(X;{}^b\Omega^{\frac12})$.

It is convenient to have at hand the negative order Sobolev spaces as well. For integral orders a similar, but dual, definition to (5.42) can be used:

(5.43)

$$H_b^m(X;{}^b\Omega^{\frac12}) = L^2(X;{}^b\Omega^{\frac12}) + \operatorname{Diff}_b^{-m}(X;{}^b\Omega^{\frac12})\cdot L^2(X;{}^b\Omega^{\frac12}),\ m\in -\mathbb{N}.$$

The general case will be discussed later using the continuity properties of b-pseudodifferential operators.

The elementary considerations of Chapter 1 underscore the importance of considering *weighted* versions of spaces of this type. Let $\rho_i\in\mathcal{C}^\infty(X)$ for $i=1,\dots,J$ be defining functions for the bounding hypersurfaces (i.e.

boundary components) of a compact manifold with boundary and for $\mathfrak{a} = (\alpha_1, \dots, \alpha_J) \in \mathbb{R}^J$ set

$$\begin{aligned} \rho^{\mathfrak{a}} H_b^m(X; {}^b\Omega^{\frac{1}{2}}) = \Big\{ & u \in \mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac{1}{2}}); \\ & u = \rho_1^{\alpha_1} \cdots \rho_J^{\alpha_J} v, v \in H_b^m(X; {}^b\Omega^{\frac{1}{2}}) \Big\}, \ m \in \mathbb{Z}. \end{aligned} \tag{5.44}$$

EXERCISE 5.13. There is a choice of ordering involved in (5.44) since, for $m > 0$, it says exactly:

$$u \in \rho^{\mathfrak{a}} H_b^m(X; {}^b\Omega^{\frac{1}{2}}) \Longleftrightarrow \operatorname{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})[\rho^{-\mathfrak{a}} u] \subset L^2(X; {}^b\Omega^{\frac{1}{2}}),$$

where ρ^{α} is written for $\rho_1^{\alpha_1} \cdots \rho_J^{\alpha_J}$. Show that in fact the order is immaterial and

$$\begin{gathered} u \in \rho^{\mathfrak{a}} H_b^m(X; {}^b\Omega^{\frac{1}{2}}) \Longleftrightarrow \\ \operatorname{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}}) u \subset \rho^{\mathfrak{a}} L^2(X; {}^b\Omega^{\frac{1}{2}}) = \rho^{\mathfrak{a}} H_b^0(X; {}^b\Omega^{\frac{1}{2}}). \end{gathered}$$

If $m < 0$ show that similarly (5.44) is just

$$\begin{gathered} u \in \rho^{\mathfrak{a}} H_b^m \Longleftrightarrow \\ \rho^{-\mathfrak{a}} u \in L^2(X; {}^b\Omega^{\frac{1}{2}}) + \operatorname{Diff}_b^{-m}(X; {}^b\Omega^{\frac{1}{2}}) \cdot L^2(X; {}^b\Omega^{\frac{1}{2}}), \ m \in -\mathbb{Z}, \end{gathered}$$

and that alternatively

$$\begin{gathered} u \in \rho^{\mathfrak{a}} H_b^m \Longleftrightarrow \\ u \in \rho^{\mathfrak{a}} L^2(X; {}^b\Omega^{\frac{1}{2}}) + \operatorname{Diff}_b^{-m}(X; {}^b\Omega^{\frac{1}{2}}) \cdot \rho^{\mathfrak{a}} L^2(X; {}^b\Omega^{\frac{1}{2}}), \ m \in -\mathbb{Z}. \end{gathered}$$

As is to be expected b-differential operators are bounded on these spaces:

LEMMA 5.14. *For any $k \in \mathbb{N}$ each $P \in \operatorname{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ defines a continuous linear map*

$$P : \rho^{\mathfrak{a}} H_b^m(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \rho^{\mathfrak{a}} H_b^{m-k}(X; {}^b\Omega^{\frac{1}{2}}) \tag{5.45}$$

for any $m \in \mathbb{Z}$ and any $\mathfrak{a} \in \mathbb{R}^J$ where X has J boundary components.

PROOF: If $\mathfrak{a} \in \mathbb{R}^J$ then conjugation by $\rho^{\mathfrak{a}}$ gives an isomorphism (as in (5.29))

$$\operatorname{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}}) \ni P \longmapsto \rho^{\mathfrak{a}} P \rho^{-\mathfrak{a}} \in \operatorname{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}}).$$

Thus it suffices to consider the case $\mathfrak{a} = 0$. First suppose $m \geq k$. Then if $u \in H_b^m(X; {}^b\Omega^{\frac{1}{2}})$ and $Q \in \operatorname{Diff}_b^{m-k}(X; {}^b\Omega^{\frac{1}{2}})$ certainly $Q \circ P \in \operatorname{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$, so $Q(Pu) \in L^2(X; {}^b\Omega^{\frac{1}{2}})$ and hence $Pu \in H_b^{m-k}(X; {}^b\Omega^{\frac{1}{2}})$. Continuity is immediate from the same argument, so (5.45) follows. The case $m < 0$ is similar and if $0 \le m \le k$ then it is enough to use (2.20) to decompose P into a finite sum

$$P = \sum_r P_r Q_r, \ P_r \in \mathrm{Diff}^{k-m}(X; {}^b\Omega^{\frac{1}{2}}), \ Q_r \in \mathrm{Diff}_b^m(X; {}^b\Omega^{\frac{1}{2}})$$

since then each $Q_r u \in L^2(X; {}^b\Omega^{\frac{1}{2}})$, so

$$Pu = \sum_r P_r(Q_r u) \in H_b^{m-k}(X; {}^b\Omega^{\frac{1}{2}}).$$

Recall that the boundary of the 'model' space, $\widetilde{X}$, is divided into two parts. Each is isomorphic to ∂X, which will be assumed connected to reduce the notational overhead. Let respective defining functions be ρ_0 and ρ_∞, corresponding to the zero section of $N\partial X$ and infinity in this vector bundle. If $\alpha \in \mathbb{R}$ let $\rho^\alpha H_b^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ denote the space with weighting factor $\rho_0^\alpha \rho_\infty^{-\alpha}$, with 'opposite' weighing at the two boundary faces. From a functional analytic point of view the main result concerning indicial operators is:

PROPOSITION 5.15. *If $Q \in \mathrm{Diff}_{b,I}^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ is elliptic then*

$$Q : \rho^\alpha H_b^m(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}) \longleftrightarrow \rho^\alpha H_b^{m-k}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}) \text{ is an isomorphism} \tag{5.46}$$

for some (and equivalently all) m if and only if

$$\alpha \notin -\operatorname{Im} \operatorname{spec}_b(Q) \text{ i.e. } \alpha \in \mathbb{R}, \ \alpha \neq -\operatorname{Im}\lambda \ \forall \ \lambda \in \operatorname{spec}_b(Q). \tag{5.47}$$

PROOF: By conjugation it suffices to consider the case $\alpha = 0$; for simplicity suppose also that $m = k$. Thus, under the assumption

$$\operatorname{spec}_b(Q) \cap \mathbb{R} = \emptyset \tag{5.48}$$

it needs to be shown that Q is an isomorphism from $H_b^m(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ to $L^2(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$. From (5.6) it follows that, after Mellin transformation, Q acts as the indicial family, $Q(\lambda)$. Combining (5.40) and (5.41) it suffices to show that

$$\begin{aligned} v \in L^2(\mathbb{R} \times \partial X; |d\lambda|^{\frac{1}{2}} \cdot \Omega^{\frac{1}{2}}) &\Longrightarrow \\ (1 + |\lambda|^2)^{\frac{k}{2}} Q(\lambda)^{-1} v \in & L^2(\mathbb{R} \times \partial X; |d\lambda|^{\frac{1}{2}} \cdot \Omega^{\frac{1}{2}}), \\ Q(\lambda)^{-1} v \in & L^2(\mathbb{R}; H^m(\partial X; \Omega^{\frac{1}{2}})). \end{aligned} \tag{5.49}$$

Consider the decomposition of the inverse of the indicial family given by (5.28). The first term is the indicial family of an element of $\Psi^{-k}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12})$, so for this part the analogues of (5.49) follow directly from (5.15) and (5.16). The second term in (5.28) is non-singular precisely because of (5.48) and consists therefore of a family of smoothing operators on ∂X, depending smoothly on λ and decreasing rapidly as $|\lambda| \longrightarrow \infty$. This proves the proposition when $k = m$. The general case is similar.

5.7. Kernel of the inverse of the indicial operator.

Although this is quite satisfactory from a functional analytic point of view, it is important to see the specific nature of the Schwartz kernel of the inverse of Q i. e. to consider the action of Q^{-1} on Dirac delta 'functions.' In (5.13) and (5.28) the inverse of Q is presented as the sum of a term in the small calculus (the parametrix constructed in Chapter 4) and the inverse Mellin transform of the second term in (5.28). So consider the smooth kernel, reverting to compact coordinates:

$$R'_B(\tau, y, y') = \frac{1}{2\pi} \int\limits_{\mathrm{Im}\,\lambda = -\alpha} \left(\frac{1+\tau}{1-\tau}\right)^{i\lambda} R'(\lambda, y, y') d\lambda |dy dy'|^{\frac12} \Big|\frac{dx}{x}\Big|^{\frac12}, \tag{5.50}$$

with (5.47) assumed so that the integral is defined.

LEMMA 5.16. *For any $\alpha \notin -\mathrm{Im}\,\mathrm{spec}_b(Q)$ where $Q \in \mathrm{Diff}^k_{b,I}(\widetilde{X}; {}^b\Omega^{\frac12})$ is elliptic, the integral (5.50), with R' as in (5.28), gives $R'_B \in \mathcal{C}^\infty((-1,1) \times (\partial X)^2; {}^b\Omega^{\frac12})$ having asymptotic expansions at the boundaries $\tau = \pm 1$:*

$$R'_B(\tau, y, y') \sim \sum_{\substack{z \in \mathrm{spec}_b(Q) \\ k \le \mathrm{ord}(z) \\ \pm\,\mathrm{Im}\, z > \mp\alpha}} \big(\frac{1+\tau}{1-\tau}\big)^{iz} \big[\log\big(\frac{1+\tau}{1-\tau}\big)\big]^k A_{z,k}(y, y'), \tag{5.51}$$

where the coefficients $A_{z,k}(y, y')$ are finite rank smoothing operators.

PROOF: Recall the meaning of the asymptotic expansion in (5.51). If $\phi \in \mathcal{C}^\infty(\mathbb{R})$ takes the value 1 near 1 and vanishes near -1 then (5.51) just means that for any $N \in \mathbb{N}$

$$\begin{aligned} &\phi(\pm\tau)\Bigg[R'_B(\tau, y, y') - \\ &\qquad \sum_{N \ge \pm\,\mathrm{Im}\, z > \mp\alpha} \big(\frac{1+\tau}{1-\tau}\big)^{iz} \big[\log\big(\frac{1+\tau}{1-\tau}\big)\Bigg]^k A_{z,k}(y, y') \\ &\qquad\qquad \in \dot{\mathcal{C}}^N([-1,1] \times (\partial X)^2), \end{aligned} \tag{5.52}$$

i.e. the difference must be N times continuously differentiable with all N derivatives vanishing at the boundary. The sum in (5.52) is still limited to $z \in \mathrm{spec}_b(Q)$ and $k \leq \mathrm{ord}(z)$. Of course the regularity statement is really two separate statements, near $\tau = 1$ and $\tau = -1$. It is simpler to rewrite (5.52) in terms of s near $\tau = -1$, corresponding to $s = 0$.

Thus it needs to be shown that

(5.53)
$$\begin{aligned} R'_B(s, y, y') &= \frac{1}{2\pi} \int\limits_{\mathrm{Im}\,\lambda = -\alpha} s^{i\lambda} R'(\lambda, y, y') \\ &\sim \sum_{\substack{z \in \mathrm{spec}_b(Q) \\ k \leq \mathrm{ord}(z) \\ \mathrm{Im}\, z < -\alpha}} s^{iz} [\log(s)]^k A_{z,k}(y, y'), \text{ as } s \downarrow 0. \end{aligned}$$

Simply estimating the integral near $s = 0$ gives

$$\left| D_s^p D_{y,y'}^\delta R'_B(s, y, y') \right| \leq C|s|^{\alpha - p},\ s < 1.$$

If $\alpha > N$ then $R'_B \in \dot{\mathcal{C}}^N([0,1) \times (\partial X)^2)$. Since the integrand in (5.53) is rapidly decreasing at real infinity, the contour can always be moved from $\mathrm{Im}\,\lambda = -\alpha$ to $\mathrm{Im}\,\lambda = -N - \epsilon$, $\epsilon > 0$, and hence ensure this regularity, except that by Cauchy's formula the residues at all the poles

$$z \in \mathrm{spec}_b(Q),\ -\alpha > \mathrm{Im}\, z \geq -N$$

need to be added. These residues give precisely the sum in (5.53), by virtue of the properties of $R'(\lambda)$ in (5.28). This shows the existence and form of the expansion near $s = 0$, i.e. $\tau = -1$. Changing variable in the integral from λ to $-\lambda$ and replacing s by $1/s$, the same argument can be applied to give the expansion at $\tau = 1$, i.e. $s = \infty$. This proves the proposition.

Notice that the expansion at $\mathrm{lb}(X_b^2)$, corresponding to $\tau = -1$ arises from the poles of $R'(\lambda)$, i.e the points of $\mathrm{spec}_b(Q)$, in $\mathrm{Im}\,\lambda < -\alpha$, whereas the expansion near $\mathrm{rb}(X_b^2)$ corresponds to the poles in $\mathrm{Im}\,\lambda > -\alpha$. One can think of the choice of weight, i.e. the choice of $\alpha \in \mathbb{R}$, as splitting the λ-plane in which $\mathrm{spec}_b(Q)$ is defined, into the upper part, corresponding to rb and the lower part, corresponding to lb.

5.8. Index formula for invariant operators.

Using these results on the indicial operator a simple index formula in the $\mathbb{R}^+$-invariant case can be derived. It is quite analogous to that for the half line, as described in Chapter 1. Consider the space $\widetilde{X}$ and let ρ_0 and ρ_∞ be defining functions for the zero section and ∞.

PROPOSITION 5.17. *Let $Q \in \mathrm{Diff}^m_{b,I}(\widetilde{X}; {}^b\Omega^{\frac12})$ be an elliptic differential operator and suppose α, $\beta \in \mathbb{R}$ satisfy*

$$\alpha \notin -\operatorname{Im}\operatorname{spec}_b(Q),\ \beta \notin \operatorname{Im}\operatorname{spec}_b(Q) \tag{5.54}$$

then

$$\begin{gathered}\beta > -\alpha \Longrightarrow Q : \rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac12}) \longrightarrow \rho_0^\alpha \rho_\infty^\beta H_b^{k-m}(\widetilde{X}; {}^b\Omega^{\frac12}) \\ \textit{is injective with closed range of codimension} \sum_{\substack{z\in\operatorname{spec}_b(Q)\\ \operatorname{Im} z\in[-\alpha,\beta]}} \operatorname{rank}(z)\end{gathered} \tag{5.55}$$

and

$$\begin{gathered}\beta < -\alpha \Longrightarrow Q : \rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac12}) \twoheadrightarrow \rho_0^\alpha \rho_\infty^\beta H_b^{k-m}(\widetilde{X}; {}^b\Omega^{\frac12}) \\ \textit{is surjective with null space of dimension} \sum_{\substack{z\in\operatorname{spec}_b(Q)\\ \operatorname{Im} z\in[\beta,-\alpha]}} \operatorname{rank}(z).\end{gathered} \tag{5.56}$$

In particular the index is, up to sign, the sum of the ranks of the points in $\operatorname{spec}_b(Q)$ with imaginary parts between $-\alpha$ and β :

$$\operatorname{ind}(Q) = \operatorname{sgn}(\alpha+\beta) \sum_{\substack{z\in\operatorname{spec}_b(Q)\\ \operatorname{Im} z\in[-\alpha,\beta]\cup[\beta,-\alpha]}} \operatorname{rank}(z).$$

PROOF: Consider (5.55) first. Since $\beta > -\alpha$ the weighting at infinity is 'smaller' than that at zero in the sense that

$$\rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac12}) \subset \rho^\alpha H_b^k(\widetilde{X}; {}^b\Omega^{\frac12}),$$

where the space on the right is the one that appears in Proposition 5.15. Thus the injectivity follows from (5.46). It remains then to show that the range is closed and to compute its codimension. As in the proof of Proposition 5.15, the Mellin transform and the inverse of the indicial family can be used. To do so a characterization of the range of the Mellin transform on the mixed-weight space is required. This is again a standard result of Paley-Wiener type for the Fourier transform:

LEMMA 5.18. *If $\alpha+\beta > 0$ then the Mellin transform gives an isomorphism*

$$\begin{gathered}\rho_0^\alpha \rho_\infty^\beta L^2(\widetilde{X}; {}^b\Omega^{\frac12}) \longrightarrow \Big\{ u_M : \mathbb{R}\times i(-\alpha,\beta) \ni \lambda \longmapsto u_M(\lambda) \in L^2(\partial X; \Omega^{\frac12}); \\ u_M(\lambda) \textit{ is holomorphic in } \lambda \textit{ and } \sup_{-\alpha<r<\beta} \|u_M(\cdot+ir,\cdot)\|_{L^2} < \infty \Big\}.\end{gathered} \tag{5.57}$$

A similar result can easily be deduced for the Mellin transforms of the weighted Sobolev spaces. Thus the range of (5.55) can be found, for simplicity when $m = k$. Namely it consists of those elements in the range of (5.57) which are of the form $u_M = Q(\lambda)v_M$ where $v \in \rho_0^\alpha \rho_\infty^\beta H_b^m(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$. This means that

$$v_M(\lambda) = Q(\lambda)^{-1}u_M(\lambda) \text{ is holomorphic in } -\alpha < \operatorname{Im}\lambda < \beta \tag{5.58}$$

since the regularity follows directly from the mapping properties of $Q(\lambda)^{-1}$ as before. In fact (5.58) means that all the singular terms in $Q(\lambda)^{-1}u_M(\lambda)$ must vanish. At each point $\lambda \in \operatorname{spec}_b(Q)$ with $-\alpha < \operatorname{Im}\lambda < \beta$ this imposes, by definition, exactly $\operatorname{rank}(\lambda)$ linearly independent conditions on $u_M(\lambda)$. The conditions for different points in $\operatorname{spec}_b(Q)$ are independent. To see this it is enough to show that there exists u_M having a given finite Taylor series at each of a finite number of points λ_i, which in turn follows by superposition with polynomial coefficients, since one can always choose $u \in \dot{\mathcal{C}}^\infty([-1,1])$ with u_M non-zero at a given point. So the range has the finite codimension given in (5.55), and is closed.

Note that it could not be concluded that the range was closed if (5.54) did not hold, and in fact it is not closed unless these conditions hold.

The other case, (5.56), can be analyzed by duality. The details are left as an exercise. However, note that the null space in (5.56) can be computed precisely in terms of the generalized null spaces of $Q(\lambda)$ for the appropriate elements of $\operatorname{spec}_b(Q)$:

$$\begin{aligned} &\left\{u \in \rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}); Qu = 0\right\} = \\ &\left\{u = \sum_{\substack{z \in \operatorname{spec}_b(Q),\ k \le \operatorname{ord}(z) \\ -\alpha < \operatorname{Im} z < -\beta}} s^{iz}(\log s)^k u_{z,k};\right. \\ &\qquad\qquad \left.\sum_{p=r}^{k} D_z^{p-r} Q(z) u_{z,p} = 0,\ 0 \le r \le k\right\}. \end{aligned} \tag{5.59}$$

Moreover the range of Q in (5.55) can be written

$$\begin{aligned} \big\{f \in \rho_0^\alpha \rho_\infty^\beta &H_b^{k-m}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}); v(f) = 0 \\ &\forall\ v \in \rho_0^{-\alpha}\rho_\infty^{-\beta} H_b^\infty(\widetilde{X}, {}^b\Omega^{\frac{1}{2}}) \text{ satisfying } Q^t v = 0\big\}. \end{aligned}$$

Here Q^t is the transpose with respect to the intrinsic (real) pairing between the spaces $\rho_0^\alpha \rho_\infty^\beta H_b^k(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$ and $\rho_0^{-\alpha}\rho_\infty^{-\beta} H_b^{-k}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}})$. Thus (5.59) can be used to express these linear constraints in a precise form.

Exercise 5.19. Extend Proposition 5.17 to the pseudodifferential case, i.e. to elliptic elements of $\Psi^m_{b,I,\mathrm{os}}(\widetilde{X}, {}^b\Omega^{\frac12})$.

5.9. Composition in the small calculus.

An argument will now be given which proves, using the standard composition properties of pseudodifferential operators on compact manifolds without boundary, the composition properties for b-pseudodifferential operators already hinted at. It should noted, again, that there is a somewhat 'better' proof (see for example [63]) which is more geometric and which generalizes more readily.[7]

Proposition 5.20. *If X is a compact manifold with boundary then*

$$\Psi^m_b(X; {}^b\Omega^{\frac12}) \circ \Psi^{m'}_b(X; {}^b\Omega^{\frac12}) \subset \Psi^{m+m'}_b(X; {}^b\Omega^{\frac12}).$$

Proof: First it can be assumed that the operators, $A \in \Psi^m_b(X; {}^b\Omega^{\frac12})$ and $B \in \Psi^{m'}_b(X; {}^b\Omega^{\frac12})$, have Schwartz kernels supported in a small neighbourhood of the corner of X^2. Indeed this can be arranged by appropriately decomposing

$$A = A' + A'',\ B = B' + B''$$

where A', and B' have the support property and A'' and B'' have kernels which vanish near the corner. Then A'' and B'' have kernels which are $\mathcal{C}^\infty$ in a neighbourhood of the complete boundary of X^2, $\partial X \times X \cup X \times \partial X$. It follows, as in (4.71), that they in turn are the sums of a pseudodifferential operator, in the usual sense, with kernel supported strictly in the interior of X^2 plus a smooth kernel vanishing to infinite order at the boundary of X^2. The latter maximally residual operator maps $\mathcal{C}^{-\infty}(X; {}^b\Omega^{\frac12})$ into $\dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac12})$, and conversely this is a characterization of these operators. From the standard theory of pseudodifferential operators, the product of two b-pseudodifferential operators with kernels supported strictly in the interior of X^2 is of the same type. If either operator is maximally residual then from the regularity properties of pseudodifferential operators so is the product. Thus it follows that $A'' \circ B''$ is the sum of a pseudodifferential operator supported in the interior of X^2 and a maximally residual operator. In fact the same is true for the products $A' \circ B''$ and $A'' \circ B'$ since if B'' or A'' is maximally residual, so is the product (using (4.70)) and if B'' or A'' has kernel supported in the interior then the kernels of A' and B' near the corner are irrelevant, i.e. they can be replaced by kernels which vanish near the corner. Since

$$A \circ B = (A' + A'') \circ (B' + B'') = A' \circ B' + A' \circ B'' + A'' \circ B' + A'' \circ B'' \tag{5.60}$$

[7] Such a proof can be based on the notion of a b-fibration discussed in Exercise 4.7 and the construction of a stretched triple product.

it remains only to consider the product $A' \circ B'$.

Let x be a boundary defining function. A neighbourhood of the boundary of X can be identified with $[0,\epsilon)_x \times \partial X$, and hence a neighbourhood of the corner of X^2 with $[0,\epsilon)_x \times \partial X \times [0,\epsilon)_{x'} \times \partial X$. As discussed in Chapter 4 a neighbourhood of the front face of X_b^2 can be identified with the closure of

$$[0,\infty)_s \times [0,\epsilon)_x \times \partial X \times \partial X,\ \ s = x/x'. \tag{5.61}$$

Then the kernel of A is, in view of (4.58), of the form

$$\kappa_A = k(x,s,y,y')\left|\frac{ds}{s}\frac{dx}{x}dydy'\right|^{\frac12},$$

where k is singular only at $s=1, y=y'$ and is rapidly vanishing with all derivatives as $s \downarrow 0$ or $s \uparrow \infty$.

The action of the operator can then be written, as in (4.68),

$$A\phi(x,y)\left|\frac{dx}{x}dy\right|^{\frac12} = \int_0^\infty \int_{\partial X} k(s,x,y,y')\phi(x/s,y')\frac{ds}{s}dy'\left|\frac{dx}{x}dy\right|^{\frac12}. \tag{5.62}$$

Writing ϕ in terms of its Mellin transform gives

$$A\phi(x,y)\left|\frac{dx}{x}dy\right|^{\frac12} = \frac{1}{2\pi}\int_{\mathbb{R}} \int_{\partial X} k_M(x,\lambda,y,y')\phi_M(\lambda,y')x^{i\lambda}\frac{ds}{s}dy'\left|\frac{dx}{x}dy\right|^{\frac12}, \tag{5.63}$$

where k_M is the Mellin transform of k with respect to the variable s.

Using the variable x' in place of x in (5.61) for the kernel of B gives

$$B\phi(x,y)\left|\frac{dx}{x}dy\right|^{\frac12} = \int_0^\infty \int_{\partial X} k'(x/s,s,y,y')\phi(x/s,y')\frac{ds}{s}dy'\left|\frac{dx}{x}dy\right|^{\frac12}. \tag{5.64}$$

Taking the Mellin transform this becomes

$$(B\phi)_M(\lambda,y)|dy|^{\frac12} = \int_{\partial X} k'_M(x',\lambda,y,y')(x')^{-i\lambda}\phi(x',y')\frac{dx'}{x'}dy'\,|dy|^{\frac12}. \tag{5.65}$$

Combining (5.63) and (5.65) expresses the composite operator as

$$AB\phi(x,y)\left|\frac{dx}{x}dy\right|^{\frac12} = \int_0^\infty \int_{\partial X} k''(x,x/s,s,y,y')\phi(x/s,y')\frac{ds}{s}dy'\left|\frac{dx}{x}dy\right|^{\frac12},$$

where the kernel is
(5.66)
$$k''(x,x',s,y,y') = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\partial X} k_M(x,\lambda,y,y'')k'_M(x',\lambda,y'',y')s^{i\lambda}d\lambda dy''.$$

In this integral x and x' are simply smooth parameters. From the properties of the indicial families, as in (5.14) and the composition formula for pseudodifferential operators on compact manifolds without boundary ([47]), it follows that k'' is indeed the kernel of an element of $\Psi_b^{m+m'}(X;{}^b\Omega^{\frac12})$, proving the proposition.

EXERCISE 5.21. Note that this analysis is directly modeled on one of the standard methods of proof of the composition formula for pseudodifferential operators on $\mathbb{R}^n$, by writing one of the operators in 'left reduced' form and the other in 'right reduced' form. See if you can directly deduce the composition formula from the composition properties of pseudodifferential operators on $\mathbb{R}^n$, by introducing $\log x$ as a variable, and localizing in the tangential variables.

COROLLARY. *The product formula* (5.18) *extends to show that the map* (5.60) *is a homomorphism; hence* (4.101) *also extends to show that* (4.98) *is a homomorphism.*

It is useful to record an extension of the argument used to prove Proposition 5.20; this will be rather helpful in handling remainder terms later.

LEMMA 5.22. *There exists an integer p such that composition of operators defines a bilinear map*

$$\begin{gathered}\dot{\mathcal{C}}^N(X_b^2;{}^b\Omega^{\frac12})\times\dot{\mathcal{C}}^N(X_b^2;{}^b\Omega^{\frac12})\subset\dot{\mathcal{C}}^{N-p}(X_b^2;{}^b\Omega^{\frac12})\ \forall\ N>p\\ \Psi_b^m(X;{}^b\Omega^{\frac12})\times\dot{\mathcal{C}}^N(X_b^2;{}^b\Omega^{\frac12})\subset\dot{\mathcal{C}}^{N-p-m}(X_b^2;{}^b\Omega^{\frac12})\ \forall\ N>p+m.\end{gathered}$$

PROOF: See (5.62) - (5.66).

5.10. Polyhomogeneous conormal distributions.

It has been shown above that the inverse of the indicial operator of an elliptic b-differential operator has a kernel which is the sum of a term in the small calculus, plus a $\mathcal{C}^\infty$ term which has a complete asymptotic expansion at the boundaries of $\mathrm{bf}(X_b^2)$. This can be taken as a strong indication of the structure of the kernels of the generalized inverses of elliptic elements of $\mathrm{Diff}_b^m(X;{}^b\Omega^{\frac12})$. The notion of a b-pseudodifferential operator will therefore be expanded accordingly. A parametrix, modulo compact errors, can then be found in this enlarged calculus just using the inverse of the indicial operator. Subsequently it will be shown that a much finer parametrix can be obtained by using iteration, but at some cost in terms of complexity.

Recall that X_b^2 has three distinct pieces of boundary: the front face, $\mathrm{bf}(X_b^2)$, and the left and right boundary faces. The extra terms added to the calculus (for the moment) will be $\mathcal{C}^\infty$ sections of ${}^b\Omega^{\frac{1}{2}}$ up to $\mathrm{bf}(X_b^2)$, but having expansions at lb and rb. These will be polyhomogeneous conormal distributions with respect to the boundary. This notion will be considered in some detail, first with respect to a boundary hypersurface of a compact manifold with boundary, X.

If $\rho \in \mathcal{C}^\infty(X)$ is a boundary defining function, *define*

$$u \in \mathcal{A}^E_{\mathrm{phg}}(X) \Longleftrightarrow u \sim \sum_{(z,k)\in E} \rho^z (\log \rho)^k a_{k,z}. \tag{5.67}$$

Here the coefficients are smooth, $a_{k,z} \in \mathcal{C}^\infty(X)$. The precise meaning of $\sim$ is discussed below.

Consider first the conditions to be imposed on E. Clearly

$$E \subset \mathbb{C} \times \mathbb{N}_0, \quad \mathbb{N}_0 = \{0, 1, \dots\} \tag{5.68}$$

is a sort of 'divisor' (because it will correspond to the singularities of the Mellin transform, see Proposition 5.27). In order for the sum in (5.67) to be reasonably sensible it is surely necessary that

$$E \text{ is discrete.} \tag{5.69}$$

Furthermore all but a finite number of terms should vanish to any fixed order at $\rho = 0$. Thus it is natural to require

$$(z_j, k_j) \in E,\ |(z_j, k_j)| \longrightarrow \infty \implies \operatorname{Re} z_j \longrightarrow \infty. \tag{5.70}$$

Multiplying ρ by a positive $\mathcal{C}^\infty$ function clearly produces from any power of $\log \rho$ the lower powers of $\log \rho$. Since the space is supposed to be independent of the choice of ρ it is also natural to require

$$(z, k) \in E \Longrightarrow (z, l) \in E,\ l \in \mathbb{N}_0,\ 0 \le l \le k. \tag{5.71}$$

Generally the space is to be a $\mathcal{C}^\infty(X)$-module, i.e. to be preserved under multiplication by $\mathcal{C}^\infty(X)$, and this will follow from:

$$(z, k) \in E \Longrightarrow (z + j, k) \in E,\ j \in \mathbb{N}_0. \tag{5.72}$$

DEFINITION 5.23. An index set (or sometimes a $\mathcal{C}^\infty$ index set for emphasis) is a subset (5.68) which satisfies (5.69), (5.70), (5.71) and (5.72).

The last condition, (5.72) appears because the coefficients in (5.67) are $\mathcal{C}^\infty$ functions. They therefore have Taylor expansions in powers of ρ. Thus (5.72) might as well be imposed since it corresponds to the 'lower order terms' in such an expansion. The discussion of the model problem above takes place on a space (the normal bundle) with an $\mathbb{R}^+$-action. Then it makes sense to choose a homogeneous defining function and demand that the coefficients in (5.67) be homogeneous of degree zero. In that case it is not necessary to impose the condition that the space be a $\mathcal{C}^\infty(X)$-module, and then (5.72) will be dropped. A set satisfying all the conditions except this last one will be called an *absolute* index set. As will become clear below, $\mathcal{C}^\infty$ index sets often arise simply by adding exponents to an absolute index set to get (5.72). In doing so rather unpleasant complications arise with (5.71). This is the conundrum of accidental multiplicities.

If E is an index set then (5.67) is to be interpreted as requiring

$$\begin{aligned} &u \in \mathcal{A}^E_{\mathrm{phg}}(X) \text{ iff } \exists\ a_{z,k} \in \mathcal{C}^\infty(X)\ \forall\ (z,k) \in E \\ &\text{s.t. } \forall\ N,\ u - \sum_{\substack{(z,k)\in E \\ \operatorname{Re} z \le N}} \rho^z (\log \rho)^k a_{z,k} \in \dot{\mathcal{C}}^N(X), \end{aligned} \tag{5.73}$$

where $\dot{\mathcal{C}}^N(X)$ is the space of functions which are N times differentiable on X and vanish at ∂X with all derivatives up to order N. The condition (5.70) means that the sum in (5.73) is always finite. Clearly $\mathcal{C}^\infty(X) \cdot \dot{\mathcal{C}}^N(X) = \dot{\mathcal{C}}^N(X)$ so indeed

$$\mathcal{C}^\infty(X) \cdot \mathcal{A}^E_{\mathrm{phg}}(X) = \mathcal{A}^E_{\mathrm{phg}}(X),$$

for any index set because of (5.72). Moreover the same definition can be used for the space $\mathcal{A}^E_{\mathrm{phg}}(X;F)$, where F is any vector bundle F over X, just replacing $a_{z,k} \in \mathcal{C}^\infty(X)$ with $a_{z,k} \in \mathcal{C}^\infty(X;F)$.

It is reassuring to know that there are many functions with non-trivial expansions and this is shown by a lemma of Borel which in present circumstances takes the form:

LEMMA 5.24. *Let X be a compact manifold with boundary, let E be an index set and suppose $a_{z,k} \in \mathcal{C}^\infty(X)$ is given for each $(z,k) \in E$ then there exists $u \in \mathcal{A}^E_{\mathrm{phg}}(X)$ satisfying (5.67) and if $u' \in \mathcal{A}^E_{\mathrm{phg}}(X)$ is any other element with the same expansion then $u' - u \in \dot{\mathcal{C}}^\infty(X)$.*

PROOF: Choose $\mu \in \mathcal{C}^\infty_c([0,\infty))$ with $\mu(x) = 1$ on $[0,1]$. It is straightforward to show that if $\epsilon_{z,k} \in (0,1)$ are positive constants then for each $N \in \mathbb{N}$ the series

$$\sum_{\substack{(z,k)\in E \\ \operatorname{Re} z > N}} \mu\Big(\frac{\rho}{\epsilon_{z,k}}\Big) \rho^z (\log \rho)^k a_{z,k} \text{ converges absolutely in } \dot{\mathcal{C}}^N(X) \tag{5.74}$$

provided the $\epsilon_{z,k}$ decrease fast enough. More precisely there is, for each N, a sequence $\epsilon^{(N)}_{z,k}$ such that (5.74) holds for that N provided

$$\epsilon_{z,k} < \epsilon^{(N)}_{z,k} \ \forall \ \operatorname{Re} z > N.$$

Since $\operatorname{Re} z < N$ eventually because of (5.70), all these conditions taken together still reduce to only a finite number of conditions on each $\epsilon_{z,k}$. Therefore these constants can be chosen in such a way that (5.74) holds for all N. Then the series converges,

$$u = \sum_{(z,k)\in E} \mu(\frac{\rho}{\epsilon_{z,k}})\rho^z (\log\rho)^k a_{z,k} \in \mathcal{A}^E_{\mathrm{phg}}(X),$$

to an element satisfying (5.67).

The essential uniqueness of this asymptotic sum follows directly from the definition and the fact that

$$\dot{\mathcal{C}}^\infty(X) = \bigcap_N \dot{\mathcal{C}}^N(X).$$

These polyhomogeneous conormal distributions are intimately associated to the b-pseudodifferential operators, as will be seen below (see Proposition 5.59) since each element of the null space of a b-elliptic operator is polyhomogeneous. For the moment consider the more prosaic fact that the b-pseudodifferential operators act on the polyhomogeneous conormal distributions:

Proposition 5.25. *If X is a compact manifold with boundary then for any index set E each $A \in \Psi^*_b(X; {}^b\Omega^{\frac12})$ defines an operator*

$$A\colon \mathcal{A}^E_{\mathrm{phg}}(X; {}^b\Omega^{\frac12}) \longrightarrow \mathcal{A}^E_{\mathrm{phg}}(X; {}^b\Omega^{\frac12}).$$

Proof: The proof of (4.70) in Proposition 4.29 actually shows that

$$A\colon \dot{\mathcal{C}}^N(X; {}^b\Omega^{\frac12}) \longrightarrow \dot{\mathcal{C}}^{N'}(X; {}^b\Omega^{\frac12}) \text{ where } N' \longrightarrow \infty \text{ as } N \longrightarrow \infty.$$

Thus it suffices to show that the structure of the finite sum in (5.73) is preserved, i.e.

$$(5.75)\qquad \begin{gathered} A\left(x^z \sum_{k\le l}(\log x)^l a_k\right) = x^z \sum_{k\le l}(\log x)^l b_l, \\ a_l \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac12}) \Longrightarrow b_l \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac12}). \end{gathered}$$

In case $l = 0$ (5.75) just reduces to

$$x^{-z} A x^{z} : \mathcal{C}^{\infty}(X; {}^{b}\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{\infty}(X; {}^{b}\Omega^{\frac{1}{2}}) \tag{5.76}$$

and so follows from the invariance of $\Psi_{b}^{m}(X; {}^{b}\Omega^{\frac{1}{2}})$, for each m, under conjugation by complex powers of x, as shown in Proposition 5.7 and from the fact that the elements of the small calculus preserve $\mathcal{C}^{\infty}$ regularity, shown in Proposition 4.34. The general case follows by differentiating (5.76) with respect to z.

If X is a manifold with corners various extensions of $\mathcal{A}_{\mathrm{phg}}^{E}(X)$ can be defined, although fortunately only particular cases occur here (see [63] for the general case). Let $H_1, \dots, H_k$ be the boundary hypersurfaces of X. By an *index family* for X is meant $\mathcal{E} = (E_1, \dots E_k)$, where each E_j is an index set associated to H_j. If all but one of the E_j are the special index set $E = \{(l, 0); l \in \mathbb{N}_0\}$, which will simply be written '0' and corresponds to

$$\mathcal{A}_{\mathrm{phg}}^{E}(X) = \mathcal{A}_{\mathrm{phg}}^{0}(X) = \mathcal{C}^{\infty}(X),$$

then the definition (5.73) can still be used, with ρ replaced by ρ_p, corresponding to the one non-trivial index set:

$$\begin{gathered} \mathcal{E} = (E_1, \dots E_k), E_j = 0, j \neq p, \text{ then } u \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(X) \Longleftrightarrow \\ \exists\ a_{z,k} \in \mathcal{C}^{\infty}(X)\ \forall\ (z,k) \in E_p \text{ s.t. } u \sim \sum_{(z,k) \in E_p} \rho_p^{z} (\log \rho_p)^{k} a_{z,k}. \end{gathered} \tag{5.77}$$

In the definition, (5.73), of $\sim$ the error term should be taken as the space of functions which are N times differentiable and vanish to order N at H_p. More generally the definition can be extended to the case of an index family $\mathcal{E} = (E_1, \dots, E_k)$ with the property that the boundary hypersurfaces which correspond to index sets other than 0 are all disjoint. That is,

$$\begin{gathered} \mathcal{E} = (E_1, \dots, E_k) \text{ where } E_j = 0 \text{ for } j \notin I \subset \{1, \dots, k\}, \text{ and} \\ j, k \in I,\ j \neq k \Longrightarrow H_j \cap H_k = \emptyset. \end{gathered} \tag{5.78}$$

The expansions at the 'non-trivial' boundary hypersurfaces are then independent of one another. Choose a partition of unity

$$\begin{gathered} \phi_j \in \mathcal{C}^{\infty}(X),\ j \in I,\ \sum_{j \in I} \phi_j = 1 \quad \text{ on } X \\ \operatorname{supp}(\phi_j) \cap H_p = \emptyset \quad \text{ if } j, p \in I,\ j \neq p. \end{gathered} \tag{5.79}$$

Then set

$$\mathcal{E}_j = (E_1^{(j)}, \dots, E_k^{(j)}),\ E_l^{(j)} = \begin{cases} 0 & l \neq j, \\ E_j & l = j. \end{cases} \quad \text{for each } j \in I.$$

This is an index family of the type (5.77) for each $j \in I$, so the previous definition applies and

$$u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X; F) \Longleftrightarrow \phi_j u \in \mathcal{A}^{\mathcal{E}_j}_{\mathrm{phg}}(X; F)\ \forall\ j \in I.$$

This is clearly independent of the choice of partition of unity satisfying (5.79) since for another such partition, ϕ'_j, $(\phi'_j - \phi_j)u$ is always $\mathcal{C}^\infty$ and supported away from the boundary hypersurfaces H_j, $j \in I$. Later a case where the index sets on two boundary hypersurfaces which meet are both non-trivial will have to be considered.

If $\mathcal{E}$ is an index family as in (5.78) for a manifold with corners, X, let $Y = H_j$ be a boundary hypersurface with $j \notin I$, i.e. the index set for $Y = H_j$ is 0. Then each element of $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X; F)$ is smooth up to Y. Now Y is also a manifold with corners where each boundary hypersurface is a component of the intersection of a boundary hypersurface of X with Y. Let $\mathcal{F}$ be the index family for Y, which associates to each boundary hypersurface the index set of the boundary hypersurface of X from which it arises.

Lemma 5.26. *If $\mathcal{E}$ is an index family satisfying (5.78) for X and $Y = H_j$, for some $j \notin I$, is a boundary hypersurface of X with index set 0 then with $\mathcal{F}$ defined as above,*

$$\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X; F) \ni u \longmapsto u_{|Y} \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(Y; F) \tag{5.80}$$

is surjective.

Proof: The expansion for $u_{|Y}$ required for (5.80) follows immediately from that for u. To see the surjectivity first extend a given element $v \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(Y; F)$ off Y into X to be independent of a normal variable near Y. Multiplying by $\phi \in \mathcal{C}^\infty(X)$ which is 1 near Y and has support sufficiently close to Y clearly gives an element of $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X; F)$ which restricts to Y to give v.

5.11. Mellin transform and polyhomogeneity.

There is an intimate connection between the Mellin transform and polyhomogeneity at a boundary. In Lemma 5.16 the Mellin transform of elements of weighted Sobolev spaces on $\widetilde{X}$ was considered. By choosing a trivialization of the normal bundle these are just results about the Mellin transform of functions on $[-1, 1]_r \times \partial X$ in which ∂X is really a space of parameters. A useful characterization of polyhomogeneous conormal functions is easily obtained. It has really already been used in the proof of Lemma 5.14 and has therefore guided the definition of polyhomogeneity, so should not come as a surprise:

Proposition 5.27. *Let Y be a compact manifold with corners and let $\mathcal{E} = (E, \emptyset, 0, \dots, 0)$ be the index family for $[-1,1]\times Y$ which assigns index set E to $\{-1\}\times Y$, the trivial index set $\emptyset$ to $\{1\}\times Y$ and the $\mathcal{C}^\infty$ index set to all boundary faces of Y, then the Mellin transform*

$$u_M(\lambda, y) = \int_0^\infty x^{-i\lambda} u(\frac{x-1}{x+1}, y)\frac{dx}{x}$$

gives an isomorphism from $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}([-1,1]\times Y)$ to the space of meromorphic functions with values in $\mathcal{C}^\infty(Y)$ having poles of order k only at points $\lambda = -iz \in \mathbb{C}$ such that $(z, k-1) \in E$ and satisfying for each large N

$$\|u_M(\lambda,\cdot)\|_N \le C_N(1+|\lambda|)^{-N} \text{ in } |\operatorname{Im}\lambda| \le N, |\operatorname{Re}\lambda| \ge C_N, \tag{5.81}$$

where $\|\cdot\|_N$ is a norm on $\mathcal{C}^N(Y)$.

The estimates (5.81) are consistent with the meromorphy of u_M since there are only finitely many poles in any strip $\operatorname{Im}\lambda \ge -N$.

Proof: If $u \in \mathcal{C}^N([-1,1]\times Y)$ vanishes with its first N derivatives at $\{-1\}\times Y$ and $\{1\}\times Y$ then the integral defining u_M converges absolutely for $|\operatorname{Im} z| < N$ and, using the identities as in (5.5), satisfies (5.81) with N replaced by $N-1$. In particular the desired estimates hold if $E = \emptyset$ with u_M entire. It can therefore be assumed that u has support near $\{-1\}\times Y$ and the variable x can be used, so that $[-1,1]\times Y$ is replaced by $[0,\infty)\times Y$, with u supported in $x < 1$. The remainder terms in the expansion (5.77) become arbitrarily smooth and vanish with their derivatives at $x = 0$, so to prove (5.81) it suffices to check the estimates, and the meromorphy, for each term in the expansion.

Thus it is enough to suppose that u is of the form

$$u = x^z(\log x)^k\phi(x)\psi(y),\ \ \psi\in\mathcal{C}^\infty(Y),$$

where $\phi \in \mathcal{C}^\infty_c([0,\infty))$ takes the value 1 in $x \le \frac{1}{2}$. Then the Mellin transform of u is just

$$u_M(\lambda, y) = i^k\left(\frac{\partial}{\partial\lambda}\right)^k v_M(\lambda + iz, y),\ \ v = \phi(x)\psi(y).$$

Thus it is enough to show that v_M has only a simple pole at $\lambda = 0$ and satisfies (5.81). The meromorphy property follows by writing the Mellin transform as

$$v_M(\lambda, y) = \psi(y)[\frac{i}{\lambda}2^{-i\lambda} + \int_{\frac{1}{2}}^1 x^{-i\lambda}\phi(x)\frac{dx}{x}]$$

and noting that the second term is entire. The estimates (5.81) follow from integration by parts, since

$$\lambda v_M(\lambda) = w_M(\lambda),\ w(x) = -ix\frac{\partial}{\partial x}\phi(x)\psi(y).$$

This shows that the range of the Mellin transform has the properties indicated. Moreover u can be recovered from its Mellin transform by (5.3) for $r >> 0$. Thus it suffices to show that u, given by this formula from u_M with the meromorphy property and satisfying (5.81) is, in $\mathcal{A}^{\mathcal{E}}_{\text{phg}}([-1,1]\times Y)$. This is essentially a repetition of the proof of Lemma 5.14, with the terms in the expansion arising from Cauchy's formula, so the details are omitted.

5.12. Boundary terms.

With this preamble additional boundary terms can now be added to the calculus. Let E_{lb} and E_{rb} be index sets associated to $\text{lb}(X^2_b)$ and $\text{rb}(X^2_b)$ and set $\mathcal{E}' = (E_{\text{lb}}, E_{\text{rb}}, 0)$, corresponding to the ordering lb, rb, bf of the boundary hypersurfaces of X^2_b. Then define

$$\widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) \overset{\text{def}}{=} \mathcal{A}^{\mathcal{E}'}_{\text{phg}}(X^2_b;{}^b\Omega^{\frac12}). \tag{5.82}$$

In this notation, $\mathcal{E} = (E_{\text{lb}}, E_{\text{rb}})$ is thought of as an index family for X^2, $\text{lb} = \partial X \times X$, $\text{rb} = X \times \partial X$. This is only part of the full calculus; there is still just a little bit more to come. A parametrix for an elliptic differential operator will be found in the space

$$\widetilde{\Psi}^{m,\mathcal{E}}_{b,\text{os}}(X;{}^b\Omega^{\frac12}) \overset{\text{def}}{=} \Psi^m_{b,\text{os}}(X;{}^b\Omega^{\frac12}) + \widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}). \tag{5.83}$$

Notice that an element $A \in \widetilde{\Psi}^{m,\mathcal{E}}_{b,\text{os}}(X;{}^b\Omega^{\frac12})$, or rather its kernel, can be characterized by taking a partition of unity

$$\phi_{\text{lb}}, \phi_{\text{rb}}, \phi_\Delta \in \mathcal{C}^\infty(X^2_b),\ \phi_{\text{lb}} + \phi_{\text{rb}} + \phi_\Delta = 1$$
$$\text{supp}(\phi_{\text{lb}}) \cap [\text{rb} \sqcup \Delta_b] = \emptyset$$
$$\text{supp}(\phi_{\text{rb}}) \cap [\text{lb} \sqcup \Delta_b] = \emptyset$$
$$\text{supp}(\phi_\Delta) \cap [\text{lb} \sqcup \text{rb}] = \emptyset$$

and then

$$A \in \widetilde{\Psi}^{m,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) \Longleftrightarrow$$
$$\phi_\Delta A \in \Psi^m_b(X;{}^b\Omega^{\frac12}),$$
$$\phi_{\text{lb}} A \in \mathcal{A}^{(E_{\text{lb}},0,0)}_{\text{phg}}(X^2_b;{}^b\Omega^{\frac12}),$$
$$\phi_{\text{rb}} A \in \mathcal{A}^{(0,E_{\text{rb}},0)}_{\text{phg}}(X^2_b;{}^b\Omega^{\frac12}).$$

From this it follows that the intersection of the small calculus and the extra boundary calculus is just the residual space for the diagonal symbol:

$$\Psi^m_b(X;{}^b\Omega^{\frac12}) \cap \widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) = \Psi^{-\infty}_b(X;{}^b\Omega^{\frac12}) \quad \forall\ m, \mathcal{E}.$$

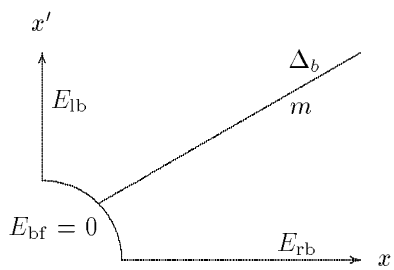

Figure 10. Singularities of $A \in \widetilde{\Psi}_b^{m,\mathcal{E}}(X; {}^b\Omega^{\frac{1}{2}})$.

5.13. True parametrix.

To get started on the parametrix construction recall what was shown above about the inverse Mellin transform of the inverse of the indicial family of an elliptic operator. Restating Lemma 5.16 in terms of the new notation for polyhomogeneous conormal distributions

(5.84)
$$Q \in \mathrm{Diff}^k_{b,I}(\widetilde{X}; {}^b\Omega^{\frac{1}{2}}) \text{ elliptic, } \alpha \notin -\operatorname{Im}\operatorname{spec}_b(Q) \Longrightarrow$$
$$K(s,\cdot) = \frac{1}{2\pi} \int\limits_{\operatorname{Im}\lambda = -\alpha} s^{i\lambda} \widehat{Q}(\lambda)^{-1} d\lambda = A + B,\ B \in \mathcal{A}^{\mathcal{E}(\alpha)}_{\mathrm{phg}}(\mathrm{bf}(X^2_b); {}^b\Omega^{\frac{1}{2}}).$$

Here A is just the 'crude' parametrix discussed in Chapter 4. The front face $\mathrm{bf}(X^2_b)$ has a natural $\mathbb{R}^+$-action and the index family

$$\mathcal{E}(\alpha) = (E_{\mathrm{lb}}, E_{\mathrm{rb}}) = (E^+(\alpha), E^-(\alpha))$$

for bf correspondingly consists of two absolute index sets:

(5.85)
$$E^{\pm}(\alpha) = \big\{(z,k); \widehat{Q}(\lambda)^{-1} \text{ has a pole at } \lambda = \mp iz$$
$$\text{of order at most } k+1 \text{ and } \pm\operatorname{Im}\lambda < \mp\alpha\big\}.$$

Recall that the *order* of a pole is the least integer l such that $(\lambda + iz)^l \widehat{Q}(\lambda)^{-1}$ is regular near $\lambda = -iz$. Using this consider the smallest $\mathcal{C}^\infty$ index sets containing $E^{\pm}(\alpha)$:

(5.86) $$\widetilde{E}^{\pm}(\alpha) = \big\{(z,k) \in \mathbb{C} \times \mathbb{N}_0; (z-r,k) \in E^{\pm}(\alpha) \text{ for some } r \in \mathbb{N}_0\big\}.$$

A little more notation is also useful for the spaces of b-pseudodifferential operators. Namely let

$$\rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \subset \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}),$$

with a formal coefficient ρ_{bf}, stand for the subspace of operators with kernel of the form $\rho_{\mathrm{bf}}B$, where B is in $\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12})$.

PROPOSITION 5.28. *If $P \in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12})$ is elliptic then for each $\alpha \in \mathbb{R}$ such that $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ there exists*

$$G_\alpha \in \widetilde{\Psi}_b^{-k,\widetilde{\mathcal{E}}(\alpha)}(X;{}^b\Omega^{\frac12}), \tag{5.87}$$

where $\widetilde{\mathcal{E}}(\alpha) = (\widetilde{E}^+(\alpha), \widetilde{E}^-(\alpha))$ consists of the index sets given by (5.86) and

$$P \circ G_\alpha = \mathrm{Id} - R_\alpha,\ R_\alpha \in \rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\widetilde{\mathcal{E}}(\alpha)}(X;{}^b\Omega^{\frac12}). \tag{5.88}$$

Clearly it is necessary to analyze $P \circ G_\alpha$ where G_α is as in (5.87). Using the decomposition (5.83) and (4.80), only the action on the new part of the calculus is needed:

LEMMA 5.29. *For any index family $\mathcal{E} = (E_{\mathrm{lb}}, E_{\mathrm{rb}})$ and $P \in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12})$ composition gives*

$$\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \ni G \longmapsto P \circ G \in \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}). \tag{5.89}$$

PROOF: Since $\mathrm{Diff}_b^k(X)$ is generated by $\mathcal{V}_b(X)$ which lifts from the left factor of X^2 into $\mathcal{V}_b(X_b^2)$ (Lemma 4.3) it suffices to show that

$$\mathcal{V}_b(X_b^2) \cdot \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(X_b^2) \subset \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(X_b^2)$$

which is rather obvious, since it amounts to the observation that if $k \ge 1$

$$x\frac{\partial}{\partial x}\left(x^z(\log x)^k\right) = zx^z(\log x)^k + kx^z(\log x)^{k-1}$$

and of course $x\partial x^z/\partial x = z\,x^z$. This, by the way, is a good reason for having the condition (5.71) on index sets.

Not only does this argument prove the lemma but it gives a little more. Namely just as in (4.89) the indicial homomorphism can be defined for the boundary terms by

$$I : \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \ni B \longmapsto B_{|\mathrm{bf}(X_b^2)} \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(\mathrm{bf}(X_b^2);{}^b\Omega^{\frac12}), \tag{5.90}$$

where on the right $\mathcal{E}$ is interpreted as an index family for $\mathrm{bf}(X_b^2)$. Then

$$I(P\circ B) = I(P)\circ I(B),\ \ P\in \mathrm{Diff}_b^k(X;{}^b\Omega^{\frac12}),\ \ B\in \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}).$$

Here the indicial operator of P is acting as a b-differential operator on the front face of X_b^2. This is enough preliminary orientation to begin the

PROOF OF PROPOSITION 5.28: The subspace

$$\rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\widetilde{\mathcal{E}}(\alpha)}(X;{}^b\Omega^{\frac12})\subset \widetilde{\Psi}_b^{-\infty,\widetilde{\mathcal{E}}(\alpha)}(X;{}^b\Omega^{\frac12})$$

is just the null space of the indicial homomorphism (5.90). Indeed the extension result in Lemma 5.24 shows that

$$\begin{aligned}0\longrightarrow \rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12})&\hookrightarrow\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12})\\ &\xrightarrow{I}\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(\mathrm{bf}(X_b^2);{}^b\Omega^{\frac12})\longrightarrow 0\end{aligned}\tag{5.91}$$

is exact. If $A\in\Psi_b^{-k}(X;{}^b\Omega^{\frac12})$ is the 'small' parametrix of Proposition 4.38 then $G=A+G'$ satisfies (5.88) provided

$$I(P)\circ I(G') = \mathrm{Id}-I(P)\circ I(A).$$

That this has a solution of the desired type is the content of (5.84), i.e. it is only necessary to take $I(G')=B$, which is possible because of (5.91).

Lemma 5.29 can be extended to the case that P itself is a b-pseudo-differential operator but to do so requires a further composition formula, at least extending (5.89) to the case $P\in\Psi_b^m(X;{}^b\Omega^{\frac12})$. This is done in Proposition 5.46 below.

5.14. Finitely residual terms.

In (5.42) the weighted b-Sobolev spaces were defined for a manifold with boundary. The definition uses the Lie algebra of smooth vector fields, $\mathcal{V}_b(X)$, in the form of its (filtered) enveloping algebra $\mathrm{Diff}_b^m(X)$, and the L^2 space of b-half densities. The elements are well defined on a manifolds with corners, such as X^2 and X_b^2. Replacing X by either of these spaces in (5.42) defines the weighted Sobolev spaces, at least of positive order. The blow-down map gives an isomorphism

$$\beta_b^*\colon L^2(X^2;{}^b\Omega^{\frac12})\longleftrightarrow L^2(X_b^2;{}^b\Omega^{\frac12})\tag{5.92}$$

because of (4.55) and this extends to all regularity orders.

Lemma 5.30. *The blow-down map* (4.1) *gives an isomorphism*

$$\beta_b^* : H_b^m(X^2; {}^b\Omega^{\frac12}) \longleftrightarrow H_b^m(X_b^2; {}^b\Omega^{\frac12}), \ \forall\ m \in \mathbb{N}. \tag{5.93}$$

Proof: This follows directly from (5.92) and Lemma 4.10.

Exercise 5.31. Show that (5.93) extends to an isomorphism for all $m \in \mathbb{Z}$.

Since X^2 has two boundary hypersurfaces (assuming that the boundary is connected) and X_b^2 has three, there is a corresponding variety of weighted spaces:

$$\begin{gathered}\rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta} H_b^m(X^2; {}^b\Omega^{\frac12}) = \\ \left\{u \in \mathcal{C}^{-\infty}(X^2; {}^b\Omega^{\frac12}); u = \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta} v, v \in H_b^m(X^2; {}^b\Omega^{\frac12})\right\}\end{gathered} \tag{5.94}$$

$$\begin{gathered}\rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}\rho_{\mathrm{bf}}^{\gamma} H_b^m(X_b^2; {}^b\Omega^{\frac12}) = \\ \left\{u \in \mathcal{C}^{-\infty}(X_b^2; {}^b\Omega^{\frac12}); u = \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}\rho_{\mathrm{bf}}^{\gamma} v, v \in H_b^m(X^2; {}^b\Omega^{\frac12})\right\},\end{gathered} \tag{5.95}$$

where α, β and c are real constants. Under the blow-down map

$$\beta_b^*\rho_{\mathrm{lb}} = \rho_{\mathrm{bf}}\rho_{\mathrm{lb}},\ \beta_b^*\rho_{\mathrm{rb}} = \rho_{\mathrm{bf}}\rho_{\mathrm{rb}},$$

so from (5.93), (5.94) and (5.95) it follows that

$$\beta_b^* : \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta} H_b^m(X^2; {}^b\Omega^{\frac12}) \longleftrightarrow \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}\rho_{\mathrm{bf}}^{\alpha+\beta} H_b^m(X_b^2; {}^b\Omega^{\frac12}). \tag{5.96}$$

The elements of $\rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta} H_b^m(X^2; {}^b\Omega^{\frac12})$ can be considered as operators. In particular the remainder term in (5.88) is of this type. For an index set let

$$\inf E = \min\{\operatorname{Re} z; (z, 0) \in E\} \tag{5.97}$$

measure the smallest power which can occur, then:

Lemma 5.32. *For any index family* $\mathcal{E} = (E_{\mathrm{lb}}, E_{\mathrm{rb}})$ *for* X^2

$$\begin{gathered}\rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X; {}^b\Omega^{\frac12}) \subset \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta} H_b^{\infty}(X^2; {}^b\Omega^{\frac12}) \textit{ provided} \\ \alpha < \inf E_{\mathrm{lb}},\ \beta < \inf E_{\mathrm{rb}} \textit{ and } \alpha + \beta < 1.\end{gathered} \tag{5.98}$$

Proof: If A is the kernel of an element of the space on the left in (5.98) then

$$\begin{gathered}\operatorname{Diff}_b^*(X_b^2; {}^b\Omega^{\frac12})A \subset \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}\rho_{\mathrm{bf}} L^{\infty}(X_b^2; {}^b\Omega^{\frac12}) \text{ provided} \\ \alpha < \inf E_{\mathrm{lb}}, \beta < \inf E_{\mathrm{rb}}.\end{gathered}$$

For any $\epsilon > 0$

$$\rho_{\mathrm{lb}}^{\epsilon}\rho_{\mathrm{rb}}^{\epsilon}\rho_{\mathrm{bf}}^{\epsilon} L^{\infty}(X_b^2; {}^b\Omega^{\frac12}) \subset L^2(X_b^2; {}^b\Omega^{\frac12}),$$

so $A \in \rho_{\mathrm{lb}}^{\alpha-\epsilon}\rho_{\mathrm{rb}}^{\beta-\epsilon}\rho_{\mathrm{bf}}^{1-\epsilon} \in H_b^{\infty}(X_b^2; {}^b\Omega^{\frac12})$ and then (5.98) follows from (5.96).

By definition the space $L^2(X^2; {}^b\Omega^{\frac12})$ consists of the Hilbert-Schmidt operators on $L^2(X; {}^b\Omega^{\frac12})$. Conjugating by powers of ρ shows that

Lemma 5.33. *The space $\rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}H_b^{\infty}(X^2;{}^b\Omega^{\frac12})$ consists of Hilbert-Schmidt operators on $\rho^{\alpha}L^2(X;{}^b\Omega^{\frac12})$, provided $\beta \ge -\alpha$*

In particular:

Corollary. *In Proposition 5.28 the remainder term for the parametrix satisfies*

$$R_\alpha \in \rho_{\mathrm{lb}}^{\alpha}\rho_{\mathrm{rb}}^{\beta}H_b^{\infty}(X^2;{}^b\Omega^{\frac12}) \text{ provided} \\ \alpha < \inf E^+(\alpha),\ \beta < \inf E^-(\alpha),\ \alpha+\beta<1 \tag{5.99}$$

and is therefore Hilbert-Schmidt on $\rho^{\alpha}L^2(X;{}^b\Omega^{\frac12})$.

5.15. Boundedness on Sobolev spaces.

To use the parametrix effectively it is important to know its boundedness properties, particularly on the b-Sobolev spaces.

Theorem 5.34. *Any $A \in \widetilde{\Psi}_b^{m,\mathcal{E}}(X;{}^b\Omega^{\frac12})$, $m \in \mathbb{Z}$, is bounded as an operator*

$$A\colon \rho^{\alpha}H_b^M(X;{}^b\Omega^{\frac12}) \longrightarrow \rho^{\beta}H_b^{M-m}(X;{}^b\Omega^{\frac12}),\ M\in\mathbb{Z}, \\ \text{provided } \beta\le\alpha,\ \alpha+\inf E_{\mathrm{rb}}>0,\ \inf E_{\mathrm{lb}}>\beta. \tag{5.100}$$

Proof: The operator can be divided into a part in the small calculus and a part of order $-\infty$. For the part in the small calculus the elegant symbolic argument of Hörmander ([46]) can be used. Since the small calculus is invariant under conjugation by complex powers of defining functions, it suffices to take $\alpha=\beta=0$ in this case and show that

$$A \in \Psi_b^m(X;{}^b\Omega^{\frac12}) \Longrightarrow A\colon H_b^M(X;{}^b\Omega^{\frac12}) \longrightarrow H_b^{M-m}(X;{}^b\Omega^{\frac12}). \tag{5.101}$$

Hörmander's argument reduces (5.101) to the case $m=-\infty$ which will be assumed for the moment, since it has to be checked separately anyway. Since parametrices have been obtained for elliptic operators it is actually enough to prove (5.101) for $M=m=0$. Then a standard symbolic argument (see [46]) allows the extraction of an approximate square root, in the sense that

$$-A^*A+C = B^*B-R,\ B\in\Psi_b^0(X;{}^b\Omega^{\frac12}),\ R\in\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12}), \tag{5.102}$$

provided $C>0$ is large enough that the symbol of $-A^*A+C$ is strictly positive near infinity (see Exercise 5.35). From (5.102) it follows that

$$\langle Au,Au\rangle \le C\|u\|^2 + \langle Ru,u\rangle.$$

Thus (5.100) needs only to be checked when $m = -\infty$.

Using the general composition result it is enough to show that

$$|\langle Ru, u\rangle| \le C\|u\|^2, u \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}), \tag{5.103}$$

for these operators. By multiplying the operator from both left and right by powers, it also suffices to consider the case $\alpha = \beta = 0$ with the assumption on the index sets transformed to

$$\inf E_{\mathrm{rb}}, \inf E_{\mathrm{lb}} > 0.$$

If the kernel vanishes at $\mathrm{bf}(X^2_b)$ then Lemmas 5.32 and 5.33 apply. Thus it can be assumed that the kernel has support near bf. It is also convenient to divide it again into two parts, one with support away from rb and the other with support away from lb. In fact it suffices to consider the first one of these since the other is the adjoint of such an operator so will also satisfy (5.103). Again using Lemma 5.32 the kernel can be supposed to have support near bf and u in (5.103) can be supposed to have its support in $x < \epsilon$. Then the representation (4.68) can be used:

$$\langle Ru, u\rangle = \int_\delta^\infty \int_0^\epsilon \int_{\partial X\times\partial X} u(x,y)\kappa(x,s,y,y')u(x/s,y')\frac{ds}{s}\frac{dx}{x}dydy'.$$

This can be estimated by the Cauchy-Schwarz inequality, using some small $r > 0$:

$$\begin{aligned}|\langle Ru, u\rangle|^2 \le &\int_\delta^\infty \int_0^\epsilon \int_{\partial X\times\partial X} |s^{-r}\kappa(x,s,y,y')u(x,y)|^2\frac{ds}{s}\frac{dx}{x}dydy' \\ &\times \int_\delta^\infty \int_0^\epsilon \int_{\partial X\times\partial X} |s^r u(x/s,y')|^2\frac{ds}{s}\frac{dx}{x}dydy'.\end{aligned}$$

Both integrals on the right are bounded by the square of the L^2 norm, so the theorem is proved.

Exercise 5.35. Go through the construction of B in (5.102). The symbol of B should be $c_0 = (C - \sigma(A)^2)^{\frac{1}{2}}$. Check that if C is large enough this is well-defined as a symbol modulo $S^{-\infty}$. Find a self-adjoint operator B_0 with symbol c_0 and check that

$$R_1 = B_0^2 + A^*A - C \in \Psi_b^{-1}(X; {}^b\Omega^{\frac{1}{2}}).$$

Next look for $B_{(1)} = B_0 + B_1$ with B_1 self-adjoint and or order -1. Deduce that the symbol of B_1 should satisfy

$$2c_0\sigma_{-1}(B_1) = \sigma_{-1}(R_1).$$

Show that this has a self-adjoint solution and that the remainder term

$$R_2 = B_1^2 + A^*A - C$$

is of order -2. Now give a similar argument which serves as the inductive step to show that for every $k \in \mathbb{N}$ there exists a self-adjoint B_k of order $-k$ such that

$$R_{k+1} = \left[\sum_{j=0}^{k} B_j\right]^2 + A^*A - C \in \Psi_b^{-k-1}(X; {}^b\Omega^{\frac{1}{2}}).$$

Finally use asymptotic summation to construct B satisfying (5.102), making sure that it is self-adjoint.

5.16. Calculus with bounds.

Although a considerable effort is expended below to keep rather detailed information on the asymptotic expansions of kernels, it is convenient for various purposes to have available operators satisfying only *conormal bounds.* This is quite analogous, for the usual calculus of pseudodifferential operators, to the relationship between the general symbol estimates (4.42) and the polyhomogeneous symbols in (4.43). Although the kernels will only satisfy conormal bounds at lb and rb, some smoothness up to $\mathrm{bf}(X_b^2)$ will be maintained.

Let $\mathcal{W}$ be the space of $\mathcal{C}^\infty$ vector fields on X_b^2 which are tangent to lb and rb but may be transversal to bf . Choose a non-vanishing section $\nu \in \mathcal{C}^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$ and then consider

(5.104)
$$\left\{K \in \rho_{\mathrm{bf}}^{-\frac{1}{2}} H_b^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}}); K = K'\nu,\ \mathcal{W}^p K' \subset \rho_{\mathrm{bf}}^{-\frac{1}{2}} H_b^\infty(X_b^2)\ \forall\ p \in \mathbb{N}\right\}.$$

The factor of $\rho_{\mathrm{bf}}^{-\frac{1}{2}}$ is included so as not to force any vanishing of the elements at $\mathrm{bf}(X_b^2)$. Away from bf, (5.104) involves no further regularity than $H_b^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$. Taking a product neighbourhood of bf with r a defining function for it, the regularity in (5.104) just requires that $K = K(r, \tau, y, y')|\frac{dr}{r}|^{\frac{1}{2}}$, with

$$K(r, \cdot) \in \mathcal{C}^\infty([0, \epsilon]; H_b^\infty([-1, 1] \times (\partial X)^2; {}^b\Omega^{\frac{1}{2}}). \tag{5.105}$$

EXERCISE 5.36. Show that (5.104) is equivalent to (5.105) together with the condition that $\phi K \in H_b^\infty(X_b^2; {}^b\Omega^{\frac{1}{2}})$ if $\phi \in \mathcal{C}^\infty(X_b^2)$ has support disjoint from $\mathrm{bf}(X_b^2)$.

Now for $\alpha,\beta\in\mathbb{R}$ define $A\in\widetilde{\Psi}^{-\infty,\alpha,\beta}_{b,\infty}(X;{}^b\Omega^{\frac12})$ as consisting of the b-half-densities on X^2_b such that for some $\epsilon>0$ (depending on A) $\rho_{\mathrm{lb}}^{-\alpha-\epsilon}\rho_{\mathrm{rb}}^{-\beta-\epsilon}A$ is in the space (5.104). The extra subscript ∞ is supposed to indicate that the space is defined by bounds and the inclusion of $\epsilon>0$ gives a little 'room' in the estimates. Notice that

$$\alpha<\inf E_{\mathrm{lb}},\ \beta<\inf E_{\mathrm{rb}}\Longrightarrow\widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12})\subset\widetilde{\Psi}^{-\infty,\alpha,\beta}_{b,\infty}(X;{}^b\Omega^{\frac12}).$$

In fact the second part of the proof of Theorem 5.34 applies unchanged to show that

$$\begin{gathered}A\in\widetilde{\Psi}^{-\infty,\alpha,\beta}_{b,\infty}(X;{}^b\Omega^{\frac12})\text{ defines a bounded operator}\\ A\colon\rho^{\alpha'}H^M_b\longrightarrow\rho^{\alpha}H^m_b\quad\forall\ m,M\text{ if }\alpha'+\beta\geq0\text{ and }\alpha\leq\alpha'.\end{gathered}\tag{5.106}$$

It suffices to have $\alpha'+\beta\geq0$ because of the inclusion of $\epsilon>0$ in the definition of the kernels. Now the general calculus with bounds is the sum of three terms:

$$\begin{gathered}\Psi^{m,\alpha,\beta}_{b,\mathrm{os},\infty}(X;{}^b\Omega^{\frac12})=\\ \Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})+\widetilde{\Psi}^{-\infty,\alpha,\beta}_{b,\infty}(X;{}^b\Omega^{\frac12})+\rho^{\alpha}_{\mathrm{lb}}\rho^{\beta}_{\mathrm{rb}}H^\infty_b(X^2;{}^b\Omega^{\frac12}).\end{gathered}\tag{5.107}$$

The L^2 boundedness in (5.106) leads to the composition properties involving the action of the first two summands on the third:

Proposition 5.37. *If $\alpha'+\beta\geq0$ and $\alpha\leq\alpha'$ then composition of operators gives*

$$\Psi^{m,\alpha,\beta}_{b,\mathrm{os},\infty}(X;{}^b\Omega^{\frac12})\cdot\rho^{\alpha'}_{\mathrm{lb}}\rho^{\beta'}_{\mathrm{rb}}H^\infty_b(X^2;{}^b\Omega^{\frac12})\subset\rho^{\alpha}_{\mathrm{lb}}\rho^{\beta'}_{\mathrm{rb}}H^\infty_b(X^2;{}^b\Omega^{\frac12}).$$

Proof: Multiplying on the left by $\rho^{-\beta'}$ it suffices to consider the case $\beta'=0$. Any $B\in\rho^{\alpha'}_{\mathrm{lb}}H^\infty_b(X^2;{}^b\Omega^{\frac12})$ can be decomposed into

$$B=B_1+B_2,\ B_1=\phi(\rho_{\mathrm{rb}})B,\tag{5.108}$$

where ρ_{rb} is the pull-back of a boundary defining function, $\rho\in\mathcal{C}^\infty(X)$, from the right factor and $\phi(\rho)\in\mathcal{C}^\infty(X)$ takes the value 1 near the boundary and has support in a collar neighbourhood. The support of B_2 is therefore disjoint from the right boundary of X^2, so it is of the form of a $\mathcal{C}^\infty$ function of the right variables with values in the weighted Sobolev space:

$$B_2=b_2\otimes\nu,\ \nu\in\mathcal{C}^\infty(X;{}^b\Omega^{\frac12}),\ b_2\in\dot{\mathcal{C}}^\infty(X;\rho^{\alpha'}H^\infty_b(X;{}^b\Omega^{\frac12})).$$

The composition can then be written as the action on the left

$$A \circ B_2 = A(b_2) \otimes \nu \in \dot{\mathcal{C}}^\infty(X; \rho^\alpha H_b^\infty(X; {}^b\Omega^{\frac{1}{2}})) = \rho_{\mathrm{lb}}^\alpha \rho_{\mathrm{rb}}^\infty H_b^\infty(X^2; {}^b\Omega^{\frac{1}{2}})$$

using (5.106). To handle $A \circ B_1$, the product decomposition of the right factor of X near the boundary and the Mellin characterization of (5.41) can be used. Taking the Mellin transform in the normal variable of a product decomposition near the boundary of the right factor allows b_1 to be written as the inverse Mellin transform of

$$b_{1,M} \in \mathcal{S}(\mathbb{R} \times \partial X; \rho^{\alpha'} H_b^\infty(X; {}^b\Omega^{\frac{1}{2}})) \otimes \nu, \tag{5.109}$$

with ν a b-half-density on the right factor as before. Here the notation $\mathcal{S}$ stands for the space with all derivatives continuous into the target and rapidly decreasing as the variable, $\lambda \to \infty$. Now the same argument as above allows A to be applied on the left, meaning in the image space in (5.109). The result, by (5.106), is in the same space, with α' replaced by α. Taking the inverse Mellin transform by (5.41) and noting the bound on the support in the right variables, it follows that

$$A \circ B_1 \in \rho_{\mathrm{lb}}^\alpha H_b^\infty(X^2; {}^b\Omega^{\frac{1}{2}}).$$

This completes the proof of the proposition.

There are obvious properties for the composition of the third summand in (5.107) using this argument. In fact it is useful to prove a stronger property. Let R be some ring and suppose $I \subset R$ is a subring. Then I will be said to be a *semi-ideal* (sometimes called a bi-ideal, see [86], [87], and also called a corner in [3] - the latter terminology being particularly unfortunate in the present context) if

$$A, B \in I,\ X \in R \Longrightarrow AXB \in I. \tag{5.110}$$

On a compact manifold without boundary the smoothing operators form such a semi-ideal in the bounded operators on L^2. An extension of this to the case of manifolds with boundary is:

Proposition 5.38. *For any $\alpha \in \mathbb{R}$ the subspace $\rho_{\mathrm{lb}}^\alpha \rho_{\mathrm{rb}}^{-\alpha} H_b^\infty(X^2; {}^b\Omega^{\frac{1}{2}})$ is a semi-ideal in the space of bounded operators on $\rho^\alpha L^2(X; {}^b\Omega^{\frac{1}{2}})$.*

Proof: As usual, by conjugation, it is enough to take $\alpha = 0$. So consider $A, B \in H_b^\infty(X^2; {}^b\Omega^{\frac{1}{2}})$ and take a decomposition as in (5.108) for B and a similar decomposition for A but with respect to the left variables (i.e. take such a decomposition of A^*). Then the composite operator is a sum of four terms, $A_j \circ X \circ B_k$ for $j, k = 1, 2$. Now

$$A_2 \circ X \circ B_2 \in \dot{\mathcal{C}}^\infty(X^2; {}^b\Omega^{\frac12})$$

is very residual. The term $A_1 \circ X \circ B_2$ is the adjoint of a term of the type of $A_2 \circ X \circ B_1$, so it suffices to consider the latter. Using the Mellin transform in the right variables, as in the proof of Proposition 5.37 this is easily seen to be an element of $\rho_{\mathrm{lb}}^\infty H_b^\infty(X^2; {}^b\Omega^{\frac12})$. The last term, $A_1 \circ X \circ B_1$ can be handled similarly by taking the Mellin transform, and its inverse, in both sets of variables. To do this, and so complete the proof, a characterization of the Mellin transform as in (5.41) is needed in two sets of variables. This is left as an exercise:

Exercise 5.39. Show that the double Mellin transform

$$u_{M,\nu}(\lambda, \lambda', y, y') = \int_0^\infty \int_0^\infty x^{-i\lambda}(x')^{-i\lambda'} u(x, y, x', y') \frac{dx}{x} \frac{dx'}{x'}$$

defines, for any $m \in \mathbb{R}$, an isomorphism

$$H_b^m(\widetilde{X}^2; {}^b\Omega^{\frac12}) = \Big\{ u \in L^2(\widetilde{X}^2; {}^b\Omega^{\frac12}); u_{M,\nu} \in L^2(\mathbb{R}^2; H^m((\partial X)^2; \Omega^{\frac12}),$$
$$(1 + |\lambda|^2 + |\lambda'|^2)^{m/2} u_{M,\nu} \in L^2(\mathbb{R}^2 \times (\partial X)^2; |d\lambda|^{\frac12} |d\lambda'|^{\frac12} \Omega^{\frac12}) \Big\}.$$

5.17. Fredholm properties.

The basic Fredholm properties for elliptic operators on weighted Sobolev spaces are now straightforward consequences of the continuity of the parametrix and the compactness of the error:

Theorem 5.40. *If $P \in \mathrm{Diff}_b^k(X; {}^b\Omega^{\frac12})$ is elliptic on a compact manifold with boundary X then for each $\alpha \in \mathbb{R}$ and $M \in \mathbb{R}$ the null space*

$$\begin{gathered}\Big\{ u \in \rho^\alpha H_b^M(X; {}^b\Omega^{\frac12}); Pu = 0 \Big\} \subset \rho^\alpha H_b^\infty(X; {}^b\Omega^{\frac12}) \\ \text{is finite dimensional}\end{gathered} \tag{5.111}$$

and

$$\begin{gathered}P \colon \rho^\alpha H_b^M(X; {}^b\Omega^{\frac12}) \longrightarrow \rho^\alpha H_b^{M-k}(X; {}^b\Omega^{\frac12}) \text{ is Fredholm} \\ \Longleftrightarrow \alpha \notin -\operatorname{Im}\operatorname{spec}_b(P).\end{gathered} \tag{5.112}$$

A 'relative index formula' relating the index of the operator in (5.112) for different values of the weight α is obtained in §6.2.

PROOF: The sufficiency of the condition $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ for P to be Fredholm follows from the existence of right and left parametrices. From Lemma 5.4 it follows that P is bounded as an operator (5.112). Moreover from Proposition 5.28 and the Corollary to Lemma 5.33, applied to P and to its adjoint, there are bounded maps

$$\begin{gathered} E_R, E_L \colon \rho^\alpha H_b^{M-k}(X; {}^b\Omega^{\frac12}) \longrightarrow \rho^\alpha H_b^M(X; {}^b\Omega^{\frac12}) \text{ s.t.} \\ P \circ E_R = \operatorname{Id} - R_R,\ E_L \circ P = \operatorname{Id} - R_L, \\ R_R, R_L \colon \rho^\alpha H_b^q(X; {}^b\Omega^{\frac12}) \longrightarrow \rho^{\alpha+\epsilon} H_b^\infty(X; {}^b\Omega^{\frac12}). \end{gathered} \tag{5.113}$$

It follows that P is Fredholm. First the null space in (5.111) is contained in the null space of $\operatorname{Id} - R_L$ and is therefore finite dimensional, since R_L is compact. This proves (5.111) when $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$. However the general case follows since the null space can only increase with decreasing α. Similarly the range of P contains the range of $\operatorname{Id} - R_R$ which is closed with finite codimension, because R_R is compact. Thus the range of P is also closed and of finite codimension. It follows that P is Fredholm as stated in (5.112).

So it remains only to prove that the condition on α is also necessary for P to be Fredholm. Since this is a direct consequence of the relative index formula proved in Theorem 6.5 below, the proof is deferred until after this has been established.

EXERCISE 5.41. Make sure that you understand why $\operatorname{Id} - R$ is Fredholm if R is a compact operator on a Hilbert space.

As a Fredholm operator P has a generalized inverse. Thus for each $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ consider the map E_α which is zero on the orthocomplement of the range of P in (5.112) for $M = k$ and maps each element f in the range of P to the unique solution u of $Pu = f$ which is in $\rho^\alpha H_b^k(X; {}^b\Omega^{\frac12})$ and is orthogonal to the null space with respect to the inner product on $\rho^\alpha H_b^0(X; {}^b\Omega^{\frac12})$. This defines an operator

$$E_\alpha \colon \rho^\alpha H_b^0(X; {}^b\Omega^{\frac12}) \longrightarrow \rho^\alpha H_b^k(X; {}^b\Omega^{\frac12}) \tag{5.114}$$

which is such that

$$P \circ E_\alpha = \operatorname{Id} - \Pi_1,\ E_\alpha \circ P = \operatorname{Id} - \Pi_0, \tag{5.115}$$

where Π_0 and Π_1 are the orthogonal projections in $\rho^\alpha H_b^0(X; {}^b\Omega^{\frac12})$ onto the null space and orthocomplement to the range of P. The inner product on the space $\rho^\alpha H_b^0(X; {}^b\Omega^{\frac12})$ depends on the choice of defining function for the boundary and so this affects the definition of E_α when $\alpha \neq 0$ but otherwise E_α is well-defined.

PROPOSITION 5.42. *The generalized inverse E_α in* (5.114), (5.115) *for an elliptic element $P \in \mathrm{Diff}_b^k(X; {}^b\Omega^{\frac12})$ and $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ is an element of the space of operators defined in* (5.107)*:*

$$E_\alpha \in \Psi_{b,\mathrm{os},\infty}^{-k,\alpha+\epsilon,\alpha+\epsilon}(X; {}^b\Omega^{\frac12}) \tag{5.116}$$

for some $\epsilon > 0$.

PROOF: By conjugation it suffices to take $\alpha = 0$ as usual. Consider the left and right parametrices in (5.113). These satisfy

$$E_L - E_R = E_L \circ (P \circ E_R + R_R) - (E_L \circ P + R_L) \circ E_R = E_L \circ R_R - R_L \circ E_R. \tag{5.117}$$

This shows that the difference is in the same space, as in (5.99), as the error terms. Thus the left parametrix is also a right parametrix with the same type of error term. Let $E = E_L$ be the two-sided parametrix and R_R and R_L the errors as a right and left parametrix.

Now consider the same computation but involving the generalized inverse, E_α. Notice first that both Π_0 and Π_1 are orthogonal projections onto finite dimensional subspaces of $\rho^\epsilon H_b^\infty(X; {}^b\Omega^{\frac12})$ for some $\epsilon > 0$. The kernels of these operators are therefore sums of products of elements of this space, so are in $\rho_{\mathrm{lb}}^\epsilon \rho_{\mathrm{rb}}^\epsilon H_b^\infty(X^2; {}^b\Omega^{\frac12})$. Now using (5.115) and (5.117)

$$E_\alpha - E = E_\alpha \circ R_R - \Pi_0 \circ E,\ \ E - E_\alpha = E \circ \Pi_0 - R_L \circ E_\alpha$$

from which it follows that

$$E_\alpha = E + E \circ R_R - \Pi_0 \circ E - E \circ \Pi_0 \circ R_R + R_L \circ E_\alpha \circ R_R.$$

All terms on the right are in the space in (5.116), with the semi-ideal property of Proposition 5.38 used on the last term. This proves the proposition.

5.18. Extended index sets.

Although a parametrix for any elliptic b-differential operator has now been obtained, the construction will be taken somewhat further, resulting in a finer parametrix and thence a more precise description of the generalized inverse. This finer description is used in the next chapter to show the existence of a meromorphic continuation of the resolvent of the Laplacian of an exact b-metric through the continuous spectrum. It is *not* used in the proof of the APS theorem.

Most of the work has already been done, in analyzing the solvability properties of the indicial operator. It remains to consider the perturbation theory which will allow this to be used to improve the parametrix constructed in Proposition 5.28 by more carefully choosing the extension off the front face of X_b^2. As will be seen below this is really the process of relating a solution of the indicial operator to an approximate solution of the operator itself.

In the construction of a finer parametrix the index sets in (5.86) will have to be replaced by somewhat bigger ones which take into account the horrors of accidental multiplicities, to wit

(5.118)
$$\widehat{E}^{\pm}(\alpha) = \Big\{ (z,k) \in \mathbb{C} \times \mathbb{N}_0; \exists\, r \in \mathbb{N}_0,\ \operatorname{Re}(z) > \pm\alpha + r,$$
$$\pm\, i(z-r) \in \operatorname{spec}_b(P) \text{ and } k+1 \le \sum_{j=0}^{r} \operatorname{ord}(\pm i(z-j)\Big\}.$$

To understand where this definition comes from, start with the absolute index sets in (5.85). To get to (5.118) consider the shifts of these sets by integral steps in the imaginary direction:

$$E^{\pm}(\alpha) + r = \left\{ (z+r,k); (z,k) \in E^{\pm}(\alpha) \right\},\ r \in \mathbb{N}.$$

Then

$$\widetilde{E}^{\pm}(\alpha) = \bigcup_{r\in\mathbb{N}_0} \left[E^{\pm}(\alpha) + r \right] \tag{5.119}$$

are the smallest $\mathcal{C}^\infty$ index set containing $E^{\pm}(\alpha)$. On the other hand to define (5.118) the 'extended' union of index sets is used:

$$E \overline{\cup} F = E \cup F \cup \{(z,k); \exists\, (z,l_1) \in E, (z,l_2) \in F,\ k = l_1 + l_2 + 1\}. \tag{5.120}$$

Thus $E\overline{\cup}F = E \cup F$ unless $E \cap F \neq \emptyset$. The extra term in (5.120) just consists of the points of $E \cap F$ with multiplicity increased to the sum of the multiplicities, plus one. Since 0 already represents a pole of some Mellin transform (see Proposition 5.27) this just arises from multiplying meromorphic functions. The operation of taking the extended union is commutative and associative. Then (5.118) is just

$$\widehat{E}^{\pm}(\alpha) = E^{\pm}(\alpha)\overline{\cup}[E^{\pm}(\alpha)+1]\overline{\cup}[E^{\pm}+1]\overline{\cup}\dots. \tag{5.121}$$

These are the index sets which appear in the parametrix or the inverse if P is invertible. If P has null space they need to be increased further to

$$\check{E}^{\pm}(\alpha) = \widehat{E}^{\pm}(\alpha)\overline{\cup}\widehat{E}^{\pm}(\alpha). \tag{5.122}$$

PROPOSITION 5.43. *If $P \in \operatorname{Diff}^k_b(X; {}^b\Omega^{\frac12})$ is elliptic and $\alpha \in \mathbb{R}$ is not an element of $-\operatorname{Im}\operatorname{spec}_b(P)$ there exists*

$$\begin{gathered} \widehat{G}_\alpha \in \widetilde{\Psi}_b^{-k,\widehat{\mathcal{E}}(\alpha)}(X; {}^b\Omega^{\frac12}) \text{ s.t.} \\ P \circ \widehat{G}_\alpha = \operatorname{Id} - \widehat{R}_\alpha,\ \widehat{R}_\alpha \in \rho_{\mathrm{bf}} \widetilde{\Psi}_b^{-\infty,(\emptyset,\widehat{E}^-(\alpha))}(X; {}^b\Omega^{\frac12}), \end{gathered} \tag{5.123}$$

where $\widehat{\mathcal{E}}(\alpha) = (\widehat{E}^+(\alpha), \widehat{E}^-(\alpha))$, is defined by (5.121) or (5.118).

Notice what is claimed here for the error term, as compared to R_α in (5.88). There is no improvement at $\mathrm{bf}(X_b^2)$; both R_α and $\widehat{R}_\alpha$ just vanish there. The improvement is all at the left boundary, $\mathrm{lb}(X_b^2)$, where $\widehat{R}_\alpha$ is to vanish to all orders, i.e. has trivial expansion.

5.19. Formal solutions.

The main ingredient in the proof of Proposition 5.43 is a simple result on the formal inversion of the operator P, i.e. relating to its action on polyhomogeneous conormal distributions.

LEMMA 5.44. *If $P \in \mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ is elliptic and E is an index set then for each $f \in \mathcal{A}^E_{\mathrm{phg}}(X; {}^b\Omega^{\frac{1}{2}})$, there exists $u \in \mathcal{A}^F_{\mathrm{phg}}(X; {}^b\Omega^{\frac{1}{2}})$ with*

$$Pu - f \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}),\ F = E\overline{\cup}\widehat{E}^+(\alpha), \tag{5.124}$$

using the notation of (5.97), *provided* $\inf E > \alpha$.

PROOF: By definition f has an expansion (5.67). To solve away each term in the expansion, a formal solution of

$$Pu = x^z(\log x)^k v,\ v \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \tag{5.125}$$

is needed for each $\lambda \in \mathbb{C}$ and each $k \in \mathbb{N}_0$. Suppose for the moment that $k = 0$. Then it is natural to look for $u = x^z w$, $w \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}})$, although this will not quite work. From (5.32) it follows that, near ∂X,

$$P(x^z w) = x^z I_\nu(P, -iz)w + x^{z+1}g(z), \tag{5.126}$$

where $g(z) \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}})$ is analytic in z. Here a product decomposition of $X = [0, \epsilon)_x \times \partial X$ has been taken to allow $I_\nu(P, -iz)$ to act on $w = w(x, \cdot)$.

So as a first step towards (5.125) take

$$w(0, \cdot) = [I_\nu(P, -iz)]^{-1}v(0, \cdot).$$

Of course there is a problem with this in that $\lambda = -iz$ might be a point of $\mathrm{spec}_b(P)$ in which case the inverse does not exist. However, in Proposition 5.3 it has been established that the inverse is meromorphic, so consider instead the contour integral:

$$u_0'(x, \cdot) = \frac{1}{2\pi} \oint\limits_{\gamma(z)} x^{i\lambda}(\lambda + iz)^{-1}[I_\nu(P, \lambda)]^{-1}v(0, \cdot)d\lambda, \tag{5.127}$$

where $\gamma(z)$ is a small circle with centre $-iz$ traversed anticlockwise. Replacing the inverse of the indicial family by its Laurent series gives

$$u_0'(x,\cdot) = \sum_{j=0}^{\operatorname{ord}(-iz)} x^z (\log x)^j u_{0,j}(\cdot),\ u_{0,j} \in \mathcal{C}^\infty(\partial X; \Omega^{\frac{1}{2}}).$$

Here, as in §5.2, the order of a regular point is taken to be 0, in which case there are no logarithmic terms. Let $u_0 = \phi(x)u_0'$, where $\phi \in \mathcal{C}^\infty(X)$ has support in $x < \epsilon$ and $\phi(x) = 1$ in $x < \frac{1}{2}\epsilon$. Applying P to (5.127) and using (5.32) and (5.126) gives

$$Pu_0 = x^z v + \sum_{j=0}^{\operatorname{ord}(-iz)} x^{z+1}(\log x)^j v_{1,j}(x),\ v_{1,j} \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}).$$

Thus, in case $k = 0$, (5.125) has been solved up to an error which is one order lower (has a factor of x) but may have higher logarithmic order. If $k > 0$ the same result can be obtained by observing that

$$x^z(\log x)^j v = (\frac{\partial}{\partial z})^j x^z v,$$

assuming v to be independent of z. The general result then follows by differentiating and proceeding inductively over the power of the logarithm. Thus (5.125) has an approximate solution of the form

$$u_0 = \sum_{j=0}^{\operatorname{ord}(-iz)+k} x^z(\log x)^j u_{0,j}(\cdot),\ u_{0,j} \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}),$$

in the sense that

$$Pu_0 = x^z(\log x)^k v + \sum_{j=0}^{\operatorname{ord}(-iz)+k} x^{z+1}(\log x)^j v_{1,j}(\cdot),\ v_{1,j} \in \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}).$$

Of course this is the basis for an inductive solution of (5.125). Applying the construction repeatedly, and in the end taking an asymptotic sum, shows that (5.125) does indeed have a formal solution

(5.128)
$$Pu = x^z(\log x)^k v + f,\ u \in \mathcal{A}^{F(z)}_{\mathrm{phg}}(X; {}^b\Omega^{\frac{1}{2}}),\ f \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}),$$
$$F(z) = \left\{ (z+r,p); r \in \mathbb{N}_0,\ p \le k + \sum_{\substack{0 \le j \le r \\ -i(z+j) \in \operatorname{spec}_b(P)}} \operatorname{ord}(-i(z+j)) \right\}.$$

To get the full result it is only necessary to sum asymptotically over the points of E and observe that the resulting index set is given by (5.124).

5.20. Finer parametrix.

This formal solution can be used to improve the error term at $\mathrm{lb}(X_b^2)$ from the parametrix.

PROOF OF PROPOSITION 5.43: To get (5.88) it was only necessary to choose G_α as in (5.87) with indicial operator, i.e. kernel restricted to the front face, being the inverse of the indicial operator of P. To get (5.123) the extension off the front face, near $\mathrm{lb}(X_b^2)$, needs to be chosen a little more carefully. Near $\mathrm{lb}(X_b^2)$ take as a boundary defining function $s = x/x'$ and then

$$\mathrm{lb}(X_b^2) = \partial X_y \times X,$$

where the factor of X is just the right factor of X in X^2. Thus the action of $P \in \mathrm{Diff}_b^k(X; {}^b\Omega^{\frac{1}{2}})$ on a kernel on X_b^2 can be written near lb as

$$P\kappa(s, y, x', y') = \sum_{r+|\alpha|\le k} p_{k,\alpha}(x's, y)(sD_s)^r D_y^\alpha \kappa(s, y, x', y'). \tag{5.129}$$

In particular, as a b-differential operator in s, y it depends smoothly on x' as a parameter, in the strong sense that the indicial family is just $I_\nu(P, \lambda)$, and is independent of x'. This is important because it means that the proof of Lemma 5.44 applies uniformly, the indicial set and indicial family being independent of the parameter.

Using this observation, Lemma 5.44 shows that (5.88) can be improved with G_α still given by (5.87). Near $\mathrm{lb}(X_b^2)$ one can use x' as a defining function for the front face. The normal operator of G_α is, by choice, a solution of the indicial equation on $\mathrm{bf}(X_b^2)$ near $\mathrm{lb}(X_b^2)$, since the kernel of the identity is supported at Δ_b. Thus the extension off $\mathrm{bf}(X_b^2)$ can simply be chosen to be, near $\mathrm{lb}(X_b^2)$, of the form $\phi(x')\kappa_\alpha(s, y, y')$, where ϕ has small support and takes the value 1 near $x' = 0$. Then the the remainder in (5.88) will improve at $\mathrm{lb}(X_b^2)$, i.e.

$$R_\alpha \in \rho_{\mathrm{bf}} \widetilde{\Psi}_b^{-\infty,(E'(\alpha),E^-(\alpha))}(X; {}^b\Omega^{\frac{1}{2}}),$$

where $E'(\alpha)$ is the smallest $\mathcal{C}^\infty$ index set containing $E^+(\alpha) \cap \{\mathrm{Re}\, z > \inf(E^+(\alpha)\}$, the leading terms at lb being absent. Also the support of the kernel of R_α meets $\mathrm{lb}(X_b^2)$ only in a small neighbourhood of $\mathrm{bf}(X_b^2)$.

Now apply Lemma 5.44 to (5.129), with x' and y' as $\mathcal{C}^\infty$ parameters. The index set of the result is at worst $E'(\alpha)\overline{\cup}\widehat{E}^+(\alpha)$ which is contained in $\widehat{E}^+(\alpha)$. This gives a correction term

$$\kappa' \in \rho_{\mathrm{bf}} \widetilde{\Psi}_b^{-\infty,(\widehat{E}^+(\alpha),\emptyset)}(X; {}^b\Omega^{\frac{1}{2}}),$$

with the index set as in the statement of the proposition and support near $\mathrm{bf}(X^2_b)\cap\mathrm{lb}(X^2_b)$, such that if G'_α is the corresponding operator then

$$P\circ G'_\alpha - R_\alpha \in \rho_{\mathrm{bf}}\widetilde{\Psi}_b^{-\infty,(\emptyset,\widehat{E}^-(\alpha))}(X;{}^b\Omega^{\frac12}) \tag{5.130}$$

since all the terms at $\mathrm{lb}(X^2_b)$ have been eliminated. Finally then (5.123) is satisfied by taking $\widehat{G}_\alpha = G_\alpha - G'_\alpha$. This completes the proof of Proposition 5.43.

Why is this parametrix significantly better than the one in Proposition 5.28? The main reason is that an iterative argument can be used to improve the remainder term even more. Consider the formal Neumann series for the inverse of $\mathrm{Id}-\widehat{R}_\alpha$:

$$\mathrm{Id}+\sum_{j>0}(\widehat{R}_\alpha)^j. \tag{5.131}$$

Theorem 5.34 shows that the compositions in the series are all defined. Of course the powers of $\widehat{R}_\alpha$ have to be examined to get much out of this. It follows from Proposition 5.38 that the vanishing of the kernel of R_α to all orders at the left boundary makes the powers much better behaved than just for R_α in (5.88). This said, it should still be recognized that the main reason that the series in (5.131) converges asymptotically at $\mathrm{bf}(X^2_b)$, which will now be shown, is that the kernel of $\widehat{R}_\alpha$ vanishes at $\mathrm{bf}(X^2_b)$ and this is already true for R_α. The composition results above for the bounded calculus will now be refined to show that the terms $(\widehat{R}_\alpha)^j$ in (5.131) are polyhomogeneous.

5.21. Composition with boundary terms.

The rather inelegant and piecemeal nature of the proofs of the series of results leading up to the general composition formula for the full, polyhomogeneous, calculus is one reason why it is preferable to have more machinery (as in [63]) which allows one to prove these sorts of results with equanimity. Here a more elementary method is used but at the expense of some obfuscation and the general composition result, (5.153) in Theorem 5.53, is proved in stages.

First Proposition 5.25 needs to be extended to include boundary terms:

PROPOSITION 5.45. *If $A\in\widetilde{\Psi}_b^{-\infty,(\emptyset,E)}(X;{}^b\Omega^{\frac12})$ and F is an index set with*

$$\inf E+\inf F>0 \tag{5.132}$$

then

$$A\colon \mathcal{A}^F_{\mathrm{phg}}(X;{}^b\Omega^{\frac12})\longrightarrow \mathcal{A}^F_{\mathrm{phg}}(X;{}^b\Omega^{\frac12}). \tag{5.133}$$

If $A \in \widetilde{\Psi}_b^{-\infty,(E,\emptyset)}(X; {}^b\Omega^{\frac{1}{2}})$ *then for any index set*

$$A\colon \mathcal{A}^F_{\mathrm{phg}}(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{A}^{E\overline{\cup}F}_{\mathrm{phg}}(X; {}^b\Omega^{\frac{1}{2}}). \tag{5.134}$$

PROOF: Consider $A \in \widetilde{\Psi}_b^{-\infty,(\emptyset,E)}(X; {}^b\Omega^{\frac{1}{2}})$. If x' is the boundary defining function on the right factor and $M > \inf E$ then multiplying the kernel

$$B = A \circ (x')^M \in \dot{\mathcal{C}}^N(X^2; \Omega^{\frac{1}{2}}),\ N + 1 < M - \inf E.$$

The operator B therefore maps $\mathcal{C}^0(X; {}^b\Omega^{\frac{1}{2}})$ into $\dot{\mathcal{C}}^N(X; {}^b\Omega^{\frac{1}{2}})$. This takes care of the remainder terms if $u \in \mathcal{A}^F_{\mathrm{phg}}(X; {}^b\Omega^{\frac{1}{2}})$ is replaced by its expansion to very high order.

It is therefore enough to check that Au is polyhomogeneous when u is of the form $x^z v$ with $v \in \mathcal{C}^\infty(X)$. By conjugating the operator with the power the problem is reduced to the action of A, with its index set shifted by z, on $\mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}})$. That is if $\inf E > 0$ it has to be shown that

$$A\colon \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}). \tag{5.135}$$

For terms in the small calculus this was shown in Proposition 4.34. Thus it can be assumed that the kernel of A has support near $\mathrm{rb}(X^2_b)$ and, by localizing the support, can be taken to be in a coordinate patch where $t = x/x'$, x and y, y' are coordinates. Since a $\mathcal{C}^\infty$ factor can be absorbed in the kernel, it is enough to consider the action on the coordinate b-density and hence just to consider the integral of the kernel:

$$\int_0^\infty \kappa'(x, \frac{x'}{x}, y, y')dy' \frac{dx'}{x'} = \int_0^\infty \kappa'(x, t, y, y')dy' \frac{dt}{t}.$$

The integral converges because of the assumption that $\inf E > 0$ and the integrand is $\mathcal{C}^\infty$ in x and y, so (5.135) follows. This proves (5.133).

The proof of (5.134), when $A \in \widetilde{\Psi}_b^{-\infty,(E,\emptyset)}(X; {}^b\Omega^{\frac{1}{2}})$ is similar but a little more involved. Using Proposition 4.34 it can be assumed that the kernel of A is supported near the left boundary, and also that the support is small. Then the coordinates $s = x/x'$, x' and y, y' can be used. The action of the operator becomes:

$$Au(x,y)|\frac{dx}{x}dy|^{\frac{1}{2}} = \int \kappa(x', x/x', y, y')u(x', y')\frac{dx'}{x'}dy'|\frac{dx}{x}dy|^{\frac{1}{2}}.$$

The kernel is $\mathcal{C}^\infty$ in the first variable, is supported in $s < 1$, where s is the second variable and has an expansion at $s = 0$. The integral certainly converges for $x > 0$. To show that the result is polyhomogeneous, Proposition 5.27 will be used. The Mellin transform of Au is

$$\int_0^\infty \int_0^\infty \int_{\partial X} x^{-i\lambda} \kappa(x', x/x', y, y') u(x', y') \frac{dx'}{x'} dy' \frac{dx}{x}.$$

Changing variable of integration from x to s this becomes

$$\int_0^\infty \int_0^\infty \int_{\partial X} s^{-i\lambda} (x')^{-i\lambda} \kappa(x', s, y, y') u(x', y') \frac{dx'}{x'} dy' \frac{ds}{s}. \tag{5.136}$$

Now $\kappa(x', s, y, y')$ is polyhomogeneous at $s = 0$ and $\mathcal{C}^\infty$ in x', y, y'. It follows from Proposition 5.27 that the s integral is meromorphic in λ with poles of order $k+1$ only at points $\lambda = -iz$ for $(z, k) \in E$. This meromorphic function takes values in $\mathcal{C}^\infty$ functions in x', y, y'. This allows the x' integral to be analyzed using Proposition 5.27 too, with the essentially trivial extension that the polyhomogeneous conormal integrand is meromorphic in the parameter λ. The result is that the integral (5.136) is meromorphic with poles only at $\lambda = -iz$, where $(z, k) \in E$ or $(z, l) \in F$ (which has been normalized to 0). If z is a power in both asymptotic expansions then the order of the pole is the sum of the orders. The normalization of the index set, where $(z, 0)$ corresponds to a simple pole of the Mellin transform, means that, now applying Proposition 5.27 in the reverse direction, the index set of Au is at worst $E \overline{\cup} F$. This proves (5.134) and Proposition 5.45.

The first composition result is for the small calculus and the operators defined in (5.83).

PROPOSITION 5.46. *The transpose of operators defines an isomorphism*

$$\Psi^{m,\mathcal{E}}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}) \longleftrightarrow \Psi^{m,\mathcal{E}'}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}),\ \mathcal{E} = (E_{\mathrm{lb}}, E_{\mathrm{rb}}),\ \mathcal{E}' = (E_{\mathrm{rb}}, E_{\mathrm{lb}}) \tag{5.137}$$

and these spaces are two-sided modules over the small calculus:

$$\begin{gathered} \widetilde{\Psi}^{m,\mathcal{E}}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}) \circ \Psi^{m'}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}), \\ \Psi^{m'}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}) \circ \widetilde{\Psi}^{m,\mathcal{E}}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}) \subset \widetilde{\Psi}^{m+m',\mathcal{E}}_{b,\mathrm{os}}(X; {}^b\Omega^{\frac12}). \end{gathered} \tag{5.138}$$

PROOF: The invariance under transpose, (5.137), just amounts to the observation that under the factor exchanging involution the appropriate polyhomogeneous conormal spaces are mapped into each other. Using this observation, and the corresponding fact for the small calculus (Exercise 4.39), reduces (5.138) to either of the two cases. Since, in Proposition 5.20, the composition properties in the small calculus have already been checked from the definition, (5.83), it suffices to show that the operators of order $-\infty$ form a module:

$$\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \circ \widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}) \subset \widetilde{\Psi}^{-\infty,\mathcal{E}}_b(X;{}^b\Omega^{\frac12}).$$

It is again convenient to split the second factor into pieces so that one of the representations (4.68) or (4.69) applies. Using the composition properties of the small calculus, it suffices to check separately that

$$\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \circ \widetilde{\Psi}^{-\infty,(E,\emptyset)}_b(X;{}^b\Omega^{\frac12}) \subset \widetilde{\Psi}^{-\infty,(E,\emptyset)}_b(X;{}^b\Omega^{\frac12}) \tag{5.139}$$

$$\Psi^m_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12}) \circ \widetilde{\Psi}^{-\infty,(\emptyset,E)}_b(X;{}^b\Omega^{\frac12}) \subset \widetilde{\Psi}^{-\infty,(\emptyset,E)}_b(X;{}^b\Omega^{\frac12}). \tag{5.140}$$

In either case it can be assumed that $\inf E > 0$ simply by conjugating by a power of a boundary defining function. Thus the representations (4.68) and (4.69) both apply and the kernel of the composite operator is given by (5.66). Consider (5.139) since it is somewhat the easier of the two cases. The kernel of the second factor, B, can be assumed to have support in a coordinate neighbourhood near $\mathrm{lb}(X^2_b)$. Thus in the representations (5.65) the kernel $k'_M(x',\lambda,y,y')$ is $\mathcal{C}^\infty$ in x',y,y' and meromorphic in λ with poles corresponding to the index set E. It follows that in (5.66) the composite kernel k_M of A in the small calculus is composed with a smoothing operator in y. The result, k'', is therefore the inverse Mellin transform of a smoothing kernel with the poles corresponding to E and rapid decrease in λ. It follows from Proposition 5.27 that k'' is polyhomogeneous at lb with index set E. This is exactly what (5.139) states.

Now consider (5.140). The difficulty is that the support of the second factor can be restricted to a neighbourhood of $\mathrm{rb}(X^2_b)$ at which the coordinates used in (5.64), namely x' and $s = x/x'$, are not admissible. The problem is x', since s can be replaced by $t = 1/s$. This makes it difficult to analyze the Mellin transform of the kernel. However the kernel can be replaced by its asymptotic expansion with error which is arbitrarily smooth and vanishes to high order at $\mathrm{rb}(X^2_b)$, so Lemma 5.18 can be used to handle it. It is therefore enough to consider B with kernel of the special form $t^z(\log t)^k\phi(t)\psi(x,y,y')$, where ψ is smooth, and has small support, and $\phi \in \mathcal{C}^\infty_c([0,\infty))$ takes the value 1 near 0. As usual the logarithmic term can be ignored because it can be recovered by differentiating in z. Then the identity

$$t^z\phi(t) = t^z + t^z(\phi(t)-1) \tag{5.141}$$

expresses $B = B_1 + B_2$ as a sum, the second term of which has already been discussed in (5.139). The first term, B_1, has kernel which is a product of the form $(x')^z x^{-z}\psi(x, y, y')$. Then Proposition 4.34, applied with parameters, shows that the kernel of the composite operator is

$$A \circ B_2 = x^{-z}\psi'(x, x', y, y')(x')^z,$$

with ψ' smooth. It follows that the kernel of $A \circ B$ is polyhomogeneous, with index set E at $\mathrm{rb}(X_b^2)$. From this argument it might seem to have additional singularities at $\mathrm{lb}(X_b^2)$ however it is clear that there are no such terms away from $\mathrm{bf}(X_b^2)$, so they must in fact vanish identically, i.e. there is cancellation between $A \circ B_1$ and $A \circ B_2$. This completes the proof of (5.140) and hence of Proposition 5.46.

Next consider composition between operators as in (5.83) but where the non-trivial boundary behaviour is only on the 'inside:'

PROPOSITION 5.47. *For index sets E and F satisfying* (5.132) *with the composite defined through Proposition 5.45*

$$\Psi_{b,\mathrm{os}}^{-\infty,(\emptyset,E)}(X; {}^b\Omega^{\frac{1}{2}}) \circ \Psi_{b,\mathrm{os}}^{-\infty,(F,\emptyset)}(X; {}^b\Omega^{\frac{1}{2}}) \subset \Psi_{b,\mathrm{os}}^{-\infty,(F,E)}(X; {}^b\Omega^{\frac{1}{2}}). \tag{5.142}$$

PROOF: Conjugating all operators by a power it can be assumed that

$$\inf E > 0,\ \inf F > 0. \tag{5.143}$$

Using Proposition 5.46 to handle terms where one factor is in the small calculus, it is enough to consider $A \circ B$ where the kernels of A and B are, respectively, supported near $\mathrm{rb}(X_b^2)$ and $\mathrm{lb}(X_b^2)$. Using a partition of unity the supports of the kernels can also be assumed to be small. Then the formulæ (5.62), (5.63), (5.64) and (5.65) can be used. Notice that these representations depend on the fact that x, $s = x'/x$, y and y' are coordinates near $\mathrm{rb}(X_b^2)$ and x', s, y and y' are coordinates near $\mathrm{lb}(X_b^2)$. Now in (5.63), using Proposition 5.27, $k_M(x, \lambda, y, y')$ is $\mathcal{C}^\infty$ in x, y, y' and meromorphic with rapid decay at real infinity in λ and poles only at $\lambda = -iz$, of order $k + 1$ if $(z, k) \in E$. Similarly $k'_M(x', \lambda, y, y')$ in (5.65) has poles only at $\lambda = -iz$ of order $l + 1$ if $(z, l) \in F$. This results in (5.66) as the formula for the kernel of the composite, expressed as the inverse Mellin transform of the product of the Mellin transforms. The product $k_M k'_M$ has poles corresponding to both index sets, with those from E in $\mathrm{Im}\,\lambda > 0$ and those from F in $\mathrm{Im}\,\lambda < 0$. As in the proof of Lemma 5.44 the contour of integration in the representation (5.66) can be moved off the real axis to find the decay as $s \to 0$ and $s \to \infty$ of k''. The conclusion is that k'' has an expansion at lb with powers from F and at rb with powers from E. This is just the content of (5.142), so the proposition is proved.

Next suppose the operators both have their non-trivial boundary terms on the same face. Using the invariance under adjoints in (5.137) it is enough to consider singularities on $\mathrm{lb}(X_b^2)$:

PROPOSITION 5.48. *For any index sets E and F*

$$\widetilde{\Psi}_b^{-\infty,(E,\emptyset)}(X;{}^b\Omega^{\frac{1}{2}})\circ\widetilde{\Psi}_b^{-\infty,(F,\emptyset)}(X;{}^b\Omega^{\frac{1}{2}})\subset\widetilde{\Psi}_b^{-\infty,(E\overline{\cup}F),\emptyset)}(X;{}^b\Omega^{\frac{1}{2}}). \tag{5.144}$$

PROOF: Conjugating by a power of a boundary defining function (5.143) can again be assumed. Without loss of generality the two factors, A and B, can be taken to have small supports near $\mathrm{lb}(X_b^2)$. The representation (5.65) is therefore available for the second factor, but (5.63) is not directly useful because the kernel is not supported in the region where x has non-vanishing differential (on X_b^2.) Initially the kernel k of A in (5.62) must be treated as a function of $x', s = x/x', y, y'$ which is $\mathcal{C}^\infty$ in $s > 0$, has support in $s < 1, x' < \epsilon$ and in a coordinate patch in y, y'.

As before the dependence on x' is the problem. Let $\phi \in \mathcal{C}_c^\infty([0,\infty))$ be such that $\phi(x)$ and $\phi(s)$ are both identically equal to 1 on the support of k, so k can be replaced by $\phi(x)\phi(s)k$. Then replace k by its Taylor series at $x' = 0$, which is the front face, $\mathrm{bf}(X_b^2)$. The individual terms are of the form

$$(x')^p k_p(0,s,y,y')\phi(x)\phi(s),\ \ k_p = \left(\frac{\partial}{\partial x'}\right)^p k. \tag{5.145}$$

The support may be larger than that of k but the explicit dependence on x' has been removed. Let $A_p = A'_p \circ (x')^p$ be the operator with kernel (5.145). The composite operator $A_p \circ B = A'_p \circ B'_p(x')^p$, $B'_p = (x)^p\phi(s)B(x')^{-p}$ can now be analyzed using (5.65), with k replaced by $k_p(0,s,y,y')\phi(x)$ and k' the kernel of B'_p. The poles of the Mellin transform, k_M, arise from the powers in E and those of k' from the powers in F shifted by k, hence also from F. Thus the product of the Mellin transforms in (5.66) has poles from both E and F. At common poles the order is at most the sum of the orders of the poles. It therefore follows from Proposition 5.45 that

$$A'_p \circ B'_p \in \widetilde{\Psi}_b^{-\infty,(E\overline{\cup}F,\emptyset)}(X;{}^b\Omega^{\frac{1}{2}}).$$

This is no less true when the extra factor of $(x')^p$ is applied to the right.

Thus the contribution to the product from the individual terms, (5.145), in the Taylor series of the kernel of A at $\mathrm{bf}(X_b^2)$ are all of the expected type. So consider the remainder term. Thus it may be assumed that the kernel of

A vanishes to some high order, N, at $\mathrm{bf}(X_b^2)$. Now replace k in (5.62) by its expansion at $\mathrm{lb}(X_b^2)$, up to terms which vanish to order N. The remainder term therefore vanishes at bf and lb to order N. Replacing N by $2N$ the product from the remainder can be written $A' \circ B'$ where, for the kernels, $B' = x^N B$ and A' vanishes to order N. Thus, remembering (5.143), both factors now have kernels in $\dot{\mathcal{C}}^N(X_b^2; {}^b\Omega^{\frac12})$. Lemma 5.18 therefore applies and shows that the composite is increasingly smooth on X_b^2 and vanishes to high order at all boundaries.

Thus it is only necessary to consider the finite terms in the expansion of the kernel of A. These can be taken to be of the form

$$(x')^N \phi(x)\phi(s) s^z (\log s)^k \psi(y, y'), \ (z, k) \in E, \tag{5.146}$$

where ψ is $\mathcal{C}^\infty$. As usual it suffices to take $k = 0$ and differentiate with respect to z to recover the general case. The identity (5.141) replaces A, with kernel (5.146), by a sum, $A' + A''$, where the kernel of A' is $(x')^N (x/x')^z \phi(x)\psi(y, y')$ and A'' has support away from $\mathrm{lb}(X_b^2)$. Proposition 5.47 applies to the composition $A'' \circ B$. The explicit x' dependence of the kernel of A' can be removed, by absorbing the factor of $(x')^N$ in B, so the composite $A'' \circ B$ can be analyzed using (5.65). The conclusion then is that all terms are polyhomogeneous on X_b^2 and smooth up to $\mathrm{bf}(X_b^2)$. All the terms in the expansion at $\mathrm{lb}(X_b^2)$ arise from $E \overline{\cup} F$. There are spurious terms at $\mathrm{rb}(X_b^2)$ for the composite kernel, but these are certainly absent since the kernel of the composite clearly has support disjoint from $\mathrm{rb}(X_b^2)$. This completes the proof of Proposition 5.27.

5.22. Residual terms.

These Propositions together handle composition of the operators defined by (5.83) except when the first factor has singularities on $\mathrm{lb}(X_b^2)$ and the second factor on $\mathrm{rb}(X_b^2)$. For this case see (5.154). However Proposition 5.47 does cover the terms in the series (5.131). Applying (5.144) repeatedly shows that

$$(\widehat{R}_\alpha)^j \in \rho_{\mathrm{bf}}^j \widetilde{\Psi}_b^{-\infty,(\emptyset,E_j)}(X; {}^b\Omega^{\frac12}),$$

where the index sets E_j are defined inductively,

$$E_1 = E^-(\alpha), \ E_{j+1} - 1 = E_j \overline{\cup} (E^-(\alpha) - 1).$$

Here, if $(z, k) \in F$ then $(z - 1, k) \in F - 1$ and conversely. It is important to note that these index sets stabilize as $j \to \infty$, indeed

$$E_j \to \widehat{E}^-(\alpha) \text{ as } j \to \infty$$

meaning that eventually $E_j \cap K = \widehat{E}^-(\alpha) \cap K$ is constant for any compact set K. The series of kernels in (5.131) can therefore be summed as a Taylor series at $\mathrm{bf}(X_b^2)$ giving

$$
\begin{gathered}
S_\alpha \in \rho_{\mathrm{bf}} \widetilde{\Psi}_b^{-\infty,(\emptyset,\widehat{E}^-(\alpha))}(X;{}^b\Omega^{\frac{1}{2}}) \text{ such that} \\
S_\alpha - \sum_{j=1}^{p} (\widehat{R}_\alpha)^j \in \rho_{\mathrm{bf}}^{p+1} \widetilde{\Psi}_b^{-\infty,(\emptyset,\widehat{E}^-(\alpha))}(X;{}^b\Omega^{\frac{1}{2}}).
\end{gathered}
\tag{5.147}
$$

This in turn means that

$$
(\mathrm{Id} - \widehat{R}_\alpha)(\mathrm{Id} + S_\alpha) - \mathrm{Id} \in \bigcap_{j>0} \rho_{\mathrm{bf}}^j \widetilde{\Psi}_b^{-\infty,(\emptyset,\widehat{E}^-(\alpha))}(X;{}^b\Omega^{\frac{1}{2}}).
\tag{5.148}
$$

Here the error term is getting rather benign, so the composite operator

$$
G''_\alpha = \widehat{G}_\alpha \circ (\mathrm{Id} + S_\alpha),
\tag{5.149}
$$

with $\widehat{G}_\alpha$ from (5.130), is a rather precise parametrix. Notice that in the composition $\widehat{G}_\alpha \circ S_\alpha$ there are non-trivial index sets on the 'outside,' meaning on the left for the first factor and the right for the second factor. Such compositions have not yet been considered. To fully understand them requires a little extra work even though the fact that the error term will be as in (5.148) is already established. This leads to the final extension of the polyhomogeneous b-pseudodifferential operator calculus that will be made in the elliptic setting.

To make this final extension, the discussion of polyhomogeneous conormal distribution above needs to be expanded to include a special case where the separation condition, (5.78), fails. Indeed the case is X^2 with its two boundaries, $\mathrm{lb}(X^2) = \partial X \times X$ and $\mathrm{rb}(X^2) = X \times \partial X$. Let $\mathcal{E} = (E, F)$ be an index family for X^2 with E the index set for $\mathrm{lb}(X^2)$ and F the index set for rb . The objective is to define $\mathcal{A}_{\mathrm{phg}}^{(E,F)}(X^2)$. The special case $\mathcal{A}_{\mathrm{phg}}^{(0,E)}(X^2)$ has already been defined and naturally the existence of an expansion

$$
u \sim \sum_{(z,k)\in E} \rho_{\mathrm{lb}}^z (\log \rho_{\mathrm{lb}})^k u_{z,k}, \;\; u_{z,k} \in \mathcal{A}_{\mathrm{phg}}^{(0,F)}(X^2)
\tag{5.150}
$$

is to be expected. However the meaning of this asymptotic expansion, i.e. the nature of the remainder terms, needs to be specified carefully. The spaces of polyhomogeneous conormal distributions have natural topologies, given by $\mathcal{C}^\infty$ norms on the coefficients and $\mathcal{C}^N$-norms on the remainders in the expansions. Thus, using the fact that X^2 is a product, the space $\dot{\mathcal{C}}^N(X;\mathcal{A}_{\mathrm{phg}}^F(X))$ of N times differentiable functions on X with values in $\mathcal{A}_{\mathrm{phg}}^F(X)$ and with all derivatives up to order N vanishing at the boundary can be defined. These are the remainder terms allowed in (5.150):

DEFINITION 5.49. The space $\mathcal{A}^{(E,F)}_{\text{phg}}(X^2)$ consists of those extendible distributions on X^2 which have an expansion (5.150) with coefficients in $\mathcal{A}^{(0,F)}_{\text{phg}}(X^2)$ in the sense that

$$u - \sum_{\substack{(z,k)\in E \\ \operatorname{Re} z\le N}} \rho_{\text{lb}}^z(\log\rho_{\text{lb}})^k u_{z,k} \in \dot{\mathcal{C}}^N(X;\mathcal{A}^F_{\text{phg}}(X)) \ \forall\ N\in\mathbb{N}.$$

Certainly these spaces are local $\mathcal{C}^\infty(X^2)$-modules, so it makes sense to define the spaces of conormal sections of any vector bundle over X^2 as the finite sums of products:

$$\begin{aligned}\mathcal{A}^{(E,F)}_{\text{phg}}(X^2;U) =& \mathcal{A}^{(E,F)}_{\text{phg}}(X^2)\cdot\mathcal{C}^\infty(X^2;U)\\ =& \mathcal{A}^{(E,F)}_{\text{phg}}(X^2)\otimes_{\mathcal{C}^\infty(X^2)}\mathcal{C}^\infty(X^2;U).\end{aligned}$$

Of course it would be reassuring to know that these spaces are invariant under diffeomorphisms and are also independent of the choice of ordering of the two boundaries. For a more detailed discussion of these niceties the interested reader is referred to [63]. It is straightforward, just as in the proof of Lemma 5.22, to show that any series of coefficients $u_{z,k}\in\mathcal{A}^{(0,F)}_{\text{phg}}(X^2)$, $(z,k)\in E$, can appear in the expansion (5.150) of an element $u\in\mathcal{A}^{(E,F)}_{\text{phg}}(X^2)$ and that this expansion determines the sum uniquely up to an element of

$$\mathcal{A}^{(\emptyset,F)}_{\text{phg}}(X^2)\subset\mathcal{A}^{(0,F)}_{\text{phg}}(X^2),$$

this being the subspace with all derivatives vanishing on $\text{lb}(X^2)$.

EXERCISE 5.50. Prove this asymptotic completeness result and use it to prove that the space is indeed independent of the ordering of the two hypersurfaces by showing that the elements always have an expansion 'at the other boundary.'

Now these spaces can be used to define the third part of the polyhomogeneous b-pseudodifferential calculus, although these kernels really have nothing much to do with the 'b' nature of the calculus but are rather universal residual terms:

$$\Psi^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}) = \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X^2;{}^b\Omega^{\frac12}),\ \mathcal{E}=(E_{\text{lb}},E_{\text{rb}}).$$

This is consistent with the notation for the maximally residual operators in Definition 4.30, provided the superscript $\emptyset$ is taken to mean that both index sets are trivial. The 'full calculus' is then the sum of the three terms considered so far.

Definition 5.51. The 'full calculus' of (one-step polyhomogeneous) b-pseudodifferential operators on a compact manifold with boundary corresponding to an index family $\mathcal{E} = (E_{\rm lb}, E_{\rm rb})$ for X^2 consists of the operators of the form

(5.151)
$$\Psi_{b,\rm os}^{m,\mathcal{E}}(X;{}^b\Omega^{\frac12}) = \Psi_{b,\rm os}^{m}(X;{}^b\Omega^{\frac12}) + \widetilde{\Psi}_b^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}) + \Psi^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}),$$

with the three terms defined in Definition 4.22, (5.82) and Definition 5.49 respectively.

Notice that in terms of the bounded calculus

$$\Psi_{b,\rm os}^{m,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \subset \Psi_{b,\rm os,\infty}^{m,\alpha,\beta}(\alpha,\beta) \Longleftrightarrow \alpha < \inf E_{\rm lb},\ \beta < \inf E_{\rm rb}.$$

5.23. Composition in general.

So far various pieces of the composition formula have been checked. Now they can be put together, and extended somewhat, to arrive at the general result. First consider the action of the operators:

Proposition 5.52. *If $\mathcal{E}$ is an index family for X^2 and F is an index set for X with $\inf E_{\rm rb} + \inf F > 0$ then*

(5.152)
$$A \in \Psi_{b,\rm os}^{m,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \Longrightarrow A\colon \mathcal{A}_{\rm phg}^{F}(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{A}_{\rm phg}^{G}(X;{}^b\Omega^{\frac12}),\ G = E_{\rm lb}\overline{\cup}F.$$

Proof: The operator A is a sum of three terms as in (5.151). The second term is a sum of two terms to which (5.133) and (5.134) apply, so this part satisfies (5.152). The first term is in an element of order m in the small calculus to which Lemma 5.24 applies. This reduces consideration to an element of the third, residual, space in (5.151). The expansion of the kernel at the left boundary then gives the expansion, i.e.

$$\begin{gathered} A \in \Psi^{-\infty,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \Longrightarrow \\ A\colon \rho^\alpha H_b^m(X;{}^b\Omega^{\frac12}) \longrightarrow \mathcal{A}_{\rm phg}^{E_{\rm lb}}(X;{}^b\Omega^{\frac12}),\ \alpha + \inf E_{\rm rb} > 0,\ \forall\ m. \end{gathered}$$

This shows when composition is defined and then

Theorem 5.53. *If $\mathcal{E}$ and $\mathcal{F}$ are index families for X^2 with $\inf E_{\rm rb} + \inf F_{\rm lb} > 0$ then*

$$\begin{gathered} \Psi_{b,\rm os}^{m,\mathcal{E}}(X;{}^b\Omega^{\frac12}) \circ \Psi_{b,\rm os}^{m',\mathcal{F}}(X;{}^b\Omega^{\frac12}) \subset \Psi_{b,\rm os}^{m+m',\mathcal{G}}(X;{}^b\Omega^{\frac12}), \\ G_{\rm lb} = E_{\rm lb}\overline{\cup}F_{\rm lb},\ G_{\rm rb} = E_{\rm rb}\overline{\cup}F_{\rm rb}. \end{gathered} \tag{5.153}$$

Proof: When the two operators are decomposed as in (5.151) nine terms result. The number is however easily reduced. First note:

LEMMA 5.54. *The space* $\Psi^{m',\mathcal{E}}_{b,\mathrm{os}}(X;{}^b\Omega^{\frac12})$ *is an order-filtered two-sided module over the corresponding small calculus* $\Psi^{-\infty}_b(X;{}^b\Omega^{\frac12})$.

PROOF: Proposition 5.46 takes care of the first two terms in the decomposition (5.151), so it suffices to consider composition of the small and residual terms. This follows from Proposition 5.25. In fact if $A\in\Psi^m_b(X;{}^b\Omega^{\frac12})$ and $B\in\Psi^{-\infty,(E,F)}(X;{}^b\Omega^{\frac12})$ then expanding B at $\mathrm{rb}(X^2)$ expresses it as a sum of terms in $\mathcal{A}^E_{\mathrm{phg}}(X;{}^b\Omega^{\frac12})$, with coefficients in $\mathcal{A}^F_{\mathrm{phg}}(X;{}^b\Omega^{\frac12})$ on the right factor of X and a remainder term which is smooth with values in the polyhomogeneous space. Thus Proposition 5.25 shows that $A\circ B$ has a similar expansion so is also in $\Psi^{-\infty,(E,F)}(X;{}^b\Omega^{\frac12})$.

Thus all the terms in (5.153) with one factor in the small calculus have been controlled, leaving only four terms. Taking into account the behaviour under adjoints, which allows the order of the factors to be changed, this is reduced to three types of terms. The terms where one factor is residual can be handled as in Lemma 5.44, with Proposition 5.45 replacing Proposition 5.46.

Taking into account Propositions 5.47 and 5.27 it only remains to consider the case where the non-trivial expansions are on the outside and to show that

$$\begin{aligned}&\widetilde{\Psi}^{-\infty,(E,\emptyset)}_b(X;{}^b\Omega^{\frac12})\circ\widetilde{\Psi}^{-\infty,(\emptyset,F)}_b(X;{}^b\Omega^{\frac12})\\&\subset\widetilde{\Psi}^{-\infty,(E,F)}_b(X;{}^b\Omega^{\frac12})+\Psi^{-\infty,(E,F)}(X;{}^b\Omega^{\frac12}).\end{aligned}\tag{5.154}$$

Note that the second, residual, term must be included here, as opposed to (5.142) and (5.144).

To prove (5.154) it can further be assumed that the two factors have kernels supported near the front face, and even near the appropriate boundary hypersurface of the front face, left for the left factor and right for the right. Proceeding as before the kernel of the right factor will be replaced by its expansion, with remainder, at the right boundary. So choose a boundary defining function and denote its lift from the left factor as x and from the right factor x', as usual. Let $\phi\in\mathcal{C}^\infty_c(\mathbb{R})$ be equal to 1 near 0 so that both kernels are supported in the region where $\phi(x)\phi(x')=1$. Now the expansion gives

$$\begin{gathered}B=B''_N+B_N+B'_N\\ \kappa''(x,1/s,y,y')=\kappa''_N(x,1/s,y,y')\\ +\sum_{\substack{\operatorname{Re}z\le N\\(z,k)\in F_{\mathrm{lb}}}}(\frac{x'}{x})^z(\log\frac{x'}{x})^k a_{z,k}(x,y,y')\phi(x)\phi(x')+\kappa'_N(x',s,y,y').\end{gathered}\tag{5.155}$$

Here the first term is supported near $\mathrm{rb}(X_b^2)$, i.e. $s = \infty$, and is N times differentiable with derivatives to order N vanishing at $\mathrm{rb}(X_b^2)$. The sum forming the second term is the expansion, extended as a power of s all the way to the left boundary. The third term corrects the fact that the sum is extended globally so in fact has support disjoint from $\mathrm{rb}(X_b^2)$; its (finite) index set at $\mathrm{lb}(X_b^2)$ reflects the fact that it has the *negatives* of the powers from the right boundary in it,

$$\kappa'_N \in \widetilde{\Psi}_b^{-\infty,(G_N,\emptyset)}(X;{}^b\Omega^{\frac{1}{2}}),\ G_N = \{(z,k); -\operatorname{Re} z \geq -N, (z,k) \in F_{\mathrm{rb}}\}.$$

The point of the decomposition (5.155) is that Proposition 5.47 can be applied to the composition with the third term to see that

$$A \circ B'_N \in \widetilde{\Psi}_b^{-\infty,(H_N,\emptyset)}(X;{}^b\Omega^{\frac{1}{2}}),\ H_N = E_{\mathrm{lb}}\overline{\cup}G_N.$$

For the second term the fact that the kernel is essentially a power of x/x' can be used. This means that the variables y' can be regarded as parameters and the composition treated as simply the action on a polyhomogeneous conormal distribution, x^{-z}, with an appropriate logarithmic factor, using Proposition 5.47, so

$$A \circ B_N = \sum_{\substack{\operatorname{Re} z \leq N \\ (z,k)\in F_{\mathrm{lb}}}} \sum_{0\leq l\leq k} C_{z,l}(x')^z(\log x')^{k-l}\phi(x'),$$
$$C_{z,l} \in \mathcal{A}_{\mathrm{phg}}^{(H'_N,0)}(X^2,{}^b\Omega^{\frac{1}{2}}),\ H'_N = E_{\mathrm{lb}}\overline{\cup}\{(-z,l)\}.$$

In particular this term is residual. Finally the first term in (5.155) contributes a remainder term, in the sense of an N times differentiable function with values in $\mathcal{A}_{\mathrm{phg}}^E(X;{}^b\Omega^{\frac{1}{2}})$, to the residual kernel. This proves (5.154) and hence completes the proof of Theorem 5.53.

5.24. General bundles and summary.

As for the small calculus, discussed in §4.16, the extension of both the calculus with bounds and the full polyhomogeneous calculus to arbitrary vector bundle coefficients is straightforward, and essentially only a matter of notation. Thus the spaces themselves are defined as in (4.108) for an index family $\mathcal{E}$ for X^2 and any two vector bundles E and F over X :

$$\Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E,F) \overset{\mathrm{def}}{=} \Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;{}^b\Omega^{\frac{1}{2}}) \otimes_{\mathcal{C}^\infty(X_b^2)} \mathcal{C}^\infty(X_b^2;\beta_b^*\operatorname{Hom}(F\otimes{}^b\Omega^{-\frac{1}{2}},E\otimes{}^b\Omega^{-\frac{1}{2}})). \tag{5.156}$$

As before the shortened notation

$$\Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E) = \Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E,E)$$

is used when the two vector bundles coincide. The calculus has the obvious transformation property under the passage to adjoints with respect to inner products on the fibres and a b-density on X :

$$\begin{gathered}\Psi^{m,\mathcal{E}}_{b,\mathrm{os}}(X;E,F)\ni A\longleftrightarrow A^*\in\Psi^{m,\mathcal{F}}_{b,\mathrm{os}}(X;F,E),\\ \mathcal{E}=(E_{\mathrm{lb}},E_{\mathrm{rb}}),\ \mathcal{F}=(E_{\mathrm{rb}},E_{\mathrm{lb}}).\end{gathered}\tag{5.157}$$

Note that the behaviour of the index sets here depends on the fact that the adjoint is taken with respect to a b-density on X.

EXERCISE 5.55. Work out the behaviour of the general calculus under conjugation by powers of a defining function and deduce what happens to the index sets if the adjoint in (5.157) is taken with respect to a non-vanishing density in the usual sense.

The definition, (5.144), is the same as requiring that on subdivision by a partition of unity on X^2_b subordinate to a covering on which the bundles are trivial the kernels become matrices with entries in the space in (5.151). This localization can be used to extend all the results above to the general case. Thus Proposition 5.52 gives the action on polyhomogeneous conormal distributions and Theorem 5.34 on weighted Sobolev spaces:

$$\begin{gathered}A\in\Psi^{m,\mathcal{E}}_{b,\mathrm{os}}(X;E,F)\Longrightarrow\\ A\colon\mathcal{A}^G_{\mathrm{phg}}(X;E)\longrightarrow\mathcal{A}^H_{\mathrm{phg}}(X;F),\ \text{if }\inf G+\inf E_{\mathrm{rb}}>0,\ H=G\overline{\cup}E_{\mathrm{lb}},\\ A\colon\rho^\alpha H^M_b(X;E)\longrightarrow\rho^\beta H^{M-m}_b(X;F),\ \text{if }\alpha+\inf E_{\mathrm{rb}}>0,\ \beta<\inf E_{lb}.\end{gathered}\tag{5.158}$$

Notice that the b-Sobolev spaces are based on the L^2 space with respect to a b-density on X.

The composition formula then follows by localizing Theorem 5.53:

$$\begin{gathered}\Psi^{m,\mathcal{E}}_{b,\mathrm{os}}(X;G;F)\circ\Psi^{m',\mathcal{F}}_{b,\mathrm{os}}(X;E;G)\subset\Psi^{m+m',\mathcal{G}}_{b,\mathrm{os}}(X;E,F)\\ \text{provided }\inf E_{\mathrm{rb}}+\inf F_{\mathrm{lb}}>0,\ \mathcal{G}=(E_{\mathrm{lb}}\overline{\cup}F_{\mathrm{lb}},E_{\mathrm{rb}}\overline{\cup}F_{\mathrm{rb}}).\end{gathered}$$

EXERCISE 5.56. Strictly speaking the mapping property on Sobolev spaces has only been proved for integral orders (including that of the operator). This is all that is needed in the discussion of the inverses of differential operators but it is worthwhile to consider the general case, to see how the b-Sobolev spaces fit with the calculus in the same way that ordinary Sobolev spaces mesh with the usual calculus. Using the small calculus (corresponding to the index family $\mathcal{E}=(\emptyset,\emptyset)$) :

$$\Psi^m_{b,\mathrm{os}}(X;E)=\Psi^{m,(\emptyset,\emptyset)}_{b,\mathrm{os}}(X;E)$$

one can set

(5.159)

$$\rho^\alpha H_b^m(X;E) = \left\{u \in \rho^\alpha H_b^0(X;E); \Psi_{b,\mathrm{os}}^m(X;E)u \subset \rho^\alpha H_b^0(X;E)\right\} \ \ m>0$$

$$\rho^\alpha H_b^m(X;E) = \rho^\alpha H_b^0(X;E) + \Psi_{b,\mathrm{os}}^m(X;E)\rho^\alpha H_b^0(X;E) \ \ m<0.$$

Using the part of (5.158) which has already been shown and the composition formula for the small calculus check that this is consistent with the definition for integral m from (5.42) and (5.43). Deduce (5.158) in general.

The symbol sequence extends from (4.109) to give a short exact sequence

$$0 \longrightarrow \Psi_{b,\mathrm{os}}^{m-1,\mathcal{E}}(X;E,F) \hookrightarrow \Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E,F) \xrightarrow{{}^b\sigma_m} S^{[m]}({}^bT^*X;\pi^*\hom(E,F)) \longrightarrow 0$$

for any index family, since the additional terms are smooth near the diagonal, at least in the interior. Similarly the indicial homomorphism extends to give a short exact sequence:

$$0 \longrightarrow \rho_{\mathrm{bf}}\Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E,F) \hookrightarrow \Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E,F) \longrightarrow \Psi_{b,I,\mathrm{os}}^{m,\mathcal{E}}(\widetilde{X};E,F) \longrightarrow 0 \tag{5.160}$$

$$\text{provided if } \inf E_{\mathrm{lb}} + \inf E_{\mathrm{rb}} > 0 \tag{5.161}$$

for each m. This is defined simply by restriction of the kernels to the front face, with $\mathcal{E}$ interpreted as an index set for $\widetilde{X}^2$. The choice of a trivialization of the normal bundle allows the indicial family to be defined by Mellin transform of the restriction of its kernel to the front face. In this case

$$I_\nu(P,\lambda) \in \Psi^m(\partial X;E,F) \text{ if } \inf E_{\mathrm{lb}} + \inf E_{\mathrm{rb}} > 0$$

is meromorphic in $\lambda \in \mathbb{C}$ with poles only at the points $-iz$, with order at most k if $(z,k-1) \in E_{\mathrm{lb}}$ or $(-z,k-1) \in E_{\mathrm{rb}}$. The residues are finite rank smoothing operators. Notice that in general the indicial family does not quite determine the indicial operator, since one has to decide which poles correspond to lb and which to rb. As already noted this splitting of the complex plane amounts to a boundary condition. As before, the symbol map and indicial operator both give multiplicative homomorphisms when the composition of operators is defined.

Exercise 5.57. The normal homomorphism

$$\Psi_{b,\mathrm{os}}^{m,\mathcal{E}}(X;E,F) \longrightarrow \Psi_{b,I,\mathrm{os}}^{m,\mathcal{E}}(\widetilde{X};E,F) \tag{5.162}$$

can be defined even without assuming (5.161), although at some expense to continuity in the index family. To do this observe that 'restriction of the kernels to the front face' can be interpreted as taking the coefficient of ρ_{bf}^0 in an asymptotic expansion around $\mathrm{bf}(X_b^2)$. With this more general definition interpret (5.162) and compute the range and null space.

The extension of the calculus with bounds to the case of general bundles is completely parallel to this discussion and is therefore left as an exercise. Since the b-trace is quite important in applications it is worth noting how it extends to the general calculus with bounds. First, for each choice of trivialization of the normal bundle, the b-integral in Lemma 4.59 extends directly to define a linear functional

$$\text{(5.163)} \qquad {}^{\nu}\!\!\int : \mathcal{C}^\infty(X; {}^b\Omega^{\frac{1}{2}}) + \rho^\alpha H_b^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathbb{C} \text{ if } \alpha > 0$$

since the additional term is integrable. Moreover (4.139) still holds. Since the restriction to the diagonal of an element of $\Psi_{b,\mathrm{os},\infty}^{-\infty,(\alpha,\beta)}(X; E)$ is just an element of $\mathcal{C}^\infty(X; \hom(E) \otimes {}^b\Omega) + \rho^{\alpha+\beta} H_b^\infty(X; \hom(E \otimes {}^b\Omega))$, it follows that the pointwise trace on $\hom(E)$ with (5.163) defines the b-trace, just as in (4.141):

$$\text{b-Tr}_\nu : \Psi_{b,\mathrm{os},\infty}^{-\infty,(\alpha,\beta)}(X; E) \longrightarrow \mathbb{C} \text{ if } \alpha, \beta > 0.$$

The continuity of this map in terms of the seminorms defining the calculus with bounds is used in the treatment of the limit of the b-trace of the heat semigroup as $t \to \infty$ in §7.7.

EXERCISE 5.58. Extend Proposition 5.9 to this setting.

5.25. Parametrices and null space.

The constructions above can now be extended to the case of general bundle coefficients. More importantly the composition in (5.149) can be analyzed.

PROPOSITION 5.59. *If $P \in \mathrm{Diff}_b^k(X; E, F)$ is elliptic there exist parametrices corresponding to any $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$*

$$E_L \in \Psi_{b,\mathrm{os}}^{-k,(\widehat{E}^+(\alpha),\check{E}^-(\alpha))}(X; F, E)$$
$$E_R \in \Psi_{b,\mathrm{os}}^{-k,(\check{E}^+(\alpha),\widehat{E}^-(\alpha))}(X; F, E),$$

where $\widehat{E}^\pm(\alpha)$ are the index sets defined by (5.121), $\check{E}^\pm(\alpha)$ those defined by (5.122) and the full polyhomogeneous calculus is given by (5.156), such that

$$P \circ E_R = \mathrm{Id} - R_R,\ R_R \in \rho_{\mathrm{bf}}^\infty \Psi_{b,\mathrm{os}}^{-\infty,(\emptyset,\widehat{E}^-(\alpha))}(X; E)$$
$$E_L \circ P = \mathrm{Id} - R_L,\ R_L \in \rho_{\mathrm{bf}}^\infty \Psi_{b,\mathrm{os}}^{-\infty,(\widehat{E}^+(\alpha),\emptyset)}(X; E)$$

as operators on $\rho^\alpha H_b^{-\infty}(X;E)$. *Moreover*

$$E_R - E_L \in \Psi_{b,\mathrm{os}}^{-\infty,\check{\mathcal{E}}(\alpha)}(X;F,E). \tag{5.164}$$

PROOF: The right parametrix is given by (5.149), extended to the case of bundles, and its properties follow directly by applying the composition formula. The left parametrix is just the adjoint of a right parametrix for P^* with respect to some inner product. That the difference of the parametrices satisfies (5.117) and (5.164) follows from the composition formula.

In particular the proof of Theorem 5.40 extends immediately to give:

THEOREM 5.60. *If X is a compact manifold with boundary any elliptic $P \in \mathrm{Diff}_b^k(X;E,F)$ is Fredholm as an operator $P\colon x^{\mathfrak{a}}H_b^{k+m}(X;E) \longrightarrow x^{\mathfrak{a}}H_b^m(X;F)$ if and only if $\mathfrak{a} \notin -\operatorname{Im}\operatorname{spec}_b(P)$ and the index is independent of m.*

Observe that if P is an elliptic b-differential operator then the polyhomogeneity of Pu implies that of u.

PROPOSITION 5.61. *For any elliptic $P \in \mathrm{Diff}_b^k(X;E,F)$*

$$\begin{gathered} u \in \rho^\alpha H_b^m(X;E),\ Pu \in \mathcal{A}^G_{\mathrm{phg}}(X;F) \\ \Longrightarrow u \in \mathcal{A}^H_{\mathrm{phg}}(X;E),\ H = G\overline{\cup}\widehat{E}^+(\alpha). \end{gathered}$$

PROOF: If $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ apply the left parametrix from Proposition 5.59 and use (5.159). If $\alpha \in -\operatorname{Im}\operatorname{spec}_b(P)$ then apply the parametrix for $\alpha - \epsilon$, $\epsilon > 0$. Since the leading terms in the expansion of u correspond to powers z with $\operatorname{Re} z = \alpha$, they must vanish by the assumption on u, thus $u \in \rho^\beta H_b^m(X;E)$ for some $\beta > \alpha$ and the result follows in this case too.

As a simple consequence of this regularity result the precise form of elements of the null space of P can be investigated. Namely

$$\begin{gathered} P \in \mathrm{Diff}_b^m(X;E,F) \text{ elliptic },\ u \in \rho^\alpha H_b^{-\infty}(X;E) \text{ and } Pu = 0 \\ \Longrightarrow u \in \mathcal{A}^{\widehat{E}^+(\alpha)}_{\mathrm{phg}}(X;E). \end{gathered} \tag{5.165}$$

Define

$$\begin{aligned} \mathrm{null}(P,\alpha) &= \left\{ u \in \rho^\alpha H_b^{-\infty}(X;{}^b\Omega^{\frac12}); Pu = 0 \right\} \\ &= \left\{ u \in \rho^\alpha H_b^{\infty}(X;{}^b\Omega^{\frac12}); Pu = 0 \right\}. \end{aligned}$$

For an $\mathbb{R}^+$-invariant elliptic operator $Q \in \mathrm{Diff}_{b,I}^k(\widetilde{X};E)$ the generalized null space associated to $z \in \operatorname{spec}_b(Q)$ is

$$F(Q,z) = \left\{ u = \sum_{0\le j\le p} x^z (\log x)^j \phi_j; Qu = 0 \right\}. \tag{5.166}$$

This is clearly independent of the choice of projective coordinate x. Moreover

LEMMA 5.62. *For any elliptic $Q \in \operatorname{Diff}^k_{b,I}(\widetilde{X}; E)$*

$$\dim F(Q,z) = \operatorname{rank}(z),\ \ z \in \operatorname{spec}_b(Q). \tag{5.167}$$

PROOF: The 'formal Mellin transform'

$$u \longmapsto \sum_{j\geq 0} (z-il)^{-1-j}(-1)^j j!\phi_j$$

identifies $F(Q,z)$ with the space in (5.9) of which $\operatorname{rank}(z)$ is by definition the dimension.

Now, suppose an identification of the compactified normal bundle to the boundary with a collar neighbourhood of the boundary in X is fixed. An element $u \in F((I(P),z)$ can be naturally identified with an element of $\rho^\alpha H^\infty_b(X; {}^b\Omega^{\frac12})/\rho^{a-\epsilon}H^\infty_b(X; {}^b\Omega^{\frac12})$ provided $\epsilon \in (0,1]$ and $\operatorname{Im} z \in (-a, -a+\epsilon)$.

PROPOSITION 5.63. *If $P \in \operatorname{Diff}^m_b(X; {}^b\Omega^{\frac12})$ is elliptic and $\alpha' > \alpha > \alpha' - 1$, the leading part of the expansion of $u \in \operatorname{null}(P,\alpha)$ at ∂X defines*

$$\operatorname{null}(P,\alpha) \longrightarrow \sum_{\substack{-iz\in\operatorname{spec}_b(P)\\ \alpha<\operatorname{Re} z\leq\alpha'}} F(I(P),z) \tag{5.168}$$

with null space $\bigcap_{\beta>\alpha'} \operatorname{null}(P,\beta)$.

This map will be used in the proof of the relative index theorem.

5.26. Generalized inverse.

Finally we can give a detailed description of the generalized inverse of an elliptic b-differential operator.

PROPOSITION 5.64. *For any $P \in \operatorname{Diff}^m_b(X; E, F)$ elliptic, any weight $\alpha \notin -\operatorname{Im}\operatorname{spec}_b(P)$ and any boundary defining function $\rho \in \mathcal{C}^\infty(X)$ the generalized inverse to P, $G_\alpha \colon \rho^\alpha H^0_b(X; F) \longrightarrow \rho^\alpha H^m_b(X; E)$ is an element of $\Psi^{-m,\check{\mathcal{E}}(\alpha)}_{b,\mathrm{os}}(X; F, E)$ fixed by*

$$\begin{aligned} P \circ G_\alpha &= \operatorname{Id} - \Pi_1 \\ G_\alpha \circ P &= \operatorname{Id} - \Pi_0, \end{aligned}$$

where Π_0 *and* Π_1 *are the orthogonal projections onto the null space and off the range, in* $\rho^\alpha H_b^0(X;E)$ *and* $\rho^\alpha H_b^0(X;F)$ *respectively, and where*

$$\check{\mathcal{E}}(\alpha) = (\check{E}^+(\alpha), \check{E}^-(\alpha))$$

is the index family for X^2 *defined by* (5.122). *If* P *is invertible then*

$$G_\alpha \in \Psi_{b,\mathrm{os}}^{-m,\widehat{\mathcal{E}}(\alpha)}(X;F,E), \tag{5.169}$$

with the index set fixed by (5.121).

Proof: The generalized inverse differs from the parametrices constructed in Proposition 5.59 by finite rank operators. Using Proposition 5.59 and (5.165) to modify the remainders leads to (5.169).

Chapter 6. Relative index, cohomology and resolvent

In this chapter three applications of the construction of parametrices will be made. The first result is the relative index formula, Theorem 6.5, relating the index of an elliptic differential operator for different values of the weighting of the b-Sobolev spaces. This was proved for pseudodifferential operators in [64] and allows the proof of the APS theorem to be reduced to the Fredholm case. To illustrate the relative index theorem, an idea from [8], extended recently by Gromov and Shubin [40], is used to deduce the Riemann-Roch theorem, for Riemann surfaces, from it. Secondly the cohomology of a compact manifold with boundary is represented in terms of harmonic forms. That is, the Hodge theory associated to an exact b-metric is developed. Finally the resolvent of a second-order elliptic and self-adjoint family of b-differential operators is analyzed. In particular the resolvent kernel is shown to have an analytic extension to an infinitely branched covering of the complex plane and this is used to give a detailed description of the spectrum. The relationship between the extended L^2 null space of a Dirac operator and its adjoint is also investigated. This applies in particular to the Dirac Laplacian, $\eth_E^2$.

6.1. Boundary pairing.

A common feature of the proofs of the results in this chapter is that they involve the determination of the dimensions of various null spaces or the differences of such dimensions. We therefore start with some results on the pairing of generalized boundary data which are helpful in these computations. By generalized boundary data we mean the spaces, $F(P,z)$, associated to $z \in \operatorname{spec}_b(P)$ in (5.166). There is a basic relationship between these spaces for P and its adjoint:

Lemma 6.1. *If E and F are vector bundles with Hermitian inner products and $\nu \in \mathcal{C}^\infty(X;{}^b\Omega)$ is a non-vanishing positive b-density then the formal adjoint, P^*, of an elliptic element $P \in \operatorname{Diff}_b^k(X;E,F)$, has the property*

$$\operatorname{rank}(P,\zeta) = \operatorname{rank}(P^*,\overline{\zeta}) \ \forall\ \zeta \in \operatorname{spec}_b(P). \tag{6.1}$$

Proof: The definition (5.32) and the identity (4.112) together show that

$$I_\nu(P^*,\zeta) = [x^{-i\zeta}I(P)^*x^{i\zeta}]_\partial = [(x^{-i\overline{\zeta}}I(P)x^{i\overline{\zeta}})^*]_\partial = I_\nu(P,\overline{\zeta})^*. \tag{6.2}$$

From this (6.1) follows.

It is useful to see (6.1) in a rather more constructive way, in terms of boundary pairing. Since the indicial families at each boundary hypersurface are unrelated it suffices to assume, for the moment, that ∂X is connected.

For simplicity of notation we also assume that $E = F = {}^b\Omega^{\frac12}$; the section ν is then not needed to define P^*.

Choose a cut-off function $\phi \in \mathcal{C}^\infty(\widetilde{X})$, with $\phi \equiv 1$ near $\partial_0 X$ and $\phi \equiv 0$ near $\partial_1 X$. Then consider the sesquilinear map

$$
\begin{gathered}
F(P, z') \times F(P^*, \overline{z}) \ni (u, v) \longmapsto \\
B(u, v) = \frac{1}{i} \int\limits_{\widetilde{X}} \left(I(P)(\phi u)\overline{\phi v} - (\phi u)\overline{I(P^*)(\phi v)} \right),
\end{gathered} \tag{6.3}
$$

where $F(P, z) = F(I(P), z) \subset \mathcal{C}^{-\infty}(\widetilde{X}; {}^b\Omega^{\frac12})$, the formal null space associated to z, is defined in (5.166). The integrand is a density with compact support in the interior of $\widetilde{X}$, so the integral converges absolutely. Then (6.1) also follows from:

Proposition 6.2. *The sesquilinear form* (6.3) *is independent of the choice of cut-off function and when $z' = z$ gives a non-degenerate pairing for each $z \in \operatorname{spec}_b(P)$. Moreover the same integral gives the trivial pairing*

$$
\begin{gathered}
B(u, v) = 0 \quad \forall\ u \in F(P, z'),\ v \in F(P^*, \overline{z}), \\
z, z' \in \operatorname{spec}_b(P),\ z \neq z'.
\end{gathered} \tag{6.4}
$$

Proof: Suppose that $z = z'$ and consider the first term in the integrand in (6.3). Since u is annihilated by $I(P)$, $I(P)(\phi u)$ is compactly supported in the interior and ϕv has support bounded above with growth determined by $x^{i\overline{z}}$ as $x \downarrow 0$. Set $a = -\operatorname{Im} z$ then $x^{-a+\epsilon}(\phi v)$ is square integrable if $\epsilon > 0$. Similarly for the second term, $x^{a+\epsilon}\phi u$ is square-integrable and $I(P)(\phi v)$ has compact support in the interior. Plancherel's formula for the Mellin transform can therefore be applied, and gives

$$
\begin{aligned}
B(u, v) = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} \int\limits_{\partial X} \Big[& \left(x^{-a-\epsilon} I(P)(\phi u)\right)_M(\lambda) \overline{(x^{a+\epsilon}\phi v)_M(\lambda)} \\
& - (x^{-a+\epsilon}\phi u)_M(\lambda) \overline{\left(x^{a-\epsilon} I(P^*)(\phi v)\right)_M(\lambda)} \Big] d\lambda.
\end{aligned}
$$

Since $(x^{a+\epsilon}\phi v)_M(\lambda) = (\phi v)_M(\lambda + ia + i\epsilon)$, and similarly for the other terms, this can be rewritten

$$
\begin{aligned}
B(u, v) = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} \int\limits_{\partial X} \Big[& \left(I(P)(\phi u)\right)_M(\lambda - ia - i\epsilon) \overline{(\phi v)_M(\lambda + ia + i\epsilon)} \\
& - (\phi u)_M(\lambda - ia + i\epsilon) \overline{\left(I(P^*)(\phi v)\right)_M(\lambda + ia - i\epsilon)} \Big] d\lambda.
\end{aligned}
$$

By assumption u is the product of x^{iz} and a polynomial in $\log x$. The Mellin transform of ϕu is therefore defined in $\operatorname{Im}\lambda > \operatorname{Im} z$ and extends to be meromorphic with a pole only at $\lambda = z$ such that $I(P)(\phi u)$ has an entire Mellin transform given by $I_\nu(P, \lambda)(\phi u)_M(\lambda)$. Similar statements hold for v, so the second term can be written as a contour integral in $\zeta = \lambda - ia - i\epsilon$ over $\operatorname{Im}\zeta = \operatorname{Im} z + \epsilon$. The same reasoning allows the first term to be written as a contour integral over $\operatorname{Im}\zeta = \operatorname{Im} z - \epsilon$:

(6.5)
$$B(u,v) = \frac{1}{2\pi i} \int_{\operatorname{Im}\zeta = \operatorname{Im} z - \epsilon} \int_{\partial X} \left(I(P)(\phi u)\right)_M(\zeta) \overline{v_M(\overline{\zeta})} d\zeta - \frac{1}{2\pi i} \int_{\operatorname{Im}\zeta = \operatorname{Im} z + \epsilon} \int_{\partial X} u_M(\zeta) \overline{\left(I(P^*)(\phi u)\right)_M(\overline{\zeta})} d\zeta.$$

Now $(I(P)(\phi u))_M(\zeta) = I_\nu(P, \zeta)(\phi u)_M$ and similarly for the second term.

The rapid vanishing of the Mellin transforms at real infinity permits integration by parts in the integral over ∂X and hence, using (6.2) and reverting to the variable λ, (6.5) becomes

(6.6)
$$B(u,v) = \frac{1}{2\pi i} \oint_{\operatorname{Im}\zeta = \operatorname{Im} z - \epsilon} \int_{\partial X} I_\nu(P,\lambda)(\phi u)_M(\lambda) \overline{(\phi v)_M(\overline{\lambda})} d\lambda - \frac{1}{2\pi i} \oint_{\operatorname{Im}\zeta = \operatorname{Im} z + \epsilon} \int_{\partial X} I_\nu(P,\lambda)(\phi u)_M(\lambda) \overline{(\phi v)_M(\overline{\lambda})} d\lambda.$$

Again the rapid vanishing of all the Mellin transforms at real infinity, and the fact that the only poles are at $\lambda = z$, means that (6.6) can be rewritten as an integral over a finite contour:

$$B(u,v) = \frac{1}{2\pi i} \oint_{\Gamma} \int_{\partial X} I_\nu(P,\lambda)(\phi u)_M(\lambda) \overline{(\phi v)_M(\overline{\lambda})} d\lambda, \tag{6.7}$$

where Γ is the simple closed contour on which $|\lambda - z| = \epsilon$, traversed counterclockwise.

If ϕ is replaced by another cut-off function, ϕ', then the difference between the two integrals in (6.7) can be written

$$\frac{1}{2\pi i} \oint_{\Gamma} \int_{\partial X} ((\phi - \phi')u)_M(\lambda) \overline{I_\nu(P^*, \overline{\lambda})(\phi v)_M(\overline{\lambda})} d\lambda - \frac{1}{2\pi i} \oint_{\Gamma} \int_{\partial X} I_\nu(P,\lambda)(\phi' u)_M(\lambda) \overline{((\phi - \phi')v)_M(\overline{\lambda})} d\lambda$$

which vanishes identically since the integrands are entire. Thus B is independent of the choice of ϕ and so is a well-defined form. Next we proceed to show that it is non-degenerate in (6.3).

If $I_\nu(P,\lambda)(\phi u)_M(\lambda)$ is replaced in (6.7) by an arbitrary function $f(\lambda)$ holomorphic in $|\lambda - z| < 2\epsilon$, with values in $\mathcal{C}^\infty(\partial X;\Omega^{\frac{1}{2}})$ then the resulting pairing certainly has no null space on $F(P^*,\overline{z})$ since the vanishing of the contour integral

$$\oint\limits_{\Gamma}\int\limits_{\partial X} f(\lambda)\overline{(\phi v)_M(\overline{\lambda})}d\lambda$$

for all f implies that $(\phi v)_M$ is regular at z and hence that $v = 0$. On the other hand if $U(\lambda) = (\phi u)_M(\lambda)$ is replaced by $\widetilde{U}(\lambda)$, just such a holomorphic function, then the integral vanishes identically since it has a holomorphic integrand. For $\epsilon > 0$, small enough, $I_\nu(P,\lambda)^{-1}$ has only a pole at $\lambda = z$ in this disk, so any holomorphic function is of the form $I_\nu(P,\lambda)[U(\lambda) + \widetilde{U}(\lambda)]$ with $\widetilde{U}$ holomorphic and $U = (\phi u)_M(\lambda)$ for some $u \in F(P,z)$. It follows that B has no null space on $F(P,z)$. Integration by parts allows (6.7) to be written

$$B(u,v) = \frac{1}{2\pi i}\oint\limits_{\Gamma}\int\limits_{\partial X}(\phi u)_M(\lambda)\overline{I_\nu(P^*,\overline{\lambda})(\phi v)_M(\overline{\lambda})}d\lambda,$$

so the same argument shows that there is no null space on $F(P^*,\overline{z})$. Thus B is non-degenerate, proving the first part of the proposition.

When $z \neq z'$ the same type of argument leads to (6.4).

Now suppose that X is a general compact manifold with boundary, where ∂X may have several components. Then if $P \in \mathrm{Diff}^k_b(X;E,F)$ is elliptic, with E and F Hermitian and a positive section $\nu \in \mathcal{C}^\infty(X;{}^b\Omega)$ is specified the same argument applies to give a non-degenerate pairing for each boundary hypersurface:

$$\begin{aligned} &F_j(P,z)\times F_j(P^*,\overline{z}) \ni (u,v) \longmapsto \\ \text{(6.8)}\qquad B_j(u,v) = &\int\limits_{\widetilde{X}_j}\big(\langle I_j(P)(\phi u),\phi v\rangle - \langle \phi u, I_j(P^*)(\phi v)\rangle\big)\nu_j. \end{aligned}$$

Here ν_j is the density induced on $\partial_j X$. The analogue of (6.4) also holds.

EXERCISE 6.3. Go through the proof of Proposition 6.2 inserting inner products and densities and so check that (6.8) is non-degenerate and independent of the choice of ϕ.

Next observe that, in case ∂X is connected, the bilinear form (6.3) can also be obtained directly from P rather than in terms of $I(P)$ as an operator on $\widetilde{X}$. Choose an identification of $\widetilde{X}$ with a collar neighbourhood of ∂X and $\phi \in \mathcal{C}^\infty(X)$ which is identically equal to 1 near ∂X with support in the collar. Using the collar identification, if $u \in F(P,z)$ then ϕu can be identified with an element of $x^{-\operatorname{Im} z-\epsilon} H_b^\infty(X;{}^b\Omega^{\frac12})$. Then

$$\begin{aligned} &F(P,z) \times F(P^*,\overline{z}) \ni (u,v) \longmapsto \\ &B(u,v) = \frac{1}{i}\int\limits_X \Big(P(\phi u)\overline{\phi v} - (\phi u)\overline{P^*(\phi v)}\Big). \end{aligned} \tag{6.9}$$

To see this it suffices to note that in the collar $P = I(P) + xP'$, with $P' \in \operatorname{Diff}_b^m(X;{}^b\Omega^{\frac12})$. The extra vanishing at $x = 0$ means that integration by parts is permissible, so replacing P by xP' in (6.9) gives zero, proving (6.9). In fact by the same reasoning if $u' \in x^{-\operatorname{Im} z+\epsilon} H_b^\infty(X;{}^b\Omega^{\frac12})$ and $v' \in x^{\operatorname{Im} z+\epsilon} H_b^\infty(X;{}^b\Omega^{\frac12})$ then

$$\begin{aligned} &F(P,z) \times F(P^*,\overline{z}) \ni (u,v) \longmapsto \\ &B(u,v) = \frac{1}{i}\int\limits_X \Big(P(\phi u + u')\overline{\phi v} - (\phi u)\overline{P^*(\phi v + v')}\Big). \end{aligned} \tag{6.10}$$

If ∂X has several components then B is the sum of the B_j in (6.8).

For any $r \in -\operatorname{Im}\operatorname{spec}_b(P)$ set

$$G(P,r) = G(I(P),r) = \bigoplus_{\substack{z\in\operatorname{spec}_b(P)\\ \operatorname{Im} z = -r}} F(P,z). \tag{6.11}$$

The bilinear form B is therefore defined, by (6.9), as a sesquilinear map

$$B\colon G(P,r) \times G(P^*,-r) \longrightarrow \mathbb{C} \quad \forall\ r \in -\operatorname{Im}\operatorname{spec}_b(P). \tag{6.12}$$

For any r we shall use Proposition 5.63 with $\alpha = r - \epsilon$ and $\alpha' = r + \epsilon$, $\epsilon > 0$ being chosen so small that r is the only point of $-\operatorname{Im}\operatorname{spec}_b(P)$ in $[\alpha,\alpha']$, to define

$$\begin{aligned} G'(P,r) = \big\{u \in G(P,r);\ \exists\ u' \in x^\alpha H_b^\infty(X;{}^b\Omega^{\frac12}) \text{ with} \\ Pu' = 0,\ u' - \phi u \in x^{\alpha'} H_b^\infty(X;{}^b\Omega^{\frac12})\big\} \end{aligned} \tag{6.13}$$

as the image of (5.168). Notice that $G'(P,r)$ may very well depend on P, not just $I(P)$.

LEMMA 6.4. *Under the pairing B in (6.12), $G'(P,r)$ and $G'(P^*,-r)$ are annihilators of each other:*

$$G'(P,r) = \{u \in G(P,r); B(u,v) = 0 \ \forall \ v \in G'(P^*,-r)\}\,. \tag{6.14}$$

PROOF: Conjugating by x^r we may assume that $r = 0$. Now suppose $u \in G'(P,0)$, i.e. $u \in G(P,0)$ and $P(\phi u) = Pu'$, with $u' \in x^\epsilon H^0_b(X;{}^b\Omega^{\frac12})$. Then $P(\phi u)$ can be replaced by Pu' in the first term in (6.9). Integration by part is justified since $u' \in x^\epsilon H^m_b(X;{}^b\Omega^{\frac12})$, so the integral reduces to

$$\int\limits_X (u' - \phi u)\overline{P^*(\phi v)}.$$

Then if $v \in G'(P,0)$, i.e. $P^*(\phi v) = P^*(v')$ with $v \in x^\epsilon H^m_b(X;{}^b\Omega^{\frac12})$ integration by parts is again possible, so the pairing vanishes. Conversely suppose $u \in G(P,0)$. Then $f = P(\phi u) \in x^\epsilon H^\infty_b(X;{}^b\Omega^{\frac12})$. The range of P on $x^\epsilon H^m_b(X;{}^b\Omega^{\frac12})$ is the annihilator of $\operatorname{null}(P^*,-\epsilon)$, the null space of P^* on $x^{-\epsilon}H^m_b(X;{}^b\Omega^{\frac12})$. Of course if $w \in \operatorname{null}(P^*,\epsilon)$ then $\langle f,w\rangle = 0$, so it suffices to check that $\langle f,v\rangle = 0$ for $w = \phi v - v' \in \operatorname{null}(P^*,-\epsilon)$, where $v \in G'(P^*,0)$ and $v' \in x^\epsilon H^m_b(X;{}^b\Omega^{\frac12})$. This is exactly what the vanishing of $B(u,v)$ for $v \in G'(P^*,0)$ shows, so (6.14) does indeed hold.

6.2. Relative index formula.

The dependence of the index of an elliptic element $P \in \operatorname{Diff}^m_b(X;E,F)$ on the parameters $\mathfrak{a} \in \mathbb{R}^p$, in Theorem 5.60, will be considered next. Here p is the number of boundary components of X. The index of a strongly continuous family of Fredholm operators is constant, so the function

$$\begin{aligned}\operatorname{ind}(P,\mathfrak{a}) = \dim&\left\{u \in \rho^{\mathfrak{a}}H^M_b(X;E); Pu = 0\right\}\\ &- \operatorname{codim}\left\{v \in \rho^{\mathfrak{a}}H^{M-m}_b(X;F); v = Pu, u \in \rho^\alpha H^m_b(X;E)\right\}\end{aligned} \tag{6.15}$$

will be constant on each of the countably many open sets, all products of open intervals, forming the components of

$$\prod_{j=1}^p \left(\mathbb{R} \setminus -\operatorname{Im}\operatorname{spec}_{b,j}(P)\right) \subset \mathbb{R}^p.$$

Recall that in (5.9) the rank of a zero of the indicial family, $\operatorname{rank}(z)$, $z \in \operatorname{spec}_b(P)$, was defined. This generalizes to the case where ∂X has several components, to define $\operatorname{rank}_j(z)$ when $z \in \operatorname{spec}_{b,j}(P)$, as the integer in (5.8) and (5.9), where $F(\lambda)$ is the inverse of the indicial family of P at the jth boundary hypersurface.

THEOREM 6.5. *(Relative Index theorem) If $P \in \operatorname{Diff}_b^m(X; E, F)$ is elliptic and $\mathfrak{a}_j, \mathfrak{a}'_j \notin -\operatorname{Im}\operatorname{spec}_{b,j}(P)$, $j = 1, \dots, p$, then the index, as given by* (6.15), *satisfies*

$$\operatorname{ind}(P, \mathfrak{a}) - \operatorname{ind}(P, \mathfrak{a}') = \sum_{j=1}^{p} \left[\operatorname{sgn}(\mathfrak{a}'_j - \mathfrak{a}_j) \sum_{\substack{z \in \operatorname{spec}_{b,j}(P) \\ -\operatorname{Im} z \in [\mathfrak{a}_j, \mathfrak{a}'_j]}} \operatorname{rank}_j(z) \right]. \tag{6.16}$$

If ∂X is connected, so $\mathfrak{a} = \alpha$, $\mathfrak{a}' = \alpha'$ are just real numbers, and $\alpha > \alpha'$ then (6.16) shows that $\operatorname{ind}(P, \alpha) < \operatorname{ind}(P, \alpha')$ and the difference is minus the sum of the ranks of the points of $\operatorname{spec}_b(P)$ between the two lines $\operatorname{Im} z = -\alpha$ and $\operatorname{Im} z = -\alpha'$, i.e. with $-\operatorname{Im} z$ in the interval $[\alpha', \alpha]$ or the open interval (α', α), since the end points cannot occur by assumption.

PROOF OF THEOREM 6.5: Again it is convenient to prove the result first under the assumption that ∂X is connected and the bundles are both ${}^b\Omega^{\frac{1}{2}}$.

Clearly it suffices to prove (6.16) under the assumption that $\alpha' > \alpha$. Furthermore by dividing the interval $[\alpha', \alpha]$ into small subintervals, with no endpoints in $-\operatorname{Im}\operatorname{spec}_b(P)$, it can be assumed that $\alpha' - \alpha$ is small and that in the interval (α', α) there is at most one point of $-\operatorname{Im}\operatorname{spec}_b(P)$. Now,

$$\begin{aligned} \operatorname{ind}(P, \alpha) &= \dim \operatorname{null}(P, \alpha) - \dim \operatorname{null}(P^*, -\alpha), \\ \operatorname{null}(P, \alpha) &= \operatorname{null}\left(P \colon x^\alpha H_b^m(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow x^\alpha H_b^0(X; {}^b\Omega^{\frac{1}{2}}) \right). \end{aligned} \tag{6.17}$$

From Proposition 5.61 it follows that if there is no point of $-\operatorname{Im}\operatorname{spec}_b(P)$ in (α', α) then $\operatorname{null}(P, \alpha) = \operatorname{null}(P, \alpha')$ and similarly for P^*. Thus $\operatorname{ind}(P, \alpha) = \operatorname{ind}(P, \alpha')$ in this case, as expected. So it may be assumed that $\alpha - \alpha'$ is small and positive and that in the interval (α', α) there is exactly one point $\alpha'' \in -\operatorname{Im}\operatorname{spec}_b(P)$. Of course this may arise from several points in $\operatorname{spec}_b(P)$. For notational simplicity the operator will be conjugated by $x^{\alpha''}$, so arranging that $\alpha' = \epsilon$, $\alpha = -\epsilon$ and $\alpha'' = 0$, for some small $\epsilon > 0$.

From (6.13), $G'(P, 0) \subset G(P, 0)$ is the subspace of those $u \in G(P, 0)$ such that $P(\phi u) = Pu'$ for some $u' \in x^\epsilon H_b^m(X; {}^b\Omega^{\frac{1}{2}})$. Thus

$$\dim \operatorname{null}(P, -\epsilon) = \dim \operatorname{null}(P, \epsilon) + \dim G'(P, 0). \tag{6.18}$$

Now (6.14) in Lemma 6.4 shows that

$$\dim G'(P, 0) + \dim G'(P^*, 0) = \dim G(P, 0) = \dim G(P^*, 0). \tag{6.19}$$

Applying (6.18) to P and P^* shows that

$$\begin{aligned}[\dim \operatorname{null}(P,-\epsilon) - \dim \operatorname{null}(P,\epsilon)] + [\dim \operatorname{null}(P^*,-\epsilon) - \dim \operatorname{null}(P^*,\epsilon)] \\ = \dim G'(P,0) + \dim G'(P^*,0) = \dim G(P,0).\end{aligned} \tag{6.20}$$

This is just (6.16) in this case, so (6.16) has actually been proved in the case of a single boundary hypersurface and operators on b-half-densities.

The case of general bundles involves only notational changes. The same is true if X has several boundary components, since the generalized null spaces of the indicial operators at different boundary hypersurfaces pair to zero under B in (6.9). Thus the theorem holds in general.

The discussion in §6.1 and this proof are quite robust. Since it is not invoked below, the case of general b-pseudodifferential operators is left as an exercise.

EXERCISE 6.6. Show that the proof above generalizes directly to give a similar result for the index of any elliptic element $A \in \Psi_b^k(X; E, F)$. [Hint: The main point at which the assumption that P is a differential operator enters into the proof above is the conclusion that $I(P)(\phi u)$ in (6.3) has compact support. This is important since it justifies the use of Plancherel's formula. Show that if P is pseudodifferential then this can be satisfactorily replaced by the fact that $I(P)(\phi u) = O(|x^{-\operatorname{Im} z+\delta}|)$ for all $0 < \delta < 1$, as $x \downarrow 0$.]

The extended index function is defined, as in (In.30), by

$$\widetilde{\operatorname{ind}}(P, \mathfrak{a}) = \frac{1}{2} \lim_{\epsilon \downarrow 0} [\operatorname{ind}(P, \mathfrak{a} - \epsilon) + \operatorname{ind}(P, \mathfrak{a} + \epsilon)], \tag{6.21}$$

where ϵ is interpreted as a multiweight with the same entry in each component. The relative index formula means that this extended index function is determined by its value for any one multiweight $\mathfrak{a}$. To write the difference law that results from Theorem 6.5 in a more compact form, consider the 'incidence function' for s, α, and $\alpha' \in \mathbb{R}$:

$$\operatorname{inc}(s; \alpha, \alpha') = \operatorname{sgn}(\alpha' - \alpha) \cdot \begin{cases} 0 & s \notin [\alpha, \alpha'] \\ \frac{1}{2} & s = \alpha \text{ or } s = \alpha' \\ 1 & s \in (\alpha, \alpha'). \end{cases} \tag{6.22}$$

Here if $\alpha = \alpha'$ then by convention $\operatorname{sgn}(\alpha - \alpha') = 0$.

EXERCISE 6.7. Show that if $[\alpha, \alpha']$ is considered as an oriented interval (with the opposite of the standard orientation of $\mathbb{R}$ if $\alpha' < \alpha$) then $\operatorname{inc}(s; \alpha, \alpha')$ is the limit as $\epsilon \downarrow 0$ of $\operatorname{Len}\big((s - \epsilon, s + \epsilon) \cap [\alpha, \alpha']\big)/\epsilon$, where Len is the signed length.

With this additional piece of notation, (6.16) becomes

$$\widetilde{\operatorname{ind}}(P,\mathfrak{a})-\widetilde{\operatorname{ind}}(P,\mathfrak{a}')=\sum_{j=1}^{p}\left[\sum_{z\in\operatorname{spec}_{b,j}(P)}\operatorname{inc}(\operatorname{Im}z,\mathfrak{a}'_j,\mathfrak{a}_j)\operatorname{rank}_j(z)\right]. \tag{6.23}$$

Note that Theorem 5 shows the necessity of (5.112). Indeed P is Fredholm on the weighted spaces in (5.112) if and only if $P_\alpha = \rho^{-\alpha}P\rho^\alpha$ is Fredholm on the unweighted spaces. The P_α vary continuously with α, so if they were Fredholm for α in some open interval containing a point of $-\operatorname{Im}\operatorname{spec}_b(P)$ then the index would be constant on that interval. Since (6.16) shows that the index is not constant the family can not be Fredholm if $\alpha\in-\operatorname{Im}\operatorname{spec}_b(P)$, since it is Fredholm everywhere else.

6.3. Riemann-Roch for surfaces.

One might be inclined to think that the relative index theorem is very soft. To see that this is not so, consider a special case of an idea of Gromov and Shubin (see [40]) which shows that one can get the standard Riemann-Roch theorem for a compact Riemann surface as a corollary of Proposition 5.59. In fact this is included as a remark in [8]. Of course the usual proof of the Riemann-Roch theorem in this generality is not terribly hard ....

A compact Riemann surface, M, is just a real two-dimensional compact manifold with a complex structure. The complex structure on M is a covering by coordinate systems with transition maps which are holomorphic. This implies that the complexified tangent bundle has a decomposition into complex line bundles:

$$\mathbb{C}TM = T^{1,0}M\oplus T^{0,1}M,\ v\in T^{1,0}_xM\Longleftrightarrow \overline{v}\in T^{0,1}_xM. \tag{6.24}$$

By duality there is a similar decomposition of the cotangent bundle:

$$\mathbb{C}T^*M=\Lambda^{1,0}M\oplus\Lambda^{0,1}M, \tag{6.25}$$

which in turn gives a decomposition of the exterior derivative on both functions and 1-forms:

$$d=\partial+\overline{\partial},\ \overline{\partial}\colon\mathcal{C}^\infty(M)\longrightarrow\mathcal{C}^\infty(M;\Lambda^{0,1})$$
$$\overline{\partial}=d\text{ on }\mathcal{C}^\infty(M;\Lambda^{1,0}),\ \partial=d\text{ on }\mathcal{C}^\infty(M;\Lambda^{0,1}).$$

The operator $\overline{\partial}$ is an elliptic differential operator and the Hodge theorem allows one to compute its index.

Suppose that M is given a conformal metric, i.e. an Hermitian metric on $T^{0,1}M$. It is therefore locally a smooth positive multiple of $dzd\overline{z}$ for any holomorphic coordinate z. Consider the adjoints of ∂ and $\overline{\partial}$ on 1-forms. These are the middle-dimensional forms on an even-dimensional manifold and it follows from (2.85) that the Hodge star isomorphism, which maps 1-forms to 1-forms, does not depend on the conformal factor in the metric. For the local metric $dzd\overline{z}$ the adjoint of $\overline{\partial}$ on $\mathcal{C}^\infty(M;\Lambda^{1,0})$ is a multiple of ∂, so from (2.88) this holds in the general case and hence

$$\partial^*\phi = 0 \Longleftrightarrow \overline{\partial}\phi = 0,\ \ \phi \in \mathcal{C}^\infty(M;\Lambda^{1,0}) \tag{6.26}$$

and similarly for $\overline{\partial}^*$ on $(0,1)$-forms.

The null space of ∂ on functions consists just of the constants. The Laplacian on 1-forms is

$$\Delta = (\partial + \overline{\partial})(\partial + \overline{\partial})^* + (\partial + \overline{\partial})^*(\partial + \overline{\partial}).$$

By integration by parts an harmonic 1-form is closed and coclosed, so is of the form $u = \phi + \psi$ where $\phi \in \mathcal{C}^\infty(M;\Lambda^{1,0})$ and $\psi \in \mathcal{C}^\infty(M;\Lambda^{0,1})$ are in the null space of $\overline{\partial}$ and ∂ respectively. Conversely, from the discussion above, any $\phi \in \mathcal{C}^\infty(M;\Lambda^{1,0})$ which is holomorphic, i.e. satisfies $\overline{\partial}\phi = 0$, is harmonic. Thus the space of harmonic 1-forms has even dimension, $2p$, where p is by definition the genus of M and the spaces of holomorphic $(1,0)$-forms and antiholomorphic $(0,1)$-forms both have dimension p. The range of $\overline{\partial}$ on functions has as complement the null space of $\overline{\partial}^*$, just the space of antiholomorphic $(0,1)$-forms. Thus on functions

$$\operatorname{ind}(\overline{\partial}) = 1 - p,\ \ 2p = \dim H^1_{\mathrm{Ho}}(M).$$

There is in fact precisely one (oriented) compact Riemann surface, up to diffeomorphism, of a given genus, $p \geq 0$.

The Riemann-Roch theorem, in its most elementary form, is concerned with the spaces of meromorphic functions and differentials (1-forms) with given sets of zeros and poles. Let D be a divisor, i.e. a finite subset

$$D = \{(z_1, m_1), \ldots, (z_N, m_N)\} \subset M \times \mathbb{Z}$$

with the z_j distinct. Associated to D is a compact manifold with boundary, M', obtained by blowing up each of the points in the projection of D onto M. Let H_j be the boundary circle added by blowing up z_j, for $j = 1, \ldots, N$. The complex structure, (6.24), lifts to a complex structure on the b-tangent bundle of M' since if z is a holomorphic coordinate near $z = z_j$ then the logarithmic differential $d(z-z_j)/(z-z_j)$ is a smooth section of ${}^bT^*M'$ near H_j. Indeed in terms of polar coordinates

$$z - z_j = re^{i\theta},\ \frac{d(z-z_j)}{z-z_j} = \frac{dr}{r} + id\theta. \tag{6.27}$$

This gives a decomposition extending (6.25):

$$\mathbb{C}\,{}^bT^*M' = {}^b\Lambda^{1,0} \oplus {}^b\Lambda^{0,1}$$

where the second summand is the complex conjugate of the first.

The Cauchy-Riemann operator is an element ${}^b\overline{\partial} \in \operatorname{Diff}^1_b(M'; \mathbb{C}, {}^b\Lambda^{0,1})$. In this sense it is elliptic with indicial family

$$I_r({}^b\overline{\partial}, s)u = -i(\frac{1}{2}\partial_\theta + \frac{i}{2}s)u\frac{d(\overline{z}-\overline{z}_j)}{\overline{z}-\overline{z}_j}$$

at H_j, with r in (6.27) used to trivialize the normal bundle. Thus the indicial roots are easily computed. For each j the indicial family has only simple zeros and these are all at imaginary integer points:

$$\operatorname{Spec}_b({}^b\overline{\partial}) = \mathbb{Z} = \{(ik, 0); k \in \mathbb{Z}\} \text{ at each } H_j.$$

These just arise from the eigenfunctions of ∂_θ on the circle, or equivalently the Laurent series at z_j of meromorphic functions. Thus

$$\begin{gathered} {}^b\overline{\partial}\colon \rho^S H^m_b(M') \longrightarrow \rho^S H^{m-1}_b(M'; {}^b\Lambda^{0,1}) \\ \text{is Fredholm whenever } S = (S_1, \dots, S_N),\ S_j \notin \mathbb{Z} \quad \forall\ j = 1, \dots, N. \end{gathered}$$

The relative index theorem gives

$$\begin{gathered} \operatorname{ind}({}^b\overline{\partial}, S) - \operatorname{ind}({}^b\overline{\partial}, S') = \sum_{j=1}^{N}([S_j] - [S'_j]), \\ [S_j] = \inf\{p \in \mathbb{Z}; p > S_j\}. \end{gathered} \tag{6.28}$$

Now (6.28) is just a reformulation of the Riemann-Roch theorem. With the divisor D one associates the weights

$$S_j(D) = m_j - \delta$$

for any fixed $\delta \in (0, 1)$. In particular for the trivial divisor, with all $m_j = 0$ and the same finite set of points in M, each $S_j(0) = -\delta$. The degree of the divisor is defined to be $\deg(D) = \sum\limits_{j=1}^{N} m_j$ and (6.28) becomes

$$\operatorname{ind}({}^b\overline{\partial}, S(D)) - \operatorname{ind}({}^b\overline{\partial}, S(0)) = \deg(D). \tag{6.29}$$

Then it only remains to identify the index of ${}^b\overline{\partial}$ on the two weighted spaces, using the elementary properties of analytic functions, to deduce the Riemann-Roch formula.

For any divisor set

$$
\begin{aligned}
r_0(D) &= \dim\{\text{meromorphic functions with divisor } D\},\\
r_1(D) &= \dim\{\text{meromorphic differentials with divisor } D\}.
\end{aligned}
\tag{6.30}
$$

The convention for divisors is slightly different to that for index sets in (5.8). A meromorphic function f has divisor D if $(\lambda - z_j)^{m_j} f(\lambda)$ is holomorphic near $\lambda = z_j$. Thus $m_j = 0$ corresponds to a regular point, whereas in our convention for index sets a point $(z, 0)$ corresponds to a simple pole at z. For meromorphic $(1,0)$-forms, which are just the complex conjugates of antimeromorphic $(0,1)$-forms, the m_j correspond to the regularity of the coefficients in terms of a local holomorphic 1-form dz. This means that $m_j = -1$ corresponds to holomorphy of the coefficients in terms of the b-form $d(z - z_j)/(z - z_j)$.

From the discussion above the trivial divisor corresponds to

$$
r_0(0) = 1,\ \ r_1(0) = p. \tag{6.31}
$$

If $-D$ is the conjugate divisor to D, i.e. $-D = \{(z_j, -m_j)\}$ if $D = \{(z_j, m_j\}$ then:

Lemma 6.8. *(Riemann-Roch) For any divisor on a compact Riemann surface of genus p the dimensions in* (6.30) *are related by*

$$
r_0(D) = r_1(-D) + \deg(D) + p - 1. \tag{6.32}
$$

Proof: The Riemann-Roch formula (6.32) follows from (6.29) and (6.31) once it is shown that for any divisor,

$$
\operatorname{ind}({}^b\overline{\partial}, S(D)) = r_0(D) - r_1(-D). \tag{6.33}
$$

This follows from the removability of singularities. Thus Proposition 5.61 shows that the null space of $\overline{\partial}$, $\left\{u \in \rho^S H^m_b(M'); {}^b\overline{\partial}u = 0\right\}$, is just the space of meromorphic functions with divisor D since $S_j = m_j - \delta$. The codimension of the range of ${}^b\overline{\partial}$ on functions is equal to the dimension of $\left\{u \in \rho^{-S} H^m_b(M'; {}^b\Lambda^{1,0}); {}^b\overline{\partial}u = 0\right\}$. Since $-S_j = -m_j + \delta$, and recalling that these are b-forms, this corresponds to the space of meromorphic differentials with divisor $-D$.

6.4. Hodge theory.

Consider next the Hodge cohomology of a compact manifold with boundary, needless to say equipped with an exact a b-metric. For orientation the case of X compact with $\partial X = \emptyset$ will first be briefly recalled.

The first basic result is the de Rham theorem which gives an identification of the de Rham cohomology spaces, the cohomology of the exterior complex (2.17):

$$\mathcal{H}^k_{\mathrm{dR}}(X) = \left\{u \in \mathcal{C}^\infty(X; \Lambda^k); du = 0\right\} / d\mathcal{C}^\infty(X; \Lambda^{k-1}), \tag{6.34}$$

with the singular cohomology spaces (with complex coefficients) of the manifold. If X is given a metric then the associated Laplacian leads to the Hodge decomposition involving the Hodge cohomology, i.e. the space of harmonic forms:

$$\mathcal{H}^k_{\mathrm{Ho}}(X) = \left\{u \in \mathcal{C}^{-\infty}(X; \Lambda^k); \Delta u = 0\right\} = \left\{u \in \mathcal{C}^\infty(X; \Lambda^k); du = \delta u = 0\right\}.$$

The second characterization follows by elliptic regularity and integration by parts, for $u \in \mathcal{C}^\infty(X; \Lambda^k)$,

$$\Delta u = 0 \Longrightarrow 0 = \langle u, \Delta u\rangle = \|du\|^2 + \|\delta u\|^2 \Longrightarrow du = \delta u = 0. \tag{6.35}$$

PROPOSITION 6.9. *On a compact Riemann manifold, X, without boundary*

$$\begin{aligned} \mathcal{C}^\infty(X; \Lambda^k) &= d\mathcal{C}^\infty(X; \Lambda^{k-1}) \oplus \delta\mathcal{C}^\infty(X; \Lambda^{k-1}) \oplus \mathcal{H}^k_{\mathrm{Ho}}(X) \\ L^2(X; \Lambda^k) &= dH^1(X; \Lambda^{k-1}) \oplus \delta H^1(X; \Lambda^{k-1}) \oplus \mathcal{H}^k_{\mathrm{Ho}}(X) \\ \mathcal{C}^{-\infty}(X; \Lambda^k) &= d\mathcal{C}^{-\infty}(X; \Lambda^{k-1}) \oplus \delta\mathcal{C}^{-\infty}(X; \Lambda^{k-1}) \oplus \mathcal{H}^k_{\mathrm{Ho}}(X). \end{aligned} \tag{6.36}$$

Here the direct summation in the first and last cases just means that the factors are closed with trivial intersections in pairs; in the middle case the sum is orthogonal for the natural Hilbert space structure on L^2.

PROOF: This follows directly from the knowledge that the generalized inverse of the Laplacian

$$\Delta = d\delta + \delta d = (d + \delta)^2$$

is a pseudodifferential operator of order -2. The null space, $\mathcal{H}^k_{\mathrm{Ho}}(X)$, is closed and of finite dimension, consisting of the harmonic k-forms. The range of Δ on $\mathcal{C}^\infty(X; \Lambda^k)$ is closed and from the self-adjointness of Δ has complement $\mathcal{H}^k_{\mathrm{Ho}}(X)$. Then each of the decompositions in (6.36) arises in the same way. Thus, for instance,

$$\begin{aligned}\mathcal{C}^\infty(X;\Lambda^k) &= \Delta\mathcal{C}^\infty(X;\Lambda^k)\oplus\mathcal{H}^k_{\mathrm{Ho}}(X)\\ &= d[\delta\mathcal{C}^\infty(X;\Lambda^k)]\oplus\delta[d\mathcal{C}^\infty(X;\Lambda^k)]\oplus\mathcal{H}^k_{\mathrm{Ho}}(X).\end{aligned}\tag{6.37}$$

That $d\delta\mathcal{C}^\infty(X;\Lambda^k) = d\mathcal{C}^\infty(X;\Lambda^{k-1})$ follows from (6.37) for $k-1$ since

$$d\delta\mathcal{C}^\infty(X;\Lambda^k)\subset d\mathcal{C}^\infty(X;\Lambda^{k-1})\subset d\delta d\mathcal{C}^\infty(X;\Lambda^{k-1})\subset d\delta\mathcal{C}^\infty(X;\Lambda^k)$$

and similarly for the second term, so this leads to the first case in (6.36). The other cases are similar.

In fact (6.37) is often more useful than (6.36). Of course the main corollary of this decomposition is the Hodge theorem (proved in this generality by Weyl, [90]):

COROLLARY. *For any compact Riemann manifold without boundary there is a natural (Hodge) isomorphism* $\mathcal{H}^k_{\mathrm{Ho}}(X)\longleftrightarrow\mathcal{H}^k_{\mathrm{dR}}(X)$ *with inverse given by projection onto the Hodge cohomology in* (6.36).

PROOF: The map from $\mathcal{H}^k_{\mathrm{Ho}}(X)$ to $\mathcal{H}^k_{\mathrm{dR}}(X)$ is well-defined since harmonic forms are closed, by (6.35). From (6.37) the harmonic space is orthogonal to the range of d, so the map is injective. To see that it is surjective, observe from (6.37) that if $u\in\mathcal{C}^\infty(X;\Lambda^k)$ then $du=0$ if and only if the second term, of the form δw, in the decomposition of u is closed. By (6.37) this term is of the form δw with $dw=0$, so $d\delta w=0$ implies $\Delta w=0$ and hence implies that $\delta w=0$. Thus a closed form is the sum of its harmonic part and an exact form.

The same argument applies in the other two cases. For instance the third, distributional, decomposition in (6.36) shows that the distributional de Rham cohomology

$$\left\{u\in\mathcal{C}^{-\infty}(X;\Lambda^k); du=0\right\}/d\mathcal{C}^{-\infty}(X;\Lambda^{k-1})\tag{6.38}$$

is canonically isomorphic to the $\mathcal{C}^\infty$ de Rham cohomology and to the Hodge cohomology.

EXERCISE 6.10. Define the L^2-de Rham cohomology of a compact manifold without boundary as

$$\left\{u\in L^2(X;\Lambda^k); du=0\right\}/\left\{u\in L^2(X;\Lambda^k); u=dv, v\in L^2(X;\Lambda^{k-1})\right\}.$$

Show that this is defined independently of the choice of Riemann metric and is canonically isomorphic to the $\mathcal{C}^\infty$ de Rham and to the Hodge cohomology of any Riemann metric.

Now to the case of more immediate interest, a compact manifold with boundary. As already discussed in §2.16 there are two basic smooth de Rham cohomology spaces:

$$
\begin{aligned}
\mathcal{H}^k_{\mathrm{dR,abs}}(X) &= \left\{u \in \mathcal{C}^\infty(X;\Lambda^k); du = 0\right\}/d\mathcal{C}^\infty(X;\Lambda^{k-1}) \\
\mathcal{H}^k_{\mathrm{dR,rel}}(X) &= \left\{u \in \dot{\mathcal{C}}^\infty(X;\Lambda^k); du = 0\right\}/d\dot{\mathcal{C}}^\infty(X;\Lambda^{k-1}),
\end{aligned}
\tag{6.39}
$$

depending on whether smooth forms are required to vanish to infinite order at the boundary or not. The de Rham theorem, Theorem 2.48, then asserts the existence of natural isomorphisms with the absolute singular cohomology and the relative singular cohomology respectively. Alternatively one could take the position that the two spaces in (6.39) *are* the absolute and relative cohomology spaces of the compact manifold with boundary.

The b-Sobolev spaces have been emphasized already and it is natural to look for analogues of the regularity result showing that the distributional de Rham space, analogous to that in (6.38), is isomorphic to $\mathcal{H}^k_{\mathrm{dR}}(X)$. Consider the conormal forms:

$$
\mathcal{A}(X;{}^b\Lambda^k) = \mathcal{A}(X;\Lambda^k) = \bigcup_{s\in\mathbb{R}} x^s H^\infty_b(X;{}^b\Lambda^k).
\tag{6.40}
$$

Since $d \in \mathrm{Diff}^1_b(X;{}^b\Lambda^{k-1},{}^b\Lambda^k)$, it acts on these spaces and there are corresponding de Rham cohomology spaces:

$$
\left\{u \in \mathcal{A}(X;{}^b\Lambda^k); du = 0\right\}/d\mathcal{A}(X;{}^b\Lambda^{k-1}),
\tag{6.41}
$$

where the same space is obtained by replacing ${}^b\Lambda^*$ by Λ^*.

LEMMA 6.11. *The conormal de Rham cohomology space* (6.41) *is canonically isomorphic to* $\mathcal{H}^k_{\mathrm{dR,abs}}(X)$.

PROOF: Choose a real vector field $V \in \mathcal{V}(X)$ which is transversal to the boundary and inward pointing, i.e. $Vx > 0$ at ∂X if $x \in \mathcal{C}^\infty(X)$ is a defining function. Such a vector field certainly exists, since it can be taken to be $\partial/\partial x$ in a collar neighbourhood of the boundary and then cut off inside. Integration of V gives a 1-parameter family of $\mathcal{C}^\infty$ maps determined by:

$$
\begin{gathered}
F_s \colon X \longrightarrow X \text{ for } s \in [0,s_0],\ s_0 > 0 \text{ small, where} \\
\frac{d}{ds}(F_s)^*\phi = (F_s)^*V\phi,\ \forall\ \phi \in \mathcal{C}^\infty(X) \text{ and } F_0 = \mathrm{Id}\,.
\end{gathered}
$$

The pull-back map F_s^* commutes with d and, always for small s, induces the identity on $\mathcal{H}^*_{\mathrm{dR,abs}}(X)$, since for any form $u \in \mathcal{C}^\infty(X;\Lambda^k)$ Cartan's identity gives

$$
F_s^*u - u = \int_0^s \frac{d}{dt}(F_t)^*u\,dt = \int_0^s (F_t)^*[di_V u + i_V du]dt,
$$

where i_V is contraction with V. Thus if u is closed

$$F_s^* u - u = dv \text{ with } v = \int_0^s (F_t)^* i_V u. \tag{6.42}$$

Now, simply because V is inward pointing and transversal to the boundary, $F_s(X) \subset \overset{\circ}{X}$ for small positive s. Thus if $u \in \mathcal{A}(X; {}^b\Lambda^k)$ then $F_s^* u \in \mathcal{C}^\infty(X; \Lambda^k)$ for small $s > 0$. This shows that pull-back under F_s gives a map from the conormal de Rham cohomology into the absolute de Rham cohomology. On the other hand there is an obvious 'inclusion' map from $\mathcal{H}^k_{\text{dR,abs}}(X)$ into the conormal cohomology. Using (6.42) these maps are easily seen to be inverses of each other.

Exercise 6.12. Use this 1-parameter family of maps to show that the de Rham space of smooth forms with compact support in the interior of X

$$\left\{ u \in \mathcal{C}^\infty_c(\overset{\circ}{X}; \Lambda^k); du = 0 \right\} / d\mathcal{C}^\infty_c(\overset{\circ}{X}; \Lambda^{k-1})$$

$$\mathcal{C}^\infty_c(\overset{\circ}{X}; \Lambda^k) = \left\{ u \in \dot{\mathcal{C}}^\infty(X; \Lambda^k); \operatorname{supp}(u) \subset \overset{\circ}{X} \right\}$$

is naturally identified with the relative de Rham cohomology.

Lemma 6.11 can be refined considerably. Observe that d acts on the spaces $x^s H_b^\infty(X; {}^b\Lambda^*)$ for any fixed $s \in \mathbb{R}$.

Proposition 6.13. *The (de Rham) cohomology of the complex*

$$0 \longrightarrow x^s H_b^\infty(X) \xrightarrow{d} x^s H_b^\infty(X; {}^b\Lambda^1) \xrightarrow{d} \dots \xrightarrow{d} x^s H_b^\infty(X; {}^b\Lambda^N) \longrightarrow 0 \tag{6.43}$$

is naturally isomorphic to $\mathcal{H}^*_{\text{dR,abs}}(X)$ if $s < 0$ and to $\mathcal{H}^*_{\text{dR,rel}}(X)$ if $s > 0$.

Proof: The proof of Lemma 6.11 applies unchanged in case $s < 0$. So suppose $s > 0$. Choose a boundary defining function x and a cut-off function $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ which is identically equal to 1 in a neighbourhood of 0. Choose a collar decomposition of X near the boundary so that any $u \in x^s H_b^\infty(X; {}^b\Lambda^k)$ decomposes into tangential and normal parts:

(6.44)

$$u = u_\tau + \frac{dx}{x} \wedge u_\nu,$$

$$u_\tau \in x^s H_b^\infty([0,\epsilon) \times \partial X; \Lambda^k(\partial X)), \ \ u_\nu \in x^s H_b^\infty([0,\epsilon) \times \partial X; \Lambda^{k-1}(\partial X)).$$

The conditions for u to be closed become, in this collar neighbourhood,

$$d_{\partial X} u_\tau = 0, \ x \frac{\partial}{\partial x} u_\tau = d_{\partial X} u_\nu.$$

Now the assumption that $s > 0$ means that the second of these equations can be integrated from $x = 0$ to give

$$u_\tau = \int_0^x d_{\partial X} u_\nu \frac{dx}{x}$$

with the integral converging absolutely. So consider the map

$$x^s H_b^\infty(X; {}^b\Lambda^k) \ni u \longmapsto \phi(x) u + \phi'(x) dx \wedge \int_0^x u_\nu \frac{dx}{x} \in \dot{\mathcal{C}}^\infty(X; \Lambda^k). \tag{6.45}$$

This maps closed forms to closed forms and exact forms to exact forms so projects to a map from the de Rham cohomology of (6.43) to $\mathcal{H}^k_{\mathrm{dR,rel}}(X)$. There is a natural map, given by inclusion, the other way and it is easy to see that these are inverses of each other, so the Proposition is proved.

Naturally this raises the question of what happens in the 0-weighted case. This is just what is usually called the L^2 cohomology. The answer, justified in Exercise 6.17 below, is that unless $\dim \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) = \dim \mathcal{H}^k_{\mathrm{Ho}}(\partial X) = 0$ the L^2 cohomology, i.e. the cohomology of the complex (6.43) in case $s = 0$, is infinite dimensional (and therefore not so interesting).

There is a natural map from the relative to the absolute cohomology, given at the level of the de Rham spaces in (6.39) by inclusion:

$$e \colon \mathcal{H}^k_{\mathrm{dR,rel}}(X) \longrightarrow \mathcal{H}^k_{\mathrm{dR,abs}}(X). \tag{6.46}$$

The image can therefore be represented, for any $s < 0$ as

$$e\left[\mathcal{H}^k_{\mathrm{dR,rel}}(X)\right] = \frac{\left\{u \in \dot{\mathcal{C}}^\infty(X; \Lambda^k); du = 0\right\}}{\left\{u \in \dot{\mathcal{C}}^\infty(X; \Lambda^k); u = dv, v \in x^s H_b^\infty(X; \Lambda^{k-1})\right\}}. \tag{6.47}$$

This is of immediate interest because of the Hodge identification given in [8].

Let the null space of the Laplacian, of an exact b-metric, acting on $H_b^0(X; {}^b\Lambda^k)$, the metric L^2 space, be denoted $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$. From Proposition 5.61 the elements of this space are polyhomogeneous conormal distributions corresponding to index sets in the right half space, since x^z for $\operatorname{Re} z \le 0$ is not locally square-integrable with respect to the volume form. Thus

$$\mathcal{H}^k_{b,\mathrm{Ho}}(X) = \left\{u \in x^\epsilon H_b^0(X; {}^b\Lambda^k); \Delta u = 0\right\} \text{ for } \epsilon > 0 \text{ small.} \tag{6.48}$$

This means that integration by parts is justified so that

$$0 = \langle u, \Delta u \rangle = \|du\|^2 + \|\delta u\|^2, \ u \in \mathcal{H}^k_{b,\mathrm{Ho}}(X).$$

In particular the elements of $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$ are square-integrable d- and δ-closed forms.

Proposition 6.14. *The null space, $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$, of the Laplacian of an exact b-metric on any compact manifold with boundary, acting on $H^2_b(X;{}^b\Lambda^k)$ is naturally identified with the de Rham space $e\left[\mathcal{H}^k_{\mathrm{dR,rel}}(X)\right]$ in (6.47).*

One would expect to proceed as with the proof of the Corollary to Proposition 6.9, using a decomposition analogous to (6.36), by looking at the action of the Laplacian on $H^m_b(X;{}^b\Lambda^k)$ for $m=\infty$, $m=2$ or $m=-\infty$ corresponding to the three cases in (6.36). However there is a problem, since the fundamental point in the boundaryless case is that the range in the appropriate space is closed. From the discussion of Fredholm properties above this is the case for the Laplacian acting on $H^2_b(X;{}^b\Lambda^k)$ if and only of there are no real indicial roots.

So consider the indicial family of the Laplacian. The space of b-forms splits at the boundary as in (6.44):

$$\begin{gathered}{}^b\Lambda^k_{\restriction\partial X}\cong\Lambda^k(\partial X)\oplus\Lambda^{k-1}(\partial X)\\ \mathcal{C}^\infty(\partial X;{}^b\Lambda^k)\ni u=u_\tau+\frac{dx}{x}\wedge u_\nu,\\ u_\tau\in\mathcal{C}^\infty(\partial X;\Lambda^k),\ u_\nu\in\mathcal{C}^\infty(\partial X;\Lambda^{k-1}).\end{gathered}$$

Here dx/x is the singular 1-form given by the exact b-metric. With respect to this splitting the indicial operators of d and δ, for the given metric, were computed in §4.17:

$$I(d)=\begin{pmatrix}d&0\\x\frac{\partial}{\partial x}&-d\end{pmatrix},\ I(\delta)=\begin{pmatrix}\delta&-x\frac{\partial}{\partial x}\\0&-\delta\end{pmatrix}\tag{6.49}$$

acting from $\Lambda^k\oplus\Lambda^{k-1}$ to $\Lambda^{k+1}\oplus\Lambda^k$ and to $\Lambda^{k-1}\oplus\Lambda^{k-2}$ over the boundary, respectively. Here d and δ also stand, somewhat ambiguously, for the exterior differential and its adjoint with respect to the metric on the boundary. The corresponding indicial families are obtained by replacing $x\partial/\partial x$ by $i\lambda$. Since the indicial homomorphism is just that, a homomorphism, this gives

$$I_\nu(\Delta,\lambda)=\begin{pmatrix}\Delta_\partial+\lambda^2&0\\0&\Delta_\partial+\lambda^2\end{pmatrix}.\tag{6.50}$$

Thus the indicial roots of the Laplacian are exactly

$$\mathrm{spec}_b(\Delta)=\left\{\pm i\sigma;\sigma^2\text{ is an eigenvalue of }\Delta_\partial\right\}.$$

Furthermore the order and rank of these points is easily established. If $\sigma\neq0$ then the order is necessarily 1 and the rank is just the dimension of the corresponding eigenspace of Δ_∂ acting on $k-1$ and k forms. However the zero eigenvalue is special since, if Δ_∂ is not invertible on either $k-1$ or k forms on ∂X, $0 \in \operatorname{spec}_b(\Delta)$ represents a pole of order 2 with rank twice the sum of the Betti numbers (the dimensions of the cohomology) in these dimensions. This has a strong bearing on the behaviour of the Hodge cohomology of the b-metric.

Notice that the only real indicial root that the Laplacian can have is 0. So the Laplacian is Fredholm from $H^2_b(X;{}^b\Lambda^k)$ to $H^0_b(X;{}^b\Lambda^k)$, the L^2 space for the metric, if and only of there is no cohomology for the boundary in dimension $k-1$ or k. Thus there are two obvious options, since

(6.51)
$$\Delta\colon x^{\pm\epsilon}H^m_b(X;{}^b\Lambda^k) \longrightarrow x^{\pm\epsilon}H^{m-2}_b(X;{}^b\Lambda^k) \text{ is Fredholm for } \epsilon > 0 \text{ small.}$$

PROOF PROPOSITION 6.14 (BEGINNING): Consider the case of the 'large' spaces in (6.51), $x^{-\epsilon}H^2_b(X;{}^b\Lambda^k)$. The range is closed and has annihilator, with respect to the continuous pairing between $x^{-\epsilon}H^0_b(X;{}^b\Lambda^k)$ and $x^{\epsilon}H^0_b(X;{}^b\Lambda^k)$ for small $\epsilon > 0$, precisely the null space $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$. Thus

$$x^{-\epsilon}H^0_b(X;{}^b\Lambda^k) = \Delta\left[Gx^{-\epsilon}H^0_b(X;{}^b\Lambda^k)\right] \oplus \mathcal{H}^k_{b,\mathrm{Ho}}(X). \tag{6.52}$$

Here $G \in \Psi_b^{-2,\mathcal{E}(-\epsilon)}(X;{}^b\Lambda^k)$ is the generalized inverse discussed in Chapter 5, where $\mathcal{E}(-\epsilon)$ is the index set corresponding to the point $-\epsilon \notin -\operatorname{Im}\operatorname{spec}_b(\Delta)$.

Applying (6.52) to $\dot{\mathcal{C}}^\infty(X;{}^b\Lambda^k)$ gives

$$\begin{gathered}\dot{\mathcal{C}}^\infty(X;{}^b\Lambda^k) = \Delta\left[G\dot{\mathcal{C}}^\infty(X;{}^b\Lambda^k)\right] \oplus \mathcal{H}^k_{b,\mathrm{Ho}}(X)\\ \text{i.e. } u = \Delta v + u',\end{gathered} \tag{6.53}$$

where u' is just the L^2-projection of u onto $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$. If $u = dw$, with $w \in x^{-\epsilon}H^\infty_b(X;{}^b\Lambda^{k-1})$ then the decomposition of u is just $u = dw$. So $u' = 0$ and by (6.47) this gives a map as desired:

$$e\left[\mathcal{H}^k_{\mathrm{dR,rel}}(X)\right] \longrightarrow \mathcal{H}^k_{b,\mathrm{Ho}}(X). \tag{6.54}$$

The deformation (6.45) shows that any element of $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$ can be approximated in L^2 by closed elements of $\dot{\mathcal{C}}^\infty(X;{}^b\Lambda^k)$ so (6.54) is surjective. To prove the injectivity we need to examine the decomposition (6.53), and in particular the structure of the null space of the Laplacian, more closely.

In completing the proof of Proposition 6.14 and so giving an harmonic representation for the image of the relative cohomology in the absolute cohomology, it is natural to search for harmonic representatives for the

absolute and relative cohomologies themselves. This leads inexorably to the long exact sequence relating them:
(6.55)
$$\cdots \longrightarrow \mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X) \longrightarrow \mathcal{H}^{k}_{\mathrm{dR,rel}}(X) \xrightarrow{e} \mathcal{H}^{k}_{\mathrm{dR,abs}}(X) \longrightarrow \mathcal{H}^{k}_{\mathrm{dR}}(\partial X) \longrightarrow \dots .$$

The place to look for 'more' harmonic forms is in the null space

$$\mathrm{null}^k_-(\Delta) = \bigcap_{\epsilon>0} \left\{u \in x^{-\epsilon} H^\infty_b(X; {}^b\Lambda^k); \Delta u = 0\right\}.$$

By Proposition 5.61 this is just the null space of Δ on $x^{-\epsilon}H^\infty_b(X;{}^b\Lambda^k)$, for $\epsilon > 0$ sufficiently small. The discussion above of the structure of the 0 indicial root shows that there is a boundary data map:
(6.56)
$$\mathrm{null}^k_-(X; {}^b\Lambda^k) \xrightarrow{\mathrm{BD}_k} \left(\mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X)\right)^2 \oplus \left(\mathcal{H}^{k}_{\mathrm{Ho}}(\partial X)\right)^2$$
$$\mathrm{null}^k_-(\Delta) \ni u = U_{11}\log x + U_{21} + \frac{dx}{x} \wedge [U_{12}\log x + U_{22}] + u'$$
$$\longmapsto \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},\ u' \in x^\epsilon H^\infty_b(X; \Lambda^k).$$

Thus the top row represents the coefficients of $\log x$ and the second row the 'smooth' boundary values. Let $L^k = \mathrm{BD}_k(\mathrm{null}^k_-(\Delta))$ denote the image of (6.56). By definition, in (6.48) $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$ is the subspace of $\mathrm{null}^k_-(\Delta)$ for which this boundary data vanishes so there is a short exact sequence:

$$0 \longrightarrow \mathcal{H}^k_{b,\mathrm{Ho}}(X) \hookrightarrow \mathrm{null}^k_-(\Delta) \longrightarrow L^k \longrightarrow 0 \tag{6.57}$$

for each k. Clearly it is important to understand L^k.

The metric on ∂X induces inner products, denoted $\langle , \rangle$, on the Hodge cohomology on the boundary, so $\mathcal{H}^k_{\mathrm{Ho}}(\partial X)$ is a Euclidean vector space. This means that on the direct sum of two copies of this space there is a natural symplectic form:

$$\begin{gathered}\left[\mathcal{H}^k_{\mathrm{Ho}}(\partial X)\right]^2 \oplus \left[\mathcal{H}^k_{\mathrm{Ho}}(\partial X)\right]^2 \ni ((U_1, U_2), (U_1', U_2')) \longrightarrow \\ \omega\left((U_1, U_2), (U_1', U_2')\right) = \langle U_1, U_2'\rangle - \langle U_2, U_1'\rangle \in \mathbb{R}.\end{gathered} \tag{6.58}$$

That is, this bilinear form is antisymmetric and non-degenerate. Of course it extends in the obvious way to the direct sum of the Hodge cohomologies in dimensions $k-1$ and k.

LEMMA 6.15. *The space* $L^k \subset \left(\mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X)\right)^2 \oplus \left(\mathcal{H}^{k}_{\mathrm{Ho}}(\partial X)\right)^2$ *is a Lagrangian subspace with respect to the symplectic form (6.58), i.e. has dimension equal to that of* $\mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \oplus \mathcal{H}^{k}_{\mathrm{Ho}}(\partial X)$ *and the symplectic form vanishes identically on it.*

Proof: Consider the bilinear form B in Proposition 6.2. Since $P = \Delta$ is (formally) self-adjoint in this case, $F(I(P), 0) = F(I(P^*), 0)$ is just the sum of the boundary cohomology spaces in (6.58). Carrying out the contour integral in (6.7) shows that B is $-i$ times the symplectic pairing in (6.58). From (6.14) this bilinear form vanishes identically on L^k. The relative index theorem itself, Theorem 6.5, shows that

$$\operatorname{ind}(\Delta, -\epsilon) - \operatorname{ind}(\Delta, \epsilon) = 2\left[\dim \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) + \dim \mathcal{H}^{k}_{\mathrm{Ho}}(\partial X)\right].$$

Since Δ is formally self-adjoint

$$\operatorname{ind}(\Delta, -\epsilon) = \dim \operatorname{null}^k_-(\Delta) - \dim \mathcal{H}^k_{b,\mathrm{Ho}}(X),$$
$$\operatorname{ind}(\Delta, \epsilon) = \dim \mathcal{H}^k_{b,\mathrm{Ho}}(X) - \dim \operatorname{null}^k_-(\Delta),$$

so Theorem 6.5 becomes

$$\dim \operatorname{null}^k_-(\Delta) - \dim \mathcal{H}^k_{b,\mathrm{Ho}}(X) = \dim \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) + \dim \mathcal{H}^{k}_{\mathrm{Ho}}(\partial X).$$

From the exactness in (6.57) this is just the statement that

$$\dim L^k = \dim \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) + \dim \mathcal{H}^{k}_{\mathrm{Ho}}(\partial X).$$

Thus L^k is Lagrangian, as claimed.

Consider the subspace

$$\widetilde{L}^k = \left\{(u_1, u_2) \in \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \oplus \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X); u = \begin{pmatrix} 0 & 0 \\ u_1 & u_2 \end{pmatrix} \in L^k\right\}$$

and the projection $\pi_{\log}$ from L^k onto the top (logarithmic) row in (6.56)

$$\pi_{\log}\colon L^k \longrightarrow \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \oplus \mathcal{H}^{k}_{\mathrm{Ho}}(\partial X).$$

Lemma 6.15 shows that the range of $\pi_{\log}$ is exactly the orthocomplement of $\widetilde{L}^k$. Using the fact that $\Delta = (d+\delta)^2$ there is a finer form of this result:

Proposition 6.16. *The subspace*

$$\mathcal{H}^k_{eb,\mathrm{Ho}}(X) = \left\{u \in \operatorname{null}^k_-(\Delta); \pi_{\log} u = 0\right\} \tag{6.59}$$

is precisely the subspace of $\operatorname{null}^k_-(\Delta)$ *which* $d+\delta$ *annihilates; it consists exactly of the* d*- and* δ*-closed elements of* $\operatorname{null}^k_-(\Delta)$*. Furthermore the subspace of boundary data corresponding to* $\mathcal{H}^k_{eb,\mathrm{Ho}}(X)$ *splits:*

$$\widetilde{L}^k = \widetilde{L}^k_{k-1} \oplus \widetilde{L}^k_k,\ \widetilde{L}^k_{k-1} \subset \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X),\ \widetilde{L}^k_k \subset \mathcal{H}^{k}_{\mathrm{Ho}}(\partial X), \tag{6.60}$$

so the 'relative and absolute' b-Hodge spaces, defined by

$$\mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) = \left\{u \in \mathcal{H}^k_{eb,\mathrm{Ho}}(X); \mathrm{BD}_k\, u \in \widetilde{L}^k_{k-1} \oplus \{0\}\right\} \tag{6.61}$$

$$\mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X) = \left\{u \in \mathcal{H}^k_{eb,\mathrm{Ho}}(X); \mathrm{BD}_k\, u \in \{0\} \oplus \widetilde{L}^k_k\right\} \tag{6.62}$$

satisfy

$$\begin{aligned} \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) &= \mathcal{H}^k_{b,\mathrm{Ho}}(X) \oplus d\left[\mathrm{null}^{k-1}_{-}(\varDelta)\right] \\ \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X) &= \mathcal{H}^k_{b,\mathrm{Ho}}(X) \oplus \delta\left[\mathrm{null}^{k+1}_{-}(X)\right] \end{aligned} \tag{6.63}$$

and are such that

$$\begin{aligned} \mathcal{H}^k_{eb,\mathrm{Ho}}(X) &= \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) + \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X), \\ \mathcal{H}^k_{b,\mathrm{Ho}}(X) &= \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) \cap \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X). \end{aligned} \tag{6.64}$$

PROOF: If $u \in \mathcal{H}^k_{eb,\mathrm{Ho}}(X)$, defined in (6.59), then $du, \delta u \in x^\epsilon H^\infty_b(X; {}^b\!\varLambda^k)$ as follows from (6.49). Thus the integration by parts in

$$\langle \varDelta u, u\rangle = \langle d(\delta u), u\rangle + \langle \delta(du), u\rangle = \|\delta u\|^2 + \|du\|^2 = 0$$

is permissible and it follows that $du = \delta u = 0$. Conversely, again from (6.49), for a general element of the formal null space of $\varDelta$ on $x^{-\epsilon} H^\infty_b(X; {}^b\!\varLambda^k)$ and for $\epsilon > 0$ small

$$\begin{gathered} \mathrm{BD}_k\, u = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \Longrightarrow \\ \mathrm{BD}_{k+1}(du) = \begin{pmatrix} 0 & 0 \\ U_{12} & 0 \end{pmatrix},\ \mathrm{BD}_{k-1}(\delta u) = \begin{pmatrix} 0 & 0 \\ 0 & -U_{11} \end{pmatrix}. \end{gathered} \tag{6.65}$$

In particular if $u \in \mathrm{null}^k_{-}(\varDelta)$, $du = \delta u = 0$ implies $\pi_{\log} u = (U_{11}, U_{12}) = 0$ which shows that $u \in \mathcal{H}^k_{eb,\mathrm{Ho}}(X)$ and hence that $\mathcal{H}^k_{eb,\mathrm{Ho}}(X)$ is exactly the subspace of the d- and δ-closed elements in $\mathrm{null}^k_{-}(X)$.

With only slight abuse of notation, $\widetilde{L}^k$ can be construed as the image of $\mathcal{H}^k_{eb,\mathrm{Ho}}(X)$ under BD_k. The pairing between $\mathcal{H}^k_{eb,\mathrm{Ho}}(X)$ and $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$, given by continuous extension of the L^2 form, is non-degenerate when restricted to $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$, so the subspace

$$\left[\mathcal{H}^k_{b,\mathrm{Ho}}(X)\right]^\perp = \left\{u \in \mathcal{H}^k_{eb,\mathrm{Ho}}(X); \langle u, \mathcal{H}^k_{b,\mathrm{Ho}}(X)\rangle = 0\right\}$$

is mapped isomorphically onto L^k by BD_k. To see that the latter space splits consider the Hodge decomposition, (6.52), applied to $\left[\mathcal{H}^k_{eb,\mathrm{Ho}}(X)\right]^\perp$. Since these forms pair to zero with $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$, they are in the range of Δ :

$$\left[\mathcal{H}^k_{b,\mathrm{Ho}}(X)\right]^\perp \ni u = \Delta v,\ v = Gu \in x^{-\epsilon}H^\infty_b(X;{}^b\Lambda^k). \tag{6.66}$$

From Proposition 5.61, v is polyhomogeneous with only non-negative powers in its expansion and the order of 0 as a point in its index set is at most 3 (since u has 0 as a point in its index set with multiplicity at most 1.) Applying d and δ to (6.66) shows that $dv \in \mathrm{null}^{k+1}_-(X)$ and $\delta v \in \mathrm{null}^{k-1}_-(X)$. Furthermore, from (6.49), $d(\delta v)$, $\delta(dv) \in \mathcal{H}^k_{eb,\mathrm{Ho}}(X)$ have boundary data in $\mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \oplus \{0\}$ and $\{0\} \oplus \mathcal{H}^k_{\mathrm{Ho}}(\partial X)$ respectively. Thus

$$u = d(\delta v) + \delta(dv).$$

Since $d\colon \mathrm{null}^{k-1}_-(X) \to \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X)$ and $\delta\colon \mathrm{null}^{k+1}_-(X) \to \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X)$, this gives the first part of (6.64), the second part being immediate.

In fact applying (6.52) to $v \in \mathrm{null}^k_-(X)$ gives $v = d\delta w + \delta dw + u'$ with $u' \in \mathcal{H}^k_{b,\mathrm{Ho}}(X)$ and w polyhomogeneous with 0 of order at most 4 as a point in the index set. Since $(d+\delta)v \in \mathcal{H}^*_{eb,\mathrm{Ho}}(X)$ is closed and coclosed it follows that $d\delta w, \delta dw \in \mathrm{null}^k_-(X)$. Thus

$$\mathrm{null}^k_-(X) = \left\{v \in \mathrm{null}^k_-(X); dv = 0\right\} + \left\{v \in \mathrm{null}^k_-(X); \delta v = 0\right\}.$$

This gives (6.63).

PROOF OR PROPOSITION 6.14 (COMPLETION): It remains only to show that the map (6.54) is injective. If $u \in \dot{\mathcal{C}}^\infty(X;{}^b\Lambda^k)$ is closed and maps to 0 in $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$ then

$$u = \Delta v,\ \Delta(dv) = 0,\ v \in x^{-\epsilon}H^\infty_b(X;{}^b\Lambda^k)\ \forall\ \epsilon > 0. \tag{6.67}$$

Thus $dv \in \mathrm{null}^{k+1}_-(\Delta)$ and from (6.65) $\pi_{\log}(dv) = 0$ so in fact $dv \in \mathcal{H}^{k+1}_{eb,\mathrm{Ho}}(X)$ and therefore, by Proposition 6.16, $\delta dv = 0$. This means that (6.67) can be written $u = d(\delta v)$ which shows, from (6.47), that the class of u in $e\left[\mathcal{H}^k_{\mathrm{dR,rel}}(X)\right]$ is 0. Thus (6.54) is an isomorphism.

EXERCISE 6.17. Show that if there is cohomology on the boundary in dimensions k or $k-1$ then the range of d on $H^1_b(X;{}^b\Lambda^{k-1})$ is not closed in L^2, i.e. $H^0_b(X;{}^b\Lambda^k)$. Hence deduce that the L^2 cohomology is infinite dimensional. In cases like this the L^2 cohomology is often replaced by the refined L^2 cohomology defined to be

$$\mathcal{H}^k_{rL^2}(X) = \\ \left\{u \in H^0_b(X; {}^b\Lambda^k); du = 0\right\} / \overline{\{dv \in H^0_b(X; {}^b\Lambda^k); v \in H^0_b(X; {}^b\Lambda^{k-1})\}}.$$

Show that with this definition

$$\mathcal{H}^k_{rL^2}(X) \cong \mathcal{H}^k_{b,\mathrm{Ho}}(X).$$

Of course the notation in (6.61) and (6.62) is supposed to suggest that these spaces are related to the relative and absolute de Rham spaces respectively. This is true, in an appropriate sense. The maps to boundary data from these spaces can be interpreted as taking values in $\mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X)$ and $\mathcal{H}^k_{\mathrm{Ho}}(\partial X)$ respectively. Using (6.63), there is also a map

$$\mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) = \mathcal{H}^k_{b,\mathrm{Ho}}(X) \oplus d\left[\mathrm{null}^{k-1}_{-}(\Delta)\right] \ni u = u_1 + u_2 \\ \longmapsto u_1 \in \mathcal{H}^k_{b,\mathrm{Ho}}(X) \subset \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X).$$

Notice that in (6.63) d gives an isomorphism of $\widetilde{L}^k_{k-1}$ and $d\left[\mathrm{null}^k_{-}(X)\right] \subset \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X)$. This allows the space $\mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X)$ to be given a natural metric inner product, with the L^2 metric on $\mathcal{H}^k_{b,\mathrm{Ho}}(X)$, making the decomposition in (6.63) orthogonal. Using this inner product to define the adjoint, these maps can be organized into the diagramme:

$$\dots \xrightarrow{\mathrm{BD}_{k-1}} \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \xrightarrow{\mathrm{BD}^*_k} \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) \\ \longrightarrow \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X) \xrightarrow{\mathrm{BD}_k} \mathcal{H}^k_{\mathrm{Ho}}(\partial X) \xrightarrow{\mathrm{BD}^*_{k+1}} \dots.$$

This is the Hodge theoretic version of the long exact sequence (6.55):

Proposition 6.18. *For any exact b-metric on a compact manifold with boundary there are natural (metrically determined) isomorphisms*

$$\mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X) \longleftrightarrow \mathcal{H}^k_{\mathrm{dR,abs}}(X),\ \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) \longleftrightarrow \mathcal{H}^k_{\mathrm{dR,rel}}(X) \tag{6.68}$$

such that the diagramme

(6.69)

$$\begin{tikzcd} {} \arrow[r] & \mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \arrow[r, "\mathrm{BD}^*_k"] \arrow[d] & \mathcal{H}^k_{b\text{-rel},\mathrm{Ho}}(X) \arrow[r] \arrow[d] & \mathcal{H}^k_{b\text{-abs},\mathrm{Ho}}(X) \arrow[r, "\mathrm{BD}_{k+1}"] \arrow[d] & {} \\ {} \arrow[r] & \mathcal{H}^{k-1}_{\mathrm{dR}}(\partial X) \arrow[r] & \mathcal{H}^k_{\mathrm{dR,rel}}(X) \arrow[r, "e"'] & \mathcal{H}^k_{\mathrm{dR,abs}}(X) \arrow[r] & {} \end{tikzcd}$$

commutes.

PROOF: From Proposition 6.13 there is a natural map

$$\mathcal{H}^k_{b\text{-abs,Ho}}(X) \longrightarrow \mathcal{H}^k_{\mathrm{dR,abs}}(X) \tag{6.70}$$

given by interpreting $u \in \mathcal{H}^k_{b\text{-abs,Ho}}(X)$ as a closed conormal form. Conversely if $u \in \mathcal{C}^\infty(X; \Lambda^k)$ then (6.52) gives

$$u = d\delta v + \delta dv + u', \;\; u' \in \mathcal{H}^k_{b,\mathrm{Ho}}(X), \;\; v \in x^{-\epsilon} H^\infty_b(X; \Lambda^k).$$

Set $w = dv$. Then $du = 0$ implies $d\delta w = 0$ and, since $dw = 0$, necessarily $w \in \mathrm{null}^{k+1}_-(\Delta)$. Thus

$$u - d\delta v = u' + \delta w \in \mathcal{H}^k_{b\text{-abs,Ho}}(X)$$

by (6.63). This provides a two-sided inverse to (6.70) and proves the first part of (6.68). Moreover the boundary map from $\mathcal{H}^k_{b\text{-abs,Ho}}(X)$ into $\mathcal{H}^k_{\mathrm{Ho}}(\partial X)$ is the same as that for the de Rham cohomology, so the (implied) square, with $\mathcal{H}^k_{b\text{-abs,Ho}}(X)$ at the top left, in (6.69) commutes.

To define the map from $\mathcal{H}^k_{b\text{-rel,Ho}}(X)$ to $\mathcal{H}^k_{\mathrm{dR,rel}}(X)$ take a collar decomposition near the boundary, with x the defining function, and choose some $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ which is 1 near 0. Any $u \in \mathcal{H}^k_{b\text{-rel,Ho}}(X)$ is of the form

$$u = u' + \frac{dx}{x} \wedge u_\nu$$

in the collar, with $u' \in x^\epsilon H^\infty_b(X; \Lambda^k)$ for some $\epsilon > 0$ and u_ν independent of x and closed. Then consider the map

$$\mathcal{H}^k_{b\text{-rel,Ho}}(X) \ni u \longmapsto u - d(\phi(x) \log x \wedge u_\nu). \tag{6.71}$$

Certainly the image form is in $x^\epsilon H^\infty_b(X; {}^b\Lambda^k)$ and is closed. It is well defined apart from the choice of ϕ and a change in ϕ merely changes it by the differential of a $\mathcal{C}^\infty$ form with compact support in the interior. Thus (6.71) gives a map into the relative de Rham cohomology. By construction this map makes the square with $\mathcal{H}^k_{b\text{-rel,Ho}}(X)$ in the upper right commute since the boundary map in de Rham cohomology is just

$$\mathcal{H}^{k-1}_{\mathrm{Ho}}(\partial X) \ni v \longmapsto \phi'(x) dx \wedge v \in \mathcal{H}^k_{\mathrm{dR,rel}}(X)$$

and this map has null space $\widetilde{L}^{k-1}_{k-1} = (\widetilde{L}^k_{k-1})^\perp$. From (6.63) and Proposition 6.13 it follows that (6.71) projects to an isomorphism onto $\mathcal{H}^k_{\mathrm{dR,rel}}(X)$. This completes the proof of the theorem.

6.5. Extended L^2 null space of Dirac operators.

The relationship between the null space, on weighted spaces, of the Dirac operator, its adjoint and the Dirac Laplacians can be investigated using the same basic approach as in the previous section.

Let X be a compact even-dimensional manifold with boundary equipped with an Hermitian Clifford module, with unitary Clifford connection, associated to an exact b-metric. Then consider the null spaces of the 'Laplacian' and the positive Dirac operator:

$$\begin{aligned} \operatorname{null}_-(\eth_E^-\eth_E^+) &= \bigcap_{s<0} \left\{u \in x^s H_b^\infty(X; E^+); \eth_E^-\eth_E^+ u = 0\right\} \\ \operatorname{null}_-(\eth_E^+) &= \bigcap_{s<0} \left\{u \in x^s H_b^\infty(X; E^+); \eth_E^+ u = 0\right\}. \end{aligned} \tag{6.72}$$

Certainly $\operatorname{null}_-(\eth_E^-\eth_E^+) \subset \operatorname{null}_-(\eth_E^+)$ and the difference between then is determined by their boundary values. Using M_+ to identify (or define) the induced Clifford module on the boundary, E_0, with (as) the restriction of E^+, (4.115) shows that the indicial family of $\eth_E^-\eth_E^+$ is just $\eth_{0,E}^2 + \lambda^2$. Thus, just as for the Laplace-Beltrami operator, the corollary to Proposition 5.61 gives a map to boundary data

$$\begin{aligned} \mathrm{BD}^+ \colon \operatorname{null}_-(\eth_E^-\eth_E^+) &\longrightarrow \operatorname{null}(\eth_{0,E})^2 \\ u = u_1 \log x + u_0 + O(x^\epsilon) &\longmapsto (u_0, u_1). \end{aligned} \tag{6.73}$$

Let $\pi_{\log}(u) = u_1$ be the evaluation of the coefficient of the logarithm. Let $L^+ \subset \operatorname{null}(\eth_{0,E})$ be the subspace defined by

$$u_0 \in L^+ \iff \exists\ u \in \operatorname{null}_-(\eth_E^+) \text{ with } \mathrm{BD}^+(u) = (u_0, 0).$$

For the adjoint $\eth_E^-$, and related Laplacian $\eth_E^+\eth_E^-$, use of M_- to identify E_0 with E^- gives a similar map to boundary data, BD^-, and subspace $L^- \subset \operatorname{null}(\eth_{0,E})$ which arises as the boundary data of elements of $\operatorname{null}_-(\eth_E^-)$.

PROPOSITION 6.19. *For the Dirac operators on an even-dimensional compact manifold with boundary, as described above*

$$\operatorname{null}(\eth_{0,E}) = L^+ \oplus L^- \tag{6.74}$$

and $\eth_E^- \colon \operatorname{null}_-(\eth_E^+\eth_E^-) \longrightarrow \operatorname{null}_-(\eth_E^+)$ *has range the annihilator (with respect to the L^2 pairing) of the L^2 null space of $\eth_E^+$ and null space mapped isomorphically by $\pi_{\log}$ onto L^-.*

PROOF: From Lemma 4.51 it follows that 0 is the only possible point in the boundary spectrum of $\eth_E^+$. The boundary pairing between the formal null space of $\eth_E^+$ and that of $\eth_E^-$, associated to 0, is given by (6.3) and, as follows from (6.7), is just a multiple of the L^2 inner product on the boundary. Thus it follows from Lemma 6.4 that L^+ and L^- are orthocomplements in $\operatorname{null}(\eth_{0,E})$, as stated in (6.74). By definition $\operatorname{null}_-(\eth_E^+\eth_E^-)$ consists precisely of those $u \in x^{-\epsilon}H_b^\infty(X;E^-)$ such that $\eth_E^+(\eth_E^- u) = 0$ so certainly $\eth_E^-$ maps $\operatorname{null}_-(\eth_E^+\eth_E^-)$ into $\operatorname{null}_-(\eth_E^+)$. If $u \in \operatorname{null}_-(\eth_E^+\eth_E^-)$ then $\eth_E^- u$ is L^2-orthogonal to the L^2 null space of $\eth_E^+$, which is contained in $x^\epsilon H_b^\infty(X;E^+)$ for some $\epsilon > 0$. The Fredholm properties of $\eth_E^-$ allow $\eth_E^- u = f$ to be solved if $f \in x^{-\epsilon}H_b^\infty(X;E^+)$, for $\epsilon > 0$ sufficiently small, is orthogonal to the L^2 null space of $\eth_E^+$ so the map is surjective as stated. The last statement follows by applying Lemma 6.4 to $\eth_E^+\eth_E^-$.

The spaces $\operatorname{null}_-(\eth_E^\pm)$ are called the 'extended L^2 null spaces' of the respective operators in [8]. Proposition 6.19 answers a question posed there, in showing that every element of the null space of the boundary Dirac operator $\eth_{0,E}$ is the sum of the boundary value of an element of $\operatorname{null}_-(\eth_E^+)$ and that of an element of $\operatorname{null}_-(\eth_E^-)$; this is just (6.74). In [8] the weaker result, that $\dim L^+ + \dim L^- = \dim \operatorname{null}(\eth_{0,E})$, is shown; this reduces to the relative index theorem, across weight 0, for $\eth_E^+$.

6.6. Resolvent family.

Next the description of the generalized inverse obtained in Chapter 5 will be applied to the resolvent family of the Laplacian of an exact b-metric on a compact manifold with boundary. In fact the discussion will be carried out so that it applies to any second-order self-adjoint elliptic b-differential operator acting on some vector bundle but with diagonal principal symbol, given by an exact b-metric and with indicial family of the form (6.88). This is the 'geometric case'. For simplicity of notation at first only the action of the Laplacian on functions will be considered:

$$^b\Delta = {}^bd^* \cdot {}^bd \in \operatorname{Diff}_b^2(X),$$

where ^bd is the b-differential in (2.21) and $^bd^*$ is its adjoint with respect to an exact b-metric. The discussion of the indicial operators in §4.17 shows, as in (6.50), that the indicial roots of $^b\Delta$, the points in $\operatorname{spec}_b({}^b\Delta)$, are just the values of λ for which

$$(\Delta_\partial + \lambda^2)v = 0, \;\; v \in \mathcal{C}^\infty(\partial X)$$

has a non-trivial solution. That is,

Lemma 6.20. *For the Laplacian of an exact b-metric*

$$\lambda \in \operatorname{spec}_b({}^b\Delta) \Longleftrightarrow -\lambda^2 \in \operatorname{spec}(\Delta_\partial). \tag{6.75}$$

Since Δ_∂ is just the Laplacian on a compact Riemann manifold without boundary,

$$\operatorname{spec}(\Delta_\partial) = \left\{0 = \sigma_1^2 < \sigma_2^2 < \dots\right\}$$

can be written as an increasing sequence, where σ_k^2 corresponds to an eigenspace of dimension $m(k) \geq 1$. As a self-adjoint operator, Δ_∂ has resolvent family $(\Delta_\partial - \tau)^{-1}$ with simple poles (i.e. of order one) at the points $\tau = \sigma_k^2$, where the residue is the orthogonal projection, of rank $m(k)$, onto the corresponding eigenspace.

Exercise 6.21. Check that if E is an Hermitian vector bundle over X, $0 < \nu \in \mathcal{C}^\infty(X; {}^b\Omega)$ and $P \in \operatorname{Diff}_b^2(X; E)$ is formally self-adjoint

$$\int_X \langle P\phi, \psi\rangle \nu = \int_X \langle \phi, P\psi\rangle \nu \ \forall \ \phi, \psi \in \dot{\mathcal{C}}^\infty(X; E), \tag{6.76}$$

then if P is elliptic it is selfadjoint with domain $H_b^2(X; E)$. [Hint: It is enough to use a parametrix in the small calculus, as in Proposition 4.38.]

This allows (6.75) to be made more precise as in (5.10):

$$\operatorname{Spec}_b({}^b\Delta) = \left\{(0,1), (\pm i\sigma_j, 0), 0 \neq \sigma_j^2 \in \operatorname{spec}(\Delta_\partial)\right\}$$

and 0 is seen to be a special point, being the only indicial root of non-trivial order.

More generally consider the resolvent family of ${}^b\Delta$, i.e. the inverse of ${}^b\Delta - \tau$. The indicial roots of ${}^b\Delta - \tau$ are the solutions of

$$\tau = \lambda^2 + \sigma_j^2 \text{ i.e. } \lambda = \pm\lambda_j(\tau),\ \lambda_j(\tau) \overset{\text{def}}{=} (\tau - \sigma_j^2)^{\frac{1}{2}}. \tag{6.77}$$

Notice that

$$\lambda_j(\tau) \in \mathbb{R} \Longleftrightarrow \tau \in [\sigma_j^2, \infty). \tag{6.78}$$

This is important since real indicial roots are obstructions to the the operator ${}^b\Delta - \tau$ being Fredholm on the metric, i.e. unweighted, Sobolev spaces.

From Theorem 5.40 it can be deduced that $({}^b\Delta - \tau)\colon H_b^2(X) \longrightarrow H_b^0(X)$ is Fredholm exactly when $\tau \in \mathbb{C}\setminus[0,\infty)$. In fact ${}^b\Delta - \tau$ is actually invertible for all $\tau \notin [0,\infty)$ since, by (5.165), the null space would have to consist of classical conormal functions for which the integration by parts formula

$$0 = \langle ({}^b\Delta - \tau)u, u\rangle = -\tau\|u\|^2 + \|{}^bdu\|^2 \tag{6.79}$$

is justified.

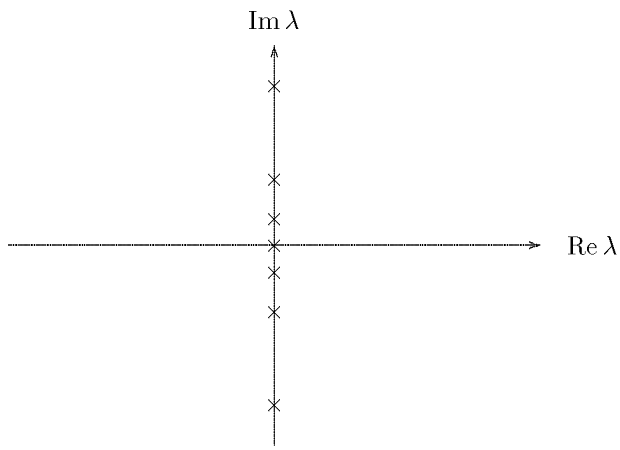

Figure 11. $\operatorname{spec}_b({}^b\Delta)$.

EXERCISE 6.22. Use an approximation argument to show that (6.79) holds if $u \in L_b^2(X)$ and ${}^b\Delta u = \tau u$.

If the square root in (6.77), $\lambda_j(\tau)$, is taken to have positive imaginary part, with cut along the positive real axis, then

$$\operatorname{Spec}_b({}^b\Delta - \tau) = \{(\pm\lambda_j(\tau), 0)\}\,,\ \tau \in \mathbb{C} \setminus [0, \infty)$$

has no points of positive order. The two absolute index sets obtained as in (5.85), by splitting $\operatorname{Spec}_b({}^b\Delta - \tau)$ into parts with negative and positive imaginary parts and changing the sign of the latter points, are the same:

$$E(\tau) = E^{\pm}(\tau) = \{(-\lambda_j(\tau), 0)\}\,,\ \tau \in \mathbb{C} \setminus [0, \infty).$$

Moreover there can be no accidental multiplicities except when τ is real, since the different roots have different real parts. Let $\widehat{\mathcal{E}}(\tau) = (\widehat{E}(\tau), \widehat{E}(\tau))$ be the corresponding $\mathcal{C}^\infty$ index sets defined by (5.118), which reduces to (5.119) when $\tau \notin \mathbb{R}$. Applying Proposition 5.61 the resolvent is therefore of the form

$$({}^b\Delta - \tau)^{-1} \in \Psi_{b,\mathrm{os}}^{-2,\widehat{\mathcal{E}}(\tau)}(X),\ \tau \in \mathbb{C} \setminus \mathbb{R}. \tag{6.80}$$

Going over the construction of the parametrix it is straightforward to check that it can be carried out holomorphically, at least locally in a set where there is no accidental multiplicity. This means that the indicial roots are holomorphic, all the terms in the expansions are holomorphic and the remainder estimates are uniform, locally, in λ. Thus near any point $\tau_0 \notin \mathbb{R}$ the inverse in (6.80) can be decomposed into the three terms of the calculus in (5.156):

$$({}^b\Delta - \tau)^{-1} = A(\tau) + B(\tau) + R(\tau), \ |\tau - \tau_0| < \epsilon$$

with each term holomorphic. For example the asymptotic expansion of $B(\tau)$ near lb and rb is therefore locally uniform in τ with holomorphic coefficients:

$$B(\tau) \sim \sum_j \rho_{\mathrm{lb}}^{-i\lambda_j(\tau)} u_j(\tau) \text{ near lb,} \tag{6.81}$$

where $u_j(\tau)$ is $\mathcal{C}^\infty$ on X_b^2 near lb and holomorphic in τ. Similarly $A(\tau)$ is holomorphic as an element of the small calculus.

This dicussion is under the assumption of the absence of accidental multiplicity. There is nothing to stop such accidents for $\tau \in (-\infty, 0)$. The resolvent certainly stays holomorphic as a bounded operator, but it cannot in general have a holomorphic expansion in the sense of (6.81). However it is important to note that the indicial roots $\pm\lambda_j(\tau)$ stay holomorphic. In fact all that happens is that in an expansion such as (6.81) the individual terms may be only meromorphic, while the whole operator remains holomorphic. To see how this can happen consider the simple model of accidental multiplicity given by the two roots $i\lambda$, $-i\lambda$ near $\lambda = 0$. The conormal distribution on $[0, \infty)$

$$u(x; \lambda) = \frac{x^{i\lambda} - x^{-i\lambda}}{2i\lambda} = \frac{\sin(x\lambda)}{\lambda}$$

is holomorphic as a distribution (i.e. weakly, when paired with a test function $\phi \in \dot{\mathcal{C}}^\infty([0, \infty)))$ but the coefficients in its expansion are meromorphic.

This is precisely what tends to happen at points of accidental multiplicity as can be seen from say (5.127) when the exponent $\lambda(\tau) + k$ crosses a pole of $I_\nu(P, z)$. Even though the expansions are then only meromorphic, not holomorphic, there is a stronger form of the holomorphy of the operator than just as a bounded operator on L^2. Namely it is holomorphic when considered as an element of the calculus with bounds in §5.16 and defined in general by (5.107). To say that a map into this space is holomorphic means that it has a decomposition

$$A(\tau) = A_1(\tau) + A_2(\tau) + A_3(\tau),$$

where $A_1(\tau)$ is holomorphic as a map into the small calculus (of fixed order), for some $\delta > 0$, $\rho_{\text{lb}}^{-a-\delta}\rho_{\text{rb}}^{-b-\delta}A_2(\tau)$ is holomorphic as a map into the space in (5.104) and $A_3(\tau)$ is holomorphic into the weighted Sobolev space in (5.107). Each of these spaces is fixed, independent of τ (unlike the polyhomogeneous space which depends on the variable of holomorphy) with Frechét topology. Thus each is a countable intersection of Hilbert spaces, and holomorphy just means holomorphy into each of these spaces. The construction above therefore gives:

LEMMA 6.23. *For any open set $G \subset \mathbb{C}$ with closure compact and contained in $\mathbb{C}\backslash[0,\infty)$ there exists $\epsilon > 0$ such that the resolvent of the Laplacian is holomorphic as a map*

$$G \ni \tau \longmapsto ({}^b\Delta - \tau)^{-1} \in \Psi_{b,\text{os},\infty}^{-2,(\epsilon,\epsilon)}(X).$$

6.7. Analytic continuation of the resolvent.

This construction can be extended outside the resolvent set, or physical space, $\mathbb{C}\setminus[0,\infty)$, in a manner typical of scattering theory. To do so first consider the roots in (6.77). Each of these corresponds to a natural Riemann surface, Z_j. Thus

$$\chi\colon Z_j \longrightarrow \mathbb{C}$$

is a double cover of $\mathbb{C}$ ramified at the singular point, $\tau = \sigma_j^2$, of the function. As a topological space, Z_j is obtained from two copies of $\mathbb{C}$ cut along $[\sigma_j^2, \infty)$, with the sides identified to their opposite in the other copy. The complex structure on Z_j is that of $\mathbb{C}$ except near the point of ramification, where it is uniformized by $\pm(\tau - \sigma_j^2)^{\frac{1}{2}}$, i.e. the two holomorphic functions outside the cuts combine to give a single holomorphic function:

$$R_j\colon Z_j \longrightarrow \mathbb{C}$$

which restricts to $\lambda_j(\tau)$ in the physical space. Although this standard construction gives a surface on which one of the indicial roots is holomorphic what is required is a surface on which *all* of them are simultaneously holomorphic. Starting with Z_0, notice that $\pm(\tau - \sigma_1^2)^{\frac{1}{2}}$ lift from $\mathbb{C}$, cut along $[\sigma_1^2, \infty)$, to Z_0 cut along the two half-lines which are the preimages of $[\sigma_1^2, \infty)$ under R_0 in Z_0. Taking two copies of Z_0, each cut in this way, and identifying the two pairs of cuts appropriately gives a four fold cover, $\widehat{Z}_1$ of $\mathbb{C}$ to which both $\pm\tau^{\frac{1}{2}}$ and $\pm(\tau - \sigma_1^2)^{\frac{1}{2}}$ extend holomorphically. Continuing indefinitely in this way successive Riemann surfaces, $\widehat{Z}_j$, are constructed with covering maps $\widehat{Z}_j \longrightarrow \mathbb{C}$. Successive constructions leave an increasingly large compact set unchanged so the Riemann surface, $\widehat{Z}$, can be defined to which all of the indicial roots extend to be holomorphic.

THEOREM 6.24. *The resolvent of the Laplacian of an exact b-metric extends to a meromorphic function*

$$\widehat{Z} \ni z \longmapsto Q(z) \in \Psi_{b,\mathrm{os}}^{-2,\widehat{\mathcal{E}}(z)}(X), \tag{6.82}$$

where $\widehat{\mathcal{E}}(z) = (E_{\mathrm{lb}}(z), E_{\mathrm{rb}}(z))$ is the smallest $\mathcal{C}^\infty$ index family containing the values of the holomorphic functions for all j :

$$E_{\mathrm{lb}}(z) = E_{\mathrm{rb}}(z) \supset \{(-R_j(z), 0); j = 0, \dots\}.$$

The notion of meromorphy for a polyhomogeneous b-pseudodifferential operator is discussed above, i.e. all expansions are meromorphic with remainder terms meromorphic in the calculus with bounds. As before there may be more poles in the representation of the expansions than in the operators themselves.

PROOF: The idea is simply to construct the parametrix for $(\Delta - z)^{-1}$ to be holomorphic in an arbitrarily large compact region of $\widehat{Z}$ and then to use analytic Fredholm theory to show that $Q(z)$, extending the resolvent, is meromorphic. From the uniqueness of analytic continuation it then follows that $Q(z)$ extends meromorphically to all of $\widehat{Z}$. It is important to note that the index sets $\widehat{\mathcal{E}}(z)$ allow, as z moves outside the physical region, increasingly large negative powers in the expansions of the kernels. In particular the extension of the resolvent will not be bounded on L^2. In order to prove the existence of this analytic extension it is therefore essential that the error term be made to vanish to high order at $\mathrm{lb}(X_b^2)$ so that the Neumann series in (5.147) makes sense.

So consider again the various steps in the construction. The first step was to construct an inverse modulo $\Psi_b^{-\infty}(X)$. This involves successive division by the principal symbol, which is independent of τ. Thus the symbols are always polynomials (of increasing order) in τ. The quantization map gives operators depending holomorphically on τ and these can be summed uniformly asymptotically, giving a parametrix in the small calculus depending holomorphically on τ in any preassigned compact domain in $\mathbb{C}$. Indeed with a little more effort it can be made entire, but this is of no particular significance here. This 'small' parametrix, $G_s(\tau)$, can then be lifted to a correspondingly large subset of $\widehat{Z}$.

The next step is to correct the indicial operator of the parametrix so that it inverts the indicial operator. This is done by using the inverse of the indicial family, see (5.84). Thus the correction term to the indicial operator of G_s is

$$G_B(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s^{i\lambda}(\Delta_\partial + \lambda^2 - \tau)^{-1} R_{s,M}(\lambda, \tau) d\lambda.$$

The integral is over the real axis when $\tau \in \mathbb{C}\setminus[0,\infty)$, the physical space for the resolvent parameter. The difficulty is that as τ approaches the positive real axis some indicial roots may approach the real axis. However, to get holomorphy in a given compact subset of $\widehat{Z}$ only a finite number of the eigenvalues of Δ_∂ need to be considered. Thus G_B can be decomposed as a sum:

$$
\begin{aligned}
(6.83)\quad G_B = G_B^{(j)} + \sum_{p=0}^{j} \frac{1}{2\pi} \sum_e \phi_e^{(p)} \int_{-\infty}^{\infty} s^{i\lambda} d\lambda \\
\times (\sigma_p^2 + \lambda^2 - \tau)^{-1} \int R_{s,M}(\lambda, y'', \cdot)\phi_e^{(p)}(y'')dy''.
\end{aligned}
$$

Here $G_B^{(j)}$ is holomorphic in $|\tau| < \sigma_j^2$ and the $\phi_e^{(p)}$ are, for $e = 1, \ldots, m(p)$, an orthonormal basis of the eigenfunctions of Δ_∂ with eigenvalue σ_p^2.

To find the analytic continuation of G_B it therefore suffices to consider the finitely many terms corresponding to the 'small' eigenvalues of Δ_∂, i.e. each integral

$$
(6.84)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} s^{i\lambda}(\sigma_j^2 + \lambda^2 - \tau)^{-1} R_{s,M}(\lambda, \cdot) d\lambda.
$$

LEMMA 6.25. *The function in* (6.84) *extends meromorphically to* Z_j, *with a pole only at the point of ramification over* $\mathbb{C}$.

PROOF: The representation (6.84) is valid for $\tau \in \mathbb{C}\setminus[0,\infty)$. Analytic continuation across the cut is given by shifting the contour of integration to $\operatorname{Im}\lambda = -N$, for N large. This crosses the pole of the integrand at $\lambda = \lambda_j(\tau)$ so, using Cauchy's formula, (6.84) decomposes into

$$
\begin{aligned}
(6.85)\qquad & -i\frac{s^{-iR_j(\tau)}}{2R_j(\tau)} R_{s,M}(-R_j(\tau), \cdot) \\
& + \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = -N} s^{i\lambda}(\sigma_j^2 + \lambda^2 - \tau)^{-1} R_{s,M}(\lambda, \cdot) d\lambda,\ N >> |\tau|.
\end{aligned}
$$

Here we have written $R_j(\tau)$ in place of $\lambda_j(\tau)$ to emphasize the analytic continuation to Z_j. Since $R_{s,M}$, coming as it does from the small calculus, is entire and rapidly decreasing as $|\operatorname{Re}\lambda| \to \infty$ when $|\operatorname{Im}\lambda|$ is bounded, the second term is holomorphic in τ. Clearly the first term is meromorphic on Z_j, with only a possible pole at $R_j(\tau) = 0$ which is the point of ramification.

In fact the decomposition (6.85) gives more since it shows that in $s < C$ the second term is not only holomorphic on Z_j but is rapidly vanishing as $s \downarrow 0$. Of course the first term corresponds to a fixed singular point, $-iR_j(\tau)$, for each value of τ. The same conclusion, as to meromorphy, can be arrived at by shifting the contour to $\operatorname{Im}\lambda = N$, for $N >> |\tau|$. Then in place of (6.85)

(6.86)
$$-i\frac{s^{iR_j(\tau)}}{2R_j(\tau)}R_{s,M}(R_j(\tau),\cdot) + \frac{1}{2\pi}\int\limits_{\operatorname{Im}\lambda=N} s^{i\lambda}(\sigma_j^2+\lambda^2-\tau)^{-1}R_{s,M}(\lambda,\cdot)d\lambda.$$

The residues in (6.85) and (6.86) are necessarily the same at the point of ramification. However in (6.86) the second term is entire on Z_j, for $s > C$, and rapidly vanishing as $s \to \infty$.

Applying Lemma 6.25 to the decomposition of the correction term in (6.83), it follows that G_B is meromorphic on $\widehat{Z}$ (or in as large a region as $R_{s,M}$ was made holomorphic as a function of τ) with poles only at the points of ramification and complete asymptotic expansion as $s \to 0, \infty$ with powers exactly the values of $R_j(\tau)$.

This is the meromorphic version of the first part of the construction of the correction term to the resolvent. Following the discussion in §5.20, G_B should next be extended off $\operatorname{bf}(X_b^2)$. That is an operator, G'_B, is to be chosen in $\widetilde{\Psi}_b^{-\infty,\widehat{\mathcal{E}}(z)}(X)$ with G_B as its normal operator in such a way that the error term in

$$(6.87)\qquad (\varDelta-\tau)(G_s+G'_B) = \operatorname{Id} - R_1 - R_2,\ I_\nu(R_1)\equiv 0$$

is to vanish to infinite order at $\operatorname{lb}(X_b^2)$. This proceeds exactly as before, with the resulting Taylor series being meromorphic in the same sense as for G_B. The only subtlety worth noting is that the simple poles at the points of ramification can be chosen to have residues in the residual calculus, so R_1 is holomorphic and $R_2 \in \Psi^{-\infty,(\emptyset,\widehat{E}^-(z))}(X)$.

This argument can be iterated, as in §5.22, to remove the Taylor series of the error term at $\operatorname{bf}(X_b^2)$. Summing again gives an improved parametrix as in (6.87), with no term R_1, in the sense that the error term is now meromorphic with values in the residual calculus and only simple poles at points of ramification. This gives a meromorphic right parametrix in the appropriate space with

$$({}^b\!\varDelta - \tau(z))G_R(z) = \operatorname{Id} - R_R(z),$$
$$R_R(z) \in \Psi_{b,\infty}^{-\infty,(\infty,-A)}(X),\ z\in K \subset\subset \widehat{Z},$$

where the constant A depends on the compact set K. In fact of course $R_R(z)$ is also meromorphic as a polyhomogeneous b-pseudodifferential operator. By analytic Fredholm theory, as discussed in §5.3, it is only necessary to construct such a parametrix with $\mathrm{Id} - R_R(z)$ invertible for one value of z to deduce that the inverse extends to be meromorphic. The space on which these operators act can be taken to be $\rho^A H_b^0(X)$, where again the constant A will have to be taken increasingly large and positive to get meromorphy on a correspondingly large compact subset of $\widehat{Z}$. It follows that the inverse

$$G_R(z)\,(\mathrm{Id} - R_R(z))^{-1}$$

is itself meromorphic as an operator from $\rho^A H_b^0(X)$ to $\rho^{-A} H_b^2(X)$. The invertibility for one value of z can be accomplished by the addition of $G(z') - G_R(z')$, where $G(z') = ({}^b\Delta - \tau(z'))^{-1}$ for some z' in the physical space. This completes the proof of Theorem 6.24.

Anticipating a little, the argument from §7.7 can be used to simplify the construction of the parametrix slightly in that a 'small' parametrix $G_s(\tau)$ can be found which is entire in $\tau \in \mathbb{C}$ and with the property that the error is small at infinity in the physical space. This approach is discussed in §7.7.

6.8. Poles of the resolvent.

Although Theorem 6.24 is stated for the Laplacian on functions, only the behaviour of the indicial roots is really crucial to the argument. In particular it extends to any operator $P \in \mathrm{Diff}_b^2(X; E)$ which is self-adjoint with respect to an Hermitian inner product on E and the metric density of an exact b-metric, has diagonal principal given by that b-metric and has indicial family with respect to the metric trivialization of the normal bundle of the form

$$I_\nu(P, \lambda) = P_\partial + \lambda^2. \tag{6.88}$$

Here P_∂ is necessarily a self-adjoint operator on sections of E over the boundary. Examples of such operators include the Laplacian on b-forms and the square of the Dirac operator. Let $\mathrm{Eig}(j)$ be the eigenspace for P_∂ and the eigenvalue σ_j.

LEMMA 6.26. *For a self-adjoint elliptic operator satisfying* (6.88) *the generalized boundary data of $P - \sigma_j$, where $\sigma_j \in \mathrm{spec}(P_\partial)$ is*

$$G(P - \sigma_j, 0) = \{u_0 + u_1 \log x;\, u_0, u_1 \in \mathrm{Eig}(j)\} \tag{6.89}$$

and with this identification the boundary pairing (6.12) *becomes*

$$B(u_0 + u_1 \log x, v_0 + v_1 \log x) = i \int_{\partial X} (\langle u_0, v_1\rangle - \langle u_1, v_0\rangle)\, \nu_0. \tag{6.90}$$

Proof: Certainly (6.89) follows directly from (6.88). If $\phi \in \mathcal{C}^\infty(\widetilde{X})$ is 1 near $\partial_0 \widetilde{X}$ and 0 near $\partial_1 \widetilde{X}$ then the Mellin transform

$$(\phi(u_0 + \log x u_1))_M = -\frac{i}{\lambda} u_0 - \frac{1}{\lambda^2} u_1 + h(\lambda)$$

with h entire. Then the extension of (6.7) to sections of bundles gives

$$B(u_0 + \log x u_1, v_0 + \log x v_1) = \\ \frac{1}{2\pi i} \oint_\Gamma \int_{\partial X} \langle \lambda^2 (-\frac{i}{\lambda} u_0 - \frac{1}{\lambda^2} u_1), -\frac{i}{\overline{\lambda}} v_0 - \frac{1}{\overline{\lambda}^2} v_1 \rangle \nu_0 \cdot d\lambda$$

which gives (6.90).

For applications it is useful to have more information about the poles of the analytic continuation. The poles of immediate interest are those on the boundary of the physical region, the *physical poles*; those outside the closure of the physical region are poles of the scattering matrix as briefly explained in §6.10. The physical poles fall into two classes, we discuss the simpler of these first.

Proposition 6.27. *For a self-adjoint element of* $\mathrm{Diff}_b^2(X; E)$ *with diagonal principal symbol given by an exact b-metric and indicial operator of the form* (6.88), *the poles of the analytic extension* (6.82) *on the boundary of the physical region* $\mathbb{C} \setminus [\sigma_0, \infty)$ *and away from the points of ramification of* $\widehat{Z}$ *are all simple, project precisely to the points* $\tau \in \mathbb{R} \setminus \mathrm{spec}(P_\partial)$ *at which* P *has* L^2 *eigenfunctions, and have as residues the orthogonal projectors onto the* L^2 *eigenspaces.*

Thus these poles behave very much as the eigenvalues in the case of a compact manifold without boundary.

Proof: Suppose $\tau \in \widehat{Z}$ is a pole of this type, i.e. it is in the boundary of the physical region and projects to $\mathbb{R} \setminus \mathrm{spec}(P_\partial)$. Thus for $r > 0$, both $\tau + ir$ and $\tau - ir$ can be interpreted as points in the resolvent set. The self-adjointness of P means that

$$\mathrm{Im} \int_X \langle (P - \tau \pm ir)\phi, \phi \rangle \nu = r \|\phi\|^2$$

for all $\phi \in H_b^2(X; E)$ so

$$\|(P - \tau \pm ir)^{-1}\| \leq 1/r \tag{6.91}$$

as an operator on L^2. This and the meromorphy established above show that, for any $\phi \in \dot{\mathcal{C}}^\infty(X; E)$, $r(P - \tau \pm ir)^{-1}\phi$ is smooth with values in $x^{-\epsilon}H_b^2(X; E)$, down to $r = 0$ with possibly different limits depending on the sign. Then (6.91) shows that the limits must exist in $H_b^0(X; E)$. The residue

$$\operatorname{Res}(\tau) = \lim_{\pm r \downarrow 0} ir(P - (\tau + ir))^{-1} \tag{6.92}$$

therefore takes values in L^2 on $\dot{\mathcal{C}}^\infty(X; E)$. However its range certainly consists of elements of the null space $\operatorname{null}(P, \epsilon)$ on $x^\epsilon H_b^\infty(X; E)$ for some $\epsilon > 0$ and the residue therefore has finite rank. The same argument applies to the adjoint, so the residue is a finite rank self-adjoint operator on L^2, in fact mapping $x^{-\epsilon}H_b^0(X; E)$ into $x^\epsilon H_b^\infty(X; E)$ for some $\epsilon > 0$. Differentiation of the resolvent identity gives

$$\begin{aligned} \lim_{\pm r \downarrow 0} \frac{d}{dr} \left((P - (\tau + ir)) r(P - (\tau + ir))^{-1} \right) \phi = \\ (P - \tau)(A\phi) - \operatorname{Res}(\tau)\phi = \phi \ \forall \ \phi \in \dot{\mathcal{C}}^\infty(X; E), \end{aligned} \tag{6.93}$$

with A taking values in $x^{-\epsilon}H_b^2(X; E)$. Thus A is a generalized inverse of $P - \tau$, so $\operatorname{Res}(\tau)$ is necessarily a projection onto the L^2 null space.

Conversely if there is a point $\tau \in \mathbb{R} \setminus \operatorname{spec}(P_\partial)$ for which $P - \tau$ has non-trivial L^2 null space then there must be a pole of the resolvent at any point on the boundary of the physical space which projects onto τ since otherwise (6.93) would hold with $\operatorname{Res}(\tau)$ zero and hence the equation $Pu = \phi$ would have a solution for every $\phi \in \dot{\mathcal{C}}^\infty(X; E)$. This completes the proof of the proposition.

So it remains to consider the poles at the points of ramification of $\widehat{Z}$.

PROPOSITION 6.28. *Under the same conditions as Proposition 6.27 the analytic continuation of the resolvent to $\widehat{Z}$ has a at most a double pole at the ramification point, over σ_j, on the boundary of the physical region with coefficient the orthogonal projection onto the L^2 eigenspace for $\sigma_j \in \operatorname{spec}(P_\partial)$ as an eigenvalue for P (if it is) and the residue at the pole is the operator from $x^\epsilon H_b^0(X; E)$ to $x^{-\epsilon}H_b^\infty(X; E)$ for any $\epsilon > 0$ with kernel*

$$\sum_l U_l \overline{U_l} \nu, \tag{6.94}$$

where the $U_l \in \mathcal{C}^\infty(X; E) + x^\epsilon H_b^\infty(X; E)$ for some $\epsilon > 0$ are a basis of those solutions of $(P - \sigma_j)U = 0$ in this space which are orthogonal to all L^2 solutions and have boundary values orthonormal in $L^2(\partial X; E)$.

PROOF: Let us consider only the case $\sigma_0 = 0$. This simplifies the form of the indicial roots, but a general point of ramification is not essentially more complicated. Consider the resolvent evaluated on the negative real axis, in fact consider $G(r) = (P + r^2)^{-1}$ for $r \in (0, \delta)$. The meromorphic extension of the resolvent shows that $r^k G(r)\phi$ has a smooth limit as $r \downarrow 0$ for any $\phi \in \dot{\mathcal{C}}^\infty(X; E)$ and some k. As in the proof of Proposition 6.27

$$\|G(r)\| \le |r|^{-2} \tag{6.95}$$

so $k \le 2$. For $\phi \in \dot{\mathcal{C}}^\infty(X; E)$ set

$$Q\phi = \lim_{r\downarrow 0} r^2 G(r)\phi \tag{6.96}$$

$$\mathrm{Res}(0)\phi = i \lim_{r\downarrow 0} \frac{d}{dr} r^2 G(r)\phi \tag{6.97}$$

$$A\phi = \frac{1}{2} \lim_{r\downarrow 0} \frac{d^2}{dr^2} r^2 G(r)\phi. \tag{6.98}$$

The first three terms in the Taylor series, at $r = 0$, of the identity

$$(P + r^2) \circ r^2 G(r) = r^2 \,\mathrm{Id},$$

in which all terms are smooth give:

$$P \circ Q = 0 \tag{6.99}$$

$$P \circ \mathrm{Res}(0) = 0 \tag{6.100}$$

$$P \circ A = \mathrm{Id} - Q \tag{6.101}$$

with all three operators formally self-adjoint on $\dot{\mathcal{C}}^\infty(X; E)$.

From (6.95) Q takes values in L^2. Certainly A takes values in the space $x^{-\epsilon} H_b^\infty(X; E)$ for any $\epsilon > 0$ so from (6.101) Q is necessarily the projection onto the L^2 null space. From the holomorphy of the resolvent and the fact that ir is the only indicial root of $P + r^2$ approaching 0 from $\mathrm{Im}\,\lambda > 0$

$$ir \left(G(r) - r^{-2} Q\right) \phi = x^{-r} E(r)\phi + B(r)\phi, \tag{6.102}$$

where $E(r)\phi \in \mathcal{C}^\infty([0, \delta)_r \times X; E)$ and $B(r)\phi$ is $\mathcal{C}^\infty$ in r down to $r = 0$ with values in $x^\epsilon H_b^\infty(X; E)$ for some $\epsilon > 0$. Thus

$$\mathrm{Res}(0)\phi = E(0)\phi + B(0)\phi \tag{6.103}$$

takes values in $\mathcal{C}^\infty(X; E) + x^\epsilon H_b^\infty(X; E)$. Differentiating (6.102) we conclude that the generalized inverse in (6.101) is of the form

$$A\phi = -\log x \,\mathrm{Res}(0)\phi + A_0\phi + A'\phi, \tag{6.104}$$

where A_0 takes values in $\mathcal{C}^\infty(X;E)$ and A' takes values in $x^\epsilon H_b^\infty(X;E)$ for some $\epsilon > 0$.

Suppose $\phi \in \dot{\mathcal{C}}^\infty(X;E)$ and $Q\phi = 0$. Let $u \in G(P,0)$ be the generalized boundary data of $A\phi$ associated to the indicial root 0. Let U_l be a basis of the solutions of $PU = 0$ which are in $\mathcal{C}^\infty(X;E) + x^\epsilon H_b^\infty(X;E)$ for some $\epsilon > 0$ and orthogonal to the L^2 null space. Then U_l can be identified with its generalized boundary data $U_l \in G'(P,0)$ as in (6.13). Moreover

$$B(u,U_l) = \frac{1}{i}\int_X \langle Pu, U_l\rangle\nu = \frac{1}{i}\langle\phi, U_l\rangle_\partial \tag{6.105}$$

as follows from (6.10). From (6.101) and its formal self-adjointness, $\mathrm{Res}(0)$ can be written in terms of the basis as

$$\mathrm{Res}(0)\phi = \sum_{k,l} A_{kl}U_k\langle\phi, U_l\rangle.$$

Now B is given in (6.90) so applying (6.104) to (6.105), with $u = A\phi$, shows that A_{kl} is the identity matrix. This completes the proof of the proposition.

6.9. Spectral theory.

The existence of an analytic continuation of the resolvent can be used to give a rather precise description of the spectrum of the Laplacian. First recall the definition of the spectral measure:

PROPOSITION 6.29. *If P is an unbounded selfadjoint operator on a Hilbert space H then the resolvent $(P-\tau)^{-1} \in \mathcal{L}(H)$ is analytic in $\tau \in \mathbb{C}\setminus\mathbb{R}$ and the limit*

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mathbb{R}} &\left((P-r-i\epsilon)^{-1} - (P-r+i\epsilon)^{-1}\right) f(r)dr \\ &= \int_{\mathbb{R}} f(r)dE_P \in \mathcal{L}(H)\end{aligned} \tag{6.106}$$

exists for all continuous f of compact support and so defines a measure, $dE_P(r)$, on $\mathbb{R}$ with values in $\mathcal{L}(H)$, the spectral measure of P.

The spectral measure can be used to give integral representations of functions of P. In particular the support of dE_P is the spectrum of P and it is the complement in $\mathbb{C}$ of the maximal open set to which $(P-\tau)^{-1}$ extends as a bounded operator holomorphic in τ. The spectrum is divided into various pieces according to the behaviour of dE_P.

The discrete spectrum consists of the points at which $(P-\tau)^{-1}$ has a pole of finite order (necessarily simple). This can either be of finite or infinite multiplicity, just the rank of the residue which is necessarily a projection in $\mathcal{L}(H)$. Thus dE_P is equal to a Dirac measure near a point of the discrete spectrum.

The continuous spectrum consists of the points at which dE_P is the differential of a continuous function with values in the bounded operators. The rank of a point in the continuous spectrum is k if there is such a representation of dE_P locally with the continuous function having values in the operators of rank at most k. Thus k can be infinite. The spectrum is said to be continuous with embedded eigenvalues, if dE_P is the sum of Dirac masses and the differential of a continuous function.

THEOREM 6.30. *If $P \in \mathrm{Diff}_b^2(X;E)$ is self-adjoint with symbol given by an exact b-metric and indicial family of the form* (6.88) *then the spectral measure of P is of the form*

$$dE_P = \sum_k \delta(\tau - \tau_k')P_k + \sum_{\sigma_j \in \mathrm{spec}(P_\partial)} [\tau - \sigma_j]_+^{-\frac{1}{2}} F_j, \tag{6.107}$$

where the τ_k' are the L^2 eigenspaces, all finite dimensional with orthogonal projections P_k and the F_j are smooth functions of $(\tau-\sigma_j)^{\frac{1}{2}}$ with values in the null space of $(P-\tau)$ on $x^{-\epsilon}H_b^2(X;E)$, for small $\epsilon > 0$, orthogonal to the null space on L^2 and such that $F_j(\sigma_j)$ is given by (6.94).

COROLLARY. *Under the conditions of Theorem 6.30 the spectrum of P is discrete of finite multiplicity outside $[\sigma_0, \infty)$ and on $[\sigma_j, \sigma_{j+1})$, where σ_j $j = 0, 1, \dots$ are the eigenvalues of P_∂ in increasing order, P has continuous spectrum of multiplicity equal to the sum of the multiplicities of the eigenvalues $\{\sigma_0, \dots, \sigma_j\}$ of P_∂, with possibly embedded discrete spectrum of finite multiplicity.*

PROOF: Naturally we make heavy use of the analytic extension of the resolvent and the discussion in §6.8. Certainly outside $[\sigma_0, \infty)$ the resolvent family is meromorphic as a family of bounded operators on L^2, with at most poles of finite rank. The self adjointness of P forces these poles to be simple (i.e. of order 1) and to occur at real points.

Now consider a point in $\tau \in (\sigma_0, \infty)$ which is not equal to one of the σ_j. For the moment suppose as well that the analytic extension of the resolvent does not have a pole at this point, meaning as it is approached from above or below from the physical space. Then the limit in (6.106) certainly exists since the individual kernels converge. Set

$$F(r) = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \left((P - r + i\epsilon)^{-1} - (P - r - i\epsilon)^{-1} \right), \ r \in \mathbb{R}. \tag{6.108}$$

Thus $F(r)$ maps $x^\delta H_b^0(X;E)$ into $x^{-\delta}H_b^2(X;E)$ for any $\delta>0$. Since it is given as the limit of the resolvent its range is in the null space of $P-r$ on $x^{-\delta}H_b^2(X;E)$. It therefore has finite rank, by Proposition 5.61. In fact we know that the space $G(P-r,0)$ defined in (6.11) has rank exactly twice the collective multiplicity of the eigenvalues of P_∂ less than r. From (6.19) applied to $P-r=(P-r)^*$ we conclude that $G'(P-r,0)$ has dimension exactly half of that of $G(P-r,0)$. Now $G'(P,0)$ is precisely the range of the map in Proposition 5.63 for $\alpha=-\epsilon$ and $\alpha'=\epsilon$ sufficiently small. Thus we conclude that

$$\dim\operatorname{null}(P-r,-\epsilon)=\frac{1}{2}\dim G(P-r,0)+\dim\operatorname{null}(P-r,\epsilon). \tag{6.109}$$

In fact $\dim\operatorname{null}(P-r,\epsilon)=0$ since a non-trivial L^2 solution would entail a pole in $(P-\tau)^{-1}$ at $\tau=r$ (from either above or below) since it cannot be in the range of $(P-r)$ on $x^{-\delta}H_b^2(X;E)$ for small $\delta>0$. This shows that the limiting operator (6.108) has rank exactly as stated, at regular points.

The discussion at singular points is essentially the same, with Propositions 6.27 and 6.28 taken into account.

6.10. Scattering matrices.

A little of the scattering theory for an exact b-metric will now be discussed and used to refine the statements concerning F_0 in (6.107). For simplicity, at first, only the case of the Laplacian on functions for a compact manifold with ∂X having just one component will be treated, although as usual the general case presents no essential difficulties.

The scattering matrices are operators on each eigenspace of the boundary Laplacian (and between them) depending on the resolvent parameter. They capture the leading global behaviour at the boundary. These matrices can be viewed as relating the resolvent of the actual Laplacian to the resolvent of its indicial operator and it is in this sense that they are scattering matrices. The analogy is strongest with one-dimensional scattering theory on the line, but this is not examined. Rather the position taken here is that these scattering matrices are really the analogue of the Calderón projector ([25], [81]) in standard elliptic boundary problems.

If σ_j^2 is an eigenvalue of Δ_∂ and $\phi\in\operatorname{Eig}(j)$ is an associated eigenfunction consider $u(\tau)=x^{iR_j(\tau)}\phi$, where $R_j(\tau)=(\tau-\sigma_j^2)^{\frac{1}{2}}$ is the square root with positive imaginary part for $\tau\in\mathbb{C}\setminus[0,\infty)$, continued to $\widehat{Z}$, and x is an admissible boundary defining function. Then $u(\tau)$ is not square-integrable, for τ in the resolvent set, but

$$v(\tau)=(\Delta-\tau)u(\tau)=O(|x^{1+iR_j(\tau)}|).$$

In particular if $|\operatorname{Im} R_j(\tau)| < 1$, which is certainly true when τ is close to $[\sigma_j^2, \infty)$ and in the physical region, then $v(\tau)$ is square-integrable. Thus there is a unique L^2 solution of

$$(\varDelta - \tau)u'(\tau) = v(\tau), \tag{6.110}$$

so

$$U(\tau) = \frac{1}{2}[u(\tau) - u'(\tau)] \neq 0 \text{ satisfies } (\varDelta - \tau)U(\tau) = 0. \tag{6.111}$$

This shows the first part of:

Lemma 6.31. *For each eigenfunction ϕ of $\varDelta_\partial$ with eigenvalue σ_j^2 there is an open neighbourhood $O_j \subset \mathbb{C}$ of $[\sigma_j^2, \infty)$ such that to each $\tau \in O_j \setminus [0, \infty)$ there corresponds a unique element $U(\tau) \in x^{-1}H_b^\infty(X; {}^b\Omega^{\frac{1}{2}})$ satisfying*

$$(\varDelta - \tau)U(\tau) = 0,\ U(\tau) \equiv \frac{1}{2}x^{iR_j(\tau)}\phi \mod L^2. \tag{6.112}$$

Moreover, for τ sufficiently close to σ_j^2 in $\mathbb{C} \setminus [0, \infty)$

$$\begin{gathered} U(\tau) - \frac{1}{2}x^{iR_j(\tau)}\phi = \frac{1}{2}\sum_{\sigma_k^2 \le \sigma_j^2} x^{-iR_k(\tau)}\psi_{jk}(\tau)\phi + O(x^\epsilon), \\ \psi_{jk}(\tau)\phi \in \operatorname{Eig}(k), \end{gathered} \tag{6.113}$$

where $\epsilon > 0$ is independent of τ. The coefficients in this expansion extend to meromorphic matrices:

$$\psi_{jk} : \widehat{Z} \setminus D_{jk} \longrightarrow \operatorname{Hom}(\operatorname{Eig}(j), \operatorname{Eig}(k))$$

with $D_{jk} \subset \widehat{Z}$ the discrete set of poles.

After an appropriate normalization (to give unitarity) these ψ_{jk} become the scattering matrices for the Laplacian, or for the metric.

Proof: The existence and uniqueness of the solution to (6.112) has already been noted. For $\tau \in \mathbb{C} \setminus [0, \infty)$ in a small enough neighbourhood of σ_j^2 the only indicial roots of $\varDelta - \tau$ corresponding to L^2 are the $-iR_k(\tau)$ for $\sigma_k^2 \le \sigma_j^2$. Thus the expansion (6.113) follows from Proposition 5.59. Since the resolvent is holomorphic in this region it follows the the ψ_{jk} are also holomorphic in $\mathbb{C} \setminus [0, \infty)$ near σ_j^2. It remains then to show that they extend to be meromorphic functions on $\widehat{Z}$.

This is a consequence of the meromorphy of the resolvent on $\widehat{Z}$. The first step is to proceed formally. Namely, for any preassigned N there is a meromorphic solution of the formal version of (6.112):

$$\begin{gathered}(\Delta-\tau)U_N(\tau)=v_N(\tau)=O(x^N),\\ U_N(\tau)\in x^{-N}H_b^\infty(X;{}^b\Omega^{\frac12}) \text{ in } |\tau|<N\\ U_N(\tau)-U(\tau)\in H_b^0(X;{}^b\Omega^{\frac12}) \text{ for } |\tau-\sigma_j^2|<\epsilon, \tau\in\mathbb{C}\setminus[0,\infty).\end{gathered}$$

This can be constructed as in Lemma 5.44. Since the remainder term v_N is uniformly small at the boundary, the resolvent can be applied to it, using Theorem 6.24 and Theorem 5.34, for τ in a subset of $\widehat{Z}$ which becomes arbitrarily large with N. It follows that

$$U(\tau)=U_N(\tau)-(\Delta-\tau)^{-1}v_N(\tau)$$

is meromorphic in $\widehat{Z}$. Moreover the coefficients of the $x^{-iR_k(\tau)}$ can always be identified in its expansion, except for points of accidental multiplicity. Hence the ψ_{jk} are also meromorphic, as stated.

Once again essentially identical results hold for the Laplacian acting on forms and the Dirac Laplacian on the spinor bundle.

Let us consider further the behaviour near $\tau=0$ of the $U(\tau)$ corresponding to the zero eigenspace of Δ_∂. Set $\tau=z^2$ with z the local parameter on $\widehat{Z}$ (really just $\widehat{Z}_0$ locally) with $\operatorname{Im} z>0$ being the physical region. Then for an orthonormal basis $\phi_j\in\operatorname{null}(\Delta_\partial)$

$$U_j(z)=\frac12 x^{iz}\phi_j+\frac12\sum_k x^{-iz}\psi_{jk}(z)\phi_k+U_j'(z),\ |U_j'(z)|=O(x^\epsilon) \tag{6.114}$$

near $z=0$ with $\epsilon>0$. For small real $z\neq 0$ the dimension of the space of these solution is exactly $\dim\operatorname{null}(\Delta_\partial)$, i.e. they span $F'(P-z^2,0)$. It follows from the vanishing of the boundary pairing on $F'(P-z^2,0)$ that $\psi(z)$ is unitary for small real z. The uniqueness shows that

$$\psi(-z)=\psi(z)^{-1}=\psi(z)^*. \tag{6.115}$$

Thus $\psi(z)$ is regular at $z=0$ and $\psi(0)^2=\operatorname{Id}$. The $+1$ eigenspace is therefore the boundary data of the solutions in $\mathcal{C}^\infty(X;E)+x^\epsilon H_b^\infty(X;E)$, just those appearing in (6.94). Thus we conclude:

PROPOSITION 6.32. *Under the assumptions Proposition 6.27 on P, the leading part, F_0, of the spectral measure can be written in terms of the solutions in* (6.114) *as*

$$F_0v=\sum_{jk}r^{-\frac12}a_{jk}(r^{\frac12})U_j(r)\langle v,U_k(r)\rangle \text{ near } r=0, \tag{6.116}$$

where $a_{jk}(s) \in \mathcal{C}^\infty([0,\epsilon)$ *is a self-adjoint matrix and*

$$u \longmapsto \sum_{jk} a_{jk}(0)\phi_j\langle u, \phi_k\rangle$$

is the projection onto the subspace of the null space of P_∂ *spanned by the boundary values of the null space of* P *on* $\mathcal{C}^\infty(X;E) + L^2(X;E)$.

This representation of the leading part of the continuous spectrum will be used in §7.8 to analyze the long-time behaviour of the trace of the heat kernel when Δ_∂ is not invertible.

Chapter 7. Heat calculus

To carry through the proof of the APS index theorem as outlined in the introduction a reasonably good understanding of the heat kernel,

$$\exp(-tP) \text{ for } P \in \mathrm{Diff}_b^2(X;E),\ \ P \geq 0,$$

is needed. In particular the cases $P = \eth^-\eth^+$, $P = \eth^+\eth^-$ are important. The analysis of these kernels will start with the case $\partial X = \emptyset$, which is very standard. However the use of blow-up techniques to define a space (the heat space) on which the heat kernel is quite simple is not so usual, although philosophically it is just a slight extension of Hadamard's method. This blow-up approach is very much in the same style as the treatment of the b-calculus and therefore has the advantage that it generalizes very readily to the case of the heat kernel for a b-metric, which is the important case for the APS theorem. For other generalizations of this approach to the heat kernel see [57] and [30].

7.1. Heat space.

Let $P \in \mathrm{Diff}^2(X;\Omega^{\frac{1}{2}})$ be elliptic with positive principal symbol on a compact manifold without boundary. The heat kernel for P can be viewed as an operator

$$H_P \colon \mathcal{C}^\infty(X;\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty([0,\infty)\times X;\Omega^{\frac{1}{2}}),$$

which gives the unique solution to

$$(7.1)\qquad \begin{aligned}(\frac{\partial}{\partial t}+P)u(t,\cdot) &= 0 \text{ in } [0,\infty)\times X\\ u(0,\cdot) &= v \text{ in } X\end{aligned} \quad \Longleftrightarrow u = H_P v.$$

Here the natural half-density on the line, $|dt|^{\frac{1}{2}}$ has been used to identify the pull-back of $\Omega^{\frac{1}{2}}(X)$ to $[0,\infty)\times X$ with $\Omega^{\frac{1}{2}}([0,\infty)\times X)$. The kernel, h, of H_P can therefore be identified as a distribution

$$h \in \mathcal{C}^{-\infty}([0,\infty)\times X\times X;\Omega^{\frac{1}{2}}).$$

At least formally the conditions (7.1) can be transcribed to

$$(7.2)\qquad \begin{aligned}(\partial_t + P(x,D_x))h(t,x,x') &= 0\\ h(0,x,x') &= \delta(x-x').\end{aligned}$$

To show the existence, and properties, of the heat kernel an approximate solution to (7.2), i.e. a forward parametrix for $\partial/\partial t + P$, will be constructed and then used in an iterative procedure. The *model* case is simply the Laplacian on $\mathbb{R}^n$, for which the heat kernel, with density factor removed, is well known:

$$h_0(t,x,x') = \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-x'|^2}{4t}\right). \tag{7.3}$$

EXERCISE 7.1. Derive (7.3) using the Fourier transform.

As part of the usual quest to understand Schwartz kernels consider the sense in which (7.3) is a $\mathcal{C}^\infty$ function, i.e. its natural homogeneity. It is, of course, only a function of $x-x'$ and t, but more importantly it is a simple function of $t^{\frac{1}{2}}$ and $(x-x')/t^{\frac{1}{2}}$. As will be seen below it is also useful to think of it as a power of $t^{\frac{1}{2}}$ times a smooth function of $|x-x'|^2/t$. In the space $[0,\infty)_t \times \mathbb{R}^n_{x-x'}$ a *parabolic blow-up* of the origin will be performed, by introducing as polar coordinates

$$\begin{gathered} x-x' = r\omega',\ t = r^2\omega_0 \\ \omega = (\omega_0,\omega') \in \mathbb{S}^n_H \stackrel{\text{def}}{=} \left\{\omega \in \mathbb{R}^{n+1}; \omega_0 \geq 0,\ \omega_0^2 + |\omega'|^4 = 1\right\}. \end{gathered} \tag{7.4}$$

Thus if $X = \mathbb{R}^n$,

$$X^2_H = [0,\infty) \times \mathbb{S}^n_H \times \mathbb{R}^n \tag{7.5}$$

and the parabolic blow-down map is

$$\beta_H \colon X^2_H \ni (r,\omega,y) \longmapsto (r^2\omega_0, r\omega' + y, y) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n. \tag{7.6}$$

It is quite straightforward to give a general description of this process of the parabolic blow-up of a submanifold, in this case

$$\begin{gathered} B_H = \{(0,x,x) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n\} = \{0\} \times \Delta, \\ \Delta = \left\{(x,x) \in \mathbb{R}^{2n}; x \in \mathbb{R}^n\right\}, \end{gathered}$$

where the parabolic directions (in this case the span of dt) form a subbundle of the conormal bundle to the submanifold. The interested reader may consult [32] and [31]. Here only the very special cases needed will be introduced. The new manifold, X^2_H, in (7.5) is a manifold with corners which has two boundary hypersurfaces, since it is the product of two manifolds with boundary. The boundary hypersurfaces will be denoted $\mathrm{tb}(X^2_H)$ and $\mathrm{tf}(X^2_H)$. The first of these is just the lift of the old boundary $\{t=0\}$ of $[0,\infty) \times X$:

$$\mathrm{tb} = \mathrm{cl}_{X^2_H}\, \beta_H^{-1}\left[\{t=0\}\backslash B_H\right].$$

This is the *temporal boundary.* The other boundary hypersurface is the front face for the blow-up, i.e. the submanifold on which the blow-down map does not have invertible differential. In this case

$$\mathrm{tf}(X_H^2) = \{r = 0\} = \beta_H^{-1}(B_H)$$

will be called the *temporal front face,* if absolutely necessary. Defining functions for these two boundary hypersurfaces will be denoted ρ_{tb} (for example ω_0) and ρ_{tf} (for instance r). Notice that the lift of t is not a defining function for either, rather it is of the form

$$\beta_H^* t = \omega_0 r^2 = \rho_{\mathrm{tb}} \rho_{\mathrm{tf}}^2.$$

This quickly leads to:

LEMMA 7.2. *Lifted to the manifold with corners X_H^2, for $X = \mathbb{R}^n$, the heat kernel* (7.3) *is of the form*

$$h_0 \in \rho_{\mathrm{tf}}^{-n} \mathcal{C}^\infty(X_H^2),\ h_0 \equiv 0 \ \text{ at } \ \mathrm{tb}\,. \tag{7.7}$$

PROOF: Lifting (7.3) gives, explicitly,

$$\beta_H^* h_0 = r^{-n} \times \frac{1}{(2\pi\omega_0)^{\frac{n}{2}}} \exp\left(-\frac{|\omega'|^2}{4\omega_0}\right). \tag{7.8}$$

This reduces to (7.7) if it is shown that the coefficient of r^{-n} is $\mathcal{C}^\infty$ on $\mathbb{S}_H^n$. However this is immediate where $\omega_0 \neq 0$ and as $\omega_0 \downarrow 0$ the exponential vanishes rapidly (since $|\omega'| \longrightarrow 1$).

It is useful to consider the lifts to X_H^2 in (7.5) of vector fields tangent to the manifold, B_H, blown up to construct it.

LEMMA 7.3. *Under the blow-down map* (7.6) *the vector fields on $\mathbb{R}^{2n}$ tangent to the diagonal, $t\partial/\partial t$ and the vector fields of the form $f\partial/\partial t$, where $f \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ vanishes to second order at $\Delta \subset \mathbb{R}^{2n}$, lift to be smooth on X_H^2 and, over $\mathcal{C}^\infty(X_H^2)$, span the space of smooth vector fields tangent to the boundary.*

PROOF: The vector fields in question are spanned by

$$(x_j - x_j')\frac{\partial}{\partial x_k},\ \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k'},\ t\frac{\partial}{\partial t},\ \text{and}\ (x_j - x_j')(x_k - x_k')\frac{\partial}{\partial t} \tag{7.9}$$

over $\mathcal{C}^\infty([0,\infty) \times \mathbb{R}^{2n}$. Consider the lifts of these generating vector fields to X_H^2, initially away from $\mathrm{tf}(X_H^2)$. All of them are homogeneous of degree 0 under the transformation $t \longmapsto r^2 t$, $x \longmapsto x + r(x - x')$ with x' unchanged. Thus, in terms of the parabolic polar coordinates (7.4), they must all lift to be of the form

$$ar\frac{\partial}{\partial r} + v, \tag{7.10}$$

where $a \in \mathcal{C}^\infty(\mathbb{S}^n_H \times \mathbb{R}^n)$ and v is a $\mathcal{C}^\infty$ vector field on $\mathbb{S}^n_H \times \mathbb{R}^n$. Thus all the lifts are smooth up to $r = 0$, i.e. on X^2_H. The vector field $r\partial/\partial r$ is the lift of $2t\partial/\partial t + (x - x') \cdot \partial/\partial x$, so it is in the span of the lifts of the vector fields in (7.9). Moreover the v's in (7.10) must span all the smooth vector fields on $\mathbb{S}^n_H \times \mathbb{R}^n$, and tangent to the boundary, since they do so in $r > 0$. This proves the lemma.

The brave idea, due to Hadamard in this sort of generality, is to do the same thing in the variable coefficient case. The heat kernel is the inverse of an operator of order 2, and so should be thought of as having order -2. Thus (7.8) is the model for an element of the heat calculus of order -2. Proceeding in a deliberate fashion, the objective is to define the space of heat pseudodifferential operators of order k, by

(7.11)

$$\Psi^k_H(X;\Omega^{\frac12}) = \left\{K \in \rho_{\mathrm{tf}}^{-\frac12(n+3)-k}\mathcal{C}^\infty(X^2_H;\Omega^{\frac12}); K \equiv 0 \text{ at } \mathrm{tb}\right\},\ k < 0.$$

Here X^2_H needs to be defined first, of course. The apparent change in power of r between (7.8) and (7.11) arises from the density factor which has been suppressed. Lifting to X^2_H :

$$|dtdxdx'| = (\beta_H)_*\left[r^{(n+1)}|drd\omega dx'|\right]. \tag{7.12}$$

This means that (7.8) should really be written:

$$\beta^*_H(h_0|dtdxdx'|^{\frac12}) = r^{-\frac12(n+3)+2} \times \frac{1}{(2\pi\omega_0)^{\frac{n}{2}}}\exp\left(-\frac{|\omega'|^2}{4\omega_0}\right)|drd\omega dx'|^{\frac12}$$

and it is then consistent with (7.11).

To define X^2_H as a set it is only necessary to proceed in the obvious way from the point of view of blowing up:

$$\begin{aligned} X^2_H &= \left[([0,\infty) \times X^2)\backslash B_H\right] \sqcup \left[\mathrm{tf}(X^2_H)\right] \\ B_H &= \{0\} \times \Delta,\ \Delta = \{(x,x) \in X^2; X \in X) \cong X. \end{aligned} \tag{7.13}$$

What should the temporal front face, tf, be in general? From the experience gathered during the construction of X^2_b, it should be some sort of spherical normal bundle to B_H but in this case should have parabolic homogeneity in t.

Consider the curves

$$\chi\colon [0,\epsilon] \longrightarrow [0,\infty) \times X^2,\ \epsilon > 0 \tag{7.14}$$

such that

$$\chi(s) \in B_H \Longleftrightarrow s = 0 \text{ and } \chi^*(t) = O(s^2). \tag{7.15}$$

Then impose the equivalence relation

$$\chi \sim \widetilde{\chi} \Longleftrightarrow \begin{cases} \chi(0) = \widetilde{\chi}(0), \\ (\chi^* - (\widetilde{\chi})^*)f = O(s^2)\ \forall\ f \in \mathcal{C}^\infty(X^2) \text{ s.t. } f_{\restriction\Delta} = 0, \\ (\chi^* - (\widetilde{\chi})^*)t = O(s^3). \end{cases} \tag{7.16}$$

LEMMA 7.4. *The equivalence classes* $[\chi]$ *of curves* (7.14)*, satisfying* (7.15)*, under* (7.16)*, form a space with a natural additive structure and* $\mathbb{R}^+$*-action (but not a vector space structure) on the sets where* $\chi(0)$ *takes a fixed value.*

PROOF: The sum of two curves χ_1, χ_2 satisfying (7.15) with $\chi_1(0) = \chi_2(0)$ is a curve χ also satisfying (7.15), such that $\chi(0) = \chi_1(0)$ and

$$\begin{gathered} (\chi^* - \chi_1^* - \chi_2^*)f = O(s^2)\ \forall\ f \in \mathcal{C}^\infty([0,\infty)\times X^2),\ f = 0 \text{ on } B_H \\ (\chi^* - \chi_1^* - \chi_2^*)t = O(s^3). \end{gathered} \tag{7.17}$$

Clearly the equivalence class of χ is determined by (7.17). To see that such a curve exists consider local coordinates x near $p \in X$ where $\chi_1(0) = (0,p,p) \in B$. Then a curve satisfying (7.15), with $\chi(0) = (0,p,p)$, p being the origin of the coordinates, is of the form

$$\chi_i(s) = (s^2\alpha_i(s), s\gamma_i(s), s\widetilde{\gamma}_i(s)), i = 1, 2.$$

Thus a curve χ with $[\chi] = [\chi_1] + [\chi_2]$ in the sense of (7.17) only needs to satisfy

$$\begin{gathered} \chi(s) = (s^2\alpha(s), s\gamma(s), s\widetilde{\gamma}(s)), \text{ with} \\ \alpha(0) = \alpha_1(0) + \alpha_2(0),\ \gamma(0) = \gamma_1(0) + \gamma_2(0) \text{ and } \widetilde{\gamma}(0) = \widetilde{\gamma}_1(0) + \widetilde{\gamma}_2(0). \end{gathered} \tag{7.18}$$

Notice, from (7.16), that the equivalence class of the curve χ in (7.18), amongst curves with the same end point determines, and is determined by

$$[\chi] \longleftrightarrow (\alpha(0), \gamma(0) - \widetilde{\gamma}(0)) \in [0,\infty) \times \mathbb{R}^n,\ \chi(0) \in B_H \text{ fixed}. \tag{7.19}$$

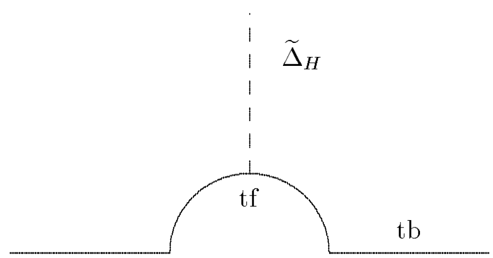

Figure 12. The heat space, X^2_H.

The $\mathbb{R}^+$-action is the obvious one:

$$a[\chi] = [\widetilde{\chi}],\ \widetilde{\chi}(s) = \chi(as) \text{ near } s = 0, a \in \mathbb{R}^+.$$

Under the local isomorphism (7.19) the $\mathbb{R}^+$-action becomes

$$a[\chi] \longleftrightarrow (a^2\alpha(0), a(\gamma(0) - \widetilde{\gamma}(0))),\ [\chi] \longleftrightarrow (\alpha(0), \gamma(0) - \widetilde{\gamma}(0)).$$

This makes it clear why the fibre over $q \in B_H$, consisting of the equivalence classes of curves with $\chi(0) = q$, is *not* a vector space. Namely the $\mathbb{R}^+$ action does not distribute over addition.

So now *define* the front face of X^2_H (the inward-pointing t-parabolic spherical normal bundle of B_H in $[0, \infty) \times X^2$) as the quotient by the $\mathbb{R}^+$-action:

$$\mathrm{tf}(X^2_H) \stackrel{\mathrm{def}}{=} \{[\chi] \neq 0\}/\mathbb{R}^+.$$

Thus $\mathrm{tf}(X^2_H)$ is a bundle over B

$$\mathrm{tf}(X^2_H) \longrightarrow B_H,\ [\chi] \longmapsto \chi(0),$$

with fibre isomorphic to $\mathbb{S}^n_H$, given in (7.4). This is diffeomorphically a half-sphere, i.e. a closed n-ball. The additive structure and $\mathbb{R}^+$-action of Lemma 7.2 give a linear structure to the interiors of these balls, since

(7.20)
$$q \in \mathrm{int}(\mathrm{tf}_p(X^2_H)) \Longleftrightarrow q \sim [\chi],\ \chi \text{ in } (7.15), \chi(0) = p, \chi^*t = s^2 + O(s^3).$$

Then the linear structure arises from that in (7.19) on the $\gamma(0) - \widetilde{\gamma}(0)$, where $\alpha(0) = 1$.

EXERCISE 7.5. Show that the $\mathcal{C}^\infty$ structure on $\mathrm{tf}(X^2_H)$ arises from the compactification of the vector bundle, being the interior of the fibres with the linear structure just described, with a linear space compactified in linear coordinates by

$$\{x \in \mathbb{R}^n; |x| \ge 1\} \ni x \longrightarrow (|x|^{-2}, \frac{x}{|x|}) \subset [0,1) \times \mathbb{S}^{n-1}.$$

EXERCISE 7.6. Show that, as a vector bundle, the interior of $\mathrm{tf}(X^2_H)$ is *canonically isomorphic* to the normal bundle to the diagonal in X^2, hence is canonically isomorphic to the tangent bundle to X. [Hint: The isomorphism is given by (7.20).]

Now that $\mathrm{tf}(X^2_H)$ has been defined, the heat space of X, X^2_H, is defined by (7.13). It still needs to be shown that it has a natural $\mathcal{C}^\infty$ structure as a manifold with corners such that $\mathrm{tf}(X^2_H)$ is a boundary hypersurface.

PROPOSITION 7.7. *There is a unique $\mathcal{C}^\infty$ structure on X^2_H in which $\mathrm{tf}(X^2_H)$ is a boundary hypersurface, the $\mathcal{C}^\infty$ structure induced on $([0,\infty)\times X^2)\backslash B_H$ by (7.13) is the same as that induced from $[0,\infty)\times X^2$ and for any coordinate patch, in $X, \beta_H^{-1}([0,\infty)\times O^2)$ is diffeomorphic to the preimage of $[0,\infty)\times O^2$ in (7.5) under (7.6).*

PROOF: The $\mathcal{C}^\infty$ structure is certainly fixed away from $\mathrm{tf}(X^2_H)$ as that inherited from $[0,\infty)\times X^2$. Moreover the sets $\beta_H^{-1}([0,\infty)\times O^2)$ clearly cover a neighbourhood of $\mathrm{tf}(X^2_H)$, so it suffices to show that the transition functions on $[0,\infty)\times \mathbb{S}^n_H \times \mathbb{R}^n$ are $\mathcal{C}^\infty$, i.e. the transformation induced on (7.5) by a change of coordinates on $\mathbb{R}^n$ is $\mathcal{C}^\infty$. Let $y = f(x)$ be a local diffeomorphism on $\mathbb{R}^n$, so $(y,y') = (f(x), f(x'))$ is the transformation on $\mathbb{R}^n\times\mathbb{R}^n$. The transformation induced on $[0,\infty)\times \mathbb{S}^n_H\times\mathbb{R}^n$ is, by (7.6),

$$(r,\omega,x) \longmapsto (\rho, w, y)$$

$$y = f(x), \rho = (t^2 + |y-y'|^4)^{\frac14},\ w = (\frac{t}{\rho^2}, \frac{y-y'}{\rho}) \tag{7.21}$$
$$r = (t^2+|x-x'|^4)^{\frac14},\ \omega = (\frac{t}{r^2}, \frac{x-x'}{r}).$$

Now, $y - y' = (x-x')\cdot G(x,x')$ with G a $\mathcal{C}^\infty$matrix and $\det G \neq 0$, for x near x' which may be assumed. Thus

$$\rho = r\left(\frac{t^2+|y-y'|^4}{t^2+|x-x'|^4}\right)^{\frac14} = r\left(\omega_0^2 + |\omega'\cdot G|^4\right)^{\frac14} \tag{7.22}$$

since $\omega_0^2 + |\omega'|^4 = 1$ this is $\mathcal{C}^\infty$. Similarly

$$w = \left(\omega_0(\frac{r}{\rho})^2, \omega'\cdot G(x,x')(\frac{r}{\rho})\right) \text{ is } \mathcal{C}^\infty.$$

Thus the transition functions are smooth and the proposition is proved.

In brief the $\mathcal{C}^\infty$ structure near tf is generated by the functions which are homogeneous of non-negative integral order under

$$(t, x, x') \longrightarrow (a^2 t, x, x + a(x' - x)),\ a > 0.$$

It turns out that there is an important subspace in the space $\mathcal{C}^\infty(X^2_H)$ of all $\mathcal{C}^\infty$ functions on X^2_H, just defined. Consider the lift under (7.6) of a $\mathcal{C}^\infty$ function on $[0,\infty) \times X^2$, $X = \mathbb{R}^n$,

$$g = \beta_H^* f = f(r^2\omega_0, r\omega' + x', x'). \tag{7.23}$$

The Taylor series expansion of g at $\mathrm{tf}(X^2_H)$ is of the form

$$g(r, \omega, x') \sim \sum_{j=0}^{\infty} r^{2j} g'_j(\omega_0, \omega', x') + \sum_{j=0}^{\infty} r^{1+2j} g''_j(\omega_0, \omega', x') \tag{7.24}$$

and from (7.23) it can be seen that the coefficients satisfy

$$\begin{aligned} g'_j(\omega_0, -\omega', x') &= g'_j(\omega_0, \omega', x') \\ g''_j(\omega_0, -\omega', x') &= -g''_j(\omega_0, \omega', x'), \end{aligned} \tag{7.25}$$

i.e. are respectively even and odd under the involution $\omega' \longmapsto -\omega'$. For $X = \mathbb{R}^n$, let $\mathcal{C}^\infty_{\mathrm{evn}}(X^2_H)$ denote the subspace of $\mathcal{C}^\infty(X^2_H)$ the elements of which have Taylor series at $\mathrm{tf}(X^2_H)$ as in (7.24) and (7.25). Similarly let $\mathcal{C}^\infty_{\mathrm{odd}}(X^2_H) \subset \mathcal{C}^\infty(X^2_H)$ denote the subspace which has Taylor series (7.24) with coefficients satisfying the opposite condition:

$$\begin{aligned} g'_j(\omega_0, -\omega', x') &= -g'_j(\omega_0, \omega', x') \\ g''_j(\omega_0, -\omega', x') &= g''_j(\omega_0, \omega', x'). \end{aligned} \tag{7.26}$$

Thus (7.23) shows that

$$\beta_H^* \mathcal{C}^\infty([0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n) \subset \mathcal{C}^\infty_{\mathrm{evn}}(X^2_H). \tag{7.27}$$

Moreover observe from (7.21) and (7.22) that the new coordinates on X^2_H induced by a change of coordinates on $\mathbb{R}^n$ satisfy

$$\rho \in \mathcal{C}^\infty_{\mathrm{odd}}(X^2_H), w_0 \in \mathcal{C}^\infty_{\mathrm{evn}}(X^2_H), w' \in \mathcal{C}^\infty_{\mathrm{odd}}(X^2_H), y \in \mathcal{C}^\infty_{\mathrm{evn}}(X^2_H)$$

as do the old coordinates. Thus the spaces are actually defined independently of coordinates, and hence on X^2_H for a general manifold X.

LEMMA 7.8. *For any compact manifold without boundary, X, the subspaces of $\mathcal{C}^\infty_{\text{evn}}(X^2_H)$, $\mathcal{C}^\infty_{\text{odd}}(X^2_H) \subset \mathcal{C}^\infty(X^2_H)$ with Taylor series at $\text{tf}(X^2_H)$, as in (7.23), satisfying (7.25) and (7.26) respectively, are well defined independent of coordinates and*

$$\begin{aligned}\mathcal{C}^\infty_{\text{evn}}(X^2_H) + \mathcal{C}^\infty_{\text{odd}}(X^2_H) &= \mathcal{C}^\infty(X^2_H)\\ \mathcal{C}^\infty_{\text{evn}}(X^2_H) \cap \mathcal{C}^\infty_{\text{odd}}(X^2_H) &= \rho_{\text{tf}}^\infty \mathcal{C}^\infty(X^2_H).\end{aligned} \tag{7.28}$$

PROOF: The first part has already been proved and (7.28) arises by splitting the terms in the Taylor series into ω'-odd and even parts.

In view of (7.27), which of course extends to the general case, the spaces of odd and even sections of the lift to X^2_H of any vector bundle over X^2 are also defined by using the lift from X^2 of a smooth basis of the bundle. More formally if E is a $\mathcal{C}^\infty$ vector bundle over $[0,\infty) \times X^2$ then the space of even sections can be written

$$\mathcal{C}^\infty_{\text{evn}}(X^2_H; \beta^*_H E) = \mathcal{C}^\infty_{\text{evn}}(X^2_H) \otimes_{\beta^*_H \mathcal{C}^\infty([0,\infty)\times X^2)} \beta^*_H \mathcal{C}^\infty([0,\infty) \times X^2; E). \tag{7.29}$$

EXERCISE 7.9. Make sure you understand exactly what (7.29) means.

Consider the half-density bundle on X^2_H. This is *not* the lift of a bundle on $[0,\infty) \times X^2$ (because of the extra factors of ρ_{tf}.) However choosing $\rho_{\text{tf}} \in \mathcal{C}^\infty_{\text{evn}}(X^2_H)$ the spaces of even and odd sections are *defined* to be

$$\begin{aligned}\mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12}) = \Big\{ & u \in \mathcal{C}^\infty(X^2_H; \Omega^{\frac12}); u = \rho_{\text{tf}}^{-\frac12(n+1)} u'\nu,\\ & u' \in \mathcal{C}^\infty_{\text{evn}}(X^2_H), 0 \neq (\beta_H)_* \nu \in \mathcal{C}^\infty([0,\infty) \times X^2; \Omega^{\frac12}) \Big\}\\ \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}) = \Big\{ & u \in \mathcal{C}^\infty(X^2_H; \Omega^{\frac12}); u = \rho_{\text{tf}}^{-\frac12(n+1)} u'\nu,\\ & u' \in \mathcal{C}^\infty_{\text{odd}}(X^2_H), 0 \neq (\beta_H)_* \nu \in \mathcal{C}^\infty([0,\infty) \times X^2; \Omega^{\frac12}) \Big\}.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{C}^\infty_{\text{evn}}(X^2_H) \cdot \mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12}) &\subset \mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12}),\\ \mathcal{C}^\infty_{\text{evn}}(X^2_H) \cdot \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}) &\subset \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}),\\ \mathcal{C}^\infty_{\text{odd}}(X^2_H) \cdot \mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12}) &\subset \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}),\\ \mathcal{C}^\infty_{\text{odd}}(X^2_H) \cdot \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}) &\subset \mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12})\end{aligned}$$

and the analogues of (7.28) hold:

$$\begin{aligned}\mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12}) + \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}) &= \mathcal{C}^\infty(X^2_H; \Omega^{\frac12})\\ \mathcal{C}^\infty_{\text{evn}}(X^2_H; \Omega^{\frac12}) \cap \mathcal{C}^\infty_{\text{odd}}(X^2_H; \Omega^{\frac12}) &= \rho_{\text{tf}}^\infty \mathcal{C}^\infty(X^2_H; \Omega^{\frac12}).\end{aligned}$$

In particular this leads to the spaces of even and odd heat pseudodifferential operators defined by

(7.30)
$$\begin{gathered}\Psi^k_{H,\mathrm{evn}}(X;\Omega^{\frac12}) = \left\{K \in \rho_{\mathrm{tf}}^{-\frac12(n+3)-k}\mathcal{C}^\infty_{\mathrm{evn}}(X^2_H;\Omega^{\frac12}); K \equiv 0 \text{ at } \mathrm{tb}\right\}\\ \Psi^k_{H,\mathrm{odd}}(X;\Omega^{\frac12}) = \left\{K \in \rho_{\mathrm{tf}}^{-\frac12(n+3)-k}\mathcal{C}^\infty_{\mathrm{odd}}(X^2_H;\Omega^{\frac12}); K \equiv 0 \text{ at } \mathrm{tb}\right\},\\ \text{for } k<0 \text{ even}\end{gathered}$$

and

(7.31)
$$\begin{gathered}\Psi^k_{H,\mathrm{evn}}(X;\Omega^{\frac12}) = \left\{K \in \rho_{\mathrm{tf}}^{-\frac12(n+3)-k}\mathcal{C}^\infty_{\mathrm{odd}}(X^2_H;\Omega^{\frac12}); K \equiv 0 \text{ at } \mathrm{tb}\right\}\\ \Psi^k_{H,\mathrm{odd}}(X;\Omega^{\frac12}) = \left\{K \in \rho_{\mathrm{tf}}^{-\frac12(n+3)-k}\mathcal{C}^\infty_{\mathrm{evn}}(X^2_H;\Omega^{\frac12}); K \equiv 0 \text{ at } \mathrm{tb}\right\},\\ \text{for } k<0 \text{ odd}\end{gathered}$$

in place of (7.11). In practice the even operators are the interesting ones.

7.2. Standard heat calculus.

Now the definition of (7.11) is complete, i.e. the elements of the heat calculus of negative integral order have been fixed. Naturally they are intended to be operators. Their action is simple enough since as distributions they are locally integrable.

LEMMA 7.10. *If $k \in -\mathbb{N}$, each $A \in \Psi^k_H(X;\Omega^{\frac12})$ pushes forward under β_H to a locally integrable half-density on $[0,\infty)\times X^2$.*

PROOF: It suffices to assume that A has support in the preimage of a coordinate patch and then sum using a partition of unity. Of course there is nothing to be proved away from B_H since the kernels are smooth there. Taking account of the factors of r in (7.12) the push-forward of A to $[0,\infty)\times X^2$ is locally of the form

$$t^{-\frac12 n-1-\frac12 k}A'|dtdxdx'|^{\frac12} \text{ with } A' \text{ bounded.} \tag{7.32}$$

Since A' vanishes to infinite order at tb the local integrability of the coefficient in (7.32) reduces to the local integrability, on $[0,\infty)\times X^2$, of the function which lifts to $\rho_{\mathrm{tf}}^{-n-2-k}$. Clearly this is locally integrable provided $k<0$.

A locally integrable function, or half-density, has a unique extension from $[0,\infty)\times X^2$ to $\mathbb{R}\times X^2$ which is locally integrable and vanishes in $t<0$. With this extension Lemma 7.10 shows that the elements $A \in \Psi^k_H(X;\Omega^{\frac12})$,

$k \in -\mathbb{N}$, define operators from $\mathcal{C}^\infty(X;\Omega^{\frac12})$ to $\mathcal{C}^{-\infty}(\mathbb{R}\times X;\Omega^{\frac12})$; clearly Au always vanishes in $t<0$. It is generally preferable to think of these heat pseudodifferential operators as acting through

$$A \in \Psi^k_H(X;\Omega^{\frac12}) \Longrightarrow A\colon \mathcal{C}^\infty(X;\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}([0,\infty)\times X;\Omega^{\frac12}), \tag{7.33}$$

by restriction. This does not result in any loss of information, since the kernel is locally integrable. In fact there is much more regularity than (7.33).

LEMMA 7.11. *Each element $A \in \Psi^k_H(X;\Omega^{\frac12})$, $k\in -\mathbb{N}$, defines a continuous operator*

$$A\colon \mathcal{C}^\infty(X;\Omega^{\frac12}) \longrightarrow t^{-\frac{k}{2}-1}\mathcal{C}^\infty([0,\infty)_{\frac12}\times X;\Omega^{\frac12}), \tag{7.34}$$

where $[0,\infty)_{\frac12}$ is the half-line with $t^{\frac12}$ as coordinate. An element $A \in \Psi^k_H(X;\Omega^{\frac12})$ is in the even part of the calculus, $A\in\Psi^k_{H,\mathrm{evn}}(X;\Omega^{\frac12})$, if and only if (7.34) actually gives a map

$$A\colon \mathcal{C}^\infty(X;\Omega^{\frac12}) \longrightarrow t^{[-\frac{k}{2}]-1}\mathcal{C}^\infty([0,\infty)\times X;\Omega^{\frac12}), \tag{7.35}$$

where $[r]$ is the integer part of r, the largest integer no larger than r.

PROOF: Using a $\mathcal{C}^\infty$ partition of unity on $[0,\infty)\times X^2$ it may be assumed that the kernel of A has small support on $[0,\infty)\times X^2$. Certainly, since the kernel is $\mathcal{C}^\infty$ and vanishes to all orders at $t=0$, any term away from $\mathrm{tf}(X^2_H)$ satisfies (7.34), for any k. Thus it can even be assumed that A has support in the preimage of a coordinate patch. Then

$$A = r^{-(n+2)-k}A'(r,\omega,x)|dtdx'dx|^{\frac12}$$

where, by assumption, A' vanishes to infinite order at $\omega_0=0$ and is $\mathcal{C}^\infty$. So if the projective coordinates

$$t^{\frac12}, X = \frac{x-x'}{t^{\frac12}}, x', \tag{7.36}$$

which are valid everywhere *except* $\omega_0=0$ (corresponding to $X=\infty$), are introduced it follows that

(7.37)
$$r^{-(n+2)-k}A'(r,\omega,x) = t^{-\frac12(n+2)-\frac12 k}\alpha(t^{\frac12},X,x), \text{ with } \alpha \text{ a } \mathcal{C}^\infty \text{ function}$$
$$\text{rapidly decreasing with all derivatives as } |X|\longrightarrow\infty.$$

The push-forward of the kernel to $\mathbb{R} \times X^2$ is given by

$$\begin{aligned} K_A(\phi) &= \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \alpha(t^{\frac{1}{2}}, \frac{x-x'}{t^{\frac{1}{2}}}, x)\phi(t,x,x')t^{-\frac{1}{2}(n+2)-\frac{1}{2}k}dtdxdx' \\ &= \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \alpha(t^{\frac{1}{2}}, X, x)\phi(t,x,x-t^{\frac{1}{2}}X)t^{-\frac{k}{2}-1}dtdXdx, \end{aligned}$$

where $\phi(t,x,x')|dtdxdx'|^{\frac{1}{2}}$ is a $\mathcal{C}^\infty$ half-density of compact support. Thus the action of A as an operator is just

$$A(\psi) = t^{-\frac{k}{2}-1} \int\limits_{\mathbb{R}^n} \alpha(t^{\frac{1}{2}}, X, x)\psi(x - t^{\frac{1}{2}}X)dX. \tag{7.38}$$

The rapid decrease of α as $|X| \longrightarrow \infty$ shows that this integral converges uniformly to $t^{-\frac{k}{2}-1}\times$ a $\mathcal{C}^\infty$ function of $t^{\frac{1}{2}}$ and x. This gives (7.34).

The decomposition of the Taylor series of the kernel at $\mathrm{tf}(X_H^2)$ corresponds to writing α as

$$\begin{aligned} \alpha(t^{\frac{1}{2}}, X, x) &= \alpha_e(t^{\frac{1}{2}}, X, x) + \alpha_o(t^{\frac{1}{2}}, X, x) + \alpha'(t^{\frac{1}{2}}, X, x) \\ \alpha_e(t^{\frac{1}{2}}, X, x) &= \alpha_{ee}(t, X, x) + t^{\frac{1}{2}}\alpha_{oe}(t, X, x) \\ \alpha_o(t^{\frac{1}{2}}, X, x) &= \alpha_{oo}(t, X, x) + t^{\frac{1}{2}}\alpha_{eo}(t, X, x), \end{aligned}$$

where $\alpha_{ee}(t,X,x)$ and $\alpha_{oe}(t,X,x)$ are even and $\alpha_{eo}(t,X,x)$ and $\alpha_{oo}(t,X,x)$ are odd under $X \longmapsto -X$. All are $\mathcal{C}^\infty$ and vanish rapidly as $|X| \longrightarrow \infty$ with all derivatives. The extra term α' vanishes to infinite order at $t = 0$. Inserting these into (7.38) one easily sees that an element of the even part of the calculus satisfies (7.35). An element of the odd part of the calculus similarly satisfies

$$A\colon \mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \longrightarrow t^{[-\frac{k}{2}]-\frac{1}{2}}\mathcal{C}^\infty([0,\infty)\times X; \Omega^{\frac{1}{2}}),\ A \in \Psi^k_{H,\mathrm{odd}}(X; \Omega^{\frac{1}{2}}).$$

Moreover the range can only be in $t^{[-\frac{k}{2}]-1}\mathcal{C}^\infty([0,\infty)\times X; \Omega^{\frac{1}{2}})$ if all the coefficients in the Taylor series at tf vanish. This proves the remainder of the lemma.

7.3. Heat kernel.

Notice in particular that if $k = -2$ then the multiplicative factor in (7.34) disappears and the restriction to $t = 0$ is well defined:

$$\begin{gathered} A \in \Psi_H^{-2}(X; \Omega^{\frac{1}{2}}) \Longrightarrow \\ A_0\colon \mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}),\ A_0\psi = (A\psi)_{\restriction t=0}. \end{gathered} \tag{7.39}$$

This allows the main result for the heat kernel to be stated:

THEOREM 7.12. *If $P \in \operatorname{Diff}^2(X;\Omega^{\frac12})$ has non-negative principal symbol and is elliptic on a compact manifold, X, without boundary then there is a unique element $H_P \in \Psi^{-2}_{H,\mathrm{evn}}(X;\Omega^{\frac12})$ satisfying*

$$(\partial_t + P)\cdot H_P = 0 \ \text{ in } t>0,\ (H_P)_{\restriction t=0} = \operatorname{Id}. \tag{7.40}$$

The proof still requires some work. First consider the initial condition, which can be examined using (7.37) and (7.38). Thus if $A \in \Psi^{-2}_H(X;\Omega^{\frac12})$ then (7.38) shows that, at least locally,

$$(A\psi)_{\restriction t=0} = \int_X \alpha(0,X,x)\psi(x)dX = \Phi(x)|dt|^{\frac12}\cdot\psi(x) \tag{7.41}$$

is actually a multiplication operator. The function Φ is just the integral, over the fibres, of the restriction of the kernel of A, on X^2_H, to the front face. To make sense of this globally requires a little thought about densities.

In fact the restriction to the temporal front face can be written

$$A_{\restriction \mathrm{tf}(X^2_H)} = A'\cdot\beta_H^*\nu,\ A' \in \dot{\mathcal{C}}^\infty(\mathrm{tf}(X^2_H)),\ 0 \neq \nu \in \mathcal{C}^\infty(B_H;\Omega^{\frac12}([0,\infty)\times X^2)).$$

Here B_H is given by (7.13) and lies above the diagonal. So as in the ordinary pseudodifferential calculus, there is an identification:

$$\nu = |dt|^{\frac12}\otimes\nu',\ \nu' \in \mathcal{C}^\infty(X;\Omega).$$

Furthermore a density on X can also be considered as a (fibre-translation invariant) fibre-density on $TX \equiv N\Delta$. Since $\mathrm{tf}(X^2_H)$ is just a compactification of the tangent bundle to X (see Exercise 7.6) this means that restriction to the temporal front face gives a map

$$\Psi^{-2}_H(X;\Omega^{\frac12}) \ni A \longmapsto A_{\restriction \mathrm{tf}(X^2_H)} \in \dot{\mathcal{C}}^\infty(\mathrm{tf}(X^2_H);\Omega_{\mathrm{fibre}}).$$

Thus integration over the fibres is invariantly defined:

LEMMA 7.13. *If $A \in \Psi^{-2}_H(X;\Omega^{\frac12})$ then A_0, defined by (7.39), is multiplication by*

$$\Phi(x)|dt|^{\frac12} = \int_{\mathrm{fibre}} A_{\restriction \mathrm{tf}(X^2_H)}. \tag{7.42}$$

The initial condition in (7.40) is therefore seen to be the requirement that on each leaf of $\mathrm{tf}(X^2_H)$ the kernel should have mean value 1. It is also necessary to arrange that $H_P v$ be a solution of the heat equation. As usual this cannot be done directly, rather a parametrix is first constructed and then iteration is used to get the precise solution. Now, $G \in \Psi_H^{-2}(X;\Omega^{\frac12})$ will be a parametrix if

$$t(\partial_t + P)\cdot G = R \in \Psi_H^{-\infty}(X;\Omega^{\frac12}) \text{ and } G_0 = \mathrm{Id}\,. \tag{7.43}$$

Notice that an element of the residual space for the heat calculus here is given by a $\mathcal{C}^\infty$ kernel on X_H^2 which vanishes to infinite order at both boundaries, tf and tb . This is just the same as a $\mathcal{C}^\infty$ function on $[0,\infty)\times X^2$ vanishing to all orders at $t=0$. Thus

$$A \in \Psi_H^{-\infty}(X;\Omega^{\frac12}) \Longleftrightarrow A \in \dot{\mathcal{C}}^\infty([0,\infty)\times X^2;\Omega^{\frac12}). \tag{7.44}$$

To arrange (7.43) the form of the operator on the left needs to be computed.

Lemma 7.14. *If $A \in \Psi_H^k(X;\Omega^{\frac12})$, for $k \in -\mathbb{N}$, then*

$$t(\partial_t + P)\cdot A \in \Psi_H^k(X;\Omega^{\frac12}) \tag{7.45}$$

for any $P \in \mathrm{Diff}^2(X;\Omega^{\frac12})$ and furthermore

$$A \in \Psi_{H,\mathrm{evn}}^k(X;\Omega^{\frac12}) \Longrightarrow t(\partial_t + P)\cdot A \in \Psi_{H,\mathrm{evn}}^k(X;\Omega^{\frac12}).$$

Proof: Recall Lemma 7.10, which defines A as an operator. Certainly

$$\begin{aligned} tP &: t^{-\frac12 k-1}\mathcal{C}^\infty([0,\infty)_{\frac12}\times X;\Omega^{\frac12}) \longrightarrow t^{-\frac12 k-1}\mathcal{C}^\infty([0,\infty)_{\frac12}\times X;\Omega^{\frac12})\\ t\partial_t &: t^{-\frac12 k-1}\mathcal{C}^\infty([0,\infty)_{\frac12}\times X;\Omega^{\frac12}) \longrightarrow t^{-\frac12 k-1}\mathcal{C}^\infty([0,\infty)_{\frac12}\times X;\Omega^{\frac12}). \end{aligned}$$

So the composite operator in (7.45) is well defined and the statement that it is an element of $\Psi_H^k(X;\Omega^{\frac12})$ is meaningful. To check its veracity it is enough to work with the local form (7.38). The local form of P,

$$P = \sum_{|\alpha|\le 2} p_\alpha(x) D_x^\alpha,$$

gives

$$\begin{aligned} tP\cdot A\psi &= t^{-\frac{k}{2}}\int P\left[\alpha(t^{\frac12},X,x)\psi(x-t^{\frac12}X)\right]dX\\ &= t^{-\frac{k}{2}-1}\int \beta(t^{\frac12},X,x)\psi(x-t^{\frac12}X)dX, \end{aligned}$$

where integration by parts in X has been used to find

$$\beta(t^{\frac{1}{2}}, X, x) = \sum_{|\alpha|=2} p_\alpha(x) D_X^\alpha \alpha(t^{\frac{1}{2}}, X, x) + t^{\frac{1}{2}} \cdot \beta', \tag{7.46}$$

with β' a $\mathcal{C}^\infty$ function. Similarly, applying $t\partial_t$ gives:

$$\begin{aligned} t\partial_t A\psi &= t^{-\frac{k}{2}-1} \int \gamma(t^{\frac{1}{2}}, X, x)\psi(x - t^{\frac{1}{2}}X)dX, \\ \gamma &= \frac{1}{2}\left\{-\partial_X(X\alpha) - (k+2)\alpha\right\} + t^{\frac{1}{2}}\gamma' \text{ with } \gamma' \ \mathcal{C}^\infty. \end{aligned} \tag{7.47}$$

All terms are rapidly decreasing as $|X| \longrightarrow \infty$, so (7.45) follows. The conservation of parity follows from the last part of Lemma 7.11 and the fact that tP and $t\partial/\partial t$ preserve the spaces on the right in (7.35).

In fact not only does this computation give (7.45) but it also results in an explicit formula for the restriction of the kernel to the front fact. Consider the restriction map (the normal operator in this context)

$$\begin{aligned} N_{H,k} : \Psi_H^k(X; \Omega^{\frac{1}{2}}) \ni A &\longmapsto \\ & t^{(k+n+2)/2} A_{|\mathrm{tf}} \in \dot{\mathcal{C}}^\infty(\mathrm{tf}(X_H^2); \Omega_{\mathrm{fibre}}),\ k \in -\mathbb{N}, \end{aligned} \tag{7.48}$$

where the normalization comes from (7.12). Certainly this map is surjective, by definition in (7.11), and its null space is clearly the space of operators of order $k-1$, i.e. there is an exact sequence

$$0 \longrightarrow \Psi_H^{k-1}(X; \Omega^{\frac{1}{2}}) \hookrightarrow \Psi_H^k(X; \Omega^{\frac{1}{2}}) \xrightarrow{N_{H,k}} \dot{\mathcal{C}}^\infty(\mathrm{tf}(X_H^2); \Omega_{\mathrm{fibre}}) \longrightarrow 0. \tag{7.49}$$

The parities in (7.30) and (7.31) are set up so that the same is true of the even part of the calculus:

$$0 \longrightarrow \Psi_{H,\mathrm{evn}}^{k-1}(X; \Omega^{\frac{1}{2}}) \hookrightarrow \Psi_{H,\mathrm{evn}}^k(X; \Omega^{\frac{1}{2}}) \xrightarrow{N_{H,k}} \dot{\mathcal{C}}_{\mathrm{evn}}^\infty(\mathrm{tf}(X_H^2); \Omega_{\mathrm{fibre}}) \longrightarrow 0 \tag{7.50}$$

where $\dot{\mathcal{C}}_{\mathrm{evn}}^\infty(\mathrm{tf}(X_H^2); \Omega_{\mathrm{fibre}}) \subset \dot{\mathcal{C}}^\infty(\mathrm{tf}(X_H^2); \Omega_{\mathrm{fibre}})$ is the subspace consisting of the elements which are invariant under the natural reflection around the origin of the fibres.

The image space in (7.49) can also be thought of as the space of $\mathcal{C}^\infty$ sections, over TX, which are rapidly vanishing at infinity. Here the identification established in Exercise 7.6 is used again. These are 'Schwartz functions,' so

$$\dot{\mathcal{C}}^\infty(\mathrm{tf}(X_H^2); \Omega^{\frac{1}{2}}(X_H^2)) \cong \mathcal{S}(TX; \Omega_{\mathrm{fibre}}) \tag{7.51}$$

is just the space of those sections over TX, of the lift of the density bundle, which vanish rapidly at infinity with all derivatives. Now from (7.46) and (7.47) the fundamental formula results.

Proposition 7.15. *If $P \in \mathrm{Diff}^2(X;\Omega^{\frac{1}{2}})$ has symbol $\sigma(P)$, thought of as a translation-invariant differential operator on the fibres of TX, then under* (7.51)

$$N_{H,k}(t(\partial_t + P)A) = [\sigma(P) - \frac{1}{2}(R+n+k+2)]N_{H,k}(A), \tag{7.52}$$

where R is the radial vector field on the fibres of TX.

Proof: It is usual to think of the symbol of a differential operator as a polynomial on T^*X, but that is completely equivalent to a constant coefficient differential operator on each fibre of TX, varying smoothly with the fibre. Then (7.46) shows that

$$N_{H,k}(tP \cdot A) = \sigma(P) \cdot N_{H,k}(A).$$

Similarly (7.47) gives the action of $t\partial_t$ as $-\frac{1}{2}[\partial_X(X) + k + 2]$, i.e. $-\frac{1}{2}(R + n + k + 2)$ as stated in (7.52).

Now we can construct a parametrix for the heat operator.

Lemma 7.16. *Under the assumptions of Theorem 7.12 the heat operator has a forward parametrix, $G \in \Psi^{-2}_{H,\mathrm{evn}}(X;\Omega^{\frac{1}{2}})$, satisfying* (7.43).

Proof: The terms in the Taylor series of G at tf will be constructed successively. The first step is to find

$$G^{(0)} \in \Psi^{-2}_H(X;\Omega^{\frac{1}{2}}) \text{ with } G^{(0)}_0 = \mathrm{Id},\ t(\partial_t + P)G^{(0)} \in \Psi^{-3}_H(X;\Omega^{\frac{1}{2}}). \tag{7.53}$$

By (7.42), (7.49) and (7.52) these are conditions only on the normal operator, *viz*

$$\begin{aligned} &[\sigma(P) - \frac{1}{2}(R+n)]N_{H,-2}(G^{(0)}) = 0 \\ &\int_{\text{fibre}} N_{H,-2}(G^{(0)}) = 1. \end{aligned} \tag{7.54}$$

These are fibre-by-fibre conditions. In fact each fibre is just $\mathbb{R}^n$, so the solution to (7.54) must be implicit in (7.3). More precisely local coordinates can always be introduced such that

$$\sigma(P) = D_1^2 + \cdots + D_n^2 \text{ on } T_xX \tag{7.55}$$

since P is assumed to have a real elliptic principal symbol. Then (7.54) becomes

$$\begin{aligned} &\left[D_1^2 + \cdots + D_n^2 - \frac{1}{2}(R+n)\right] N_{H,-2}(G^{(0)}) = 0 \\ &\int_{\text{fibre}} N_{H,-2}(G^{(0)}) = 1. \end{aligned} \tag{7.56}$$

From (7.3) it follows that

$$N_{H,-2}(G^{(0)}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{|X|^2}{4}\right) \tag{7.57}$$

must be a solution. Indeed it is the *only* solution in $\mathcal{S}(\mathbb{R}^n)$. To see this simply take the Fourier transform of (7.56), with $u = N_{H,-2}(G^0)_{\restriction T_x X}$. It becomes

$$\left(\xi \cdot \partial_\xi + 2|\xi|^2\right) \hat{u} = 0, \quad \hat{u}(0) = 1.$$

This is an ordinary differential equation, with initial condition, along each radial line and the Fourier transform of (7.57) is the only solution. Clearly then the choice (7.57), with $|X|$ now the Riemannian norm on TX induced by P, gives a solution to (7.53).

Now the order of the error term can be reduced inductively. Suppose $j \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $G^{(j)} \in \Psi_H^{-2}(X; \Omega^{\frac{1}{2}})$ has been found such that

$$t(\partial_t + P)G^{(j)} = R_j \in \Psi_H^{-3-j}(X; \Omega^{\frac{1}{2}}),\ G_0^{(j)} = \mathrm{Id}\,. \tag{7.58}$$

Then look for $G^{(j+1)} = G^{(j)} - T_j$, $T_j \in \Psi_H^{-3-j}(X; \Omega^{\frac{1}{2}})$, such that (7.58) holds for $j+1$. Of course this just means solving

$$\begin{gathered} t(\partial_t + P)T_j = R_j + R_{j+1} \in \Psi_H^{-3-j}(X; \Omega^{\frac{1}{2}}), \\ R_{j+2} \in \Psi_H^{-4-j}(X; \Omega^{\frac{1}{2}}), \end{gathered} \tag{7.59}$$

where R_j is given as the error term in (7.58). The initial condition continues to hold since $(T_j)_0 = 0$ as its order is at most -3. By (7.52) the equation (7.59) can be transformed into a condition on the normal operator, namely

$$\left[\sigma(P) - \frac{1}{2}(R + n - j - 1)\right] N_{H,-3-j}(T_j) = N_{H,-3-j}(R_j). \tag{7.60}$$

Again this has a unique solution, this time without any integral condition. In appropriate coordinates (7.55) holds. Taking Fourier transforms on a given fibre, with $u = N_{H,-3-j}(T_{j+1})$, $f = N_{H,-3-j}(R_{j+1})$, equation (7.60) becomes

$$(\xi\partial_\xi + 2|\xi|^2 + j + 1)\hat{u} = \hat{f} \in \mathcal{S}(\mathbb{R}^n). \tag{7.61}$$

The solution to this is:

$$\hat{u}(\xi) = \int_0^1 \exp((r-1)|\xi|^2)\hat{f}(r\xi)r^{j+1}dr. \tag{7.62}$$

This is clearly an element of $\mathcal{S}(\mathbb{R}^n)$, and easily seen to be the only solution to (7.61). This completes the inductive proof of (7.58), for all j.

Now the successive $T_j = G^{(j+1)} - G^{(j)}$ give a formal power series at $\mathrm{tf}(X^2_H)$. By Borel's lemma this can be summed, so $G \in \Psi_H^{-2}(X;\Omega^{\frac{1}{2}})$ satisfying (7.43) can indeed be found. Since the normal problems (7.56) and (7.60) are clearly invariant under the reflection in the fibres, using (7.50) instead of (7.49), it follows that $G \in \Psi^{-2}_{H,\mathrm{evn}}(X;\Omega^{\frac{1}{2}})$.

It still remains to complete the proof of Theorem 7.12. So far we have considered the elements of the heat calculus as operators from X to $\mathbb{R}\times X$, as in (7.34), but in order to remove the error term in (7.43) it is more convenient to consider them as t-convolution operators. If $A \in \Psi^k_H(X;\Omega^{\frac{1}{2}})$, its action as a t-convolution operator will be denoted

$$A*\colon \dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac{1}{2}}) \longrightarrow \dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac{1}{2}}). \tag{7.63}$$

This can be expressed in terms of (7.34) by

$$A*u(t) = \int_0^t [Au(t-s)](s)ds, \tag{7.64}$$

where the dependence on the spatial variables is suppressed. Notice that as a function of $r \geq 0$ and $s \geq 0$

$$[Au(r)](s) \in s^{-\frac{1}{2}k-1}\mathcal{C}^\infty([0,\infty)\times[0,\infty)_{\frac{1}{2}}\times X;\Omega^{\frac{1}{2}})$$

vanishes to infinite order at $r=0$. Restricting to $r = t-s$ gives

$$[Au(t-s)](s) = s^{-\frac{1}{2}k-1}(t-s)^j u_j(t-s,s^{\frac{1}{2}},x)$$

for any j, with u_j a $\mathcal{C}^\infty$ half-density. Since $k<0$ this is integrable in s and $\mathcal{C}^\infty$ in t. Thus $A*u$, defined by (7.64), is $\mathcal{C}^\infty$ in t and vanishes rapidly as $t\downarrow 0$. This gives the mapping property (7.63).

The elements of $\Psi_H^{-\infty}(X;\Omega^{\frac{1}{2}})$ give rise to Volterra operators in (7.63) and the removal of the error term in (7.43) reduces to standard invertibility results for such operators.

PROPOSITION 7.17. *If $A \in \Psi_H^{-\infty}(X;\Omega^{\frac{1}{2}})$ then $\mathrm{Id}-A*$ is invertible as an operator on $\dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac{1}{2}})$, with inverse $\mathrm{Id}-S*$ for some $S \in \Psi_H^{-\infty}(X;\Omega^{\frac{1}{2}})$.*

PROOF: If $A,\ B \in \Psi_H^{-\infty}(X;\Omega^{\frac12})$ the composite operator $A * B* = C*$, where $C \in \Psi_H^{-\infty}(X;\Omega^{\frac12})$ has kernel

$$\gamma(t,x,x')|dt|^{\frac12}\nu(x)\nu(x') = \int_0^t \int_X \alpha(t-s,x,x'')\beta(s,x'',x')\nu^2(x'')ds\nu(x)\nu(x') \tag{7.65}$$

in terms of the kernels $\alpha(t,x,x')|dt|^{\frac12}\nu(x)\nu(x')$ and $\beta(t,x,x')|dt|^{\frac12}\nu(x)\nu(x')$ of A and B, with ν a fixed non-vanishing smooth half-density on X. Direct estimation shows that there is a constant K, depending only on ν and X such that

$$\begin{aligned} &|\alpha(t,x,x')| \le C\frac{t^k}{k!},\ |\beta(t,x,x')| \le C' \text{ in } t<T \\ &\qquad\Longrightarrow |\gamma(t,x,x')| \in CC'K\frac{t^{k+1}}{(k+1)!} \text{ in } t<T. \end{aligned} \tag{7.66}$$

This estimate can be applied iteratively to $A \in \Psi_H^{-\infty}(X;\Omega^{\frac12})$. For fixed $T>0$ let $C=C_k$ be such that the estimate on α in (7.66) holds and let $C'=C_0$. Let the kernel of $A_j \in \Psi_H^{-\infty}(X;\Omega^{\frac12})$, fixed by $A*_j = (A*)^j$, be α_j. Then for any k and j

$$|\alpha_j(t,x,x')| \le (KC_0)^{j-1}C_k\frac{t^{k+j-1}}{(k+j-1)!} \text{ in } t<T.$$

It follows that the Neumann series for the inverse of $\operatorname{Id}-A*$,

$$(\operatorname{Id}-A*)^{-1} = \operatorname{Id}+\sum_{j=1}^{\infty}(A*)^j = \operatorname{Id}+\sum_{j=1}^{\infty}(A_j)* = \operatorname{Id}-S*,$$

converges. The same estimates hold on the derivatives, which shows that $S \in \Psi_H^{-\infty}(X;\Omega^{\frac12})$ and so proves the lemma.

PROOF OF THEOREM 7.12: If the parametrix of Lemma 7.16 is considered as a convolution operator and R is given by (7.43) then

$$(\partial_t+P)\cdot G* = \operatorname{Id}-A*,\ A=-R/t \in \Psi_H^{-\infty}(X). \tag{7.67}$$

To see this note first that for any $u \in \mathcal{C}^\infty(X;\Omega^{\frac12})$ we know from (7.35) that $v(t)=Gu(t) \in \mathcal{C}^\infty([0,\infty)\times X;\Omega^{\frac12})$ and from (7.43) that if $f(t)=(\partial v/\partial t+$

$Pv)(t)$ then $f \in \dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac12})$. Thus if $g \in \dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac12})$ then in $t>0$

$$
\begin{aligned}
(\frac{\partial}{\partial t}+P)G*g(t) &= \left(\frac{\partial}{\partial t}+P\right)\int_0^t [Gg(s)](t-s)ds \\
&= [Gg(t)](0)+\int_0^t [\frac{1}{s}(\frac{\partial}{\partial s}+P)g(s)](t-s)ds
\end{aligned}
$$

giving (7.67). Since $R \in \Psi_H^{-\infty}(X;\Omega^{\frac12})$ its kernel vanishes to all orders at $t=0$, so dividing by t gives another kernel of the same type.

Thus Proposition 7.17 can be applied to (7.67). It follows that if $\mathrm{Id}-S*$ is the inverse of $\mathrm{Id}-A*$ then the composite operator $H* = G*-G*S*$ is a right inverse of the heat operator:

$$
v = H*g,\ g\in\dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac12}) \Longrightarrow (\frac{\partial}{\partial t}+P)v = g \text{ in } \mathbb{R}\times X. \tag{7.68}
$$

From the regularity, (7.35), for the action of G it follows that $G*S* = B*$, with $B\in\Psi_H^{-\infty}(X;\Omega^{\frac12})$. Thus $H\in\Psi_{H,\mathrm{evn}}^{-2}(X;\Omega^{\frac12})$ differs from G by an element of $\Psi_H^{-\infty}(X;\Omega^{\frac12})$. In fact H must satisfy

$$
t(\partial_t+P)H = 0 \text{ and } H_0 = \mathrm{Id} \tag{7.69}
$$

since H satisfies (7.43) in place of G and hence (7.67), but the error term is necessarily zero.

It remains only to show the uniqueness of H. To see this it suffices to show that there are no solutions to

$$
(\frac{\partial}{\partial t}+P)u = 0,\ u_{\restriction t=0}=0,\ u\in\mathcal{C}^\infty([0,\infty)\times X;\Omega^{\frac12}).
$$

Formal differentiation of the equation shows that all the t-derivatives of u must vanish at $t=0$, so $u\in\dot{\mathcal{C}}^\infty([0,\infty)\times X;\Omega^{\frac12})$. Thus u can be extended as 0 to $t<0$ and then it satisfies $(\partial/\partial t+P)u=0$ on the whole of $\mathbb{R}\times X$. Now notice that the formal adjoint of $\partial/\partial t+P$ is just $-\partial/\partial t+P^*$. This is the heat operator, with time reversed, for P^*. Since P^* satisfies the hypotheses of the theorem if P does, the construction of H above means that for any $\phi\in\mathcal{C}_c^\infty(\mathbb{R}\times X)$, vanishing in $t>T$, we can find a solution of $(-\partial/\partial t+P^*)v=\phi$, where $v\in\mathcal{C}^\infty(\mathbb{R}\times X;\Omega^{\frac12})$ vanishes in $t>T$. This means that

$$
\int_{\mathbb{R}\times X} u\overline{\phi} = \int_{\mathbb{R}\times X} u\overline{(-\frac{\partial}{\partial t}+P^*)v} = \int_{\mathbb{R}\times X}(\frac{\partial}{\partial t}+P)u\overline{v} = 0.
$$

Here the integration by parts is justified by the fact that u vanishes in $t < 0$ and v and ϕ vanish in $t > T$. This shows that $u = 0$ and hence completes the proof of Theorem 7.12.

The uniqueness of the solution to the heat equation shown in the course of the proof of Theorem 7.12 implies that the one-parameter family of operators defined by the forward fundamental solution H forms a semigroup. This is the justification for the standard exponential notation

$$\begin{gathered}\exp(-tP)u = (Hu)(t)\ \forall\ u \in \mathcal{C}^\infty(X;\Omega^{\frac12}),\ t > 0 \Longrightarrow \\ \exp(-tP)\exp(-sP) = \exp(-(s+t)P)\ \forall\ s,t > 0.\end{gathered} \tag{7.70}$$

Since the kernel of $H \in \Psi_H^{-2}(X;\Omega^{\frac12})$ is a $\mathcal{C}^\infty$ half-density in $t > 0$, it follows that this is a semigroup of smoothing operators; such smoothing operators are trace class.

The local index theorem involves computing the limit, as $t \downarrow 0$, of the difference of the traces of heat kernels, see (In.27). This computation is carried out in the next chapter since it involves cancellation in the short-time asymptotics. Except for such cancellation the short-time behaviour of the heat kernel can be read off directly from the definition of the heat space. Thus let

$$\widetilde{\Delta} = \left\{(t,x,x) \in [0,\infty) \times X^2\right\} = \{x = x'\} = [0,\infty) \times \Delta$$

be the diagonal with time parameter. In the local variables from (7.4) in X_H^2 this is given by $\omega' = 0$. It is therefore an embedded submanifold. Denote its lift under β_H as $\widetilde{\Delta}_H \subset X_H^2$. In fact the restriction $\beta_H : \widetilde{\Delta}_H \longleftrightarrow \widetilde{\Delta} \cong [0,\infty) \times X$ is a diffeomorphism. Lidskii's theorem, Proposition 4.55, then takes the form

$$\operatorname{tr}(A(t)) = \int_X A_{\restriction \widetilde{\Delta}_H},\ t > 0,$$

where $A(t)$ is the map (7.34) at fixed t. From (7.11), (7.30) and (7.31) it follows that

LEMMA 7.18. *For any $A \in \Psi_H^k(X;\Omega^{\frac12})$ the restriction of the kernel to the diagonal is a polyhomogeneous conormal distribution at $t = 0$ on $[0,\infty) \times X$ with an expansion*

$$A_{\restriction \widetilde{\Delta}_H} \sim \sum_{j=0}^{\infty} A_j t^{-\frac12(n+k+2)+\frac12 j},\ A_j \in \mathcal{C}^\infty(X;\Omega).$$

If $A \in \Psi_{H,\mathrm{evn}}^k(X;\Omega^{\frac12})$ the terms with j odd vanish identically.

As a direct corollary the trace itself has a complete asymptotic expansion at $t = 0$. The case of most interest is:

$$A \in \Psi^{-2}_{H,\mathrm{evn}}(X;\Omega^{\frac{1}{2}}) \Longrightarrow \operatorname{tr}(A(t)) \sim \sum_{j=0}^{\infty} t^{-\frac{1}{2}n+j} a_j \text{ as } t \downarrow 0. \tag{7.71}$$

The cancellation for the supertrace is discussed in Chapter 8.

The long-time behaviour of the heat kernel is discussed below after the finite time analysis is extended to cover b-differential operators.

7.4. b-heat space.

The description of the heat calculus above is in a form which is quite straightforward to generalize to the b-category. Indeed the b-heat calculus which will be considered next is little more than a direct combination of the constructions above and those from Chapter 4. None of the complications having to do with the boundary terms of the type constructed in Chapter 5 arise for finite times, only in the long-time asymptotics. Thus, most of this section amounts to putting together the various pieces already examined. The discussion proceeds in the usual way, first defining the space on which it is reasonable to expect the heat kernel to be simplest and then showing that it is indeed an element of the calculus so defined. No composition formula is proved for this calculus, although it is easy enough to do so (see for example [30]). The parametrix is constructed by symbolic means rather than iteration.

Given $P \in \operatorname{Diff}^2_b(X;{}^b\Omega^{\frac{1}{2}})$, an elliptic b-differential operator with non-negative principal symbol corresponding to an exact b-metric, the objective is to construct the heat semigroup of P, $\exp(-tP)$, with kernel $H_P \in \mathcal{C}^{-\infty}([0,\infty) \times X^2, {}^b\Omega^{\frac{1}{2}})$. Again the densities can be freely played with on the factor $[0,\infty)$, using $|dt|$ or $|dt/t|$ and their powers. Thus H_P should define an operator

$$H_P \colon \dot{\mathcal{C}}^\infty(X;{}^b\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^{-\infty}([0,\infty) \times X;{}^b\Omega^{\frac{1}{2}})$$

which is intended to have the properties

$$u = H_P v \text{ satisfies } \begin{cases} (\partial_t + P)u = 0 & \text{in } t > 0 \\ u\big|_{t=0} = v. \end{cases}$$

Of course the initial condition does not make sense *a priori*; some regularity is needed to define $u\big|_{t=0}$ as before. Since P is a b-differential operator it is natural to expect the b-stretched product, X^2_b, defined in Chapter 5 to play a rôle. Similarly away from the boundary, it is the just the ordinary heat kernel being considered, so the heat space, X^2_H, of (7.13) should be involved.

The question then is how to combine these spaces when X is a compact manifold with boundary. Certainly there are two obvious submanifolds on which something subtle will happen:

$$\begin{aligned} B_b &= \{(t,z,z') \in [0,\infty) \times X^2; z,z' \in \partial X\} \cong [0,\infty) \times (\partial X)^2, \\ B_H &= \{(0,z,z) \in [0,\infty) \times X^2\} \cong X. \end{aligned} \tag{7.72}$$

Both these submanifolds will be blown up. The question then arises as to the order in which to proceed. The obvious way (for some) to approach such a problem is to guess, and analyze the options geometrically. It is easily seen that blowing up B_H first is *not* a good option. In any case it is probably more convincing to look at a simple example and see how the kernel actually behaves.

EXERCISE 7.19. Check that the order of blow-up does make a difference.

So consider an $\mathbb{R}^+$-invariant operator, $P = (xD_x)^2$, on $[0,\infty)$ (so really $X = [-1,1]$ with variable $(1-x)/(1+x)$ etc. as in Chapter 1). Then P is self-adjoint with respect to $|dx/x|$ and in terms of the variable

$$r = \log x, \quad P = D_r^2.$$

The heat kernel is therefore

$$\frac{1}{(2\pi t)^{\frac{1}{2}}} \exp\left(-\frac{|r-r'|^2}{4t}\right) = \frac{1}{(2\pi t)^{\frac{1}{2}}} \exp\left(-\frac{|\log\frac{x}{x'}|^2}{4t}\right).$$

Notice how this can be made into a smooth function for finite times. First introduce $s = x/x'$ as a $\mathcal{C}^\infty$ variable. This is the b-stretching construction. Then make the t-parabolic blow-up of the submanifold $s=1, t=0$; finally remove the simple singular factor at $t=0$. The conclusion is that the 'b-' blow-up should be done first and then the 'H-' blow-up.

DEFINITION 7.20. The b-heat space, X^2_η, of a compact manifold with boundary, X, is defined from $[0,\infty) \times X^2$ by first blowing up B_b, defined in (7.72), giving $[0,\infty) \times X^2_b$ and then making the t-parabolic blow-up of $\{0\} \times \Delta_b \subset [0,\infty) \times X^2_b$, which is the lift of B_H in (7.72). The overall blow-down map will be written

$$\beta_\eta : X^2_\eta \longrightarrow [0,\infty) \times X^2.$$

It is worthwhile making the last step more explicit. The b-heat space can be written as a union

$$X^2_\eta = \big[([0,\infty)\times X^2_b)\backslash(\{0\}\times\Delta_b)\big]\sqcup \mathrm{tf}(X^2_\eta), \tag{7.73}$$

where $\mathrm{tf}(X^2_\eta)$ is the 'temporal front face' which replaces the lift of B_H under the b-blow-down map, $\beta_b^*(B_H)=\{0\}\times\Delta_b$. Geometrically $\mathrm{tf}(X^2_\eta)$ is the 't-parabolic, inward-pointing spherical normal bundle' to $\{0\}\times\Delta_b$ in $[0,\infty)\times X^2_b$. Following the discussion above this is a bundle, over $\Delta_b\cong X$, with points given by equivalence classes of curves.

Thus, consider curves ($\mathcal{C}^\infty$ of course)

$$\chi\colon[0,\epsilon)\longrightarrow[0,\infty)\times X^2_b,\ \chi(s)\in\{0\}\times\Delta_b \text{ iff } s=0. \tag{7.74}$$

The t-parabolic curves are those satisfying in addition

$$\chi^*(t)=O(s^2) \text{ as } s\downarrow 0. \tag{7.75}$$

The equivalence relation defining the inward pointing t-parabolic normal bundle to $\{0\}\times\Delta_b$ is

$$\chi_1\sim\chi_2 \Longleftrightarrow \begin{cases} \chi_1^*f-\chi_2^*f=O(s)\ \forall\ f\in\mathcal{C}^\infty(X^2_b)\\ \chi_1^*g-\chi_2^*g=O(s^2)\ \forall\ g\in\mathcal{C}^\infty(X^2_b) \text{ with } g\big|_{\Delta_b}=0\\ \chi_1^*t-\chi_2^*t=O(s^3).\end{cases} \tag{7.76}$$

Set $\chi_i=(T_i,C_i)$, $T_i\in\mathcal{C}^\infty([0,1])$, $C_i\colon[0,1]\longrightarrow X^2_b$. Then the first condition in (7.76) demands $C_1(0)=C_2(0)\in\Delta_b$ by (7.74). The second two conditions require, respectively

$$\begin{gathered} C_1'(0)=C_2'(0) \text{ in } N_{C_1(0)}\Delta_b=T_{C_1(0)}X^2_b/T_{C_1(0)}\Delta_b\\ T_1''(0)=T_2''(0), \end{gathered} \tag{7.77}$$

where $T_1'(0)=T_2'(0)=0$ by (7.75). If the equivalence classes under (7.76) are written $[\chi]$, and $[\chi]=0$ means the vanishing of $C_1'(0),T_1''(0)$ in (7.77), then

$$\mathrm{tf}(X^2_\eta)=\big\{[\chi]/\mathbb{R}^+\big\},\ a[\chi]=[\chi(a\cdot)],\ a>0.$$

In this $\mathbb{R}^+$-action the fact that χ only needs to be defined on $[0,\epsilon)$ for some $\epsilon>0$ is used. To show that X^2_η, given by (7.73), has a natural $\mathcal{C}^\infty$ structure and to consider the nature of its boundary hypersurfaces, particularly the two 'front faces,' the discussion in §7.1 above can be followed essentially *verbatim*. In particular the fact that X^2_η is a $\mathcal{C}^\infty$ manifold with corners follows directly from the proof of Proposition 7.7.

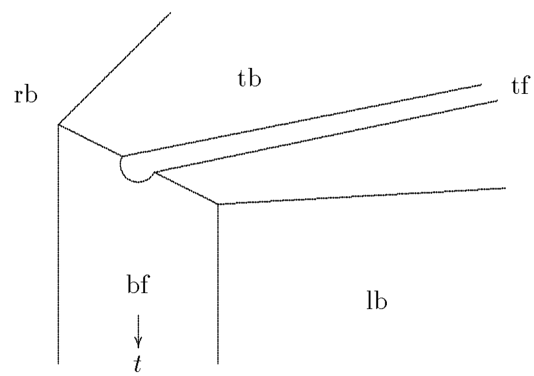

Figure 13. The b-heat space X^2_η.

EXERCISE 7.21. Extend Exercise 7.6 to show that the interior of $\mathrm{tf}(X^2_\eta)$ has a vector bundle structure, over Δ_b, and that as a vector bundle it is *canonically isomorphic* to ${}^bTX$.

EXERCISE 7.22. Give an abstract definition of a compactification of a vector bundle, V, over a compact manifold with boundary, X, which applied to ${}^bTX$ gives $\mathrm{tf}(X^2_\eta)$.

The boundary hypersurfaces will be named as indicated in Figure 13. The left, right and temporal boundaries, lb, rb and tb are respectively the lifts of $[0,\infty)\times\partial X\times X$, $[0,\infty)\times X\times\partial X$ and $\{0\}\times X^2$ to X^2_η. The two front faces introduced by the blow-up are the b-front face, bf, and the temporal front face, tf . As usual a defining function for the face F will be written ρ_F.

As in Lemma 7.8 there are well-defined even and odd subspaces $\mathcal{C}^\infty_{\mathrm{evn}}(X^2_\eta)$, $\mathcal{C}^\infty_{\mathrm{odd}}(X^2_\eta)\subset\mathcal{C}^\infty(X^2_\eta)$ and these extend to define the spaces of even and odd sections of any vector bundle on X^2_η which is the lift of a $\mathcal{C}^\infty$ vector bundle on $[0,\infty)\times X^2_b$.

7.5. b-heat calculus.

It should be more or less clear by now how the b-heat calculus will be defined.

DEFINITION 7.23. For any compact manifold with boundary, X, and $k\in-\mathbb{N}$ the space of b-heat pseudodifferential operators acting on b-half-densitites is

(7.78)
$$\Psi^k_\eta(X;{}^b\Omega^{\frac12})=\left\{K\in\rho_{\mathrm{tf}}^{-(\frac12(n+3)-k}\mathcal{C}^\infty(X^2_\eta;{}^b\Omega^{\frac12});K\equiv0\text{ at }\ \mathrm{tb}\cup\mathrm{lb}\cup\mathrm{rb}\right\}.$$

The discussion of the density bundle following Exercise 7.9 also extends to b-metrics with only notational changes. Thus the even and odd parts of the b-heat calculus can be defined as in (7.31).

THEOREM 7.24. *Suppose $P \in \mathrm{Diff}_b^2(X; {}^b\Omega^{\frac{1}{2}})$ is elliptic with positive principal symbol, then there exists a unique operator $H_P \in \Psi_{\eta,\mathrm{evn}}^{-2}(X; {}^b\Omega^{\frac{1}{2}})$ such that*

$$(\partial_t + P)H_P = 0 \text{ in } t > 0,\ H_{P\restriction t=0} = \mathrm{Id}\,. \tag{7.79}$$

Of course as before it is necessary to make sense of the initial condition in (7.79). To do so, direct extensions of Lemma 7.11 and Lemma 7.13 will be used.

LEMMA 7.25. *Each $A \in \Psi_\eta^k(X; {}^b\Omega^{\frac{1}{2}})$, $k \in -\mathbb{N}$, defines a continuous linear operator*

$$A\colon \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow t^{-1-\frac{k}{2}} \left\{ u \in \mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X; {}^b\Omega^{\frac{1}{2}}), u \equiv 0 \text{ at } \partial X \right\} \tag{7.80}$$

and if $A \in \Psi_{\eta,\mathrm{evn}}^k(X; {}^b\Omega^{\frac{1}{2}})$ then

$$A\colon \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}) \longrightarrow t^{-1-\frac{k}{2}} \left\{ u \in \mathcal{C}^\infty([0,\infty) \times X; {}^b\Omega^{\frac{1}{2}}), u \equiv 0 \text{ at } \partial X \right\}.$$

If $k = -2$ restriction to $t = 0$ is given by a multiplication operator

$$v \longmapsto Av\big|_{t=0} = \Phi_A \cdot v,\ \Phi_A = \int_{\text{fibre}} A_{\restriction \mathrm{tf}(X_\eta^2)},\ \forall\ v \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}}). \tag{7.81}$$

PROOF: One can follow the proofs of Lemma 7.11 and Lemma 7.13 closely. The operator A in (7.80) acts on $v \in \dot{\mathcal{C}}^\infty(X; {}^b\Omega^{\frac{1}{2}})$ by lifting v from X to X_η^2, from the right factor, multiplying by the kernel and then pushing forward to $[0,\infty) \times X$. Since v vanishes to all orders at ∂X, when lifted to X_η^2 it vanishes to all orders at $\mathrm{bf}(X_\eta^2)$. Thus the product with the kernel vanishes to all orders at all boundary faces of X_η^2 except $\mathrm{tf}(X_\eta^2)$. The argument of Lemma 7.11 now applies directly. Similarly (7.81) follows from the proof of Lemma 7.13.

This makes sense of the initial condition in (7.79). The construction of H_P proceeds in the expected three main steps. First the Taylor series of the kernel at $\mathrm{tf}(X_\eta^2)$ is constructed. Then the Taylor series at $\mathrm{bf}(X_\eta^2)$ is found and finally a convergence argument is used to remove the error term. The first step is just a direct extension of Lemma 7.13. To state it succinctly consider the normal operators by extension from (7.48). Thus

$$\begin{aligned} N_{k,\eta} : \Psi^k_\eta(X; {}^b\Omega^{\frac12}) &\longrightarrow \dot{\mathcal{C}}^\infty(\mathrm{tf}(X^2_\eta); {}^b\Omega^{\frac12}) \cong \mathcal{S}({}^bTX; {}^b\Omega^{\frac12}), \\ N_{k,\eta}(A) &= t^{\frac12(n+2+k)} A_{|\mathrm{tf}}. \end{aligned} \tag{7.82}$$

The surjectivity here follows directly from the definition, (7.78), and in Exercise 7.21 the interior of $\mathrm{tf}(X^2_\eta)$ is identified with ${}^bTX$ in such a way that the spaces on the right in (7.82) are identified. In fact, as in (7.49) there is a short exact sequence for each $k \in -\mathbb{N}$:

$$0 \longrightarrow \Psi^{k-1}_\eta(X; {}^b\Omega^{\frac12}) \hookrightarrow \Psi^k_\eta(X; {}^b\Omega^{\frac12}) \xrightarrow{N_{k,\eta}} \mathcal{S}({}^bTX; {}^b\Omega^{\frac12}) \longrightarrow 0. \tag{7.83}$$

EXERCISE 7.26. Write down the sequence analogous to (7.83) for the even part of the calculus and check its exactness.

As in Proposition 7.15 this leads to:

PROPOSITION 7.27. *If $P \in \mathrm{Diff}^2_b(X; {}^b\Omega^{\frac12})$ has symbol ${}^b\sigma(P)$, as a translation-invariant operator on the fibres of ${}^bTX$, then for each $k \in -\mathbb{N}$*

$$\begin{aligned} t(\partial_t + P) \cdot A \in \Psi^k_\eta(X; {}^b\Omega^{\frac12}) \ \forall\ A \in \Psi^k_\eta(X; {}^b\Omega^{\frac12}) \\ N_{k,\eta}(t(\partial_t + P)A) = [{}^b\sigma(P) - \frac12(R + n + 2 + k)] N_{k,\eta}(A), \end{aligned} \tag{7.84}$$

where R is the radial vector field on the fibres of ${}^bTX$.

PROOF: From Proposition 4.4 it follows that $P \in \mathrm{Diff}^2_b(X; {}^b\Omega^{\frac12})$, acting on the left factor of X^2, lifts to $\tilde P \in \mathrm{Diff}^2_b(X^2_b; {}^b\Omega^{\frac12})$. Then the further lift of $t(\partial_t + \tilde P)$ from $[0,\infty) \times X^2_b$ to X^2_η exactly parallels the proof of Proposition 7.15. It is even possible to think of the formula for the normal operator in (7.84) as following by continuity.

Now notice that the first step in the construction of H_P, as desired in Theorem 7.24, follows *exactly* the construction starting with (7.53) and finishing with (7.62) in the boundaryless case; where ${}^b\sigma(P)$ *is* elliptic on the fibres of ${}^bTX$. This gives the analogue of (7.43). Since it is only the first step in the construction of a parametrix, in this case we denote this partial parametrix $G^{(1)}$. Thus

$$G^{(1)} \in \Psi^{-2}_\eta(X; {}^b\Omega^{\frac12}) \text{ satifies } \begin{cases} (\partial_t + P)G^{(1)} = R^{(1)} \in \Psi^{-\infty}_\eta(X; {}^b\Omega^{\frac12}) \\ G^{(1)}\big|_{t=0} = \mathrm{Id}\,. \end{cases} \tag{7.85}$$

The serious difference is that the remainder term here, in $\Psi_\eta^{-\infty}(X;{}^b\Omega^{\frac12})$, is *not* a smoothing operator. Rather directly from (7.78) it is clear that

$$\Psi_\eta^{-\infty}(X;{}^b\Omega^{\frac12}) = \dot{\mathcal{C}}^\infty([0,\infty);\Psi_b^{-\infty}(X;{}^b\Omega^{\frac12})) \tag{7.86}$$

is (using the freedom to cancel off $|dt/t|^{\frac12}$) just a b-pseudodifferential operator, in the small calculus of order $-\infty$, depending smoothly on t and vanishing to infinite order at $t=0$.

The next, new, step is to remove the Taylor series at $\mathrm{bf}(X^2_\eta)$. As is implicit in (7.86) it is natural to think of these kernels as defined on $[0,\infty)\times X^2_b$. Thus the problem is to find

$$\begin{gathered} G^{(2)} \in \Psi_\eta^{-\infty}(X;{}^b\Omega^{\frac12}) \text{ s.t. } (\partial_t + P)G^{(2)} = R^{(1)} + R^{(2)}, \\ R^{(2)} \in \dot{\mathcal{C}}^\infty([0,\infty)\times X^2;{}^b\Omega^{\frac12}). \end{gathered} \tag{7.87}$$

Here $R^{(1)}$ is the remainder term in (7.85) and the remainder $R^{(2)}$ is just an element of $\Psi_\eta^{-\infty}(X;{}^b\Omega^{\frac12})$ with kernel vanishing to infinite order at $\mathrm{bf}(X^2_\eta)$, as well as $\mathrm{tf}(X^2_\eta)$, and therefore $\mathcal{C}^\infty$ on $[0,\infty)\times X^2$ and vanishing to infinite order at all (finite) boundaries.

The first stage in (7.87) is clear enough. Namely the indicial equation on $[0,\infty)\times \mathrm{bf}(X^2_b)$,

$$(\partial_t + I(P))I(G^{(2)}) = I(R^{(1)}), \tag{7.88}$$

should be solved. Here a boundary defining function can be chosen giving a decomposition

$$\mathrm{bf}(X^2_b) \simeq \widetilde{X}\times\partial X$$

so that (7.88) is just the heat equation for the indicial operator of P.

PROPOSITION 7.28. *If $Q\in \mathrm{Diff}^2_{b,I}(\widetilde{X};{}^b\Omega^{\frac12})$ is elliptic with positive principal symbol then the equation*

$$(\frac{\partial}{\partial t} + Q)u = f \quad t>0 \tag{7.89}$$

has, for each forcing function $f\in t^\infty\mathcal{C}^\infty([0,\infty)\times\widetilde{X};{}^b\Omega^{\frac12})$, a unique solution $u\in t^\infty\mathcal{C}^\infty([0,\infty)\times\widetilde{X};{}^b\Omega^{\frac12})$.

PROOF: As in the elliptic case, examined in Chapter 5, it is natural to take the Mellin transform. Then (7.89) becomes

$$(\partial_t + Q(\lambda))u_M(\lambda) = f_M(\lambda) \in \dot{\mathcal{C}}^\infty([0,\infty)\times\partial X), \tag{7.90}$$

where $f_M(\lambda)$ is entire in λ, and vanishes rapidly with all derivatives as $|\operatorname{Re}\lambda| \longrightarrow \infty$ with $|\operatorname{Im}\lambda|$ bounded. A solution of the same type is needed. Certainly the solution exists because $Q(\lambda)$ is an elliptic operator on ∂X with principal symbol ${}^b\sigma(Q)_{\restriction\lambda=0}$ independent of λ. Thus the existence of a unique solution to (7.90), for each λ, follows from the construction of the heat kernel above, i.e. Theorem 7.12. The construction of the heat kernel $\exp(-tQ(\lambda))$ is clearly locally uniform in λ, so the solution is holomorphic in λ. It remains to show that $u_M(\lambda)$ is rapidly decreasing at real infinity in λ. However to see this it suffices to apply the discussion above to a parametrix for $\exp(-tQ(\lambda))$ which has an error term which vanishes rapidly at real infinity. This is exactly what $I(G^{(1)})$ gives. The proof of Proposition 7.28 is therefore complete.

PROOF OF THEOREM 7.24: Thus, by taking the Mellin transform, (7.88) can be solved with solution the indicial operator of $G^{(2)}_1 \in \Psi^{-\infty}_{\eta}(X; {}^b\Omega^{\frac12})$. This first step towards the construction of $G^{(2)}$ can be iterated. Thus $G^{(2)}_1$ should satisfy

$$(\partial_t + P)G^{(2)}_1 = R^{(1)} + R^{(2)}_1 \text{ with}$$
$$R^{(2)}_1 \in \dot{\mathcal{C}}^\infty([0,\infty); \rho_{\mathrm{bf}}\Psi^{-\infty}_b(X; {}^b\Omega^{\frac12})),\ G^{(2)}_1 \in \dot{\mathcal{C}}^\infty([0,\infty); \Psi^{-\infty}_b(X; {}^b\Omega^{\frac12})).$$

Since $R^{(2)}_1$ is an element of the small calculus its kernel vanishes to infinite order at the left and right boundaries of X^2_b. Thus ρ_{bf} can be replaced just as well by x' and, this being a parameter in the equation, division by x' is permissbile the equation for $G^{(2)}_2$ takes the form

$$(\partial_t + I(P))I(G^{(2)}_2) = I(R^{(2)}_1/x'),$$

which is just (7.88) again. Proceeding iteratively this constructs the Taylor series of $G^{(2)}$ at $\mathrm{bf}(X^2_b)$ exactly as required to give (7.87). This completes the Taylor series part of the construction of H_P and $G = G^{(1)} - G^{(2)} \in \Psi^{-2}_{\eta}(X; {}^b\Omega^{\frac12})$ has been found such that

$$(\partial_t + P)G = R \in \dot{\mathcal{C}}^\infty([0,\infty) \times X^2; {}^b\Omega^{\frac12}),\ G\big|_{t=0} = \mathrm{Id}\,.$$

The remainder term here is a Volterra operator in a very strong sense, since its kernel vanishes to infinite order not only at $t = 0$ but at both boundaries of X^2. The usual iteration procedure, as discussed above, therefore gives a convolution inverse to $\mathrm{Id} + R$, of the form $\mathrm{Id} + S$ with $S \in \dot{\mathcal{C}}^\infty([0,\infty) \times X^2; {}^b\Omega^{\frac12})$ as well. Then the operator, the existence of which is claimed in Theorem 7.24, is

$$H_P = G^{(1)} - G^{(2)} + (G^{(1)} - G^{(2)}) * S.$$

The first two terms are, by construction, in the b-heat calculus. The final term is easily seen, from Lemma 7.18 and the properties of S, to have kernel in $\dot{\mathcal{C}}^\infty([0,\infty)\times X^2; {}^b\Omega^{\frac12})$, so it is in the b-heat calculus as well.

This proves the existence part of Theorem 7.24. The uniqueness follows as before, so is left as an exercise. It is most important to note, for use in the proof of the APS theorem, that the third equation in (In.23) has now been proved. Namely

$$I(\exp(-tP))(\lambda) = \exp(-tI(P)(\lambda)). \tag{7.91}$$

The relative simplicity of the construction of the heat kernel for P a b-differential operator should not be lost in the detail here. It amounts to essentially no more than the constructions in the b-calculus plus the construction in the heat calculus. This is the main point of presenting them in a reasonably unified form.

7.6. Bundle coefficients.

As usual the discussion above has been limited to operators on half-densities for simplicity. There are two extensions that will now be made. The first is the addition of bundle coefficients. As with the other calculi this is mainly a matter of notation, since by construction the calculus is coordinate-invariant so can be localized easily. Thus the main space of heat pseudodifferential operators is, for a bundle E, obtained as in (5.156):

$$\begin{gathered}\Psi_H^{-k}(X;E) \overset{\text{def}}{=} \\ \Psi_H^{-k}(X;{}^b\Omega^{\frac12}) \otimes_{\mathcal{C}^\infty(X^2_H)} \mathcal{C}^\infty(X^2_H; \pi^*_{H,X}\operatorname{Hom}(E\otimes\Omega^{-\frac12}).\end{gathered} \tag{7.92}$$

Notice that $\operatorname{Hom}(E\otimes B)$ is naturally isomorphic to $[E\otimes B]\boxtimes[E'\otimes B']$, for any line bundle B. For $B=\Omega^{-\frac12}$ the extra factor of $E'\otimes\Omega^{\frac12}$ on the right in (7.92) has the effect of turning a section of E into a section of $\Omega^{\frac12}$. The factor of $E\otimes\Omega^{-\frac12}$ on the left turns a section of $\Omega^{\frac12}$ back into a section of E. Thus (7.34) is replaced by

$$\Psi_H^{-k}(X;E)\ni A\colon \mathcal{C}^\infty(X;E)\longrightarrow t^{-\frac{k}{2}-1}\mathcal{C}^\infty([0,\infty)_{\frac12}\times X;E). \tag{7.93}$$

The spaces of even and odd operators have similar definitions.

All the results above extend to the case of bundle coefficients. The main result, Theorem 7.12, extends immediately under the assumption that the principal symbol is diagonal since then the inversion of the normal operator proceeds exactly as in the scalar case. Similar remarks apply to b-metrics with the general space being

$$\begin{gathered}\Psi_\eta^{-k}(X;E) \overset{\text{def}}{=} \\ \Psi_\eta^{-k}(X;{}^b\Omega^{\frac12}) \otimes_{\mathcal{C}^\infty(X^2_\eta)} \mathcal{C}^\infty(X^2_\eta; \beta^*_\eta\operatorname{Hom}(E\otimes {}^b\Omega^{-\frac12}; E\otimes {}^b\Omega^{-\frac12}).\end{gathered}$$

For purposes of the next section it is convenient to extend the main result a little further by the introduction of some complex scaling.

THEOREM 7.29. *If $P \in \mathrm{Diff}_b^2(X; E)$ has diagonal principal symbol given by an exact b-metric on X then for any $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$ the heat equation*

$$(\partial_t + aP)u = 0 \text{ in } (0, \infty) \times X$$
$$u_{\restriction t=0} = u_0 \in \dot{\mathcal{C}}^\infty(X; E)$$

has a unique solution $u \in \mathcal{C}^\infty([0, \infty) \times X; E)$ and the operator so defined,

$$\mathcal{C}^\infty(X; E) \ni u_0 \longmapsto u = \exp(-taP),$$

is an element of $\Psi^{-2}_{\eta,\mathrm{evn}}(X; E)$.

PROOF: If $a = 1$ this is a straightforward extension of Theorem 7.24 as in the boundaryless case just discussed. So consider the more general case of $a \in \mathbb{C}$ with positive real part but with $\partial X = \emptyset$. Again reviewing the proof above, the only point at which the constant a appears significantly is in the inversion of the normal operator. Since the symbol is diagonal this reduces the question to the scalar case with P the standard Laplacian on $\mathbb{R}^n$. The constant can be viewed as complexification of the time variable, i.e. t is replaced by at by continuity from the case $a = 1$ through $\operatorname{Re} a > 0$. Thus (7.57) should be replaced by

$$N_{H,-2}(G^{(0)}) = \frac{1}{(2\pi a)^{\frac{n}{2}}} \exp\left(-\frac{|X|^2}{4a}\right)$$

which is the unique tempered solution of

$$\left[a(D_1^2 + \cdots + D_n^2) - \frac{1}{2}(R + n)\right] N_{H,-2}(G^{(0)}) = 0$$
$$\int_{\text{fibre}} N_{H,-2}(G^{(0)}) = 1,$$

the replacement for (7.56). The iterative equation replacing (7.60) again has a unique solution in the Schwartz space. Thus the construction proceeds essentially as before.

The extension of this to b-metrics reduces to the same normal problem, together with the solvability of the indicial family, but this is of the same type for the boundaryless case, just discussed. Thus the theorem follows from a review of the proof of Theorem 7.12.

Proposition 7.30. *If $A \in \mathrm{Diff}_b^1(X; E, F)$ is elliptic and its adjoint, A^*, with respect to Hermitian structures on the bundles E, F and a non-vanishing b-density on X, is such that A^*A and AA^* have diagonal principal symbols then the heat kernels of A^*A and AA^* are related by*

$$A\exp(-tA^*A) = \exp(-tAA^*)A \text{ in } t > 0. \tag{7.94}$$

Proof: If $u \in \dot{\mathcal{C}}^\infty(X; E)$ then $u(t) = \exp(-tA^*A)u$ is the unique $\mathcal{C}^\infty$ solution to

$$\partial_t u(t) + A^*Au(t) = 0 \text{ in } t > 0,\ u(0) = u.$$

Applying A to this equation shows that $Au(t)$ satisfies

$$\frac{\partial}{\partial t}Au(t) + AA^*(Au(t)) = 0,\ Au(0) = Au.$$

From the uniqueness of the solution to the initial value problem for this heat equation (7.94) follows.

The identity (In.15) is an immediate consequence of this result.

Exercise 7.31. Show that even the assumption that the principal symbol is diagonal is by no means necessary for the construction, i.e. any $P \in \mathrm{Diff}_b^2(X; E)$ with the property that $P^* + P$ is elliptic of order 2 has a well-defined heat kernel $\exp(-tP) \in \Psi_b^{-2}(X; E)$. How would you go about extending this to operators of even order greater than 2?

7.7. Long-time behaviour, Fredholm case.

It remains to analyze the behaviour of the heat kernel as $t \to \infty$. In the boundaryless case this is quite straightforward, but in the case of b-metrics a little more effort is needed to get a reasonably precise description of the kernel. As usual we prove much more than is strictly necessary for a minimal proof of the APS theorem. For the boundaryless case the heat kernel has a complete asymptotic expansion in exponentials as $t \to \infty$ arising from the discrete spectrum. Following Seeley [80] this can be deduced from appropriate asymptotic information about the resolvent. The same approach will be used in the case of a b-differential operator.

Since the heat kernel has already been constructed for short times it is convenient to use it to find a uniform parametrix for the resolvent for large values of the parameter. In the first instance for the boundaryless case, taking $P \in \mathrm{Diff}^2(X; E)$, consider the Laplace transform of the heat kernel, cut off at a finite time to ensure convergence:

$$G(\lambda) = \int_0^\infty e^{t\lambda} \exp(-tP)\phi(t)dt, \tag{7.95}$$

where $\phi \in \mathcal{C}_c^\infty([0,\infty))$ has $\phi(t) = 1$ in $t < 1$. Certainly $G(\lambda)\psi$ is well-defined by (7.95), for each $\psi \in \mathcal{C}^\infty(X; E)$, with the integral absolutely convergent. Thus $G(\lambda)$ is weakly entire in λ as an operator on $\mathcal{C}^\infty(X; E)$. Applying P and using (7.40) gives

$$(7.96) \qquad \begin{gathered} (P-\lambda)G(\lambda)\psi = \psi - R(\lambda)\psi \text{ with} \\ R(\lambda) = \int_0^\infty e^{t\lambda} \exp(-tP)\phi'(t)dt. \end{gathered}$$

Since $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ the remainder here is an entire function of λ with values in the smoothing operators. Moreover in the conic region

$$(7.97) \qquad \operatorname{Re}\lambda < -\epsilon|\lambda|, \ \epsilon > 0$$

it is rapidly decreasing as a function of λ, in the uniform sense that all the $\mathcal{C}^N$ norms on the kernel decay rapidly as $|\lambda| \to \infty$. This implies in particular that the norm as an operator on L^2 is small for $|\lambda|$ large, so the Neumann series converges and gives

$$(7.98) \qquad \begin{gathered} (\operatorname{Id} - R(\lambda))^{-1} = \operatorname{Id} - S(\lambda) \text{ with} \\ S(\lambda) = \sum_1^\infty R^j(\lambda) = R(\lambda) + R^2(\lambda) + R(\lambda)S(\lambda)R(\lambda). \end{gathered}$$

The second identity here, together with the fact that the smoothing operators are a semi-ideal in the bounded operators on L^2 (see (5.110)), shows that $S(\lambda)$ is also a smoothing operator which is rapidly decreasing as $|\lambda| \to \infty$ in the region (7.97). Then

$$(7.99) \qquad (P-\lambda)^{-1} = G(\lambda) - G(\lambda) \cdot S(\lambda)$$

gives a useful representation of the resolvent.

Although the representation (7.99) is restricted to the region (7.97), Theorem 7.29, in the boundaryless case, gives a similar representation, in the complement of some disc in $\mathbb{C}$ intersected with any closed sector of angle strictly less than π which does not contain $[0,\infty)$. The composite operator $G(\lambda) \cdot S(\lambda)$ in (7.99) is again a smoothing operator.

To bound the last term in (7.99) some bound on $G(\lambda)$ is needed. The uniqueness of the resolvent means that $G(\lambda)$ is a pseudodifferential operator. It is useful to see this directly, in particular to get a uniform statement near infinity. So consider the definition (7.95) as a formula for the Schwartz kernel of $G(\lambda)$:

$$G(y,y',\lambda) = \int_0^\infty e^{t\lambda} H_P(t,y,y')\phi(t)dt. \tag{7.100}$$

Given the structure of H_P as an element of the heat calculus it is natural to consider the lift of the right side of (7.100) to X^2_H, the heat space. Away from the diagonal, (7.100) shows that $G(y,y',\lambda)$ is uniformly rapidly decreasing as a smoothing operator as $|\lambda| \to \infty$. Near the diagonal the projective, parabolic, coordinates (7.36) can be used. Suppressing both the bundle coefficients and the density factors this means that

$$G(y,y',\lambda) = \int_0^\infty e^{t\lambda} t^{-\frac{n}{2}} h(t^{\frac12}, \frac{y-y'}{t^{\frac12}}, y')\phi(t)dt, \tag{7.101}$$

where h is $\mathcal{C}^\infty$ in all arguments and rapidly decreasing with all derivatives in the second argument. Localizing in $y-y'$, by multiplying by a compactly-supported cut-off factor $\chi(y-y')$ does not change the structure of (7.101), so the Fourier transform can be written:

$$\begin{aligned} a(\eta,\lambda,y') &= \int e^{i(y-y')\cdot\eta} G(y,y',\lambda)\chi(y-y')dy \\ &= \int e^{t\lambda + i(y-y')\cdot\eta} t^{-\frac{n}{2}} h(t^{\frac12}, \frac{y-y'}{t^{\frac12}}, y')\phi(t)dtdy. \end{aligned}$$

Changing the variable of integration from y to $(y-y')/t^{\frac12}$ this becomes

$$a(\eta,\lambda,y') = \int e^{t\lambda}\hat{h}(t^{\frac12}, t^{\frac12}\eta, y')\phi(t)dt,$$

where the Fourier transform of h is taken in the second set of variables. Now set $\lambda = r^2\alpha$ with $|\alpha| = 1$ and change variable of integration to $s = rt^{\frac12}$ giving

$$a(\eta,\lambda,y') = r^{-2}\int e^{\alpha s^2}\hat{h}(s/r, s\eta/r, y')\phi(\frac{s^2}{r^2})2sds. \tag{7.102}$$

In this form it is easy to see that the full symbol of G satisfies symbol estimates in η uniformly as $r = |\lambda|^{\frac12} \to \infty$. Indeed from (7.102), $|\lambda| a(\eta,\lambda,y')$

is uniformly bounded in a region where $\operatorname{Re}\alpha < 0$. Applying any one of the vector fields

$$\eta_j \frac{\partial}{\partial \eta_k}, \frac{\partial}{\partial y'_k}, r\frac{\partial}{\partial r}$$

to (7.102) gives an integral of the same type. This means that a satisfies the uniform symbol estimates

$$\begin{gathered}|D_\eta^\beta D_r^p D_{y'}^\gamma a(\eta, r^2\alpha, y')| \le C_{\beta,p,\gamma,\delta} r^{-2}(1+|\eta|)^{-|\beta|} \\ \text{in } r > 1,\ \operatorname{Re}\alpha < -\delta \text{ for any } \delta > 0.\end{gathered}$$

Thus $G(\lambda)$ is uniformly $O(|\lambda|^{-1})$ as $|\lambda| \to \infty$ as a pseudodifferential operator of order 0. In particular the second term in (7.99) is rapidly decreasing as a smoothly operator as $|\lambda| \to \infty$. In consequence, $(P-\lambda)^{-1}$ satisfies the same estimates as $G(\lambda)$, in any subset of $\mathbb{C}$ which does not meet the spectrum of P and in which $|\arg\lambda| \ge \epsilon > 0$ for $|\lambda|$ large.

Exercise 7.32. Using this argument show that the resolvent is, in the same sets, uniformly a pseudodifferential operator of order -2 (without any vanishing in the estimates as $|\lambda| \to \infty$.)

Having obtained uniform estimates, as a pseudodifferential operator, on the resolvent of P the argument above can be reversed to study the long-time behaviour of the heat kernel. From the fact that $\exp(-tP)$ is a semi-group and that it is bounded for some positive $t = T$ as an operator on L^2, the L^2 operator norm is at most exponential:

$$\|\exp(-tP)\| \le C\exp(At).$$

Thus if λ in (7.95) is taken to have real part strictly less than $-A$ the integral converges, without the cut-off function, to the resolvent:

$$(P-\lambda)^{-1} = \int_0^\infty e^{t\lambda}\exp(-tP)dt,\ \operatorname{Re}\lambda = -A-1. \tag{7.103}$$

Now as a function of $\operatorname{Im}\lambda$ this is just the Fourier transform of a function (even if operator valued) which is tempered, even square integrable. Thus this Fourier-Laplace transform can be inverted to give:

$$\exp(-tP) = \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = -A-1} e^{-t\lambda}(P-\lambda)^{-1}d\lambda.$$

Alternatively one can justify this representation directly.

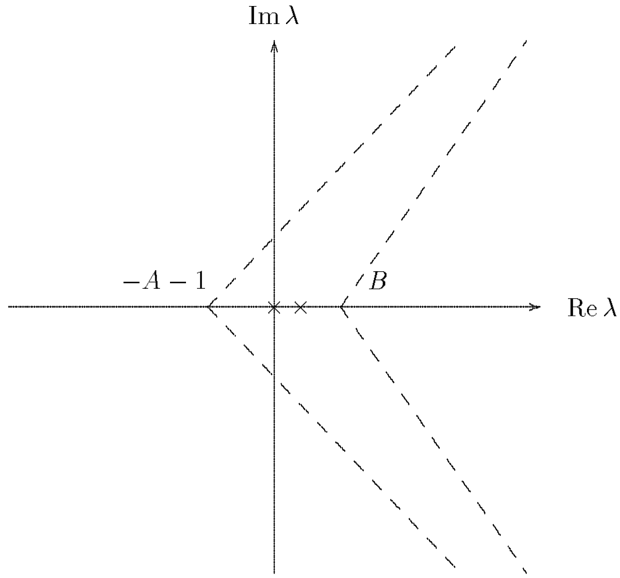

Figure 14. The contours in (7.104) and (7.106).

Using the estimates on the resolvent the integral in (7.103) can be moved, for $t > 0$, to a contour, γ_A, on which the exponential is decreasing at infinity, so giving absolute convergence:

$$\exp(-tP) = \frac{1}{2\pi i} \int_{\gamma_A} e^{-t\lambda}(P-\lambda)^{-1} d\lambda. \tag{7.104}$$

For example γ_A can be taken as in Figure 14, inward along a line segment of argument $-\delta$, $\frac{1}{2} > \delta > 0$ with end point at $(-A-1, 0)$ and outward along a segment of argument δ from this point. Now the contour can be moved to a similar one, γ_B, where the end point is at $(B, 0)$ for some $B > 0$ except for the finite number of poles of the resolvent, i.e. eigenvalues, of P crossed in between (choosing $(B, 0)$ so that γ_B does not itself hit any poles). Thus, noting that the poles have order 1, if P is self-adjoint

$$\exp(-tP) = \sum_{\text{finite}} e^{-t\lambda_j} P_j + R_B(t), \tag{7.105}$$

where the P_j are the finite rank self-adjoint projectors onto the eigenspaces for eigenvalues less than B and

$$R_B(t) = \frac{1}{2\pi i} \int\limits_{\gamma_B} e^{-t\lambda}(P-\lambda)^{-1} d\lambda. \tag{7.106}$$

The estimates on the resolvent show that $R_B(t)$ is exponentially decreasing as a smoothing operator, i.e. its kernel satisfies

$$|\partial_t^k Q R_B(t, y, y')| \le C_{k,Q} e^{-tB'}, B' < B \ \forall\ k \in \mathbb{N},\ Q \in \mathrm{Diff}^*(X; E).$$

This proves a result which is more than strong enough to handle the behaviour of the trace of the heat semi-group as $t \to \infty$:

PROPOSITION 7.33. *If $\partial X = \emptyset$ and $P \in \mathrm{Diff}^2(X; E)$ has diagonal principal symbol given by a metric and is self-adjoint then the heat kernel has a complete expansion* (7.105)*, as $t \to \infty$, with remainders exponentially decreasing smoothing operators.*

EXERCISE 7.34. State and prove an extension of Proposition 7.33 when the assumption of self-adjointness is dropped.

In fact this argument is a rather cumbersome way to prove the existence of the expansion. However it has the advantage of extending relatively easily to b-metrics, for an operator P as in Theorem 7.29. Indeed the argument leading to the representation (7.96) is valid without essential change.

Following the notation of §6.7, write the formula (7.95), in the case of a b-metric, as

$$G_s(\lambda) = \int\limits_0^\infty e^{t\lambda} \exp(-tP)\phi(t) dt. \tag{7.107}$$

In any finite interval $[T, T']$, with $0 < T < T'$, the heat kernel is a smooth function with values in the small-residual space $\Psi_b^{-\infty}(X; E)$. Thus it follows as before that the remainder term, now denoted $R_s(\lambda)$, in (7.96), is entire in λ and rapidly decreasing at infinity in the set (7.97), with values in $\Psi_b^{-\infty}(X; E)$. The estimates on $G_s(\lambda)$ are also very similar. The heat kernel near $t = 0$ needs to be lifted to X_η^2 in (7.100) but the remainder of the analysis proceeds unaltered to show that $\lambda G_s(\lambda)$ is uniformly an element of $\Psi_b^0(X; E)$ in the set (7.97). The complex scaling argument also allows this sector to be rotated, provided it does not meet the positive real axis.

The crucial difference in the case of a b-metric is, as always, that the remainder is not compact. To make it so the indicial equation needs to be solved uniformly.

LEMMA 7.35. *Let Y be compact without boundary and suppose $Q \in \operatorname{Diff}^2(Y;E)$ is self-adjoint with diagonal principal symbol given by a metric then for any $\epsilon, r > 0$ there exists R such that if*

$$Z_{r,\epsilon} = \{\lambda \in \mathbb{C}; |\lambda| > R,\ \arg z \notin [-\epsilon, \epsilon]\} \tag{7.108}$$

then $(Q - \lambda + z^2)^{-1}$ is holomorphic for

$$|\operatorname{Im} z| < r, \lambda \in Z_{r,\epsilon} \tag{7.109}$$

and uniformly bounded in $\Psi^0(Y;E)$.

PROOF: If $|\operatorname{Im} z| < r$ then for a given $\delta > 0$ there exists $R > 0$ such that $z^2 + R$ lies in the sector $|\arg(z^2 + R)| \le \delta$. Thus for R in (7.108) large enough (7.109) implies that $\lambda - z^2$ does not have argument in $[-\delta, \delta]$ so the uniformity of the resolvent in (7.109) follows from the discussion above.

Now, choosing a boundary defining function, consider a correction term, $G_B(\lambda)$, chosen to remove the indicial family of the remainder in (7.96):

$$I_\nu(G_B(\lambda), z) = (Q - \lambda + z^2)^{-1} I_\nu(R_s(\lambda), z), \tag{7.110}$$

where z is the Mellin variable and λ the spectral variable. The estimates on $R_s(\lambda)$ imply that the right side of (7.110) is holomorphic in the intersection of the set (7.109) with the sector in which $R_s(\lambda)$ is rapidly decreasing, takes values in the smoothing operators and is rapidly decreasing as $|\lambda|+|z| \to \infty$ in this set. Taking the inverse Mellin transform in z it follows that there exists a b-pseudodifferential operator in the class (5.104), i.e.

$$G_B(\lambda) \in \widetilde{\Psi}_{b,\infty}^{-\infty,(r,r)}(X;E),\ |\lambda| > R,\ \arg \lambda \in [\alpha, \beta] \tag{7.111}$$

satisfying (7.110). Here the weight, r, in the calculus with bounds, arises from the width in (7.109) of the domain of holomorphy. The sector in (7.111) has angle less than π and does not meet the positive real axis; R depends on r and the family is uniformly rapidly decreasing as $|\lambda| \to \infty$ in the sector, as an element of the calculus with bounds.

Now (7.96) is replaced by

$$(P - \lambda)[G_s(\lambda) + G_B(\lambda)] = \operatorname{Id} - R_r(\lambda), \tag{7.112}$$

where for $|\lambda| > R$ in the appropriate sector the remainder term is rapidly decreasing as $|\lambda| \to \infty$ as an element of the finitely residual calculus:

$$R_r(\lambda) \in \rho_{\mathrm{lb}}^{\epsilon} \rho_{\mathrm{rb}}^{\epsilon} H_b^{\infty}(X^2; {}^b\Omega^{\frac12}) \tag{7.113}$$

in case $E = {}^b\Omega^{\frac12}$ and similarly in general, see §5.24. It is now straightforward to prove an analogue of Proposition 7.33:

PROPOSITION 7.36. *If $P \in \mathrm{Diff}_b^2(X;E)$ is self-adjoint with diagonal principal symbol given by an exact b-metric and has indicial family with respect to the metric trivialization of the form $Q+z^2$ with Q having least eigenvalue $\sigma_0 > 0$, then the heat kernel of P has a decomposition for large t :*

$$\exp(-tP) = \sum_{\text{finite}} e^{-t\lambda_j} P_j + R_\infty(t), \tag{7.114}$$

where the P_j are the finite rank projections onto the eigenspaces of P with $\lambda_j < \sigma_0$ and there exist ϵ, $\delta > 0$ such that $\exp(\delta t) R_\infty(t)$ is uniformly bounded, with all its t-derivatives, with values in

$$\rho_{\mathrm{lb}}^{\epsilon} \rho_{\mathrm{rb}}^{\epsilon} H_b^{\infty}(X^2; \mathrm{Hom}(E \otimes {}^b\Omega^{-\frac{1}{2}}) \otimes {}^b\Omega^{-\frac{1}{2}}). \tag{7.115}$$

PROOF: The proof of Proposition 7.33 can be followed quite closely. First the identity (7.112) gives a uniform bound on the resolvent for λ large, in any sector away from the positive real axis, using (7.98) to estimate the correction term and noting Proposition 5.38 which shows that the (finitely) residual space of the calculus with bounds does form a semi-ideal in the bounded operators. Thus we conclude that in any closed sector not meeting the positive real axis and of angle less than π the resolvent has a decomposition

$$(P - \lambda)^{-1} = G_s(\lambda) + G_B(\lambda) + G_r(\lambda) \tag{7.116}$$

where $G_s(\lambda)$ is defined by (7.107), $G_B(\lambda)$ is chosen to satisfy (7.110) and (7.111) and the third term is

$$\begin{aligned} G_r(\lambda) &= (P-\lambda)^{-1} S(\lambda) \text{ where} \\ (\mathrm{Id} - R_r(\lambda))^{-1} &= \mathrm{Id} + S(\lambda). \end{aligned} \tag{7.117}$$

Thus $G_r(\lambda)$ takes values in the space (7.115) and is rapidly decreasing, in this space, as $|\lambda| \to \infty$ in the sector.

The behaviour of the resolvent in any finite part of the complex plane is discussed in §6.7. In particular the representation (7.104) can be recovered. The contour cannot be moved across the bottom of the continuous spectrum $[\sigma_0, \infty)$ of P, so one should take $B < \sigma_0$ in (7.106). Cauchy's formula gives the finite rank part of (7.114) and the estimates on the remainder follow as before.

Notice that the heat kernel itself is in the small-residual calculus, uniformly as a function of t in any finite closed subinterval of $(0, \infty)$. In particular it is uniformly rapidly decreasing at the left and right boundaries of X_b^2. However as $t \to \infty$ the 'heat' eventually starts to arrive at these boundaries and the exponential expansion at infinity in terms of the eigenvalues below the continuous spectrum gives only finite order decay at these boundaries.

7.8. Long-time behaviour, non-Fredholm case.

The results of this section are not needed in the proof of the APS theorem but they are used, in Chapter 9, in the extension of the definition of global invariants, such as the analytic torsion and the eta invariant, from the boundaryless case to the case of manifolds with exact b-metrics.

In case the hypotheses of Proposition 7.36 are fulfilled, except that P_∂ has least eigenvalue 0, the behaviour of the heat kernel as $t \to \infty$ is not as simple as (7.114). In particular there is considerable non-uniformity near the boundary. The main result needed below concerns the b-trace of the heat kernel.

PROPOSITION 7.37. *Under the hypotheses of Proposition 7.36, except that P_∂ has smallest eigenvalue* 0,

$$[0,1) \ni s \longmapsto \text{b-Tr}_\nu(\exp(-s^{-2}P)) \text{ is } \mathcal{C}^\infty$$

down to $s=0$ with

$$\lim_{t\to\infty} \text{b-Tr}_\nu(\exp(-tP)) = \frac{1}{2}(N_1+N_2), \tag{7.118}$$

where $N_1 = \dim \operatorname{null}(P,0)$ is the dimension of the null space of P acting on $H_b^\infty(X;E)$ and N_2 is the dimension of the null space of P acting on $\mathcal{C}^\infty(X;E)+H_b^\infty(X;E)$.

Note that if $P = A^*A$ with $A \in \operatorname{Diff}_b^1(X;E,F)$ then the integers in (7.118) are, respectively, the dimensions of the null space of A acting on $H_b^\infty(X;E)$ and $x^{-\epsilon}H_b^\infty(X;E)$ for $\epsilon > 0$ sufficiently small.

PROOF: The representation (7.104) remains valid, as does the discussion of the asymptotic behaviour of the resolvent. In particular the decomposition (7.116) will be used. Let $\widetilde{G}(\lambda) = (P_\lambda)^{-1} - G_s(\lambda)$ be sum of the second two terms in (7.116). Since $G_s(\lambda)$ was obtained, in (7.107), by Laplace transformation of the cut-off heat kernel it case it makes no contribution to the kernel for large t. Thus we only need to consider

$$H'_P(t) = \frac{1}{2\pi i}\int\limits_{\gamma_A} e^{-t\tau}\widetilde{G}(\tau)d\tau. \tag{7.119}$$

Choose $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\chi(r) = 1$ in $r < \frac{1}{2}B$ and $\chi(r) = 0$ in $r > \frac{3}{4}B$. Inserting this into (7.119) gives

$$\begin{aligned} H'_P(t) &= H_1(t) + H_2(t) \text{ where} \\ H_1(t) &= \frac{1}{2\pi i}\int\limits_{\gamma_A} \chi(\operatorname{Re}\tau)e^{-t\tau}\widetilde{G}(\tau)d\tau \text{ and} \\ H_2(t) &= \frac{1}{2\pi i}\int\limits_{\gamma_A} (1-\chi(\operatorname{Re}\tau))\, e^{-t\tau}\widetilde{G}(\tau)d\tau. \end{aligned} \tag{7.120}$$

In the second term the integrand is uniformly in the calculus with bounds, with a uniformly positive exponent and the real part of τ is strictly positive so, just as in the proof of Proposition 36, its b-trace is exponentially decreasing in t with all derivatives; in particular

$$\text{b-Tr}_\nu\left(H_2(s^{-2})\right) \in \mathcal{C}^\infty([0,1))$$

vanishes with all its derivatives at $s = 0$. Thus it is only necessary to examine $H_1(t)$, involving the part of the integral near $\tau = 0$.

We proceed as before, to move the contour. Of course the integrand is no longer holomorphic and convergence near the spectrum needs to be considered, so we replace γ_B by the simpler contour $\operatorname{Im} z = \delta > 0$, where $\tau = z^2$ and $\operatorname{Im} z > 0$ is the physical region. The non-holomorphic form of Cauchy's formula gives:

$$\begin{aligned} H_1(t) &= H_1'(t) + H_1''(t) \\ H_1'(t) &= \frac{1}{2\pi i} \int\limits_{\operatorname{Im} z=\delta} \chi(\operatorname{Re}\tau) e^{-t\tau} \widetilde{G}(\tau) d\tau \\ H_1''(t) &= \frac{1}{2\pi i} \iint\limits_{S(A,\delta)} \overline{\partial}\chi(\operatorname{Re}\tau) e^{-t\tau} \widetilde{G}(\tau) d\tau \wedge d\overline{\tau}. \end{aligned} \tag{7.121}$$

Here $S(A,\delta)$ is the region between γ_A and $\operatorname{Im} z = \delta$, where δ is taken small.

First consider the limit as $\delta \downarrow 0$ of the b-trace of $H_1'(t)$. To do so recall the structure of the kernel of $\widetilde{G}(z^2)$ from the construction in §6.7 and its revision in §7.7, see in particular (7.110) and (7.111). For the moment we shall suppose that P has no null space on L^2, so the resolvent has only a single pole at $z = 0$. We only need to consider $\widetilde{G}$ near the diagonal. There it decomposes into two terms:

$$\widetilde{G} = G_B + G_r \Longrightarrow H_1'(t) = H_B'(t) + H_r'(t). \tag{7.122}$$

First consider G_B which arises from the inversion of the indicial operator. It is $\mathcal{C}^\infty$ near the lifted diagonal in X_b^2 and meromorphic near $z = 0$ with just a simple pole at $z = 0$. Introducing z as variable of integration, the first term in the corresponding decomposition of the b-trace is

$$\text{b-Tr}_\nu\left(H_B'(t)\right) = \frac{1}{\pi i} \int\limits_{\operatorname{Im} z=\delta} e^{-tz^2} f_1(z) dz. \tag{7.123}$$

Here $f_1(z) = z\chi(z)\,\text{b-Tr}_\nu(G_1')$ is $\mathcal{C}^\infty$ in $\operatorname{Re} z$ uniformly as $\delta \downarrow 0$ and has uniformly compact support. Thus the integral converges as $\delta \downarrow 0$ and changing variable to $z = sZ$, $t = 1/s^2$, gives

$$\lim_{\delta\downarrow 0} \text{b-Tr}_\nu \left(H'_B(s^{-2})\right) = s\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-Z^2} f_1(sZ)dZ. \tag{7.124}$$

Clearly this is s times a $\mathcal{C}^\infty$ function of s^2.

Next consider the term in the b-trace which arises from G_r. This is the leading part of the term in the residual calculus and is of the form $\Phi'_2(z)(xx')^{-iz}\chi'(x)/z$, with Φ'_2 taking values in the smoothing operators on the boundary and having a simple pole at $z = 0$ and where $\chi'(x)$ localizes near the boundary. This term is trace class for $\operatorname{Im} z = \delta > 0$ but not uniformly down to $\delta = 0$. As in (7.123)

$$\text{b-Tr}_\nu \left(H'_r(t)\right) = \frac{1}{\pi i}\int_{\operatorname{Im} z=\delta} e^{-tz^2}\chi(\operatorname{Re} z)f'_2(z)\int \chi'(x)x^{-2iz}\frac{dx}{x}dz. \tag{7.125}$$

For $\delta > 0$ the integrals converge absolutely, so the order can be freely changed. Introducing $z = sZ$ as before and $\Xi = 2s\log x$

$$\text{b-Tr}_\nu \left(H'_r(s^{-2})\right) = \frac{1}{2\pi i}\int \chi'(\exp(\frac{\Xi}{2s})B_2(s,\Xi)d\Xi \tag{7.126}$$

$$B_2(s,\Xi) = \int_{\operatorname{Im} Z=\delta/s} e^{-Z^2} f_2(sZ)e^{-iZ\Xi}dZ. \tag{7.127}$$

Here again f_2 is uniformly $\mathcal{C}^\infty$ and of compact support as $\delta \downarrow 0$. Taking the limit $\delta \to 0$ (7.127) can be regarded as the Fourier transform of $\exp(-Z^2)f_2(sZ)$ which is uniformly in Schwartz' space $\mathcal{S}(\mathbb{R})$ and depends smoothly on s. Thus $B_2(s,\Xi)$ is uniformly rapidly decreasing in the second variable, with all derivatives. Inserting this into (7.126) shows that $\text{b-Tr}_\nu \left(H'_r(s^{-2}\right)$ is a $\mathcal{C}^\infty$ function of s down to $s = 0$. Note that the extra factor of s has been absorbed in the change of variable from $\log x$ to Ξ.

Consider next the part $H''_1(t)$ of the heat kernel near $t = \infty$ in (7.121). This can be treated in exactly the same way, by dividing considering the two terms. The important difference is that the complex derivative of $\chi(\operatorname{Re}\tau)$ has support in a region $|\operatorname{Re} z| \geq B/4 > 0$. It follows that both three terms are smooth in $s = t^{-\frac{1}{2}}$ but rapidly decreasing (in fact exponentially decreasing) as $t \to \infty$, so in particular they vanish to infinite order at $s = 0$.

This completes the proof of the regularity statement on the b-trace of the heat kernel under the assumption that there is no L^2 null space. If there is such a term then, by Proposition 6.28, it contributes a simple pole in τ at $\tau = 0$ with residue the projector onto the eigenspace. This just adds a constant term, equal to the dimension of the L^2 null space, to the b-trace. The only other source of a constant term as $t \to \infty$ is $H'_r(t)$ in the decomposition (7.122). From (7.126) and (7.127)

$$\lim_{t\to\infty} \text{b-Tr}_\nu \left(H'_r(t)\right) = \frac{1}{2\pi i}\int B'(0,\Xi)d\Xi = \frac{1}{2}f_2(0). \tag{7.128}$$

From Proposition 6.32 if follows that $f_2(0) = N_2 - N_1$ is the dimension of the boundary data of the null space of P on $\mathcal{C}^\infty(X;E) + H_b^\infty(X;E)$.

Chapter 8. Local index theorem

The cancellation between the expansions at $t = 0$ of the traces of the two heat kernels in (In.18) will be discussed using a rescaling argument due to Getzler. Although, as in [34], this can be carried out in local coordinates it is interpreted globally here. To do so the process of scaling, or rescaling, a vector bundle at a boundary hypersurface of a compact manifold with corners is examined. Such rescaling takes into account some jet information at the hypersurface. Getzler's rescaling of the homomorphism bundle of the spinor bundle, lifted to the heat space, is of this type and will be used to prove the local index theorem. As usual the point of giving a general treatment (rather than just a computation in local coordinates) is that it can be carried over to other settings. In particular the extension to the b-setting is immediate. The integral defining the eta invariant is interpreted as a b-integral in this way (in fact it converges absolutely). These rescaling arguments are used in the next chapter to discuss the convergence of the integral for the eta invariant on an odd dimensional exact b-spin manifold and also to extend the analytic torsion of Ray and Singer (see [30]) to this class of complete Riemann manifolds.

8.1. Simple rescaling.

Suppose first that X is a manifold with boundary and E is a vector bundle over X. One can easily recover the fibres of the vector bundle from the space of all $\mathcal{C}^\infty$ sections. Indeed if $p \in X$ consider

$$\mathcal{I}_p \cdot \mathcal{C}^\infty(X;E) = \{u \in \mathcal{C}^\infty(X;E); u(p) = 0\}.$$

The notation here suggests that if $\mathcal{I}_p = \{f \in \mathcal{C}^\infty(X); f(p) = 0\}$ then

$$u \in \mathcal{C}^\infty(X;E),\ u(p) = 0 \Longleftrightarrow u = \sum_{\text{finite}} f_j u_j,\ f_j \in \mathcal{I}_p, u_j \in \mathcal{C}^\infty(X;E)$$

and this is easily seen to be the case. Clearly then

$$E_p = \mathcal{C}^\infty(X;E)/\mathcal{I}_p \cdot \mathcal{C}^\infty(X;E). \tag{8.1}$$

It turns out that this is rather a useful way to construct vector bundles, starting from the putative space of all sections. The following construction is a special (and frequently encountered) case of the general rescaling process described below. Let E be a vector bundle over X and suppose that *at the boundary* E has a subbundle:

$$F \subset E_{\restriction \partial X} \text{ is a subbundle.} \tag{8.2}$$

Then consider the space

$$\mathcal{D} = \{u \in \mathcal{C}^\infty(X; E); u_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; F)\}, \tag{8.3}$$

just consisting of the sections of E which take values in F at the boundary. Observe that $\mathcal{D}$ has many of the properties of $\mathcal{C}^\infty(X; E)$. For instance it is a $\mathcal{C}^\infty(X)$-module, $\mathcal{C}^\infty(X) \cdot \mathcal{D} = \mathcal{D}$. In fact it *is* the space of all sections of a vector bundle:

PROPOSITION 8.1. *If* (8.2) *holds then* $\mathcal{D}$, *given by* (8.3), *defines a vector bundle through*

$$^F E_p = \mathcal{D}/\mathcal{I}_p \cdot \mathcal{D} \quad \forall\, p \in X, \quad {}^F E = \bigsqcup_{p \in X} {}^F E_p \tag{8.4}$$

with a natural $\mathcal{C}^\infty$ *structure and bundle map* $\iota_F \colon {}^F E \longrightarrow E$ *which is an isomorphism over the interior and such that*

$$\iota_F^* \mathcal{D} = \mathcal{C}^\infty(X; {}^F E). \tag{8.5}$$

PROOF: The vector bundle, $^F E$, with its $\mathcal{C}^\infty$ structure is defined by analogy with (8.1). With the fibres defined by (8.4) there is an obvious map

$$\iota_F \colon \mathcal{D}/\mathcal{I}_p \cdot \mathcal{D} = {}^F E_p \longrightarrow E_p = \mathcal{C}^\infty(X; E)/\mathcal{I}_p \cdot \mathcal{C}^\infty(X; E) \tag{8.6}$$

which is just evaluation of $u \in \mathcal{D}$ at p. Over the interior of X, (8.6) is an isomorphism. Suppose $p \in \partial X$ and let $u_1, \dots, u_N$ be a local, $\mathcal{C}^\infty$, basis of E such that $u_1, \dots, u_k$ is a basis of F near p in ∂X. Such a basis can be obtained simply by taking $u_1, \dots, u_k$ as a basis of F near p, extending these sections smoothly off ∂X and then completing them to a basis of E.

Any element of $\mathcal{D}$ is locally of the form

$$u = \sum_{j=1}^{k} f_j u_j + \sum_{j>k} f_j (x u_j), \tag{8.7}$$

where x is a defining function for ∂X since, by definition, it must be a section of F over ∂X. The coefficients $f_1, \dots, f_N$ in (8.7) give a local trivialization of $^F E$, which is to say that $u_1, \dots, u_k, x u_{k+1}, \dots, x u_N$ is a local basis. A change of admissible basis of E from $u_1, \dots, u_N$ to $u'_1, \dots, u'_N$ must be such that

$$u'_j = \sum_{l=1}^{k} a_{jl} u_l + x \sum_{l=k+1}^{N} a_{jl} u_l, \ j \le k$$

since the $(u'_j)_{\restriction \partial X}$, for $j \le k$, span F. Thus the basis elements u'_r, for $1 \le r \le k$, and $x u_r$ for $k < r \le N$ are smooth linear combinations of the $u_1, \dots, u_k, x u_{k+1}, \dots, x u_N$, so this induces a $\mathcal{C}^\infty$ transformation amongst the f_i. The $\mathcal{C}^\infty$ bundle structure is therefore natural. Clearly the bundle map ι_F is $\mathcal{C}^\infty$ and (8.5) follows directly from (8.7).

EXERCISE 8.2. This construction of the bundle ${}^F E$ should be at least vaguely familiar, since it occurred in the definition of the b-tangent bundle. Take

$$E = TX, \ F = T\partial X \subset T_{\partial X} X$$

and show that resulting bundle ${}^F E$ is canonically isomorphic to ${}^b TX$.

EXERCISE 8.3. Show that the bundle ${}^F E$ constructed above is always bundle-isomorphic to E. (There is generally no *natural* isomorphism.)

8.2. Rescaling bundles.

A considerable generalization of this construction will now be made where, rather than just one subbundle, there is a sequence of subbundles, i.e. a filtration

$$F^k \subset F^{k-1} \subset \cdots \subset F^0 \subset E_{\restriction \partial X}. \tag{8.8}$$

It turns out that this bundle filtration is not in itself enough information to give an unambiguous rescaling. It will be required that

$$\begin{aligned} F^j &\text{ is a } j\text{-jet of subbundle of } E \text{ at } \partial X,\\ &\text{consistent with } F^{j-1} \text{ as a } (j-1)\text{-jet.} \end{aligned} \tag{8.9}$$

Of course this needs to be explained. In practice in the cases which arise below, (8.9) follows from the more obvious condition that

(8.10) the filtration (8.8) has an extension to a neighbourhood of ∂X.

The jet conditions are just the weakened version of (8.10) which suffices for the construction. Suppose that $\tilde{F}$ and $\tilde{G}$ are subbundles of E near ∂X. They will be said to be equal, as p-jets at ∂X, if

$$\begin{aligned} &\{u \in \mathcal{C}^\infty(X;E); u = u' + x^{p+1}u'', u' \in \mathcal{C}^\infty(X;\tilde{F}), u'' \in \mathcal{C}^\infty(X;E)\}\\ &\quad = \{v \in \mathcal{C}^\infty(X;E); v = v' + x^{p+1}v'', v' \in \mathcal{C}^\infty(X;\tilde{G}), v'' \in \mathcal{C}^\infty(X;E)\}. \end{aligned} \tag{8.11}$$

That is, each section of $\tilde{F}$ is, near ∂X, the sum of a section of $\tilde{G}$ and x^{p+1} times a section of E, and conversely with $\widetilde{F}$ and $\widetilde{G}$ interchanged.

By a p-jet of subbundle of E at ∂X is meant an equivalence class of subbundles near ∂X, up to equality as p-jets. Clearly a 0-jet is just a subbundle of $E_{\restriction \partial X}$. If F and G are p-jets of subbundles of E then the condition

$$\begin{aligned} \mathcal{C}^\infty(X;\widetilde{F}) \subset \big\{u \in \mathcal{C}^\infty(X;E); u = u' + x^{p+1}u'',&\\ u' \in \mathcal{C}^\infty(X;\widetilde{G}), u'' \in \mathcal{C}^\infty(X;E)\big\}& \end{aligned} \tag{8.12}$$

is easily seen to be independent of the choice of representative subbundles, $\widetilde{F}$ and $\widetilde{G}$. It is equivalent to the existence of representatives $\widetilde{F}$ and $\widetilde{G}$ with $\widetilde{F} \subset \widetilde{G}$. The condition (8.12) on F and G will be indicated by writing $F \subset G$ as p-jets of subbundles. Certainly any p-jet of subbundle, F, defines (if $p > 0$) a $(p-1)$-jet of subbundle. Then a p-jet F and a $(p-1)$-jet G are said to be consistent if $F \subset G$ as $(p-1)$-jets of subbundles. This explains the meaning of the assumption in (8.9). Such a consistent system of jets will be called a *jet filtration.* The integer $k+1$ is the *length* of the jet filtration. Thus the simple rescaling of §8.1 corresponds to a jet filtration of length 1.

PROPOSITION 8.4. *Let E be a $\mathcal{C}^\infty$ vector bundle over a manifold with boundary, X, and let F be a jet filtration in the sense that the F^j are, for $j = 0, \dots, k$, j-jets of subbundle of E at ∂X satisfying* (8.9), *in the sense of* (8.12), *then there is a $\mathcal{C}^\infty$ vector bundle ${}^FE$ over X and a bundle map $\iota_F : {}^FE \longrightarrow E$ which is an isomorphism over the interior and is such that*

$$\mathcal{C}^\infty(X; {}^FE) = \iota_F^* \mathcal{D},$$

$$\mathcal{D} = \Big\{u \in \mathcal{C}^\infty(X;E); u \in \sum_{j=0}^{k} x^{k-j}\mathcal{C}^\infty(X;\widetilde{F}^j) + x^{k+1}\mathcal{C}^\infty(X;E) \text{ near } \partial X\Big\}, \tag{8.13}$$

the $\widetilde{F}^j$ being representatives of the F^j.

PROOF: This is just the same as the proof of Proposition 8.1, once the existence of a reasonable basis for E is shown. First note that the space $\mathcal{D}$ in (8.13) is independent of the choice of representatives $\widetilde{F}^j$. Indeed, from the definition of equality of p-jets in (8.11), if F is a p-jet of subbundle of ∂X then

$$\Big\{u \in \mathcal{C}^\infty(X;E); u = u' + x^{p+1}u'', u' \in \mathcal{C}^\infty(X;\widetilde{F}), u'' \in \mathcal{C}^\infty(X;E) \text{ near } \partial X\Big\} \tag{8.14}$$

is independent of the choice of representative subbundle. Thus for any l the space obtained from (8.14) by multiplying by x^l is also independent of choices. Summing over $l = k - j$ with $p = j$ shows that $\mathcal{D}$ in (8.13) is independent of the choice of representatives of the F^j.

Now observe that if $F \subset G$ are a j-jet and a $(j-1)$-jet which are consistent in the sense of (8.12) then near a point $p \in \partial X$ any basis $u_1, \dots, u_f$ of a representative of F can be extended to a basis $u_1, \dots, u_f, u_{f+1}, \dots, u_g$ of a representative of G. Applying this construction repeatedly gives, near p, a basis of E of the form

$$u_1, \dots, u_{f_k}, u_{f_k+1}, \dots, u_{f_{k-1}}, \dots, u_N,$$

where $f_j = \operatorname{ran}(F^j)$ and $u_1, \dots u_{f_j}$ is a basis of some representative of F^j. Thus

$$u \in \mathcal{D} \Longleftrightarrow u = \sum_{l=1}^{N} g_l x^{o(l)} u_l,$$

where $o(l) = k - j$ if $f_{j+1} < l \le f_j$, $(f_{k+1} = 0)$. This gives the desired $\mathcal{C}^\infty$ basis, $x^{o(l)}u_l$, for ${}^F E$. Clearly the bundle structure is independent of the basis chosen so the proposition is proved.

Notice that associated to the filtration of $E_{\restriction \partial X}$ induced by the jet filtration there is a natural *graded* vector bundle, namely

$$\bigoplus_{j=-1}^{k} (N^*\partial X)^{k-j} \otimes \left(F^j_{\restriction \partial X} / F^{j+1}_{\restriction \partial X} \right), \ F^{k+1} = \{0\}, \ F^{-1} = E_{\restriction \partial X}. \tag{8.15}$$

The powers of the conormal bundle here just arise from the factors of x in the definition of $\mathcal{D}$. This bundle has the same rank as E and in fact:

LEMMA 8.5. *If ${}^F E$ is the bundle obtained by rescaling E with respect to a jet filtration at ∂X then ${}^F E_{\restriction \partial X}$ is canonically isomorphic to the graded bundle,* (8.15), *associated to the filtration induced on $E_{\restriction \partial X}$ by the jet filtration.*

PROOF: The natural isomorphism can be defined directly at the level of sections. Thus the space of all $\mathcal{C}^\infty$ sections of ${}^F E_{\restriction \partial X}$ is naturally identified with the sections of ${}^F E$ modulo those which vanish at the boundary, i.e.

$$\mathcal{C}^\infty(\partial X; {}^F E) \equiv \mathcal{D}/x\mathcal{D}.$$

From (8.13) the map

(8.16)

$$\begin{aligned} \mathcal{D} \ni u = \sum_{j=0}^{k} x^{k-j} u_j + x^{k+1} u' \longmapsto \\ ([u_k], (dx) \otimes [u_{k-1}], \dots, (dx)^k \otimes [u_0], (dx)^{k+1} \otimes [u']) \\ \in \mathcal{C}^\infty \left(\partial X; \bigoplus_{j=-1}^{k} (N^*\partial X)^{k-j} \otimes \left(F^j_{\restriction \partial X} / F^{j+1}_{\restriction \partial X} \right) \right) \end{aligned}$$

is well defined, where $[u_j]$ denotes the section of $F^j_{\restriction\partial X}/F^{j+1}_{\restriction\partial X}$ defined by u_j restricted to ∂X. In fact the map (8.16) is clearly surjective, because of the compatibility condition between the jets, and has null space exactly $x\mathcal{D}$.

There is also the possibility of making an overall conformal rescaling of a bundle, by multiplying by any complex power of a boundary defining function.

LEMMA 8.6. *If E is a vector bundle over a compact manifold with boundary then for any $z \in \mathbb{C}$ the space*

$$x^z\mathcal{C}^\infty(X;E) = \{u \in \mathcal{C}^{-\infty}(X;E); u = x^z v, v \in \mathcal{C}^\infty(X;E)\}$$

is the space of all sections of a vector bundle, denoted x^zE, i.e. there is a $\mathcal{C}^\infty$ bundle map over the interior of X, $\iota\colon x^zE \longrightarrow E$ such that $\mathcal{C}^\infty(X;x^zE) = \iota^*\left(x^z\mathcal{C}^\infty(X;E)\right)$.

Of course these two types of scaling are very closely related and can be combined.

PROOF: Clearly if $u_1, \dots, u_k$ is any local basis for E then $x^zu_1, \dots, x^zu_k$ is a local basis for x^zE and the transition matrices are the same for E and x^zE.

If X is a compact manifold with corners and H is a fixed boundary hypersurface then the notions of a j-jet of a subbundle, F^j, at H of a fixed bundle is an immediate generalization of (8.11), where x is interpreted as a defining function for H and the notion of a jet filtration is defined in the natural way. Then Proposition 8.4 extends immediately too. Thus one can associate in a completely natural way a *rescaled bundle* ${}^FE$ with a jet filtration, F, of E at H :

$$\begin{gathered}\text{For } j = 0, \dots, k,\ F^j \text{ is a } j\text{-jet of subbundle of } E \text{ at } H,\\ \text{with } F^j \text{ consistent with } F^{j-1} \text{ as a } (j-1)\text{-jet.}\end{gathered}$$

For this rescaled bundle (8.5) holds so certainly

$$x^{k+1}\mathcal{C}^\infty(X;E) \subset \mathcal{C}^\infty(X;{}^FE) \subset \mathcal{C}^\infty(X;E). \tag{8.17}$$

One consequence of (8.17) is that the space of sections which vanish to infinite order at the boundary is the same for the bundle E and the rescaled bundle ${}^FE$:

$$\dot{\mathcal{C}}^\infty(X;E) \cong \dot{\mathcal{C}}^\infty(X;{}^FE)$$

whatever the jet filtration. The rescaling procedure also has a simple stability property under tensor products. If E is a bundle with jet filtration F^j at the boundary and G is another bundle then

$${}^L(E\otimes G) \equiv {}^FE \otimes G \text{ if } L \text{ is the jet filtration } L^j = F^j \otimes G. \tag{8.18}$$

Similarly if E has jet filtration F then there is a natural jet filtration of the dual bundle E' given by taking the annihilators of representatives

$$\widetilde{L}^j = (\widetilde{F}^{k-j})^\circ, \text{ i.e.}$$
$$\mathcal{C}^\infty(U;\widetilde{L}^j) = \left\{u \in \mathcal{C}^\infty(U;E'); u(v) = 0 \ \forall\ v \in \mathcal{C}^\infty(U;\widetilde{F}^{k-j})\right\}, \tag{8.19}$$

in a neighbourhood $U \supset \partial X$ of the boundary. This dual filtration gives rise to a rescaling, ${}^L(E')$, of the dual bundle.

EXERCISE 8.7. Check that (8.19) does indeed define a jet filtration of the dual bundle and in particular that it is independent of the choice of representative filtration.

LEMMA 8.8. *If ${}^FE$ is the bundle defined by rescaling E with respect to a jet filtration of length $k+1$ at the boundary and ${}^L(E')$ is obtained by rescaling the dual bundle with respect to the dual jet filtration given in* (8.19) *then there is a natural identification of $({}^FE)'$ and the bundle $x^{-k-1}{}^L(E')$.*

EXERCISE 8.9. Write down a local basis of E, adapted to the jet filtration F, and show that the dual basis of E' is adapted to the dual filtration L; deduce that there is an overall factor of x^{k+1} in the pairing and so prove the lemma.

As a corollary of this lemma and the stability under tensor products the space of extendible distributional sections of ${}^FE$ is canonically the same as that of E :

$$\mathcal{C}^{-\infty}(X;E) \cong \mathcal{C}^{-\infty}(X;{}^FE) \text{ for any rescaling by a jet filtration at } \partial X. \tag{8.20}$$

Indeed the first space is the dual of $\dot{\mathcal{C}}^\infty(X;E'\otimes\Omega)$ and the second is the dual of

$$\dot{\mathcal{C}}^\infty(X;({}^FE)'\otimes\Omega) \cong \dot{\mathcal{C}}^\infty(X;{}^L(E')\otimes\Omega) \cong \dot{\mathcal{C}}^\infty(X;E'\otimes\Omega)$$

with the topologies also the same.

8.3. Rescaling and connections.

If E is a $\mathcal{C}^\infty$ vector bundle over a manifold with boundary and $E_{\restriction\partial X}$ has a filtration, as in (8.8), a (true) connection on E can be used to extend the filtration off the boundary and hence to give a jet filtration. For simplicity let us suppose initially that ∂X has only one component.

Let N be an inward-pointing vector field, i.e. $N \in \mathcal{V}(X)$ is such that $Nx_{\restriction \partial X} > 0$ for any boundary defining function x. Simply define the space $\mathcal{D}$ by

(8.21)

$$\text{if } u \in \mathcal{C}^\infty(X;E) \text{ then } u \in \mathcal{D} \Longleftrightarrow \nabla_N^p u_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; F^{k-p}),\ 0 \le p \le k.$$

To see that this space does actually arise from an extension of the filtration off the boundary consider a defining function $x \in \mathcal{C}^\infty(X)$ such that $Nx = 1$ near ∂X. If $\epsilon > 0$ then $U = \{x < \epsilon\}$ is a neighbourhood of the boundary. If ϵ is small enough any section u of E over ∂X can be extended to a unique section $\widetilde{u} \in \mathcal{C}^\infty(U;E)$ which is covariant constant along N :

$$\nabla_N \widetilde{u} = 0,\ \widetilde{u}_{\restriction \partial X} = u. \tag{8.22}$$

Indeed, (8.22) is a system of ordinary differential equation along the integral curves of N. If x is used as a normal coordinate, so $U = [0,\epsilon)_x \times \partial X$ then $N = \partial/\partial x$ and (8.22) becomes

$$\frac{\partial}{\partial x}\widetilde{u}_i + \sum_{j=1}^{N} \gamma_{i,j}\widetilde{u}_j = 0,\ (\widetilde{u}_j)_{\restriction x=0} = u_j,$$

where $\widetilde{u}_j$ are the coefficients of $\widetilde{u}$ with respect to some basis e_j of E, the $\gamma_{i,j}$ arise from covariant differentiation of the basis, $\nabla_N e_j = \sum_l \gamma_{l,j} e_l$ and the u_i are coefficients of u with respect to this basis.

Thus the choice of N extends the filtration (8.8) to a filtration over U by taking $\mathcal{C}^\infty(U;\widetilde{F}^j)$ to be the span, over $\mathcal{C}^\infty(U)$, of the solutions to (8.22) with initial data in F^j. One can think of the filtration (8.8) as being extended by parallel transport along N. The Taylor series of any $u \in \mathcal{C}^\infty(U;E)$ at ∂X can be written to any order, r, as

$$u = u_0 + xu_1 + \cdots + x^r u_r + x^{r+1} u_{(r+1)},\ u_{(r+1)} \in \mathcal{C}^\infty(U;E) \tag{8.23}$$

where the $u_j \in \mathcal{C}^\infty(U;E)$ satisfy $\nabla_N u_j = 0$, $j!(u_j)_{\restriction \partial X} = (\nabla_N^j u)_{\restriction \partial X}$. The definition of $\mathcal{D}$ in (8.21) therefore reduces to (8.13) for the extension of the filtration.

In general this filtration and the induced jet filtration depend on the choice of N although replacing N by ϕN, where $\phi \in \mathcal{C}^\infty(X)$ is positive, leaves (8.22), and hence the filtration, unchanged. Next we consider a sufficient condition for vector fields to act on the rescaled bundle through covariant differentiation.

LEMMA 8.10. *Let E be a vector bundle with connection on a manifold with boundary, X, and let $\mathcal{D}$ be defined, by (8.21), from some filtration, (8.8), over the boundary and some inward-pointing vector field N. Then, for a given $V \in \mathcal{V}_b(X)$, ∇_V acts on $\mathcal{D}$ provided the induced connection on ∂X preserves the filtration and the curvature operator satisfies*

$$K_E(N,V)_{\restriction \partial X} : F^l \longrightarrow F^{l-1} \ \forall \ 0 \le l \le k \tag{8.24}$$

and for every $W \in \mathcal{V}_b(X)$, $1 \le l \le k$ and $1 \le p \le l$

$$\nabla^p_N \left(K_E(N,W)\right)_{\restriction \partial X} : F^l \longrightarrow F^{l-2}. \tag{8.25}$$

PROOF: If $u \in \mathcal{D}$ consider the Taylor series of $\nabla_V u$ at ∂X. Since $V \in \mathcal{V}_b(X)$ and the induced connection on the boundary is assumed to preserve the filtration it follows that $(\nabla_V u)_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; F^k)$.

For the first normal derivative

$$\nabla_N \nabla_V u = K_E(N,V)u + \nabla_{[N,V]}u + \nabla_V \nabla_N u. \tag{8.26}$$

At ∂X the first term is a section of F^{k-1} since $u_{\restriction \partial X}$ is a section of F^k and by (8.24) the curvature operator maps this into F^{k-1}. In the second term in (8.26) the commutator can be decomposed into the sum of a multiple of N and a term in $\mathcal{V}_b(X)$ giving, over the boundary, sections of F^{k-1} and F^k respectively. The last term is also in F^{k-1} over the boundary, hence so is $\nabla_N \nabla_V u$.

Next consider the action of the covariant derivative of a general element $W \in \mathcal{V}_b(X)$. It is convenient to use an inductive argument on the length of the filtration. Generalize (8.21) by defining

$$u \in \mathcal{D}_r \iff \nabla^p_N u_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; F^{r-p}), \ 0 \le p \le r. \tag{8.27}$$

Thus $\mathcal{D}_r$ just corresponds to shortening the filtration to length $r+1$. Clearly $\mathcal{D}_0 = \mathcal{C}^\infty(X; {}^{F^0}E)$ corresponds to the simple rescaling of E with respect to F^0, $\mathcal{D}_k = \mathcal{D}$ and for $r < k$

$$u \in \mathcal{D}_{r+1} \iff u, \nabla_N u \in \mathcal{D}_r, \ u_{\restriction \partial X} \in \mathcal{C}^\infty(\partial X; F^{r+1}). \tag{8.28}$$

From (8.25) we shall deduce that for any $\mathcal{C}^\infty$ vector field W on X

$$\nabla_W : \mathcal{D}_{r+1} \longrightarrow \mathcal{D}_r. \tag{8.29}$$

Since this follows for N from the definition we can assume that $W \in \mathcal{V}_b(X)$. Then (8.29) certainly holds for $r = 0$, so we proceed inductively over r. Replacing V by W in (8.26) gives

$$u \in \mathcal{D}_{r+1} \Longrightarrow \nabla_N \nabla_W u = K_E(N,W)u + \nabla_{[N,W]}u + \nabla_W \nabla_N u. \tag{8.30}$$

The first term on the right is in $\mathcal{D}_{r-1}$ because of (8.25), the second is in $\mathcal{D}_r$ and the third in $\mathcal{D}_{r-1}$ by the inductive hypothesis. Since $(\nabla_W u)_{\restriction\partial X}$ is a section of $F^{r+1} \subset F^r$ we conclude that $\nabla_W u \in \mathcal{D}_r$, by (8.28). This proves (8.29) in general.

So consider the higher normal derivatives of $\nabla_V u$ where now $V \in \mathcal{V}_b(X)$ is the vector field satisfying (8.24). Again induction can be used to show that

$$\nabla_V : \mathcal{D}_r \longrightarrow \mathcal{D}_r \ \forall\ r. \tag{8.31}$$

Assuming $u \in \mathcal{D}_r$ it follows that $\nabla_V u \in \mathcal{D}_{r-1}$ from (8.29) and in (8.26) the second term is in $\mathcal{D}_{r-1}$, again by (8.29), and the first is in $\mathcal{D}_{r-1}$ by (8.24). Since $(\nabla_V u)_{\restriction\partial X}$ is a section of F^r this proves (8.31). The statement of the lemma is the special case $r = k$.

EXERCISE 8.11. Show that if $V \in \mathcal{V}_b(X)$ acts on $\mathcal{D}$, defined by parallel transport of the filtration along N, then the same jet filtration arises by parallel transport along $N + V$.

Lemma 8.10 shows that, assuming that the connection preserves the filtration over the boundary and (8.25), for any vector field $V \in \mathcal{V}_b(X)$ satisfying (8.24) ∇_V acts on the rescaled bundle. The identity (2.58) holds by continuity, so $(\nabla_V u)_{\restriction\partial X}$ can be determined from $u_{\restriction\partial X}$, i.e. ∇_V induces an operator on the rescaled bundle over the boundary, (8.15). To obtain a formula for this operator consider, as in the proof above, the normal covariant derivatives. Thus, for $u \in \mathcal{D}$,

$$\nabla_N^p \nabla_V u = \nabla_V \nabla_N^p u + \sum_{j=1}^{p} \nabla_N^{j-1}\left((K_E(N,V) + \nabla_{[N,V]})\nabla_N^{p-j} u\right). \tag{8.32}$$

Decomposing $[N, V] = aN + W$, with $W \in \mathcal{V}_b(X)$, it follows by further commutation that

$$\nabla_N^{j-1}\nabla_{[N,V]}\nabla_N^{p-j} u_{\restriction\partial X} \equiv a_{\restriction\partial X}(\nabla_N^p u)_{\restriction\partial X} \quad (\mathrm{mod}\ \mathcal{C}^\infty(\partial X; F^{k-p+1})). \tag{8.33}$$

Similarly further commutation on the part of (8.32) involving the curvature operator gives

$$\begin{gathered}\left(\nabla_N^{j-1}(K_E(N,V)\nabla_N^{p-j} u\right)_{\restriction\partial X} \equiv \left(K_E(N,V)\nabla_N^{p-1} u\right)_{\restriction\partial X} \\ +(j-1)\left(\nabla_N K_E(N,V)\right)_{\restriction\partial X}\left(\nabla_N^{p-1} u\right)_{\restriction\partial X} \quad (\mathrm{mod}\ \mathcal{C}^\infty(\partial X; F^{k-p+1}))\end{gathered} \tag{8.34}$$

since, in view of (8.25), it is readily seen that terms involving covariant derivatives of the curvature operator of order higher that one cannot contribute to the leading part. From these formulæ and a slight generalization we easily deduce:

PROPOSITION 8.12. *If X is a compact manifold with corners and a $\mathcal{C}^\infty$ vector bundle, E, with connection over X has a filtration, F, over a boundary hypersurface, H,*

$$F^k \subset F^{k-1} \subset \cdots \subset F^0 \subset E_H, \tag{8.35}$$

which is preserved by the induced connection and is such that the curvature operator satisfies (8.25) at H for some inward-pointing vector field, N, then covariant differentiation by any $V \in \mathcal{V}_b(X)$ satisfying (8.24) at H acts on the rescaled bundle, ${}^F E$, obtained by parallel transport along N from H

(8.36)
$$\begin{aligned}\nabla_V v_j = [\nabla_{V\restriction H} v_j + (k-j) n_V v_j]_j + [K_E(N,V)_{\restriction H} v_j]_{j-1} \\ + \frac{1}{2}[(\nabla_N K_E(N,V))_{\restriction H} v_j]_{j-2},\ \forall\ v_j \in \mathcal{C}^\infty(H; F^j/F^{j+1}),\end{aligned}$$

where n_V is the normal component of $[N,V]$.

PROOF: First suppose that X is a manifold with connected boundary, H. Then (8.36) follows from (8.32), (8.33) and (8.34) since if N is used to trivialize the normal, and hence conormal bundle of H, the pieces of the map (8.16) are given by

$$\mathcal{D} \ni u \longmapsto \frac{1}{p!}(\nabla_N^p u)_{\restriction H} \in \mathcal{C}^\infty(H; F^{k-p}). \tag{8.37}$$

The extension to a manifold with corners requires only modest reinterpretation of the discussion, provided the vector field N is chosen to be tangent to the boundary hypersurfaces other than H.

8.4. Getzler's rescaling.

The example to which these constructions are directed comes from [34]. Recall, from Chapter 3, the basic properties of the Clifford bundle of a Riemann manifold. This bundle is naturally filtered by the degree, as in (3.13):

$$\mathbb{C} = \mathbb{C}l^{(0)} \subset \mathbb{C}l^{(1)} \subset \cdots \subset \mathbb{C}l^{(N)} = \mathbb{C}l \quad N = \dim X. \tag{8.38}$$

These are real filtrations, over the whole manifold. If X is an even-dimensional spin manifold then (3.65) shows that there is a natural identification

$$\mathbb{C}l_x \longleftrightarrow \hom(S_x) \quad \forall\ x \in X. \tag{8.39}$$

That is, the Clifford bundle is naturally identified with the homomorphism (or endomorphism) bundle of the spinor bundle. Here the little homomorphism bundle, $\hom(S)$, is being considered.

As a first, very simple, example of rescaling consider the lift of $\hom(S)$ from X to $[0,\infty)_{\frac{1}{2}} \times X$. Let $\pi\colon [0,\infty)_{\frac{1}{2}} \times X \longrightarrow X$ be the projection. The filtration (8.38) gives a filtration of $\pi^* \hom(S)$, and so in particular a jet filtration at $\{0\} \times X$. This allows a rescaled bundle, denoted ${}^G\hom(S)$, to be defined by application of Proposition 8.4. It will reappear below when the trace of the heat operator is considered.

Operators from sections of S to sections of S (which are typically what is being studied here) have as kernels distributional sections of the big homomorphism bundle over X^2 :

$$\mathrm{Hom}_{x,x'}(S) = \hom(S_{x'}, S_x) = \{h\colon S_x \longrightarrow S_{x'},\ \text{linear}\}\,,\ (x,x') \in X^2.$$

Clearly then $\mathrm{Hom}(S)_{\restriction\Delta} \cong \hom(S)$ as bundles over $\Delta \cong X$. Thus $\mathrm{Hom}(S)$ has, initially, a natural filtration, (8.38), only over the diagonal.

For the heat kernels of operators one needs to consider the pull-back of $\mathrm{Hom}(S)$ under blow-down and projection:

$$\pi_{H,X} : X^2_H \xrightarrow{\beta_H} [0,\infty) \times X^2 \longrightarrow X^2.$$

Directly from the definition of the pull-back of a vector bundle, see §2.11, $\pi^*_{H,X} \mathrm{Hom}(S)$ has a natural filtration over $\mathrm{tf}(X^2_H) \cup \widetilde{\Delta}_H$ where $\widetilde{\Delta}_H$ is the closure of the lift of $\mathbb{R}^+ \times \Delta$. We shall apply (8.21) to get a jet filtration at $\mathrm{tf}(X^2_H)$ and hence define a rescaled version of $\pi^*_{H,X} \mathrm{Hom}(S)$. To do so we need to consider the connection on $\pi^*_{H,X} \mathrm{Hom}(S)$.

The connection on $\mathrm{Hom}(S)$ is discussed in §2.11 as is the pull-back operation of connections. Thus $\pi^*_{H,X} \mathrm{Hom}(S)$ has a natural (Levi-Civita) connection. Lemma 2.32 allows the curvature operator on the lifted bundle to be computed in terms of the curvature operator on $\mathrm{Hom}(S)$.

LEMMA 8.13. *On an even-dimensional spin manifold, X, the pull-back to $\pi^*_{H,X} \mathrm{Hom}(S)$ of the Levi-Civita connection preserves the filtration* (8.38) *over* $\mathrm{tf}(X^2_H)$, (8.25) *holds for any inward-pointing vector field N for* $\mathrm{tf}(X^2_H)$ *and* (8.24) *holds for all $V \in \mathcal{V}_b(X^2_H)$ which are tangent to the fibres of* $\mathrm{tf}(X^2_H)$ *over Δ.*

PROOF: The restriction of the connection on $\mathrm{Hom}(S)$ to Δ reduces to the connection on $\hom(S)$ and hence coincides with that on the Clifford bundle under (8.39). Since $\mathrm{tf}(X^2_H)$ projects to Δ under $\pi_{H,X}$ it follows that the connection over $\mathrm{tf}(X^2_H)$ preserves the filtration (8.38).

Now from Lemma 2.32 it follows that the curvature operator on the bundle $\pi^*_{H,X}\operatorname{Hom}(S)$ is given directly in terms of the curvature operator on $\operatorname{Hom}(S)$ and hence ultimately, through (2.84), in terms of the curvature operator on S; this in turn is given by (3.68). Thus $K_S(V,W)$ is Clifford multiplication by a 2-form. The curvature operator on $\operatorname{Hom}(S)$ is therefore of the form

$$K_{\operatorname{Hom}(S)}(V,W)A = \operatorname{cl}(\omega)\circ A - A\circ\operatorname{cl}(\omega') \tag{8.40}$$

for some $\mathcal{C}^\infty$ 2-forms ω and ω' on X (but depending parametrically on all variables of X^2.) The covariant derivative induced on $\operatorname{Hom}(S)$ satisfies, (2.77) so $\nabla^p_N(K_{\operatorname{Hom}(S)}(V,W))$ is also given in terms of Clifford multiplication by 2-forms on the left and right. This moves the filtration (8.38) at most two steps, so (8.25) follows for every p.

The vector fields $V\in\mathcal{V}_b(X)$ which are tangent to the fibres of $\operatorname{tf}(X^2_H)$ have the property that $(\pi_{H,X})_*(V)=0$ at each point of Δ. Thus, from (2.84), the curvature operator on $E=\pi^*_{H,X}\operatorname{Hom}(S)$ in (8.24) vanishes over $\operatorname{tf}(X^2_H)$ so this condition holds trivially.

So to define the rescaling of the homomorphism bundle of the spinor bundle over X^2_H it is only necessary to make a choice of normal vector field to $\operatorname{tf}(X^2_H)$. One condition that it is convenient to demand is:

$$N \text{ is tangent to } \widetilde{\Delta}_H\subset X^2_H. \tag{8.41}$$

In fact there is rather a natural choice for N. At each point $q\in\operatorname{tf}(X^2_H)$, push-forward under the blow-down map sends N_q to a normal vector at $\beta_H(q)$ to Δ. From the definition of the blow-up in, see in particular (7.19), the image must project to a multiple of the tangent vector to the curve defining q, plus a term tangent to Δ. There is a natural choice for the normal space to Δ, at $\beta_H(q)=(p,p)$ as the span of $(v,-v)$ for $v\in T_pX$. Thus in addition to (and consistent with) (8.41) we demand

$$(\beta_H)_*(N_q)=(v_q,-v_q),\ v\in T_pX\ \forall\ q\in\operatorname{tf}(X^2_H),\beta_H(q)=(p,p)\in\Delta. \tag{8.42}$$

This will be called a radial choice of N.

EXERCISE 8.14. Show that (8.42) can be arranged by taking N to be the lift of the radial vector field in local coordinates:

$$2t\frac{\partial}{\partial t}+\frac{1}{2}\sum_{j=1}^N(x_j-x'_j)\left(\frac{\partial}{\partial x_j}+\frac{\partial}{\partial x'_j}\right) \tag{8.43}$$

divided by ρ_{tf}.

DEFINITION 8.15. If X is an even-dimensional spin manifold without boundary the bundle ${}^G\mathrm{Hom}(S)$ over X^2_H is defined by applying the rescaling procedure of Proposition 8.4 to the jet filtration of $\pi^*_{H,X}\,\mathrm{Hom}(S)$ at $\mathrm{tf}(X^2_H)$ fixed by parallel transport of (8.38) along an inward-pointing vector field satisfying (8.41) and (8.42).

This is Getzler's rescaling ([35]).

EXERCISE 8.16. Show that the imposition of (8.42) fixes N up to a positive multiple and an additive element $V \in \mathcal{V}_b(X^2_H)$ which is tangent to the fibres of β_H on $\mathrm{tf}(X^2_H)$ and hence conclude, using Exercise 8.11 and Lemma 8.13, that this rescaling is independent of the choice of N satisfying (8.42).

The lifted diagonal $\widetilde{\Delta}_H$ is naturally identified with $[0,\infty)_{\frac{1}{2}} \times X$ by β_H. The simple case of rescaling considered above therefore reappears through a natural bundle isomorphism:

$$ {}^G\mathrm{Hom}(S)_{\restriction \widetilde{\Delta}_H} \equiv {}^G\mathrm{hom}(S). \tag{8.44} $$

Notice that as a corollary of Lemma 8.5 the rescaled bundle restricts to $\mathrm{tf}(X^2_H)$ to

$$ {}^G\mathrm{Hom}(S)_{\restriction \mathrm{tf}} \equiv \pi^*_{H,X}\Lambda^*(X), \tag{8.45} $$

provided N is used to trivialize the conormal factors in (8.15). Indeed the maps from the filtration of the Clifford algebra, (8.38), induce an isomorphism from the graded bundle associated to the filtration to the exterior form bundle, i.e. give (8.45).

This needs to be done in the b-category as well. As usual this is a straightforward generalization. Certainly ${}^bS$, $\mathrm{Hom}({}^bS)$ and the filtration of $\mathrm{Hom}({}^bS)_{\restriction\Delta} = \mathrm{hom}({}^bS)$, all make sense, just assuming that X is an even-dimensional b-spin manifold. If X is an exact b-spin manifold the Levi-Civita connection is actually a true connection on ${}^bT^*X$ and hence on the spin bundle ${}^bS$. Let

$$ \pi_{H,X} : X^2_\eta \longrightarrow X^2_b,\ \ \pi_{\eta,X} : X^2_\eta \longrightarrow X^2 $$

be, respectively the composites of the heat blow down and projection to X^2_b and the combined blow down and projection to X^2. Thus $\pi_{\eta,X} = \beta_b \circ \pi_{H,X}$. The proof of Lemma 8.13 applies essentially unchanged to give:

LEMMA 8.17. *If X is an even-dimensional exact b-spin manifold with boundary the pull-back of the Levi-Civita connection to $\pi^*_{\eta,X}\,\mathrm{Hom}({}^bS)$ preserves the filtration by Clifford order over $\mathrm{tf}(X^2_H)$, (8.25) holds for any inward-pointing vector field N for $\mathrm{tf}(X^2_\eta)$ and (8.24) holds for all $V \in \mathcal{V}_b(X^2_\eta)$ which are tangent to the fibres of $\mathrm{tf}(X^2_\eta)$ over Δ.*

Thus the rescaled bundle ${}^G\mathrm{Hom}({}^bS)$ on X^2_η can be defined, just as in Definition 8.15, by choosing an inward-pointing vector field for $\mathrm{tf}(X^2_\eta)$. It will always be assumed that the analogue of (8.41) holds, i.e. N is tangent to $\widetilde{\Delta}_\eta = \mathrm{cl}\,\beta_H^{-1}(\mathbb{R}^+ \times \Delta_b)$. Then, as in (8.44)

$$(8.46) \qquad {}^G\mathrm{Hom}({}^bS)_{\restriction \widetilde{\Delta}_\eta} \equiv {}^G\mathrm{hom}({}^bS),$$

the latter bundle being the rescaling of $\mathrm{hom}({}^bS)$ over $[0,\infty)_{\frac12} \times X \equiv \widetilde{\Delta}_\eta$ with respect to the Clifford filtration. As in (8.43) we shall take N to be the lift of $2t\partial/\partial t$ plus a radial vector field for Δ_b. Using this to trivialize the conormal bundles as before give the obvious extension of (8.45), namely the natural identification

$$(8.47) \qquad {}^G\mathrm{Hom}({}^bS)_{\restriction \mathrm{tf}} \equiv \beta_\eta^*({}^b\Lambda^*(X)).$$

8.5. Rescaled trace.

The main reason for introducing this rescaling is its relationship to the supertrace functional on the homomorphism bundle of the spinor bundle, which we proceed to consider. Recall that the 'little trace' is always a linear functional on the homomorphism space of any finite dimensional vector space and hence defines a map from $\mathcal{C}^\infty$ sections of the homomorphism bundle of any vector bundle to $\mathcal{C}^\infty$ functions on the base. In particular consider this for the spinor bundle:

$$(8.48) \qquad \mathrm{tr}\colon \mathcal{C}^\infty(X;\mathrm{hom}(S)) \longrightarrow \mathcal{C}^\infty(X).$$

Here X is assumed to be a spin manifold and S is the spinor bundle. If X is even-dimensional (for the moment without boundary) then the spinor bundle decomposes into a direct sum, as in (3.64):

$$(8.49) \qquad S = {}^+S \oplus {}^-S,\ \dim X = 2k.$$

This means that $\mathrm{hom}(S)$ decomposes into four pieces, corresponding to linear maps from ${}^\pm S$ to ${}^\pm S$ for all signs. Thus

$$\mathrm{hom}(S) \ni A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}.$$

Then the trace of $A \in \mathrm{hom}(S)$ is just

$$\mathrm{tr}(A) = \mathrm{tr}_+(A_{++}) + \mathrm{tr}_-(A_{--}),$$

where $\mathrm{tr}_\pm$ denote the traces on $\hom({}^\pm S)$. The supertrace is defined as the difference instead:

$$\mathrm{str}(A) = \mathrm{tr}_+(A_{++}) - \mathrm{tr}_-(A_{--}).$$

The reason for considering the supertrace is immediately apparent from (In.16), which is really the 'big supertrace' of the heat kernel of $\eth^2$, since the Dirac operator satisfies (3.69). Consider the parity involution

$$R \in \hom(S),\ R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ R^2 = \mathrm{Id},$$

then the supertrace can be written in terms of the trace:

$$\mathrm{str}(A) = \mathrm{tr}(RA). \tag{8.50}$$

Using this the fundamental observation of Berezin [15] and Patodi [73] can be deduced:

LEMMA 8.18. *(Berezin-Patodi) Using the natural identification of* $\hom(S)$ *and the complexified Clifford bundle* $\mathbb{C}\mathrm{l}$ *for a* $2k$*-dimensional spin manifold*

$$\mathrm{str} = 0 \text{ on } \mathbb{C}\mathrm{l}^{(2k-1)},\ \mathrm{str}(\phi^1 \cdot \phi^2 \cdots \phi^{2k}) = 2^k(-i)^k \tag{8.51}$$

for any oriented orthonormal basis $\phi^1, \dots, \phi^{2k}$ *of* T^*X *or* ${}^bT^*X$.

PROOF: Recall that the decomposition of S arises from the grading of $\mathbb{C}\mathrm{l}$ as in (3.29). Namely ${}^\pm S$ are maximal proper subspaces of S which are invariant under $\mathbb{C}\mathrm{l}^+$ and then $\mathbb{C}\mathrm{l}^-$ maps ${}^\pm S$ to ${}^\mp S$. Under the decomposition (8.49), odd elements are of the form

$$\mathbb{C}\mathrm{l}^- \ni A = \begin{pmatrix} 0 & A_{+-} \\ A_{-+} & 0 \end{pmatrix} \Longrightarrow \mathrm{str}(A) = 0.$$

Thus only elements of $\mathbb{C}\mathrm{l}^+$, i.e. elements of even order, need to be considered. Choosing an oriented basis it suffices to consider the basis of $\mathbb{C}\mathrm{l}^+$

$$\phi^{i_1} \cdot \phi^{i_2} \cdots \phi^{i_{2p}},\ i_1 < i_2 < \cdots < i_{2p}.$$

For a fixed string $i_1, \dots, i_{2p}$, with $p < k$, choose $l \neq i_j$ for all $j = 1, \dots, 2p$. Clifford multiplication by ϕ^l is an isomorphism and involution:

$$E_l^\pm : {}^\pm S \overset{\phi^l \cdot}{\longleftrightarrow} {}^\mp S,\ E_l^- = (E_l^+)^{-1}.$$

Moreover, from the basic anticommutation property, (3.12), of the Clifford algebra

$$u = \phi^{i_1} \cdots \phi^{i_{2p}} \Longrightarrow u = \phi^l \cdot u \cdot \phi^l.$$

Thus the Clifford action of u under (8.49) can be written as

$$u = \begin{pmatrix} A_{++} & 0 \\ 0 & E_l A_{++} E_l^{-1} \end{pmatrix} \Longrightarrow \operatorname{str}(u) = 0$$

using the invariance of the trace under conjugation.

To complete the proof of the lemma it only remains to compute the supertrace of the action of the 'volume element' $\phi^1 \cdots \phi^{2k}$ on S. Here induction can be used with the isomorphism (3.24) providing the inductive step. In terms of (3.24) and (3.25) the sign-reversing involution R_{2k} in dimension $2k$ is represented by

$$R_{2k} = \begin{pmatrix} R_{2k-2} & 0 \\ 0 & -R_{2k-2} \end{pmatrix}.$$

Thus, from (3.26),

$$\operatorname{str}(\phi^1 \cdots \phi^{2k}) = \operatorname{tr}\left(\phi^1 \cdots \phi^{2k-2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} R_{2k} \right) = -2i \operatorname{str}(\phi^1 \cdots \phi^{2k-2}).$$

This gives the normalization condition in (8.51).

The effect of rescaling $\hom(S)$, lifted to $[0,\infty)_{\frac{1}{2}} \times X$, as discussed following (8.39) can now be seen easily:

LEMMA 8.19. *For a $2l$-dimensional spin manifold the linear functional defined through* (8.50) *and* (8.48), *restricts from* $\mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X; \hom(S))$ *to*

$$\operatorname{str}\colon \mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X; {}^G\!\hom(S)) \longrightarrow t^l \mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X). \tag{8.52}$$

PROOF: By definition $\mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X; {}^G\!\hom(S)) = \mathcal{D}$ is given by (8.13). Thus

$$u \in \mathcal{D} \Longrightarrow u = \sum_{j=0}^{2l} t^{j/2} u_j,\ \ u_j \in \mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X; \mathbb{C}\mathrm{l}^{(j)}).$$

Then (8.52) is an immediate consequence of (8.51).

This applies equally well in the case of a b-spin structure, since the argument is algebraic. Using (8.46) we find for a $2l$-dimensional exact b-spin manifold

$$\operatorname{str}\colon \mathcal{C}^\infty(\widetilde{\Delta}_\eta; {}^G\mathrm{Hom}({}^bS)) \longrightarrow t^l\mathcal{C}^\infty([0,\infty)_{\frac{1}{2}} \times X). \tag{8.53}$$

8.6. Rescaled heat calculus.

Consider the definition of the heat calculus for sections of a general bundle E, first for a manifold without boundary, in (7.92). If $\pi^*_{H,X}\operatorname{Hom}(E)$ has a rescaling at $\mathrm{tf}(X^2_H)$, as is the case for the spinor bundle on an even-dimensional spin manifold, then $\pi^*_{H,X}\operatorname{Hom}(E\otimes\Omega^{-\frac{1}{2}})$ has the tensor rescaling (8.18). For the spin manifold the rescaled bundle will be denoted ${}^G\mathrm{Hom}(S\otimes\Omega^{-\frac{1}{2}})$. We can therefore consider the rescaled heat calculus, as usual for simplicity limiting the order to be negative:

$$\begin{aligned}\Psi^j_G(X;S) &= \Psi^j_H(X;\Omega^{\frac{1}{2}})\otimes_{\mathcal{C}^\infty(X^2_H)}\mathcal{C}^\infty(X^2_H;{}^G\mathrm{Hom}(S\otimes\Omega^{-\frac{1}{2}})\\ &= \rho_{\mathrm{tf}}^{-\frac{1}{2}(n+3)-j}\big\{K\in\mathcal{C}^\infty(X^2_H;{}^G\mathrm{Hom}(S\otimes\Omega^{-\frac{1}{2}})\otimes\Omega^{\frac{1}{2}});\\ &\qquad\qquad K\equiv 0 \text{ at } \mathrm{tb}(X^2_H)\big\}.\end{aligned} \tag{8.54}$$

The second formulation follows from (7.11) and the first.

Since rescaling decreases the space of smooth sections this is a refinement of the heat calculus as defined earlier:

$$\Psi^j_G(X;S)\subset\Psi^j_H(X;S). \tag{8.55}$$

We will show below that the heat kernel of the Dirac operator, in the even-dimensional case, is in the rescaled calculus. To see why this is important consider the supertrace analogue of Lemma 7.18.

LEMMA 8.20. *If S is the spinor bundle on an even-dimensional spin manifold and $A\in\Psi^j_G(X;S)$ is an element of the rescaled calculus of order $j<0$ then the supertrace functional applied to the restriction of the kernel to the diagonal in $t>0$ gives*

$$\operatorname{str}(A_{\restriction\widetilde{\Delta}_H})\in t^{-\frac{1}{2}j-1}\mathcal{C}^\infty([0,\infty)_{\frac{1}{2}}\times X;\Omega). \tag{8.56}$$

If $A\in\Psi^j_{H,\mathrm{evn}}(X;S)\cap\Psi^j_G(X;S)$ then

$$\operatorname{str}(A_{\restriction\widetilde{\Delta}_H})\in t^{-\frac{1}{2}j-1}\mathcal{C}^\infty([0,\infty)\times X;\Omega). \tag{8.57}$$

PROOF: This is just Lemma 8.19 applied to (8.54).

In the exact b-spin case we proceed in the same way, defining the rescaled b-heat calculus by

(8.58)
$$\begin{aligned}\Psi^j_{b,G}(X;{}^bS) &= \Psi^j_\eta(X;{}^b\Omega^{\frac12})\otimes_{\mathcal{C}^\infty(X^2_\eta)}\mathcal{C}^\infty(X^2_\eta;{}^G\mathrm{Hom}({}^bS\otimes{}^b\Omega^{-\frac12})\\ &= \rho_{\mathrm{tf}}^{-\frac12(n+3)-j}\big\{K\in\mathcal{C}^\infty(X^2_\eta;{}^G\mathrm{Hom}({}^bS\otimes{}^b\Omega^{-\frac12})\otimes{}^b\Omega^{\frac12});\\ &\qquad K\equiv 0 \text{ at } \mathrm{tb}(X^2_\eta)\cup\mathrm{lb}(X^2_\eta)\cup\mathrm{rb}(X^2_\eta)\big\}.\end{aligned}$$

This is again a refinement of the b-heat calculus and both (8.56) and (8.57) extend trivially. In fact (8.58) depends only on the existence of the rescaling, so can be generalized further as in §8.12 below.

8.7. Rescaled normal operator.

It remains to show that $\exp(-t\eth^2)$ is, in the even-dimensional case, an element of the rescaled heat calculus; this will be done by re-constructing it. The main tool in the construction of the heat kernel in Chapter 7 is the normal operator, so we proceed to show that there is a rescaled normal homomorphism.

As in (7.53) powers of $t^{\frac12}$ can be used to remove factors of ρ_{tf} as needed to define the restriction of a kernel to $\mathrm{tf}(X^2_H)$. From (7.48) we see that

$$\begin{aligned}\Psi^j_G(X;S)\ni A\longmapsto t^{\frac12(n+j+2)}A_{\restriction\mathrm{tf}}\in\\ \dot{\mathcal{C}}^\infty(\mathrm{tf}(X^2_H);{}^G\mathrm{Hom}(S\otimes\Omega^{-\frac12})\otimes\pi^*_{H,X}(\Omega^{\frac12}(X^2)))\end{aligned}$$

and as before $\pi^*_{H,X}(\Omega^{\frac12}(X^2))\cong\Omega(X)\cong\Omega_{\mathrm{fibre}}(TX)$. Since the kernels all vanish to infinite order at the boundary of $\mathrm{tf}(X^2_H)$ we can also use $t^{\frac12}$, in place of ρ_{tf}, to trivialize the normal bundle to $\mathrm{tf}(X^2_H)$ in (8.15); the singular factors of $\rho_{\mathrm{tf}}/t^{\frac12}$ can be absorbed into the kernels. This gives, with the identification of $\mathrm{tf}(X^2_H)$ as a compactification of TX,

$$N_{G,j}\colon\Psi^j_G(X;S)\longrightarrow\mathcal{S}(TX;(\pi^*_{H,X}\Lambda^*X)\otimes\Omega_{\mathrm{fibre}}).\tag{8.59}$$

Clearly this is surjective and has null space exactly the operators of order $j-1$.

The inclusion (8.55) means that the normal operator on $\Psi^j_H(X;S)$ acts on the rescaled calculus. It is important to understand the relationship between the two normal operators. Corresponding to the fact that the elements of $\mathcal{D}$ in (8.13) restrict to the boundary to sections of F^k,

$$N_{H,j}\colon\Psi^j_G(X;S)\longrightarrow\mathcal{S}(TX;(\pi^*_{H,X}\mathbb{C}l^{(0)})\otimes\Omega_{\mathrm{fibre}})\cong\mathcal{S}(TX;\Omega_{\mathrm{fibre}})$$

since $\mathbb{Cl}^{(0)} \cong \mathbb{C}\mathrm{Id}$. In the rescaling this just corresponds to the 0-form part, i.e.

$$N_{H,j}(A) = \mathrm{Ev}_0(N_{G,j}(A)) \in \mathcal{S}(TX; \Omega_{\mathrm{fibre}}). \tag{8.60}$$

For operators of order -2 it therefore follows, as in Lemma 7.13, that

$$Au_{\restriction t=0}(x) = A_0(x)u,\ A_0 = \int\limits_{\mathrm{fibre}} \mathrm{Ev}_0\left(N_{G,-2}(A)\right). \tag{8.61}$$

This allows the initial condition for the heat kernel to be expressed in terms of the rescaled normal operator.

To proceed with the construction of $\exp(-t\eth^2)$ in the rescaled calculus it is clearly necessary to show that $\eth^2$ itself preserves the rescaling in an appropriate sense. Except for the small point that elements of positive order in the heat calculus have not been, and will not be, defined this amounts to showing that the heat operator $\partial/\partial t + \eth^2$ is an element of order 2 in the rescaled heat calculus. More prosaically this is the first part of:

PROPOSITION 8.21. *The Dirac Laplacian on an even-dimensional spin manifold satisfies*

$$t\eth^2 \colon \Psi_G^k(X;S) \longrightarrow \Psi_G^k(X;S),\ N_{G,-2}(t\eth^2 A) = N_G(t\eth^2)\cdot N_{G,-2}(A), \tag{8.62}$$

$N_G(t\eth^2)$ *being the operator on the lift to* TX *of the form bundle*

$$N_G(t\eth^2)(Y) = -\sum_{j=1}^{2k}\left(\partial_j + \frac{1}{4}R^\wedge(p;\partial_j,Y)\right)^2,\ Y \in T_pX \tag{8.63}$$

with respect to any orthonormal basis of T_pX *where*

$$R^\wedge(p;\partial_j,Y) = \sum_{kpq} R_{jkpq}Y^k dx^p dx^q \tag{8.64}$$

is the curvature operator acting on $\Lambda_p^*(X)$ *by exterior multiplication.*

The proof consists in examining the square of the Dirac operator, in terms of Lichnerowicz' formula, and the action of its lift to X_H^2 on the rescaled homomorphism bundle. Since the Dirac operator is a sum of products of Clifford multiplication and covariant differentiation these operators will be examined first.

LEMMA 8.22. *For any smooth* 1*-form on an even-dimensional spin manifold Clifford multiplication gives an operator*

$$\begin{gathered}\operatorname{cl}(t^{\frac{1}{2}}\omega)\colon \Psi^{p}_{G}(X;S) \longrightarrow \Psi^{p}_{G}(X;S),\ \omega\in\mathcal{C}^{\infty}(X;\Lambda^{1})\\ N_{G,p}(\operatorname{cl}(t^{\frac{1}{2}}\omega)A) = \pi^{*}_{H,X}(\omega)\wedge N_{G,p}(A).\end{gathered}\tag{8.65}$$

PROOF: Consider the lift of $\Lambda^1 X$ from X to X^2, under projection onto the left factor, and thence to X^2_H. Since $t^{\frac{1}{2}}\omega$, for $\omega \in \mathcal{C}^\infty(X;\Lambda^1 X)$, is not a smooth section of $(\pi_{H,L})^*\Lambda^1$ consider instead $\rho_{\mathrm{tf}}(\pi_{H,L})^*\omega$. The lifted bundle acts by Clifford multiplication (on the left) on $(\pi_{H,X})^*\operatorname{Hom}(S)$ and hence on the space of heat kernels. Clearly

$$\operatorname{cl}(\rho_{\mathrm{tf}}\omega)\colon \Psi^{j}_{H}(X;S) \longrightarrow \Psi^{j-1}_{H}(X;S).$$

To see that it acts on the rescaled calculus consider the Taylor series of $\rho_{\mathrm{tf}}\omega$, with respect to the Levi-Civita connection and an inward-pointing vector field, N, to $\mathrm{tf}(X^2_H)$. With the corresponding defining function, ρ_{tf}, chosen to satisfy $N\rho_{\mathrm{tf}} = 1$ near $\mathrm{tf}(X^2_H)$:

$$\begin{gathered}\rho_{\mathrm{tf}}\omega = \sum_{p=1}^{2k-1}\rho^{p}_{\mathrm{tf}}\omega_p + \rho^{2k}_{\mathrm{tf}}\,\omega_{(2k)} \text{ with}\\ \omega_{(2k)} \in \mathcal{C}^{\infty}(X^2_H;(\pi_{H,L})^*\Lambda^1 X) \text{ and } \nabla_N\omega_p = 0 \text{ near } \mathrm{tf}(X^2_H).\end{gathered}\tag{8.66}$$

Since

$$\rho^{2k}_{\mathrm{tf}}\,\Psi^{j}_{H}(X;S) \subset \Psi^{j}_{G}(X;S),\tag{8.67}$$

it suffices to consider the finite sum in (8.66). Now $\operatorname{cl}(\omega_1)$ shifts the filtration, extended by parallel transport, by one step and multiplication by ρ_{tf} shifts it back. Thus $\operatorname{cl}(\rho_{\mathrm{tf}}\omega)$ maps $\Psi^{j}_{G}(X;S)$ into itself. Moreover

$$\rho_{\mathrm{tf}}\colon \Psi^{j}_{G}(X;S) \longrightarrow \Psi^{j-1}_{G}(X;S),\tag{8.68}$$

so only the first term in (8.66) affects the normal operator. Multiplication by $t^{\frac{1}{2}}/\rho_{\mathrm{tf}}$ preserves the heat calculus so (8.65) follows, when it is recalled that $t^{\frac{1}{2}}$ is used to trivialize $N^*\,\mathrm{tf}(X^2_H)$.

For covariant differentiation there is a similar formula. Recall that the interior of $\mathrm{tf}(X^2_H)$ is naturally identified with TX. The symbol, $\sigma(V)$ (really just iV), of $V \in \mathcal{V}(X)$ can be considered as a vector field on the fibres of TX and as such it acts on sections of any bundle pulled back from X to TX, since such a bundle is canonically trivial along the fibres.

LEMMA 8.23. *For an even-dimensional spin manifold, X, if a radial choice of inward-pointing vector field to* $\mathrm{tf}(X^2_H)$ *is used to extend the filtration then* $V \in \mathcal{V}(X)$ *defines an operator*

$$t^{\frac{1}{2}}\nabla_V : \Psi^j_G(X;S) \longrightarrow \Psi^j_G(X;S) \tag{8.69}$$

and at $Y \in TX$, *the interior of* $\mathrm{tf}(X^2_H)$,

$$N_{G,j}(t^{\frac{1}{2}}\nabla_V A) = \left(-i\sigma(V) - \frac{1}{8}R^\wedge(Y,V)\right)N_{G,j}(A), \tag{8.70}$$

for all $A \in \Psi^j_G(X;S)$, *with the curvature term given by* (8.64).

PROOF: Since $t^{\frac{1}{2}}V$ is not smooth when lifted to X^2_H consider instead V' which is ρ_{tf} times the lift of V. Thus $V' \in \mathcal{V}_b(X^2_H)$ satisfies the hypothesis of Lemma 8.13, so the covariant derivative $\nabla_{V'}$ acts on ${}^G\mathrm{Hom}(S)$ and (8.36) gives a formula for the restriction to $\mathrm{tf}(X^2_H)$. The curvature operator on $\mathrm{Hom}(S)$ is given by (2.84), so the second term on the right in (8.36) vanishes, as V' pushes forward to zero from $\mathrm{tf}(X^2_H)$. For the same reason the third term can be written $\frac{1}{2}K_E(N,\nabla_N V')$ with $E=\mathrm{Hom}(S)$. By assumption $N\rho_{\mathrm{tf}}=1$, so $\nabla_N V'$ pushes forward to V as a vector field on the left factor of X in X^2. Thus with V'' being the restriction of V' to a vector field on the fibres of $\mathrm{tf}(X^2_H)$,

$$\nabla_{V'}v_j = V''v_j + (k-j)n_{V'}v_j + [\frac{1}{2}K_S((\pi_{H,L})_*N,V)v_{j-2}]_j. \tag{8.71}$$

Here ρ_{tf} has been used to trivialize the conormal factors in (8.15).

Our convention for the rescaled (as for the original) heat calculus is to use the singular defining function $t^{\frac{1}{2}}$ instead of the ρ_{tf} associated to the choice of N. This means multiplying v_j by $(\rho_{\mathrm{tf}}/t^{\frac{1}{2}})^{k-j}$. This effectively multiplies the curvature operator in (8.71) by t/ρ^2_{tf}. Observe that

$$n_{V'} = [N,V']\rho_{\mathrm{tf}} = NV'\rho_{\mathrm{tf}} \text{ on } \mathrm{tf}(X^2_H)$$

since $N\rho_{\mathrm{tf}}=1$, so commutation of this multiplicative factor through V'' cancels the second term in (8.71). Thus, multiplying by one factor of $t^{\frac{1}{2}}/\rho_{\mathrm{tf}}$ to change V' to the lift of $t^{\frac{1}{2}}V$ gives

$$N_{G,j}(\nabla_{t^{\frac{1}{2}}V}A) = \left(-i\sigma(V) + \frac{1}{2}\frac{t^{\frac{1}{2}}}{\rho_{\mathrm{tf}}}K_S((\pi_{H,L})_*N,V)\right)N_{G,j}(A). \tag{8.72}$$

Finally this proves (8.70) since, in view of (8.42), at a point Y in the interior of $\mathrm{tf}(X^2_H)$

$$\frac{t^{\frac12}}{\rho_{\mathrm{tf}}}(\pi_{H,L})_* N = \frac12 Y.$$

The action of $t\partial/\partial t$ can be deduced in the same way, namely

$$t\frac{\partial}{\partial t}\colon \Psi^j_G(X;S) \longrightarrow \Psi^j_G(X;S). \tag{8.73}$$

Indeed there are no curvature terms, so the rescaled normal operator can be deduced from the normal operator on the heat calculus in (7.52). It preserves the degree of a form and the extra factors of $t^{\frac12}$ introduced in the rescaling give

$$\mathrm{Ev}_p\left(N_{G,j}(t\frac{\partial}{\partial t}\circ A)\right) = -\frac12(R+n+j+p+2)\,\mathrm{Ev}_p(N_{G,j}A). \tag{8.74}$$

A formula for the leading term in (8.57) in terms of the rescaled normal operator is easily deduced from Lemma 8.18 and Lemma 8.20. Namely (8.75)

$$A\in\Psi^j_G(X;S) \Longrightarrow \left(t^{1+\frac12 j}\,\mathrm{str}(A_{\widetilde{\Delta}_H})\right)_{\restriction t=0} = (-2i)^k\,\mathrm{Ev}_{2k}(N_{G,j}A)_{\restriction X},$$

where restriction to X on the right is restriction to the zero section of TX, i.e. to $\widetilde{\Delta}_H \cap \mathrm{tf}(X^2_H)$.

8.8. Lichnerowicz' formula.

Lichnerowicz' formula expresses the square of the Dirac operator as the sum of the connection Laplacian and a scalar term. If E has an Hermitian inner product and X is Riemannian then $T^*X\otimes E$ has a natural inner product and the adjoint of a connection, ∇, is a first order differential operator

$$\nabla^*\colon \mathcal{C}^\infty(T^*X\otimes E) \longrightarrow \mathcal{C}^\infty(X;E).$$

The connection Laplacian is just

$$\Delta^\nabla = \nabla^*\nabla \in \mathrm{Diff}^2(X;E).$$

If υ is a non-vanishing volume form the divergence of a vector field with respect to υ is defined by the identity:

$$d(\mathrm{int}(w)\upsilon) = (\mathrm{div}\, w)\upsilon.$$

Since it is invariant under change of sign of υ, the divergence is actually defined with respect to a non-vanishing smooth density even if the manifold is not oriented (or orientable). In particular the divergence of a vector field is defined on any Riemann manifold.

EXERCISE 8.24. Show that the divergence $\operatorname{div}_\nu v$ of a vector field with respect to a non-vanishing density $\nu \in \mathcal{C}^\infty(X;\Omega)$ is fixed by the integration by parts formula

$$\int_X vf\nu = \int_X f(\operatorname{div}_\nu v)\nu \quad \forall\ f \in \mathcal{C}^\infty(X).$$

LEMMA 8.25. *If v_i, $i = 1, \dots, N$ is a local orthonormal frame for TX then for any Hermitian connection on a vector bundle E*

$$\begin{aligned} \Delta^\nabla u &= \sum_{i=1}^N \nabla_{v_i}^* \nabla_{v_i} \\ &= -\sum_{i=1}^N \left(\nabla_{v_i}^2 - \operatorname{div} v_i \nabla_{v_i}\right) u \quad \forall\ u \in \mathcal{C}^\infty(X;E). \end{aligned} \tag{8.76}$$

Notice that at any point where the frame is covariant constant the first order terms on the right vanish.

PROOF: Let ϕ^i be the dual coframe to the v_i. The connection is

$$\nabla u = \sum_{i=1}^N \phi^i \otimes \nabla_{v_i}.$$

The adjoint, acting on a section $\phi^i \otimes v$ must therefore satisfy

$$\langle \nabla^*(\phi^i \otimes v), v' \rangle = \langle \phi^i \otimes v, \nabla v' \rangle = \langle v, \nabla_{v_i} v' \rangle.$$

Thus $\nabla^*(\phi^i \otimes v) = \nabla_{v_i}^* v$, where the adjoint is with respect to the inner product on E. This gives the first formula. Since the connection is Hermitian, $\nabla_{v_i}^* = -\nabla_{v_i} + \operatorname{div} v_i$. This gives (8.76).

The square of the Dirac operator has principal symbol given by the metric tensor and the same is true of the connection Laplacian. The difference between them is therefore a differential operator of first order. In fact the difference is of order zero and was computed by Lichnerowicz:

PROPOSITION 8.26. *The Dirac operator on the spinor bundle satisfies*

$$\eth^2 = \Delta^\nabla + \frac{1}{4} S, \tag{8.77}$$

where S is the scalar curvature of the metric.

PROOF: Let v_i be a local orthonormal frame for the metric, with ϕ^i the dual coframe. Then the Dirac operator is given by (3.41), so

$$\eth^2 = -\sum_{i=1}^N \mathrm{cl}(\phi^i)\nabla_{v_i} \circ \sum_{j=1}^N \mathrm{cl}(\phi^j)\nabla_{v_j}.$$

Dividing by 2 and exchanging the dummy variables on one sum gives

$$\begin{aligned}\eth^2 &= -\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \left[\mathrm{cl}(\phi^i)\nabla_{v_i}\circ \mathrm{cl}(\phi^j)\nabla_{v_j} + \mathrm{cl}(\phi^j)\nabla_{v_j}\circ \mathrm{cl}(\phi^i)\nabla_{v_i}\right] \\ &= -\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \left[\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^j)\nabla_{v_i}\nabla_{v_j} + \mathrm{cl}(\phi^j)\,\mathrm{cl}(\phi^i)\nabla_{v_j}\nabla_{v_i}\right] \\ &\qquad - \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \left[\mathrm{cl}(\phi^i)\,\mathrm{cl}(\nabla_{v_i}\phi^j)\nabla_{v_j} + \mathrm{cl}(\phi^j)\,\mathrm{cl}(\nabla_{v_j}\phi^i)\nabla_{v_i}\right],\end{aligned}$$

where the fact that the connection is Clifford has been used. Commuting differentiation gives

$$\begin{aligned}\eth^2 &= -\sum_{i=1}^N \nabla_{v_i}^2 + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \mathrm{cl}(\phi^j)\,\mathrm{cl}(\phi^i)[\nabla_{v_i},\nabla_{v_j}] \\ &\qquad - \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \left[\mathrm{cl}(\phi^i)\,\mathrm{cl}(\nabla_{v_i}\phi^j)\nabla_{v_j} + \mathrm{cl}(\phi^j)\,\mathrm{cl}(\nabla_{v_j}\phi^i)\nabla_{v_i}\right]\end{aligned}$$

by the anticommutation property of Clifford multiplication. Further expressing the commutator of the covariant derivatives in terms of the curvature operator of the connection gives

(8.78)

$$\begin{aligned}\eth^2 &= -\sum_{i=1}^N \nabla_{v_i}^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \mathrm{cl}(\phi^j)\,\mathrm{cl}(\phi^i)\nabla_{[v_i,v_j]} + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \mathrm{cl}(\phi^j)\,\mathrm{cl}(\phi^i)Q(v_i,v_j) \\ &\quad - \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \left[\mathrm{cl}(\phi^i)\,\mathrm{cl}(\nabla_{v_i}\phi^j)\nabla_{v_j} + \mathrm{cl}(\phi^j)\,\mathrm{cl}(\nabla_{v_j}\phi^i)\nabla_{v_i}\right].\end{aligned}$$

Using (8.76) this shows the difference $\eth^2 - \Delta^\nabla$ to be a first order differential operator. Consider the coefficients of the first order terms. Choose a point

$p \in X$ and then choose the orthonormal basis to be covariant constant at p and equal to a coordinate basis to second order. Then the commutators $[v_i, v_j]$, the covariant derivatives of the coframe and the divergence terms in (8.76) all vanish at p. Since the remaining term in (8.78) is of order zero, and independent of the basis, the first order terms must in fact vanish identically and

$$\eth^2 = \Delta^\nabla + \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \operatorname{cl}(\phi^j)\operatorname{cl}(\phi^i) Q(v_i, v_j). \tag{8.79}$$

The action of the curvature operator on the spinor bundle is discussed in Chapter 3. In particular for an orthonormal basis

$$Q(v_i, v_j) = \frac{1}{4}\sum_{pq=1}^N R_{pqij}\operatorname{cl}(\phi^p)\operatorname{cl}(\phi^q).$$

Inserting this into (8.79) gives the somewhat forbidding expression for the tensorial term:

$$-\frac{1}{8}\sum_{ijpq} R_{pqij}\operatorname{cl}(\phi^i)\operatorname{cl}(\phi^j)\operatorname{cl}(\phi^p)\operatorname{cl}(\phi^q), \tag{8.80}$$

where the sign change comes from anticommutation of $\operatorname{cl}(\phi^i)$ and $\operatorname{cl}(\phi^j)$. This expression considerably simplifies because of the symmetry properties of the Riemann curvature tensor discussed in Lemma 2.29. In particular total antisymmetrization in any three indices gives zero. Subtracting the total antisymmetrization of the product of the last three Clifford factors and then using the anticommutation property of Clifford multiplication gives

$$\begin{aligned}
&\operatorname{cl}(\phi^j)\operatorname{cl}(\phi^p)\operatorname{cl}(\phi^q)\\
&\quad -\frac{1}{6}\big[\operatorname{cl}(\phi^j)\operatorname{cl}(\phi^p)\operatorname{cl}(\phi^q) + \operatorname{cl}(\phi^p)\operatorname{cl}(\phi^q)\operatorname{cl}(\phi^j) + \operatorname{cl}(\phi^q)\operatorname{cl}(\phi^j)\operatorname{cl}(\phi^p)\\
&\qquad - \operatorname{cl}(\phi^j)\operatorname{cl}(\phi^q)\operatorname{cl}(\phi^p) - \operatorname{cl}(\phi^q)\operatorname{cl}(\phi^p)\operatorname{cl}(\phi^j) - \operatorname{cl}(\phi^p)\operatorname{cl}(\phi^j)\operatorname{cl}(\phi^q)\big]\\
&= \frac{1}{6}\big[2\delta_{jp}\operatorname{cl}(\phi^q) - 2\delta_{jq}\operatorname{cl}(\phi^p) + 2\delta_{pq}\operatorname{cl}(\phi^j) - 2\delta_{jq}\operatorname{cl}(\phi^p) + 2\delta_{pq}\operatorname{cl}(\phi^j)\\
&\qquad + 2\delta_{pq}\operatorname{cl}(\phi^j) - 2\delta_{jq}\operatorname{cl}(\phi^p) + 2\delta_{jp}\operatorname{cl}(\phi^q) + 2\delta_{jp}\operatorname{cl}(\phi^q)\big]\\
&= \delta_{jp}\operatorname{cl}(\phi^q) - \delta_{jq}\operatorname{cl}(\phi^p) + \delta_{pq}\operatorname{cl}(\phi^j).
\end{aligned}$$

Inserting this into (8.80) therefore reduces it to

$$(8.81)\qquad -\frac{1}{8}\sum_{ijp} R_{jpij}\,\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^p) + \frac{1}{8}\sum_{ijp} R_{pjij}\,\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^p) \\ - \frac{1}{8}\sum_{ijp} R_{ppij}\,\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^j).$$

The last sum is zero and the first two are equal, because the Riemann curvature tensor is antisymmetric in the first and last pair of indices. Moreover the curvature tensor is unchanged under exchange of the two pairs of indices, so (8.81) can be rewritten

$$\frac{1}{8}\sum_{ijp} R_{pjij}\left[\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^p) + \mathrm{cl}(\phi^p)\,\mathrm{cl}(\phi^i)\right] = \frac{1}{4}\sum_{ij} R_{ijij}.$$

Thus the term of order zero is simply multiplication by a scalar as (8.77) claims, with

$$S = \sum_{ij} R_{ijij}$$

by definition the scalar curvature of the Riemann manifold.

Lichnerowicz' formula extends easily to the case of a twisted Dirac operator.

PROPOSITION 8.27. *The twisted Dirac operator on $S \otimes E$, where E has an Hermitian connection, satisfies*

$$(8.82)\qquad \eth^2 = \Delta^{\nabla} + \frac{1}{4}S - \frac{1}{2}\,\mathrm{cl}(K_E),$$

where S is the scalar curvature of the metric and

$$K_E \in \mathcal{C}^\infty(X; T^*X \otimes T^*X \otimes \hom(E))$$

is the curvature of the coefficient bundle acting through Clifford multiplication in the sense that for any orthonormal frame v_i and dual coframe ϕ^i

$$(8.83)\qquad \mathrm{cl}(K_E) = \sum_{ij} K_E(v_i, v_j)\,\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^j).$$

PROOF: The formula (8.83) follows exactly as in the proof of Proposition 8.26, except that now Q is the total curvature operator acting on the tensor product bundle $S \otimes E$. As shown in (2.76) the curvature splits into a sum

$$Q = Q_S + K_E,$$

where K_E is the curvature operator for the connection on E and Q_S is the curvature on the spinor bundle. This gives an extra term in (8.79) and leads directly to (8.83).

PROOF OF PROPOSITION 8.21: Applying (8.77) to $t\eth^2$ and writing it locally as

$$t\eth^2 = -\sum_i (t^{\frac{1}{2}}\nabla_{v_i})^2 + \sum_i (t^{\frac{1}{2}} \operatorname{div} v_i t^{\frac{1}{2}}\nabla_{v_i}) + t\frac{1}{4}S$$

in terms of an orthonormal basis of vector fields allows Lemmas 8.22 and 8.23 to be used to show that $t\eth^2$ preserves the space defining the rescaled heat calculus, i.e. proves (8.62). The formulæ (8.65) and (8.70) then give (8.63).

8.9. Mehler's formula.

Consider the representation (8.63) for the rescaled normal operator. This can be used to obtain an explicit formula for the normal operator of the (putative) rescaled heat kernel, using (8.62), and then to show that the Dirac operator on an even-dimensional manifold does indeed satisfy

$$\exp\left(-t\eth^2\right) \in \Psi_G^{-2}(X;S), \tag{8.84}$$

as has been anticipated.

Consider the normal operator in (8.63), acting on functions on T_yX with values in Λ_y^*X. Thus

$$P = -\sum_{i=1}^{n}(\partial_{x_i} + \frac{1}{4}\sum_k Q_{ik}x^k)^2. \tag{8.85}$$

Here the Q_{ik} are 2-forms acting by exterior product. This means that as operators on Λ_y^*X they are nilpotent:

$$(Q_{ik})^j = 0,\ j > k,\ \dim X = 2k.$$

This makes it plausible that the heat kernel for P should have an explicit representation, since it could be obtained from the heat kernel for the flat case ($Q_{ik} \equiv 0$) by a finite number of iterative steps. By exploiting the analogy between (8.85) and the harmonic oscillator and by generalizing Mehler's formula ([52]) for the latter, Getzler observed that there is in fact a closed form solution.

Thus consider the harmonic oscillator on the line:

$$P_1 = -\frac{d^2}{dx^2} + x^2.$$

One variant of Mehler's formula gives the fundamental solution to the heat equation for P_1 as

$$E_1 = \frac{1}{(2\pi \sinh 2t)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2\sinh 2t}(x^2\cosh 2t - 2xy + y^2\cosh 2t)\right\}. \tag{8.86}$$

Uniqueness follows rather easily from the existence, which can be checked by differentiation. Direct derivations of (8.86) can be found in [20] and [48].

By scaling the variables $t \longmapsto at$, $x, y \longmapsto a^{\frac{1}{2}}x, a^{\frac{1}{2}}y$ for $a > 0$, (8.86) can be generalized to obtain the heat kernel for $P_a = -\frac{d^2}{dx^2} + a^2x^2$; namely

$$\begin{aligned} E_a &= \frac{1}{(4\pi t)^{\frac{1}{2}}}\left(\frac{2at}{\sinh 2at}\right)^{\frac{1}{2}} \\ &\times \exp\left\{-\frac{a}{2\sinh 2at}(x^2\cosh 2at - 2xy + y^2\cosh 2at)\right\}. \end{aligned} \tag{8.87}$$

The fundamental solutions for the harmonic oscillators in higher dimensions can be obtained by taking products of the one-dimensional fundamental solution.

LEMMA 8.28. *If B is a non-negative symmetric matrix then*

$$\begin{aligned} &H = \Delta + \langle Bx, x\rangle \Longrightarrow \\ e^{-tH}(x,y) = &\frac{1}{(4\pi t)^{\frac{n}{2}}}\left(\det{}^{\frac{1}{2}}\alpha(t)\right)^{-1} \\ &\times \exp\left(-\frac{1}{4\pi t}\left[\langle\beta(t)x,x\rangle - 2\langle\alpha(t)x,y\rangle + \langle\beta(t)y,y\rangle\right]\right), \end{aligned} \tag{8.88}$$

where the symmetric matrices $\alpha(t)$ and $\beta(t)$ are defined by

$$\alpha(t) = F(2tB^{\frac{1}{2}}),\ \beta(t) = G(2tB^{\frac{1}{2}}),\ F(r) = \frac{r}{\sinh(r)},\ G(r) = \frac{r\cosh r}{\sinh r}. \tag{8.89}$$

PROOF: If $b_{ij} = \delta_{ij}a_i^2$ is diagonal and positive semidefinite then just taking products of (8.87) gives

$$\begin{aligned} E &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\prod_{i=1}^{n}\left(\frac{2a_it}{\sinh 2a_it}\right)^{\frac{1}{2}} \\ &\times \exp\left\{-\sum_{i=1}^{n}\frac{a_i}{2\sinh 2a_it}(x_i^2\cosh 2a_it - 2x_iy_i + y^2\cosh 2a_it)\right\}. \end{aligned} \tag{8.90}$$

If the diagonal matrices

$$\alpha_{ij}(t) = \delta_{ij}\frac{\sinh(2a_i t)}{2a_i t}, \ \beta_{ij}(t) = \delta_{ij}\frac{2a_i t\cosh 2a_i t}{\sinh 2a_i t} \tag{8.91}$$

are considered then (8.90) can be written in the form

$$\begin{aligned} E &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \\ &\times \left(\det{}^{\frac{1}{2}}\alpha(t)\right)\exp\left(-\frac{1}{4t}\left[\langle\beta(t)x,x\rangle - 2\langle\alpha(t)x,y\rangle + \langle\beta(t)y,y\rangle\right]\right). \end{aligned} \tag{8.92}$$

Of course (8.91) and (8.92) just reduce to (8.88) and (8.89) when B is diagonal. Moreover the general case follows by making an orthogonal transformation in the independent variables. Thus if $O \in \mathrm{SO}(N)$ is such that OBO^t is diagonal, changing variables from x and y to Ox and Oy reduces H in (8.88) to the form (8.91). Changing variable back from (8.92) gives the general case.

Still for operators acting on functions, suppose that $A_{ij} = -A_{ji}$ is an antisymmetric matrix. Then, thinking of it as a 'vector potential,' consider the operator

$$H_A = -\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i} - \sum_{j=1}^{n} A_{ij}x^j\right)^2 =$$
$$\Delta + \sum_{ij} A_{ij}(x^j\frac{\partial}{\partial x_i} - x^i\frac{\partial}{\partial x_j}) + \langle Ax, Ax\rangle.$$

The linear vector field

$$V = \sum_{ij} A_{ij}(x^j\frac{\partial}{\partial x_i} - x^i\frac{\partial}{\partial x_j})$$

is a sum of infinitesimal rotations, so commutes with the Laplacian, and

$$V\langle Ax, Ax\rangle = 0.$$

Thus if $H = -\Delta + \langle Ax, Ax\rangle$ then $H_A = H + V$, $[H, V] = 0$. The heat kernel can therefore be written

$$\exp(-tH_A) = \exp(-\frac{1}{2}tV)\exp(-tH)\exp(-\frac{1}{2}tV).$$

The middle term here is given by Lemma 8.28 and the exponential of the vector field is an orthogonal transformation, namely the exponential of $-\frac{1}{2}tA$ as a matrix. Thus the kernel of $\exp(-tH_A)$ can be obtained from the kernel of $\exp(-tH)$ by replacing x and y by $\exp(-t\frac{1}{2}A)x$ and $\exp(t\frac{1}{2}A)y$ respectively in (8.88).

Notice first that the functions F and G in (8.89) are actually even functions of the argument r, i.e. are actually analytic functions of r^2. Thus they can be expressed in terms of $-t^2A^2$ and hence, with conjugation by $\exp(\frac{1}{2}tA)$, the fundamental solution can be expressed as a matrix function of tA :

(8.93)

$$\begin{aligned} e^{-tH_A} &= \frac{1}{(4t)^{\frac{n}{2}}}\left(\det{}^{\frac{1}{2}}\alpha(t)\right) \\ &\quad \times \exp\left(-\frac{1}{4\pi t}\left[\langle\beta(t)x,x\rangle - 2\langle\alpha(t)x,y\rangle + \langle\beta(t)y,y\rangle\right]\right) \\ \alpha(t) &= F(2itA),\ \beta(t) = G(2itA),\ F(r) = \frac{r}{\sinh(r)},\ G(r) = \frac{r\cosh r}{\sinh r}. \end{aligned}$$

To apply this discussion to the operator (8.85) a further generalization to the case of a system is needed. This would be rather daunting, except that it is a very special system. The A_{ij} in (8.85) are antisymmetric real matrices with values in the 2-forms, acting by exterior multiplication on Λ_y^*X, which is commutative. It is also nilpotent, i.e. any $(k+1)$-fold product (where the dimension of X is $2k$) vanishes. From an algebraic point of view it is therefore obvious that (8.93) continues to give the fundamental solution in this case.

Those who are not algebraists may need to check the claim. To do so consider (8.93) as a function of the matrix A. It is certainly real-analytic, hence it is complex analytic as a function of complex (still antisymmetric) matrices near the real subspace. This allows polynomial variables z_{pq} to be introduced into the formula for all pairs p,q where $p,q = 1,\ldots,k,\ q>p$, by replacing A by

$$A(z) = \sum_{q>p} z_{qp}A^{p,q},$$

where $A^{p,q}$ is the matrix with all entries zero, except for a 1 at (p,q) and -1 at (q,p). The formula (8.93) still gives

$$(\partial_t + H_A)\exp(-tH_A) = \delta(t)\delta(x-y) \tag{8.94}$$

provided the z_{pq} are small. Let E_{pq} be the action of $dx_p \wedge dx_q$ on the exterior algebra. Using the nilpotence, which means that 0 is the only eigenvalue, it follows that for any function g analytic in z near 0

$$\frac{1}{2\pi i}\oint g(z)(z-E_{pq})^{-1}dz = g(E_{pq}).$$

Applying this to (8.94) in each of the variables z_{pq} and using the commutativity gives:

LEMMA 8.29. *The heat kernel for the operator* (8.85)*, acting on functions with values in* T_y^*X*, is given by* (8.93) *with* $A=-\frac{1}{4}R^\wedge$.

8.10. Local index formula.

Now the proof of the local index theorem for Dirac operators can be completed since (8.75) leads directly to the local index theorem, proved originally by Patodi [72] and Gilkey [37]:

THEOREM 8.30. *If* $P=\eth^2$ *is the Dirac Laplacian on a compact* $2k$*-dimensional spin manifold* X *then* (8.84) *holds and*

$$\operatorname{str}[\exp(-tP)]_{\restriction t=0} = \operatorname{Ev}_{2k}\left(\det{}^{\frac{1}{2}}(\frac{R/4\pi i}{\sinh(R4\pi i)})\right), \tag{8.95}$$

where R *is the antisymmetric matrix of 2-forms defined by the curvature on the orthonormal frame bundle and* Ev_{2k} *evaluates a form to its component of maximal degree.*

PROOF: Proceed, as previously announced, to reconstruct the heat kernel for the Dirac Laplacian. From (8.63) the first step is to take for a parametrix, $E_1 \in \Psi_G^{-2}(X;S)$, an operator which has rescaled normal operator satisfying

$$N_G(t\frac{\partial}{\partial t}+t\eth^*\eth)N_G(E_1)=0 \tag{8.96}$$

and the appropriate initial condition which follows directly from (8.60), namely

$$\int_{\text{fibre}} \operatorname{Ev}_0\left(N_{G,-2}(E_1)\right) = \operatorname{Id}. \tag{8.97}$$

In fact the solution to (8.96) with initial condition (8.97) follows directly from Lemma 8.29 and in particular (8.93). First take $y=0$ in (8.93), since that is where the diagonal is. Now rescale the form bundle and the linear coordinates on T_pX for a given fibre, setting $Z = x/t^{\frac{1}{2}}$ and $dx^j = t^{\frac{1}{2}} dZ^j$. Multiplying by an overall factor of $t^{\frac{n}{2}}$, this gives

$$N_G(E_1) = \frac{1}{(4\pi)^{\frac{n}{2}}} \left(\det{}^{\frac{1}{2}} \alpha\right) \exp\left(-\frac{1}{4}\langle \beta Z, Z\rangle\right), \tag{8.98}$$

where α and β are the matrices in (8.93) evaluated at $t = 1$. Certainly (8.96) holds, just by rescaling the equation expressing the fact that (8.93) gives the fundamental solution of (8.85). Then (8.97) follows from (8.61).

The leading part of the rescaled heat kernel has therefore been obtained. As in the scalar case discussed in Chapter 7 successive correction terms can be found, which are now lower order in the rescaled heat calculus, by using the fundamental solution (8.93). Summing the Taylor series at $\mathrm{tf}(X^2_H)$ gives an error term vanishing rapidly at $t = 0$ which is therefore a Volterra operator, the terms of order $-\infty$ in the rescaled calculus being the same as in the unscaled case. Thus E_1 can be extended to an element

$$E \in \Psi_G^{-2}(X; S) \text{ s.t. } t(\partial_t + ð^2)E = 0. \tag{8.99}$$

Furthermore from (8.55) it follows that in this way the heat kernel has simply been reconstructed. Thus (8.84) has been proved for the Dirac Laplacian.

The local index formula (8.95) is just (8.75) applied to (8.98), since the lifted diagonal meets the temporal front face at $Z = 0$ in the fibre.

The formula for the integrand is in terms of a characteristic class, namely

$$\mathrm{Ev}_{2k}\left(2^k(-i)^k \frac{1}{(4\pi)^k}\left[\det{}^{\frac{1}{2}}\left(\frac{R^\wedge/2}{\sinh(R^\wedge/2)}\right)\right]\right). \tag{8.100}$$

It is given by an invariant polynomial in the curvature as discussed in §2.14. Since only the volume part, i.e. the component of form degree $2k$, is involved the function in (8.100) is necessarily homogeneous of degree k in R. Thus the constant can be absorbed to give precisely the volume part of the $\widehat{A}$-genus as defined in (2.106) and written again in (8.95).

Now it is easy to go back and add in the effect of twisting by the Hermitian bundle E, with Hermitian connection:

THEOREM 8.31. *If $P = ð_E^2$ is the twisted Dirac Laplacian for a bundle with Hermitian connection on a compact $2k$-dimensional spin manifold X then the conclusions of Theorem 8.30 hold with (8.95) replaced by*

$$\mathrm{str}[\exp(-tP)]_{\restriction t=0} = \mathrm{Ev}_{2k}\left(\det{}^{\frac{1}{2}}\left(\frac{R/4\pi i}{\sinh(R/4\pi i)}\right) \mathrm{tr}\exp\left(\frac{iK_E}{2\pi}\right)\right), \tag{8.101}$$

where now K_E is the curvature of E.

PROOF: The formula (8.82) should be used in place of (8.77), to compute the normal operator on the rescaled bundle. Thus (8.85) is replaced by

$$P = -\sum_{i=1}^{n}(\partial_{x_i} + \frac{1}{4}\sum_{k} Q_{ik}x^k)^2 - \frac{1}{2}K_E. \tag{8.102}$$

Since this just involves the addition of a constant, nilpotent, matrix of 2-forms acting on $\Lambda_y^* X \otimes E_y$ at each point (and therefore commuting with the action of the Riemann curvature) the fundamental solution can be constructed precisely as before, with an additional exponential factor of $\exp(K_E/2)$ in (8.93). On rescaling to give the analogue of the normal operator in (8.98) this gives an extra factor of $\exp(-iK_E)$. Then (8.101) follows after taking the supertrace, which acts simply as the trace on the operators on E, and using the homogeneity to absorb factors of 2 and π as before.

This extra factor

$$\mathrm{Ch}(E) = \mathrm{tr}\exp(\frac{iK_E}{2\pi})$$

is the Chern character of the bundle E discussed briefly in §2.14. Again it is a differential form on X but is independent of the choice of trivialization used to compute it. Finally then the derivation of (In.2) is complete, since (8.101) is just the definition of the Atiyah-Singer integrand for the twisted Dirac operator:

$$\mathrm{AS} = \mathrm{Ev}_{\dim X}\left(\widehat{A}(X)\cdot \mathrm{Ch}(E)\right).$$

This is very close to the completion of the proof of the Atiyah-Singer theorem for twisted Dirac operators as outlined in the Introduction; it only remains to put the pieces together. This is done in the next chapter.

8.11. The b case.

The case of a b-metric still needs to be examined; fortunately not much has to be done apart from a review of the discussion above.

The rescaling of ${}^G\mathrm{Hom}({}^bS)$ is fixed by Proposition 7.36 and satisfies (8.47). The rescaled heat calculus is defined by (8.58). Moreover the rescaling of the trace functional, the rescaling of the connection and Lichnerowicz' formula all follow by continuity since they are true away from the boundary. The construction of the rescaled normal operator proceeds exactly as in the boundaryless case and then the remainder of the construction proceeds without difficulty.

This leads directly to the local index theorem for the twisted Dirac operator:

THEOREM 8.32. *If $P = {}^b\eth_E^2$ is the twisted Dirac Laplacian for a bundle with Hermitian connection on a compact $2k$-dimensional exact b-spin manifold X then the heat kernel is an element of the rescaled b-heat calculus, $\exp(-tP) \in \Psi_{b,G}^{-2}(X; S \otimes E)$ and* (8.101) *continues to hold for the fibre trace:*

$$\operatorname{str}[\exp(-tP)]_{|t=0} = \operatorname{Ev}_{2k}\left[\det{}^{\frac{1}{2}}\left(\frac{R/4\pi i}{\sinh(R/4\pi i)}\right)\right]\operatorname{tr}\exp\left(\frac{iK_E}{2\pi}\right), \tag{8.103}$$

where K_E is the curvature of E.

8.12. Graded Hermitian Clifford modules.

Although the results above have been discussed in the context of twisted Dirac operators they all generalize directly to the case of a generalized Dirac operator associated to a graded Hermitian Clifford module, with graded unitary Clifford connection, over an even-dimensional compact manifold with boundary with an exact b-metric. It is for this general case that the APS theorem is proved in the next Chapter.

Let E be the bundle which is a Clifford module over X, $\dim X = 2k$. The homomorphism bundle decomposes, as discussed in Lemma 3.6, giving the filtration

$$\hom^{(k)}(E) = \mathbb{C}\mathrm{l}^{(k)}(X) \otimes \hom'_{\mathbb{C}\mathrm{l}}(E). \tag{8.104}$$

This filtration can be used to define the rescaled bundle, ${}^G\mathrm{Hom}(E)$, as before. That is, the subbundles $\hom^{(k)}(E)$ lift to a filtration of $\pi_{H,X}^* \mathrm{Hom}(E)$ over $\mathrm{tf}(X_\eta^2)$ and this can be extended by parallel transport along some chosen radial normal vector field. Now the rescaled heat calculus for the bundle E, is defined by (8.58). Notice that (8.45) is replaced by

$${}^G\mathrm{Hom}(E)_{|\mathrm{tf}} \equiv \beta_\eta^*(\Lambda^* X \otimes \hom'_{\mathbb{C}\mathrm{l}}(E)).$$

Then Lemma 8.22 can be extended to give

$$\begin{gathered}\mathrm{cl}(t^{\frac{1}{2}}\gamma)\colon \Psi_{b,G}^k(X;S) \longrightarrow \Psi_{b,G}^k(X;S),\ \gamma \in {}^b\Lambda^1(X)\\ N_{b,G}(\mathrm{cl}(t^{\frac{1}{2}}\gamma)A) = \beta_\eta^*\gamma \wedge N_G(A).\end{gathered}$$

To get the analogue of Lemma 8.23, and hence Proposition 8.21, a formula for the curvature operator on E, and hence on $\mathrm{Hom}(E)$, is needed.

LEMMA 8.33. *If E is a Clifford module, with Clifford b-connection, over an even-dimensional compact manifold with boundary equipped with an exact b-metric then under* (3.37) *the curvature operator decomposes*

$$K_E(V,W) = \frac{1}{4}\operatorname{cl}(R(V,W)) + K'_E(V,W),\ \forall\ V,W \in \mathcal{V}_b(X), \tag{8.105}$$

where $K_{E'}(V,W)$ commutes with the Clifford action and for any orthonormal coframe

$$\mathrm{cl}(R(V,W)) = \sum_{pq} R_{pq}(V,W)\,\mathrm{cl}(\phi^p)\,\mathrm{cl}(\phi^q).$$

PROOF: By definition the curvature operator is

$$K_E(V,W)s = \left([\nabla_V,\nabla_W] - \nabla_{[V,W]}\right)s \;\forall\; s \in \mathcal{C}^\infty(X;E).$$

Now suppose that $\alpha \in \mathcal{C}^\infty(X;T^*X)$ is a smooth 1-form. Then the commutator of the curvature operator and Clifford multiplication by α can be evaluated using (3.36):

$$K_E(V,W)\,\mathrm{cl}(\alpha)s = \mathrm{cl}(\alpha)K_E(V,W)s + \mathrm{cl}(R(V,W)\alpha)s, \tag{8.106}$$

where $R(V,W)$ is the curvature operator on 1-forms:

$$R(V,W)\phi^p = \sum_q R_{qp}(V,W)\phi^q.$$

This in turn can be written as a commutator

$$\mathrm{cl}(R(V,W)\alpha)s = \frac{1}{4}\,\mathrm{cl}(R(V,W))\cdot\mathrm{cl}(\alpha)s - \frac{1}{4}\,\mathrm{cl}(\alpha)\cdot\mathrm{cl}(R(V,W)s. \tag{8.107}$$

Indeed, for any orthonormal coframe

$$\begin{aligned}
&\mathrm{cl}(R(V,W))\,\mathrm{cl}(\phi^p) - \mathrm{cl}(\phi^p)\,\mathrm{cl}(R(V,W)) \\
&= \sum_{ij} R_{ij}(V,W)\,\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^j)\,\mathrm{cl}(\phi^p) - \sum_{ij} R_{ij}(V,W)\,\mathrm{cl}(\phi^p)\,\mathrm{cl}(\phi^i)\,\mathrm{cl}(\phi^j) \\
&= 2\sum_i R_{ip}(V,W)\,\mathrm{cl}(\phi^i) - \sum_j R_{pj}(V,W)\,\mathrm{cl}(\phi^j) \\
&= 4\sum_i R_{ip}(V,W)\,\mathrm{cl}(\phi^i).
\end{aligned}$$

Combining (8.106) and (8.107) shows that

$$K_E(V,W) - \frac{1}{4}\,\mathrm{cl}(R(V,W)) \tag{8.108}$$

commutes with Clifford multiplication by 1-forms. It therefore commutes with all Clifford multiplication, on E. With $K_{E'}(V,W)$ defined to be the difference, (8.108), this proves the lemma.

The extensions of Lemmas 8.22 and 8.23 to this wider setting are the same, since the rescaling takes place only in the Clifford part. This then allows the heat kernel for $\eth_E^2$ to be constructed in the rescaled heat calculus, the formula replacing (8.63) for the action in (8.62) being as in (8.102) except that K_E should now be replaced by $K_{E'}$ from (8.105). It remains to analyze the supertrace of the restriction to the diagonal of the resulting kernel.

Let R_E be the involution defining the $\mathbb{Z}_2$-grading of E. By assumption this is consistent with the grading of the Clifford action, i.e. commutes with Clifford multiplication by any element of $\mathbb{Cl}^+$. Since the involution on the Clifford algebra itself, R, is given by multiplication by the volume form, it follows that R_E and R commute. Moreover R_E must anticommute with $\mathbb{Cl}^-$, as does R. The product therefore commutes with the Clifford action, so in the decomposition (3.37)

$$R_E = R \cdot R_E',\ (R_E')^2 = \mathrm{Id}\,.$$

This allows a supertrace functional to be defined on $\hom'_{\mathbb{Cl}}(E)$ by

$$\mathrm{str}'(A) = \mathrm{tr}(R_E' A),\ A \in \hom'_{\mathbb{Cl}}(E). \tag{8.109}$$

Then

$$\mathrm{str}(A) = \mathrm{tr}(R_E A) = \mathrm{str}(\alpha)\,\mathrm{str}'(A'),\ A = \alpha \otimes A',\ \alpha \in \mathbb{Cl},\ A' \in \hom'_{\mathbb{Cl}}\,.$$

In particular Lemma 8.18 applies to the supertrace on the Clifford factor, as before.

With these minor modifications the local index theorem extends to the case of a graded Hermitian Clifford module for an exact b-metric:

THEOREM 8.34. *If $P = {}^b\eth_E^2$ is the twisted Dirac Laplacian for an Hermitian Clifford module with Hermitian Clifford connection on a compact $2k$-dimensional manifold, with an exact b-metric, then the heat kernel is an element of the rescaled b-heat calculus, $\exp(-tP) \in \Psi_{b,G}^{-2}(X; S \otimes E)$ and the fibre supertrace satisfies*

$$\mathrm{str}[\exp(-tP)]_{\restriction t=0} = \mathrm{Ev}_{2k}\left[\det{}^{\frac{1}{2}}\big(\frac{R/4\pi i}{\sinh(R/4\pi i)}\big)\right]\mathrm{str}'\exp(\frac{iK_E'}{2\pi}) \tag{8.110}$$

where $K_{E'}$ is the difference in (8.108), the part of the curvature of E commuting with the Clifford action and the modified supertrace is given by (8.109).

8.13. The eta integrand.

Consider the formula (In.28) for the eta invariant. It follows from (In.27) and the local index formula just discussed that the integral converges absolutely. In fact

$$\operatorname{Tr}\left(\eth_{0,E}\exp(-t\eth_{0,E}^2)\right) \in t^{\frac{1}{2}}\mathcal{C}^\infty([0,\infty)). \tag{8.111}$$

This is for the Dirac operator on a compact spin manifold of odd dimension, assuming of course that it bounds a compact exact b-spin manifold. In fact (8.111) always holds for the Dirac operator in odd dimensions. Even more is true and there is an analogue of the local index theorem, due to Patodi and Gilkey. We shall prove this using a scaling argument given by Bismut and Freed [19].

THEOREM 8.35. *If $\eth$ is the Dirac on an odd-dimensional compact spin manifold without boundary, Y, then application of the fibre trace to the restriction to the diagonal gives*

$$\operatorname{tr}\left(\eth\exp(-t\eth^2)\right) \in t^{\frac{1}{2}}\mathcal{C}^\infty([0,\infty)\times Y;\Omega). \tag{8.112}$$

PROOF: We shall extend the spinor bundle to a product, so denote the Dirac operator on Y as $\eth_0$. As in §2.10 and §3.14 consider $X = \mathbb{S}^1\times Y$, at first with product metric

$$g = d\theta^2 + h,$$

where h is the metric on Y. The spin structure on X is discussed in §3.14 and in particular the Dirac operator takes the form (3.75).

The short-time asymptotics of the integrand for the eta invariant of $\eth_0$ can be recovered from the heat kernel for $\eth$ through

$$\begin{aligned}&\operatorname{tr}\left(\eth_0\exp(-t\eth_0^2)\right)\\ &\sim \frac{\sqrt{2\pi t}}{2i}\operatorname{tr}\left(R\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\eth_0\exp(-t^b\eth_E^2)\right)_{\theta=\theta',\Delta(Y)}.\end{aligned} \tag{8.113}$$

Here the kernel on the right acts on $S_0\oplus S_0$ and is restricted to the diagonal in Y^2 and to any one point $\theta=\theta'$ in the diagonal of $(\mathbb{S}^1)^2$. From (3.75) the square of the Dirac operator on the product is just

$$\eth^2 = \left(D_\theta^2+\eth_0^2\right)\operatorname{Id},\ D_\theta = \frac{1}{i}\frac{\partial}{\partial\theta}. \tag{8.114}$$

The heat kernel is therefore just the product of the heat kernels on $\mathbb{S}^1$ and on Y. Since $\mathbb{S}^1$ is locally diffeomorphic to $\mathbb{R}$ the heat kernel on the circle has the same short time asymptotics as on the line, so on the diagonal reduces to $1/\sqrt{2\pi t}$ up to all orders as $t \downarrow 0$. The matrices in (8.113) multiply to iR since R is just Id on the first factor of S_0 and $-\mathrm{Id}$ on the second. Thus (8.113) does indeed hold.

Consider the square of the Dirac operator for the warped product in (3.78)

$$\widetilde{\eth}^2 = \left(\left(D_\theta + F(\theta)\right)^2 + e^{-2\phi}\eth_0^2\right)\mathrm{Id} + i\phi' e^{-\phi}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eth_0.$$

Suppose that $\phi(\theta)$ depends smoothly on a parameter s then

(8.115)
$$\begin{aligned}\frac{d}{ds}\widetilde{\eth}^2 &= \dot{F}(\theta)(D_\theta + F(\theta)) + (D_\theta + F(\theta))\dot{F}(\theta) \\ &\quad + i[\dot{\phi}' - \phi'\dot{\phi}]e^{-\phi}R\eth_0 - 2\dot{\phi}e^{-2\phi}\eth_0^2,\end{aligned}$$

where $\dot{F}$ and $\dot{\phi}$ denote the derivatives with respect to the parameter s. We can choose $\phi \equiv 0$ at $s = 0$ and then (8.115) reduces to

$$\frac{d}{ds}\widetilde{\eth}^2(0) = \frac{2k-1}{4i}\left(\Phi(\theta)D_\theta + D_\theta\Phi(\theta)\right)\mathrm{Id} + i\Phi(\theta)R\eth_0 - 2\phi'\eth_0^2, \tag{8.116}$$

where $\Phi(\theta) = \dot{\phi}'(\theta)$ at $s = 0$ and (3.79) has also been used.

By Duhamel's principle the derivative of the heat kernel with respect to the parameter can be written:

$$\frac{d}{ds}\exp(-t\widetilde{\eth}^2) = -t\int_0^t \exp(-(t-r)\widetilde{\eth}^2)\frac{d\widetilde{\eth}^2}{ds}\exp(-r\widetilde{\eth}^2)dr. \tag{8.117}$$

Notice that the parity involution R is the volume element, i.e. the product of Clifford multiplication by an oriented orthonormal basis. By (3.76) this is independent of ϕ, and hence the parameter, in the identification of spinor bundles. Thus the supertrace can be taken in (8.117) at any point on the diagonal. Setting $s = 0$ and using (8.116)

(8.118)
$$\begin{aligned}&\frac{d}{ds}\operatorname{str}\left(\exp(-t\widetilde{\eth}^2)\right)_{\restriction\theta=\theta',\Delta(Y),s=0} \\ &= ct\operatorname{str}\left(\int_0^t \exp(-(t-r)\eth^2)\frac{d\widetilde{\eth}^2}{ds}(0)\exp(-r\eth^2)dr\right)_{\restriction\theta=\theta',\Delta(Y),s=0}.\end{aligned}$$

Here $c \neq 0$ is constant and $\eth^2$ is the square of the product Dirac operator given in (8.114). In (8.116) only the middle term is not a multiple of the identity 2×2 matrix on $S_0 \oplus S_0$, so it gives the only contribution to the supertrace in (8.118). Thus

(8.119)

$$\frac{d}{ds} \operatorname{str}\left(\exp(-t\widetilde{\eth}^2)\right)_{\restriction \theta=\theta', \Delta(Y), s=0}$$
$$= ct \operatorname{str}\left(\int_0^t \exp(-(t-r)\eth^2)\Phi(\theta)R\eth_0 \exp(-r\eth^2)dr\right)_{\restriction \theta=\theta', \Delta(Y), s=0}.$$

Now we are free to choose Φ as a real-valued function on the circle. Choosing it to be 1 near some value of θ and again recalling the locality of the short-time asymptotics of the heat kernels, it follows from (8.119) and (8.113) that the short-time asymptotics of the eta integrand are a constant multiple of those of

$$t^{-\frac{1}{2}} \frac{d}{ds} \operatorname{str}\left(\exp(-t\widetilde{\eth}^2)\right)_{\restriction \theta=\theta', \Delta(Y), s=0} \tag{8.120}$$

near a point where Φ is constant. The local index theorem applies to the heat kernel here, so (8.112) holds with $t^{\frac{1}{2}}$ replaced by $t^{-\frac{1}{2}}$. The leading term can be computed using the local index formula, (8.101). In this case only the $\widehat{A}$-genus appears. Consider the formula (2.74) for the curvature of the warped product. Differentiating with respect to s and evaluating at $s = 0$ and at a point $\theta = \theta'$ near which $\phi = 1 + s(\theta - \theta')$, all terms vanish.

It follows that the leading term in (8.120) vanishes. The trace can only have odd powers of $t^{\frac{1}{2}}$ in its expansion, so (8.112) follows and the theorem is proved.

Notice that this proof extends to show that (8.112) holds for the Dirac operator on any Hermitian Clifford bundle with unitary Clifford connection. In fact:

THEOREM 8.36. *If $\eth_E$ is the Dirac operator for an Hermitian Clifford bundle with unitary Clifford connection on an odd-dimensional exact b-manifold with boundary, Y, then application of the fibre trace to the restriction to the diagonal gives*

$$\operatorname{tr}\left(\eth_E \exp(-t\eth_E^2)\right) \in t^{\frac{1}{2}}\mathcal{C}^\infty([0,\infty) \times Y; \Omega). \tag{8.121}$$

EXERCISE 8.37. Go through the details of the proof. [Hint. The warped product is only used to twist the Clifford action on the bundle.]

8.14. The modified eta invariant.

Next consider the dependence of the modified η-invariant, in (In.32), on the parameter s.

PROPOSITION 8.38. *For a generalized Dirac operator, $\eth_{0,E}$, associated to an Hermitian Clifford module on an odd-dimensional compact manifold without boundary the modified η-invariant* (In.32) *is $\mathcal{C}^\infty$ near any $s \notin -\operatorname{spec}(\eth_{0,E})$ with derivative*

$$\begin{aligned} &\frac{d}{ds}\eta_s(\eth_{0,E}) \\ &= -\frac{2}{\sqrt{\pi}} \text{ coeff of } t^{-\frac{1}{2}} \text{ in } \operatorname{Tr}\exp(-t(\eth_{0,E}+s)^2) \text{ as } t \downarrow 0 \end{aligned} \tag{8.122}$$

and satisfies, for $s \in -\operatorname{spec}(\eth_{0,E})$,

$$\begin{aligned} \eta_s(\eth_{0,E}) &= \frac{1}{2}\left[\lim_{\epsilon\downarrow 0}\eta_{s+\epsilon}(\eth_{0,E}) + \lim_{\epsilon\uparrow 0}\eta_{s-\epsilon}(\eth_{0,E})\right] \\ \dim\operatorname{null}(\eth_{0,E}+s) &= \frac{1}{2}\lim_{\epsilon\downarrow 0}\left[\eta_{s+\epsilon}(\eth_{0,E}) - \eta_{s-\epsilon}(\eth_{0,E})\right]. \end{aligned} \tag{8.123}$$

PROOF: From the construction in Chapter 7 the heat kernel

$$\exp(-t(\eth_{0,E}+s)^2)$$

depends smoothly on s as an element of the heat calculus, for all finite times. In particular for any $T > 0$

$$\operatorname{Tr}\left((\eth_{0,E}+s)\exp(-t(\eth_{0,E}+s)^2)\right) \in t^{-\frac{1}{2}n}\mathcal{C}^\infty([0,T]_{\frac{1}{2}} \times \mathbb{R}_s).$$

The derivative of the finite integral can be computed by differentiation under the integral and by integration by parts in t :

$$\begin{aligned} &\frac{d}{ds}\int_\epsilon^T t^{-\frac{1}{2}}\operatorname{Tr}\left[(\eth_{0,E}+s)\exp(-t(\eth_{0,E}+s)^2)\right]dt \\ &= \int_0^\infty t^{-\frac{1}{2}}\operatorname{Tr}\left[\left(1-2t(\eth_{0,E}+s)^2\right)\exp(-t(\eth_{0,E}+s)^2)\right]dt \\ &= 2\left[t^{\frac{1}{2}}\operatorname{Tr}(\exp(-t(\eth_{0,E}+s)^2)\right]_\epsilon^T. \end{aligned}$$

If $s \notin -\operatorname{spec}(\eth_{0,E})$ the trace of this heat kernel converges exponentially to zero as $T \to \infty$. This proves (8.122). Since the right side of (8.122) is smooth for all s it follows that $\eta_s(\eth_{0,E})$ is smooth up to any point $s \in -\operatorname{spec}(\eth_{0,E})$ from above and below.

If $s \in -\operatorname{spec}(\eth_{0,E})$ then the smoothness of the integrand remains, but not the uniform decay at ∞. The finite integral is therefore smooth, and the same integration by parts gives

$$\lim_{\epsilon\downarrow 0}[\eta_{s+\epsilon}(\eth_{0,E}) - \eta_{s-\epsilon}(\eth_{0,E})] =$$

$$(8.124)\quad \begin{aligned}&\lim_{\epsilon\downarrow 0}\lim_{T\to\infty}\int_T^\infty t^{-\frac12}\left[\operatorname{Tr}\big((\eth_{0,E}+s)\exp(-t(\eth_{0,E}+r)^2)\big)\right]_{r=s-\epsilon}^{r=s+\epsilon}dt\\ &=\lim_{\epsilon\downarrow 0}\lim_{T\to\infty}\int_{s-\epsilon}^{s+\epsilon}2T^{\frac12}\operatorname{Tr}\exp(-T(\eth_{0,E}+r)^2)dr.\end{aligned}$$

The expansion of the heat kernel at infinity in (7.105) shows that only the eigenvalue $-s$ of $\eth_{0,E}$ contributes to this limit. If m is the dimension of $-s$ as an eigenvalue then changing the variable of integration to $(r+s)T^{\frac12}$ shows

$$\lim_{\epsilon\downarrow 0}[\eta_{s+\epsilon}(\eth_{0,E}) - \eta_{s-\epsilon}(\eth_{0,E})] =$$

$$= 2m\lim_{\epsilon\downarrow 0}\lim_{T\to\infty}\int_{-T^{\frac12}\epsilon}^{T^{\frac12}\epsilon}\exp(-R^2)dR.$$

This gives the second part of (8.123).

Essentially the same argument shows that if $s \in -\operatorname{spec}(\eth_{0,E})$ then

$$\eta_s(\eth_{0,E}) - \lim_{\epsilon\downarrow 0}\eta_{s+\epsilon}(\eth_{0,E}) = \dim\operatorname{null}(\eth_{0,E}+s).$$

From this the remainder of (8.123) follows, completing the proof of the Proposition.

8.15. Variation of eta.

Although the eta invariant is not a local invariant of the geometry its variation is local, as long as the dimension of the null space of $\eth$ does not vary.

Proposition 8.39. *Let Y be an odd-dimensional compact spin manifold and let g_s be a 1-parameter family of Riemann metrics on Y then, provided the null space of the Dirac operator $\eth_s$, defined by g_s, has constant dimension $\eta(\eth_s)$ is a smooth function of s and*

$$(8.125)\quad \begin{aligned}&\frac{d}{ds}\eta(\eth_s)\\ &= -\frac{2}{\sqrt{\pi}}\ \text{coeff of } t^{-\frac12} \text{ in } \operatorname{Tr}\left(\dot{\eth}_s\exp(-t\eth_s^2)\right) \text{ as } t\downarrow 0,\end{aligned}$$

with $\dot{\eth}_s$ the derivative of $\eth_s$ with respect to s.

PROOF: The spinor bundle over Y can be considered to be independent of the parameter and this allows the heat kernel for $\eth_s^2$ to be constructed uniformly. It is therefore smooth in s as an element of the heat calculus for all finite times. Thus

$$\gamma(s,\epsilon,T) = \int_\epsilon^T t^{-\frac{1}{2}} \operatorname{Tr}\left(\eth_s \exp(-t\eth_s^2)\right) dt \tag{8.126}$$

depends smoothly on s, for any finite $\epsilon > 0$ and T.

Differentiation of (8.126) gives

$$\frac{d}{ds}\gamma(s,\epsilon,T) = \int_\epsilon^T t^{-\frac{1}{2}} \operatorname{Tr}\left\{\dot{\eth}_s \exp(-t\eth_s^2) + \eth_s \frac{d}{ds}\exp(-t\eth_s^2)\right\} dt, \tag{8.127}$$

where $\dot{\eth}_s$ is the derivative of $\eth_s$ with respect to the parameter. The derivative of the heat kernel with respect to the parameter can be evaluated from the identity

$$\left(\frac{\partial}{\partial t} + \eth_s^2\right)\frac{d}{ds}\exp(-t\eth_s^2) + (\dot{\eth}_s\eth_s + \eth_s\dot{\eth}_s)\exp(-t\eth_s^2) = 0. \tag{8.128}$$

The initial condition is independent of s so, using the heat kernel to solve (8.128) (i.e. Duhamel's principle),

$$\frac{d}{ds}\exp(-t\eth_s^2) = -\int_0^t \exp(-(t-r)\eth_s^2)\left(\dot{\eth}_s\eth_s + \eth_s\dot{\eth}_s\right)\exp(-r\eth_s^2)dr. \tag{8.129}$$

Inserting this into the second term in (8.127) and using the trace identity, the commutativity of functions of $\eth_s$ and integration by parts

$$\begin{aligned}
&-\int_\epsilon^T t^{-\frac{1}{2}} \operatorname{Tr}\left\{\eth_s \int_0^t \exp(-(t-r)\eth_s^2)\left(\dot{\eth}_s\eth_s + \eth_s\dot{\eth}_s\right)\exp(-r\eth_s^2)\right\} dr \\
&= -2\int_\epsilon^T t^{\frac{1}{2}} \operatorname{Tr}\left(\dot{\eth}_s\eth_s^2\exp(-t\eth_s^2)\right) dt \\
&= 2\int_\epsilon^T t^{\frac{1}{2}} \operatorname{Tr}\left(\dot{\eth}_s\frac{d}{dt}\exp(-t\eth_s^2)\right) dt \\
&= \left[2t^{\frac{1}{2}} \operatorname{Tr}\left(\dot{\eth}_s\exp(-t\eth_s^2)\right)\right]_\epsilon^T - \int_\epsilon^T t^{-\frac{1}{2}} \operatorname{Tr}\left(\dot{\eth}_s\exp(-t\eth_s^2)\right).
\end{aligned}$$

The second term here cancels the first term on the right in (8.127) so

$$\frac{d}{ds}\gamma(s,\epsilon,T) = \left[t^{\frac{1}{2}}\operatorname{Tr}\left(\dot{\eth}_s\exp(-t\eth_s^2)\right)\right]_\epsilon^T. \tag{8.130}$$

Let Π_s be the orthogonal projection onto the null space of $\eth_s$. The assumption that the dimension is constant as s varies means that Π_s depends smoothly on s. Thus, by the self-adjointness of $\eth_s$, $\eth_s\Pi_s = \Pi_s\eth_s = 0$ so

$$\eth_s = (\mathrm{Id}-\Pi_s)\eth_s(\mathrm{Id}-\Pi_s) \Longrightarrow$$
$$\dot{\eth}_s = -\dot{\Pi}_s\eth_s(\mathrm{Id}-\Pi_s) + (\mathrm{Id}-\Pi_s)\dot{\eth}_s(\mathrm{Id}-\Pi_s) - (\mathrm{Id}-\Pi_s)\eth_s\dot{\Pi}_s.$$

Since $(\mathrm{Id}-\pi_s)\exp(-t\eth_s^2)$ is uniformly exponentially decreasing this shows that the contribution from the upper limit in (8.130) vanishes rapidly as $T\to\infty$. Thus (8.125) holds.

8.16. Spectral flow.

To extend Proposition 8.39 to the general case where the dimension of the null space may change we consider the notion of the spectral flow of a family of first order elliptic differential operators, including therefore Dirac operators. The definition of spectral flow applies to general families of self-adjoint operators and was introduced in this context by Atiyah and Lusztig see [10;§7].

If $A_s \in \mathrm{Diff}^m(X;E)$ is a smooth family of self-adjoint elliptic operators of fixed order, m, on a compact manifold without boundary then the eigenvalues vary continuously in the sense that the sum of the dimensions of the eigenspaces corresponding to the eigenvalues in any closed interval $[a,b]$ is constant as long as a and b are not eigenvalues of any element of the family.

EXERCISE 8.40. Prove this constancy using the resolvent family, the fact that the residues are the self-adjoint projectors on the eigenspaces and Cauchy's formula.

So suppose for the sake of definiteness that $[0,1] \longrightarrow A_s \in \mathrm{Diff}^1(X;E)$ is such a smooth family of first-order elliptic self-adjoint elliptic operators. The spectral flow across a real number t is intended to measure the net number of eigenvalues which cross t upwards. Thus we certainly want

$$A_s - t \text{ is invertible } \forall\ s\in[0,1] \Longrightarrow \mathrm{SF}(A_s,t) = 0. \tag{8.131}$$

More generally let us assume that

$$t\notin \operatorname{spec}(A_0) \text{ and } t\notin\operatorname{spec}(A_1). \tag{8.132}$$

Since the dependence of the eigenvalues on s may be rather complicated, a slightly indirect definition is needed, based on the expectation of homotopy invariance. By the discreteness and continuity of the spectrum we can choose a partition of $[0,1]$, fixed by endpoints $0 = s_0 < s_1 < \cdots < s_q = 1$ and associated real numbers t_i with $t_0 = t_{q+1} = t$ and such that

$$t_i \notin \operatorname{spec}(A_s) \ \forall\ s \in [s_{i-1}, s_i],\ i = 1, \dots, q. \tag{8.133}$$

Then set

$$\operatorname{SF}(A_s, t) = \sum_{i=0}^{q} \sum_{\sigma \in \operatorname{spec}(A_{s_i}) \cap [t_i, t_{i+1}]} \operatorname{sgn}(t_{i+1} - t_i) m(\sigma, s_i). \tag{8.134}$$

Here the convention for $\operatorname{sgn} r$ for $r \in \mathbb{R}$ is that it is -1, 0 or 1 as r is negative, 0 or positive.

Naturally it needs to be shown that this definition is independent of the partition used to define it, provided (8.133) holds. Clearly subdividing the intervals but keeping the values of the t_i the same does not change (8.134). Thus in comparing (8.134) for any two partitions it can be assumed that the intervals are the same. It therefore suffices to consider the case where just one of the t_i is changed, but then the independence of choice follows from the continuity of the eigenvalues as described above.

Next consider the dependence of $\operatorname{SF}(A_s, t)$ on t. Clearly if (8.132) holds then $\operatorname{SF}(A_s, t)$ is locally constant in the open intervals of $\mathbb{R} \setminus (\operatorname{spec}(A_0) \cup \operatorname{spec}(A_1))$ on which it is defined. On the other hand it follows directly from the definition that if t and t' both satisfy (8.132) then

$$\operatorname{SF}(P_s, t') - \operatorname{SF}(P_s, t) = \operatorname{sgn}(t' - t) \left(\sum_{\sigma \in \operatorname{spec}(A_1) \cap [t, t']} - \sum_{\sigma \in \operatorname{spec}(A_0) \cap [t, t']} \right). \tag{8.135}$$

That is, on crossing an eigenvalue of either A_0 or A_1 the spectral flow changes by the difference of the dimensions of the eigenspaces. In general, when (8.132) is not assumed we set

$$\begin{aligned} \operatorname{SF}(A_s, t) = \frac{1}{2} \left(\dim \operatorname{null}(A_1 - t) - \dim \operatorname{null}(A_0 - t) \right) \\ + \sum_{i=0}^{q} \sum_{\sigma \in \operatorname{spec}(A_{s_i}) \cap (t_i, t_{i+1})} \operatorname{sgn}(t_{i+1} - t_i) m(\sigma, s_i). \end{aligned} \tag{8.136}$$

EXERCISE 8.41. Interpret this formula in terms of the incidence function introduced in (6.22) and generalize (8.135) to cover general values of t and t'.

EXERCISE 8.42. Show that if A_s is a closed curve of operators, i.e. $A_0 = A_1$, then $\mathrm{SF}(A_s) = \mathrm{SF}(A_s, t)$ is independent of t and a homotopy invariant of the curve.

PROPOSITION 8.43. *If g_s is, for $s \in [0,1]$, a smooth family of metrics on a compact odd-dimensional spin manifold without boundary then for the corresponding family of Dirac operators*

$$\eta(\eth_1) - \eta(\eth_0) =$$
$$\dim \mathrm{null}(\eth_1) - \dim \mathrm{null}(\eth_0) + \int_0^1 F(s)ds + 2\,\mathrm{SF}(\eth_s, 0), \tag{8.137}$$

where $F(s)$ is the coefficient of $t^{-\frac{1}{2}}$ in $\mathrm{Tr}\left(\dot{\eth}_s \exp(-t\eth_s^2)\right)$ *as $t \downarrow 0$.*

To prove this result we need to examine the local term a little:

LEMMA 8.44. *Suppose $[0,1] \longrightarrow D_r \in \mathrm{Diff}^1(X;E)$ is a family of self-adjoint elliptic operators on a compact odd-dimensional manifold without boundary, then $\alpha = Fds + Gdr$, is a smooth closed 1-form on $\mathbb{R} \times [0,1]$ if*

$$F(s,r) \textit{ is the coeff of } t^{-\frac{1}{2}} \textit{ in } \mathrm{Tr}\left(\dot{\eth}_s \exp(-t(\eth_s + r)^2)\right) \textit{ as } t \downarrow 0 \tag{8.138}$$

and

$$G(s,r) \textit{ is the coeff of } t^{-\frac{1}{2}} \textit{ in } \mathrm{Tr}\left(\exp(-t(\eth_s + r)^2)\right) \textit{ as } t \downarrow 0. \tag{8.139}$$

PROOF: Consider the derivative of F with respect to r. The smoothness of the heat kernel in parameters means that $\partial F/\partial r$ is the coefficient of $t^{\frac{1}{2}}$ at $t = 0$ in the expansion of

$$-2\,\mathrm{Tr}\left(\dot{\eth}_s(\eth_s + r)\exp(-t(\eth_s + r)^2)\right). \tag{8.140}$$

Similarly the derivative of G with respect to s can be computed using Duhamel's principle, as in (8.129), and it also reduces to (8.140).

PROOF OF PROPOSITION 8.43: Consider the somewhat more general formula for an elliptic family depending on $s \in [a,b]$ and the modified eta invariants:

(8.141)
$$\eta_r(\eth_b) - \eta_r(\eth_a) =$$
$$\dim \mathrm{null}(\eth_b + r) - \dim \mathrm{null}(\eth_a + r) + \int_a^b F(s,r)ds + 2\,\mathrm{SF}(\eth_s + r, 0),$$

where now F is given by (8.138). This reduces to (8.137) when $r = 0$. In fact if (8.141) is proved for a particular family for any one value of r it holds for all other values of r for that family. This follows from Proposition 8.38, Lemma 8.44 and the properties already established of the spectral flow. Moreover (8.141) also follows if it holds for each interval of a partition of $[a, b]$. Since there is always such a partition in which $\eth_s + r$ is invertible on each subinterval, for a fixed r depending on the subinterval, it suffices to prove it under this assumption. In this case the two null spaces on the right are trivial, as is the spectral flow, and the formula then follows from the proof of Proposition 8.39.

EXERCISE 8.45. Using Exercise 8.42 and the fact that the space of metrics on any compact manifold without boundary is simply connected (it is affine) show that the total spectral flow of the family of Dirac operators corresponding to a closed curve of metrics, on a compact odd-dimensional spin manifold without boundary, is trivial. See how this arises from the formula (8.137) by using Proposition 8.38 and a generalization of Lemma 8.44 to a general family depending on two parameters.

8.17. The circle.

Using the properties of the modified eta invariant we will evaluate it for D_θ on the circle.

PROPOSITION 8.46. *For $D_\theta = -id/d\theta$ on the circle the modified eta invariant is*

(8.142)
$$\eta_s(D_\theta) = 2\left(\sqrt{2}s - j(s)\right),$$
$$j(-s) = -j(s),\ \ j(s) = \begin{cases} s & s \in \mathbb{Z} \\ \frac{1}{2} + \max\{j \in \mathbb{Z}; j < s\} & s \in (0, \infty) \setminus \mathbb{N}. \end{cases}$$

PROOF: From Proposition 8.38, $\eta_s(D_\theta)$ is smooth except for integral jumps at the points $s \in \mathbb{Z}$. The short-time asymptotics of the heat kernel on $\mathbb{S}^1$ are the same as those of the heat kernel on $\mathbb{R}$, since they are locally determined. Thus the derivative of $\eta_s(D_\theta)$ can be evaluated, using (8.122), from the heat kernel of $D_x + s$ on $\mathbb{R}$. Since $D_x e^{ixs} u(x) = e^{ixs}(D_x + s)u$ this kernel is

$$\exp(-t(D_x + s)^2) = \frac{1}{\sqrt{2\pi t}} \exp(\frac{(x - x')^2}{4t} + i(x - x')s).$$

Thus away from the integers

$$\frac{d}{ds}\eta_s(D_\theta) = \frac{2}{\sqrt{\pi}} \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} d\theta = 2\sqrt{2}. \tag{8.143}$$

This leads to (8.142) once it is shown for one value of s.

The Fourier series representation of the heat kernel of D_θ

$$\exp(-tD_\theta^2) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-tk^2}\exp(ik(\theta-\theta')),$$

shows that in $t>0$

$$\operatorname{tr} D_\theta \exp(-tD_\theta^2) = \frac{1}{2\pi}\sum_{k\neq 0} ke^{-tk^2} = 0.$$

Thus $\eta(D_\theta) = \eta_0(D_\theta) = 0$. This completes the proof of Proposition 8.46.

EXERCISE 8.47. Evaluate the modified eta invariant of D_θ directly using Fourier series to construct the heat kernel of $D_\theta + s$ on the circle.

Chapter 9. Proof revisited and applications

In this final chapter the various threads examined in the preceding chapters are pulled together to fully justify the proof outlined in the Introduction. Then the original application of the APS to the signature of a compact manifold with boundary is described. Some other applications of the description of b-geometry and analysis are made; in particular it is shown how the eta invariant and the analytic torsion can be defined on manifolds with exact b-metrics, under appropriate conditions.

9.1. The APS theorem.

The discussion of the APS theorem in the Introduction is limited to the twisted Dirac operators on exact b-spin manifolds. In fact the theorem extends readily to the generalized Dirac operators defined in §3.11. In the boundaryless case, for the Atiyah-Singer theorem, this extension is emphasized in [20]. It is in this more general context that the proof in the Introduction will be reviewed.

Let X be a compact even-dimensional manifold with boundary, equipped with an exact b-metric as defined in §2.3. Let $E = E^+ \oplus E^-$ be a graded Hermitian Clifford module on X with a graded unitary Clifford b-connection; these notions are defined in Chapter 3. Thus E is an Hermitian vector bundle over X with a (non-trivial) fibrewise action of the Clifford algebra of the b-metric such that $\mathrm{Cl}_x^+(X)$ preserves the spaces $E_x^\pm$ while $\mathrm{Cl}_x^-(X)$ interchanges them. That the connection is Clifford is the condition (3.36). Definition 3.7 fixes the generalized Dirac operator

$$\eth_E \in \mathrm{Diff}_b^1(X; E)$$

associated to this geometric data.

The symbol of such a b-differential operator is defined in (2.22) and in this case

$$^b\sigma_1(\eth_E)(\xi) = \mathrm{cl}(\xi) \in \hom(E_x) \quad \forall\, \xi \in {}^bT_x^*X,\ x \in X.$$

For $\xi \neq 0$ this is an isomorphism (since $\mathrm{cl}(\xi)^2 = |\xi|^2 \,\mathrm{Id}$) so $\eth_E$ is elliptic. It is formally self-adjoint and graded

$$\eth_E = \begin{pmatrix} 0 & \eth_E^- \\ \eth_E^+ & 0 \end{pmatrix},\ E = E^+ \oplus E^-,$$

see Lemma 3.32.

The b-metric fixes the choice of a trivialization of the normal bundle to each component of the boundary of X, up to a global $\mathbb{R}^+$-action (for each component the same constant on each fibre), as discussed in §2.3. Thus the indicial operator of $\eth_E$, given by (4.102) in local coordinates, which is in general an operator on the compactified normal bundle to ∂X, becomes a well-defined operator on $[0,\infty)\times\partial X$ on sections of $E_{\restriction\partial X}$. As in §3.13 set

$$E_0 = E^+_{\restriction\partial X}$$

with the identification denoted $M_+ \colon E^+_{\restriction\partial X} \longrightarrow E_0$.

Since $T^*\partial X \subset {}^bT^*_{\partial X}X$, Clifford multiplication by elements of $T^*\partial X$ is defined on $E_{\restriction\partial X}$. Clifford multiplication by the metrically defined section, dx/x, of ${}^bT^*_{\partial X}X$ is a self-adjoint involution on $E_{\restriction\partial X}$ so

$$\mathrm{cl}_\partial(\eta) = i\,\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta)\colon E_0 \longrightarrow E_0.$$

This defines a Clifford module structure on E_0 for the induced metric on ∂X since dx/x is orthogonal to $T^*\partial X$, so for $\eta,\eta'\in T^*_p\partial X$

$$\begin{aligned}\mathrm{cl}_\partial(\eta)\,\mathrm{cl}_\partial(\eta') + \mathrm{cl}_\partial(\eta')\,\mathrm{cl}_\partial(\eta) = &-\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta)\,\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta')\\ &-\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta')\,\mathrm{cl}(\frac{dx}{x})\,\mathrm{cl}(\eta) = 2\langle\eta,\eta'\rangle.\end{aligned}$$

The induced connection on E_0 is a Clifford connection for this action (since dx/x is covariant constant); let $\eth_{0,E}$ be the induced generalized Dirac operator on the boundary, acting on E_0. Let $M_- \colon E^-_{\restriction\partial X} \longrightarrow E_0$ be defined by $M_- = M_+\cdot\mathrm{cl}(i\frac{dx}{x})$ and consider the unitary isomorphism

$$M^{-1}\colon E_0\oplus E_0 \ni (u,v) \longmapsto M_+^{-1}u + M_-^{-1}v = u - i\,\mathrm{cl}(\frac{dx}{x})v \in E_{\restriction\partial X}. \tag{9.1}$$

Then

$$M\,i\,\mathrm{cl}(\frac{dx}{x})I(\eth_E)M^{-1} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} x\frac{\partial}{\partial x} + \eth_{0,E}\,\mathrm{Id}\,.$$

Correspondingly the indicial family of the Dirac operator satisfies

$$M\,i\,\mathrm{cl}(\frac{dx}{x})I_\nu(\eth_E,\lambda)M^{-1} = \begin{pmatrix}\eth_{0,E}+i\lambda & 0\\ 0 & \eth_{0,E}-i\lambda\end{pmatrix}$$

which is just (In.23).

Since $\eth_{0,E}$ is self-adjoint the only singular points for the inverse of the indicial family are pure imaginary:

$$\mathrm{spec}_b(\eth_E) = \{(\pm is,0);\, s\in\mathrm{spec}(\eth_{0,E})\}\,.$$

In particular Theorem 5.40, or rather its extension to operators on bundles, shows that

$$\eth_E : x^s H_b^m(X;E) \longrightarrow x^s H_b^{m-1}(X;E) \tag{9.2}$$

is Fredholm if and only if $s \notin \pm\operatorname{spec}(\eth_{0,E})$. For the partial Dirac operators it follows similarly that

$$\begin{gathered} \eth_E^{\pm} : x^s H_b^m(X;E^{\pm}) \longrightarrow x^s H_b^{m-1}(X;E^{\mp}) \\ \text{is Fredholm iff } s \notin \mp\operatorname{spec}(\eth_{0,E}) \end{gathered} \tag{9.3}$$

for the respective signs. When it is Fredholm the index of the operator $\eth_E^+$ in (9.3) is denoted $\operatorname{ind}_s(\eth_E^+)$. The index function $\widetilde{\operatorname{ind}}_s(\eth_E^+)$, defined by (In.30), extends this to all $s \in \mathbb{R}$.

Consider next the curvature of E. It is shown in (8.105) that the curvature operator decomposes:

$$K_E = \frac{1}{4}\operatorname{cl}(R) \otimes K'_E,$$

where K'_E is a 2-form on X with values in the homomorphisms of E commuting with the Clifford action. Correspondingly the Atiyah-Singer integrand becomes:

$$\mathrm{AS} = \frac{1}{(2\pi i)^k}\operatorname{Ev}_{\dim X}\left(\det{}^{\frac12}\big(\frac{R/2}{\sinh(R/2)}\big)\cdot \operatorname{str}'\exp(K'_E)\right). \tag{9.4}$$

Thus AS is a smooth b-form of maximal degree on X. The eta invariant of $\eth_{0,E}$ is discussed in §8.13.

With these preliminaries the Atiyah-Patodi-Singer index theorem can now be stated in this more general context:

THEOREM 9.1. *(APS) If $\eth_E^+$ is the Dirac operator on a compact manifold with boundary with exact b-metric fixed by a graded unitary Clifford module, E, with graded Hermitian Clifford b-connection then*

$$\widetilde{\operatorname{ind}}(\eth_E^+) = {}^{\nu}\!\!\int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}), \tag{9.5}$$

where AS is given by (9.4), the regularized integral (with respect to the metric trivialization of the normal bundle) of a b-form is defined in (4.138) and $\eta(\eth_{0,E})$ is defined in §8.13.

PROOF: We shall give two slightly different forms of the proof. In the first we extend (9.5) to

$$\widetilde{\operatorname{ind}}_s(\eth_E^+) = {}^{\nu}\!\!\int_X \mathrm{AS}(s) - \frac{1}{2}\eta_s(\eth_{0,E}) \quad \forall\ s \in \mathbb{R}, \tag{9.6}$$

where $\widetilde{\operatorname{ind}}_s(\eth_E^+)$ is defined above, $\eta_s(\eth_{0,E})$ is defined in §8.13 (as a regularized integral) and the generalized Atiyah-Singer integrand is fixed by (In.31). For the proof that (In.31) is meaningful see Chapter 7.

The dependence of $\widetilde{\operatorname{ind}}_s(\eth_E^+)$ on the parameter is discussed §6.2, see Theorem 6.5. Similarly the dependence of $\eta_s(\eth_{0,E})$ on the parameter is considered in §8.13, see in particular Proposition 8.38. These two results show that the sum $\widetilde{\operatorname{ind}}_s(\eth_E^+) + \frac{1}{2}\eta_s(\eth_{0,E})$, is independent of s. Thus it suffices to prove (9.5) for the dense set of values of s for which $\eth_E^+$ is Fredholm in (9.2).

Now we proceed to follow the proof outlined in the Introduction, starting with §4. The properties of the heat kernels for $\eth_E^-\eth_E^+$ and $\eth_E^+\eth_E^-$ are discussed in Chapter 7. In particular Theorem 7.29 applies to both operators, so (In.18) and (In.23) follow. The identity (In.15) is discussed in Proposition 7.30. The b-trace is defined and analyzed in §4.20, with (In.22) shown in Proposition 5.9. This justifies the identity (In.24) and hence also (In.25) and (In.26). Note that the limit as $t \downarrow 0$ in (In.27) is regularized. This completes the proof of the formula (9.6) for these generalized Dirac operators.

In the particular case $s = 0$ the Atiyah-Singer integrand reduces to (9.4) and the integral in t defining the eta invariant converges absolutely. This completes the first proof of (9.5), and hence of the theorem. In particular (In.6) holds for twisted Dirac operators.

Alternatively (9.5) can be proved directly by the same formalism. Of course if $s = 0$ happens to be a Fredholm value then the discussion above leads directly to (9.5). If it is not Fredholm then the proof in the Introduction can still be used, except that (In.10) is not valid, since the spectrum of $\eth_E^2$ is not discrete near 0. However this limit is examined in §7.8, see (7.118). This shows that (In.10) is replaced by

$$\begin{aligned}\lim_{t\to\infty} \text{b-Tr}_\nu \exp(-t\eth_E^-\eth_E^+) &= \frac{1}{2}\lim_{\epsilon\downarrow 0}\left[\operatorname{null}(\eth_E^+,-\epsilon) + \operatorname{null}(\eth_E^+,\epsilon)\right]\\ \lim_{t\to\infty} \text{b-Tr}_\nu \exp(-t\eth_E^+\eth_E^-) &= \frac{1}{2}\lim_{\epsilon\downarrow 0}\left[\operatorname{null}(\eth_E^-,-\epsilon) + \operatorname{null}(\eth_E^-,\epsilon)\right],\end{aligned} \tag{9.7}$$

where the $\operatorname{null}(\eth_E^\pm, r)$ are the null spaces on the weighted Sobolev spaces $x^r H_b^\infty(X; E^\pm)$. The definition of $\widetilde{\operatorname{ind}}_0$ then means that (In.11) is replaced by

$$\lim_{t\to\infty} \operatorname{Tr}[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)] = \widetilde{\operatorname{ind}}_0(\eth_E^+). \tag{9.8}$$

With this modification the proof proceeds as in the Fredholm case.

The advantage of this second, direct, proof in the non-Fredholm case is that it extends more readily; the disadvantage is that a more detailed study of the resolvent and spectral measure is required.

Notice that if the original connection on the Clifford bundle is a true connection, rather than just a b-connection, then the form AS is a $\mathcal{C}^\infty$ form in the usual sense and (9.5) becomes a true, unregularized integral. The $\widehat{A}$ part comes from the curvature of the exact b-metric, so it is always a smooth form by Proposition 2.39.

Using the variation formula for the eta invariant proved in §8.15 the general APS formula (9.6) can be cast in the more explicit form:

$$\widetilde{\operatorname{ind}}_s(\eth_E^+) = {}^{\nu}\!\!\int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}) - \widetilde{N}(\eth_{0,E}, s) \quad \forall\ s \in \mathbb{R}, \tag{9.9}$$

where the counting function, $\widetilde{N}(\eth_{0,E}, s)$, is defined in (In.35).

9.2. Euler characteristic.

One of the main direct achievements, and indeed motivations, of the papers [8] - [10] was the application of the index theorem to give an analytic formula for the signature (and twisted generalizations of it) of a compact Riemann manifold with boundary. This is discussed in the next section. The signature formula is closely related to the Gauss-Bonnet formula for the Euler characteristic of the manifold. As noted in [8] the Gauss-Bonnet formula can, and indeed will, be proved in this way, even though it is in reality much more elementary (see for example [27]) since the global 'eta' term vanishes.

For any finite dimensional complex

$$0 \xrightarrow{d} V^0 \xrightarrow{d} V^1 \xrightarrow{d} \cdots \xrightarrow{d} V^p \xrightarrow{d} 0,\ d^2 = 0$$

the Euler characteristic is

$$\chi(V, d) = \sum_{j=0}^{p} (-1)^j \dim V^j.$$

The cohomology spaces of the complex are $H^k = \{u \in V^k; du = 0\}/\{dv; v \in V^{k-1}\}$ and

$$\chi(V, d) = \sum_{j=0}^{p} (-1)^j \dim H^j.$$

This applies in particular to the cohomology arising from a simplicial decomposition of the manifold. Thus, by the de Rham theorem, for a compact manifold without boundary the Euler characteristic is

$$\chi(X) = \sum_{j=0}^{p} (-1)^j \dim H^j(X).$$

The operator $d + \delta$, of which the Laplacian is the square, changes the parity of differential forms so defines

$$D\colon \mathcal{C}^\infty(X; \Lambda^{\text{evn}}) \longrightarrow \mathcal{C}^\infty(X; \Lambda^{\text{odd}}),\ Du = (d + \delta)u. \tag{9.10}$$

This differential operator is elliptic, has null space $H^{\text{evn}}_{\text{Ho}}(X)$ and its adjoint,

$$D^* : \mathcal{C}^\infty(X; \Lambda^{\text{odd}}) \longrightarrow \mathcal{C}^\infty(X; \Lambda^{\text{evn}}),\ D^*\phi = (d + \delta)\phi$$

has null space $H^{\text{odd}}_{\text{Ho}}(X)$, so

$$\operatorname{ind}(D) = \dim H^{\text{evn}}_{\text{Ho}}(X) - \dim H^{\text{odd}}_{\text{Ho}}(X) = \chi(X).$$

The Atiyah-Singer index theorem asserts that this index is given by integration of a local differential expression over the manifold, and the resulting formula is the (generalized) Gauss-Bonnet formula of Chern. Namely

$$\operatorname{ind}(D) = \chi(X) = \int_X e, \tag{9.11}$$

where e is the Euler density of the Riemann manifold:

$$e(x) = \operatorname{Pf}(R_x). \tag{9.12}$$

To prove (9.11) and (9.12) observe that $\Lambda^*(X) = \Lambda^{\text{evn}}(X) \oplus \Lambda^{\text{odd}}(X)$ is a graded Hermitian Clifford module where the involution is given by (2.90). The bundle is Hermitian and the Levi-Civita connection is certainly graded, unitary and Clifford for the Clifford action given by (3.18). The associated Dirac operator is defined by (3.39):

$$\eth_{\text{GB}} = d + \delta.$$

Indeed this follows from the formula (2.63) for d and the corresponding formula for δ. Thus Theorem 9.1 applies, with $\partial X = \emptyset$.

The Gauss-Bonnet formula, (9.11) and (9.12), on a compact manifold without boundary will be deduced from the corresponding result for a manifold with boundary. For the moment note that

$$\chi(M) = 0 \text{ if } M \text{ is odd-dimensional.}$$

This is a direct consequence of Poincaré duality which shows that

$$\dim H^k(M) = \dim H^{n-k}(M), \ n = \dim M.$$

Passing to the case where M has boundary there are two 'obvious' cohomology theories for which the Euler characteristic can be computed, namely the absolute and relative cohomologies of the manifold. These lead to the same Euler characteristic in even dimensions, although not for odd dimensions. In fact the index function for the operator D in (9.10) has the expected behaviour of the Euler characteristic.

LEMMA 9.2. *If X is a compact manifold with an exact b-metric then $\eth_{\mathrm{GB}} \in \mathrm{Diff}_b^2(X; {}^b\Lambda^{\mathrm{evn}}, {}^b\Lambda^{\mathrm{odd}})$ is elliptic with $\widetilde{\mathrm{ind}}_0(\eth_{\mathrm{GB}}) = 0$ if $\dim X$ is odd and if $\dim X$ is even*

$$\begin{aligned} \widetilde{\mathrm{ind}}_0(\eth_{\mathrm{GB}}) &= \int_X \mathrm{Pf}(R) \\ &= \sum_{j=0}^{\dim X} (-1)^j \dim H^j_{\mathrm{abs}}(X) = \sum_{j=0}^{\dim X} (-1)^j \dim H^j_{\mathrm{rel}}(X). \end{aligned} \tag{9.13}$$

PROOF: The discussion above of the formal properties of $\eth_{\mathrm{GB}}$ on a compact manifold without boundary carries over to the case of an exact b-metric on a compact manifold with boundary. Thus the b-form bundle is a graded Hermitian Clifford module and has a true connection which is unitary and Clifford. Thus Theorem 9.1 does indeed apply.

Consider the decomposition of ${}^b\Lambda^{\mathrm{evn}}$ over the boundary of X. Using the metric b-conormal dx/x, it splits as an orthogonal direct sum:

$${}^b\Lambda^{\mathrm{evn}}_{\partial X}(X) = \Lambda^{\mathrm{evn}}(\partial X) \oplus \Lambda^{\mathrm{odd}}(\partial X) \wedge \frac{dx}{x}.$$

The map (9.1) becomes:

$$\begin{aligned} M\colon \ &\left(\Lambda^{\mathrm{evn}}\partial X \oplus \Lambda^{\mathrm{odd}}\partial X\right)^2 \ni ((\alpha,\beta),(\alpha',\beta')) \longmapsto \\ &\alpha + \beta \wedge \frac{dx}{x} + i\alpha' \wedge \frac{dx}{x} - i\beta'. \end{aligned} \tag{9.14}$$

This gives

$$\begin{aligned}\operatorname{cl}_\partial(\eta) M\left((\alpha,\beta),(\alpha',\beta')\right)\\ = -i\operatorname{cl}(\eta)\beta' + i\operatorname{cl}(\eta)\alpha' \wedge \frac{dx}{x} + \operatorname{cl}(\eta)\alpha + \operatorname{cl}(\eta)\beta\wedge\frac{dx}{x},\end{aligned} \tag{9.15}$$

where on the right $\operatorname{cl}(\eta)$ is the standard Clifford action on the boundary. Applying the inverse of (9.14) to (9.15) and inserting the result in (9.1) shows that the Clifford module on the boundary is just $\Lambda^*(\partial X)$ and the induced Clifford multiplication of $T^*\partial X$ on it is just the standard one (meaning (3.18)) for a manifold without boundary.

In particular the boundary operator of $\eth_{\mathrm{GB}}$ is just $d+\delta$ for the induced metric on the boundary, but acting on all forms. On the boundary $d+\delta$ interchanges even and odd forms and hence is off-diagonal with respect to the decomposition

$$\Lambda^*(\partial X) = \Lambda^{\mathrm{evn}}(\partial X) \oplus \Lambda^{\mathrm{odd}}(\partial X).$$

Since $(d+\delta)^2 = \Delta$ is diagonal for this decomposition it follows that the integrand, even at the level of the pointwise trace, of the eta invariant with parameter $s=0$ vanishes identically. Thus $\eta(d+\delta)=0$.

To prove (9.13) it only remains to evaluate the Atiyah-Singer integrand in this case. This follows directly from Theorem 8.34 once the induced supertrace and curvature are examined. Clifford multiplication, (3.18), on $\Lambda^*(X)$ can be modified into a right Clifford action. Thus consider

$$\operatorname{clr}(\xi) = [\operatorname{ext}(\xi) - \operatorname{int}(\xi)]R_\Lambda \colon \Lambda^* V \longrightarrow \Lambda^* V \tag{9.16}$$

with the involution given by (2.90). This is easily seen to extend to an action of the Clifford algebra since

$$\begin{gathered}\operatorname{clr}(\xi)\operatorname{clr}(\eta) + \operatorname{clr}(\eta)\operatorname{clr}(\xi) = 0 \text{ if } \eta\perp\xi\\ \operatorname{clr}(\xi)^2 = 2|\xi|^2.\end{gathered}$$

Moreover the action (9.16) commutes with the action (3.18).

Thus for the exterior algebra the decomposition (3.35) becomes

$$\hom(\Lambda^*) = \mathbb{C}\mathrm{l}(X)\otimes\mathbb{C}\mathrm{l}(X),\ \dim X \text{ even} \tag{9.17}$$

with the right factor acting through (9.16). The decomposition for the curvature of a tensor product therefore applies and the curvature of $\hom(\Lambda^*)$ decomposes according to (8.105) as

$$K_{\Lambda^*}(V,W) = \frac14\operatorname{cl}(R(V,W)) + \frac14\operatorname{clr}(R(V,W)).$$

Consider the product of left and right Clifford multiplication by the volume form. A simple computation shows that

$$\mathrm{cl}(\phi^1)\cdots\mathrm{cl}(\phi^N)\cdot\mathrm{clr}(\phi^1)\cdots\mathrm{clr}(\phi^N) = R_\Lambda.$$

That is, the involution induced on the Clifford bundle is just the usual one. Thus in (8.103) the induced supertrace is just the supertrace on the Clifford bundle to which Lemma 8.18 applies, so

$$\mathrm{str}'\big(\exp(\frac{1}{2}\,\mathrm{clr}(R))\big) = \mathrm{Pf}(R/4),$$

where by definition the Pfaffian is just the term homogeneous of degree $\frac{1}{2}\dim X$ in the exponential. Finally then (9.12) follows from (9.4) since the second factor is already of maximal form degree, so the constant term is the only contribution of the $\widehat{A}$-genus in this case.

9.3. Signature formula.

Now the signature formula of [8] can be discussed. This generalizes Hirzebruch's formula in the case of a compact manifold without boundary, which is described briefly first.

Let X be a compact Riemann manifold (for the moment without boundary) which is oriented and of even dimension, $2k$. The Hodge cohomology in the middle dimension has on it a bilinear form:

$$\mathcal{H}^k_{\mathrm{Ho}}(X)\times\mathcal{H}^k_{\mathrm{Ho}}(X)\ni(\phi,\psi)\longrightarrow B(\phi,\psi)=\phi\wedge\psi/\upsilon, \tag{9.18}$$

where υ is the oriented volume class of the manifold, $\upsilon = \star 1$. In case k is odd this is an antisymmetric bilinear form, if k is even it is symmetric. The signature of the manifold, $\mathrm{sign}(X)$, is the signature of this bilinear form, by definition zero if k is odd and otherwise the difference of the dimensions of the maximal subspaces on which it is positive and negative. The involution $\tau = i^{p(p-1)+k}\star$ on p-forms, introduced in §2.12, restricts to $\Lambda^k(X)$ to be $\star$ if k is even. Thus (9.18) becomes

$$B(\phi,\psi)=\langle\phi,\tau\psi\rangle,\ \dim X = 4l, l\in\mathbb{N}_0$$

with τ symmetric and $\tau^2 = \mathrm{Id}$. Thus the signature of X is just the signature of τ on the middle dimensional cohomology when $X = 4l$ and B is non-degenerate in this case.

In general τ gives the exterior algebra Λ^*X a $\mathbb{Z}_2$-grading and Clifford multiplication on Λ^*X is graded in this sense, i.e.

$$\mathrm{cl}(\xi)\tau = -\tau\,\mathrm{cl}(\xi).$$

Furthermore τ is, as a tensor, covariant constant, and since $d+\delta$ is the Dirac operator for this Clifford connection it also anticommutes with τ. Let $E_\pm \subset \Lambda^* X$ be the subbundles spanned by the eigenspaces of τ with eigenvalues ± 1. The signature operator of X is the positive Dirac operator for the $\mathbb{Z}_2$-grading and the Levi-Civita connection:

$$\eth^+_{\mathrm{sign}} \in \mathrm{Diff}^1(X; E_+, E_-).$$

Hirzebruch's theorem is:

PROPOSITION 9.3. *The signature of an oriented $4l$-dimensional compact manifold without boundary is equal to the index of the signature operator and given by the formula*

$$\mathrm{sign}(X) = \mathrm{ind}(\eth^+_{\mathrm{sign}}) = \int_X \mathrm{Ev}_{4l}\, L(X),$$

where $L(X)$ is Hirzebruch's L-polynomial in the Pontrjagin forms of X; it is given explicitly in terms of the Riemann curvature by (2.107).

The extension to compact manifolds with exact b-metrics is now quite straightforward and Proposition 9.3 will be deduced as a special case of it. The discussion above applies to the case of an exact b-metric to show that the bilinear form (9.18) gives, in case $\dim X = 4l$, a non-degenerate symmetric bilinear form

$$\mathcal{H}^{2l}_{b,\mathrm{Ho}}(X) \times \mathcal{H}^{2l}_{b,\mathrm{Ho}}(X) \ni (\phi, \psi) \longmapsto B(\phi,\psi) = \langle \phi, \tau\psi \rangle.$$

The symmetry follows from the fact that $\tau = \star$ on $2l$-forms and the non-degeneracy from the invertibilty of τ. The signature of the manifold is then the signature of this symmetric bilinear form.

Let B be the operator on even forms on ∂X given by

$$B_{\mathrm{evn}} = (-1)^{l+j}(d\star - \star d) \text{ on } \Lambda^{2p}(\partial X).$$

The eta invariant $\eta(B_{\mathrm{evn}})$ is defined in §8.13. The signature theorem of Atiyah, Patodi and Singer is:

THEOREM 9.4. *The signature of an oriented $4l$-dimensional compact manifold with boundary is*

$$\mathrm{sign}(X) = \widetilde{\mathrm{ind}}_0(\eth^+_{\mathrm{sign}}) = \int_X L_{4l}(X) - \eta(B_{\mathrm{evn}}), \tag{9.19}$$

where L_{4l} is given explicitly in terms of the Riemann curvature by (2.107).

PROOF: From the discussion above, the APS theorem does apply to the signature operator of any exact b-metric. In particular (9.5) gives the formula for the index function. To prove (9.19) it remains to show that the first equality holds, to show that the Atiyah-Singer integrand in (9.5) reduces to the first term on the right in (9.19) and to relate the boundary operator of the signature operator to $d+\delta$ on the boundary.

First consider the Atiyah-Singer integrand. As already noted the part of the curvature operator on $\Lambda^*(X)$ which commutes with the left Clifford action is given in terms of right Clifford action and any orthonormal frame by

$$K_{\mathrm{sign}} = \frac{1}{4}\sum_{ij} R_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j). \tag{9.20}$$

In this case the involution R_{sign} acts trivially on the right factor in (9.17), so

$$\mathrm{str}'(A) = \mathrm{tr}(A),\ \ A \in \hom'_{\mathbb{C}\mathrm{l}}(\Lambda^*(X)).$$

Of course the curvature takes values in the 2-forms but, as in the discussion of Mehler's formula, it suffices to consider the case of an antisymmetric real matrix in (9.20).

In fact for any such matrix S_{ij}

$$\mathrm{tr}\exp\big(\sum_{ij} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j)\big) = 4^k \det{}^{\frac{1}{2}}\big(\cosh(2S)\big). \tag{9.21}$$

On the left the operator acts on $\Lambda^*(\mathbb{R}^{2k})$ and on the right S is simply a matrix, i.e. acts on $\mathbb{R}^{2k}$. Since S is real and antisymmetric it has two-dimensional invariant subspaces, the sums of eigenspaces with eigenvalue it and $-it$, which are orthogonal. Thus by an orthogonal transformation of $\mathbb{R}^{2k}$ the matrix S can be reduced to 2×2 block-diagonal form. Both sides of (9.21) are invariant under such a transformation. Thus it can be assumed that the invariant subspaces are spanned by ϕ^{2p-1} and ϕ^{2p} for $p=1,\dots,k$. Consider the decomposition of $\Lambda^*(\mathbb{R}^{2k})$ arising from splitting off the first eigenspace $\mathbb{R}^{2k}=\mathbb{R}^2\oplus\mathbb{R}^{2k-2}$. Then

$$\begin{gathered}\Lambda^*(\mathbb{R}^{2k}) \equiv \big(\Lambda^*(\mathbb{R}^{2k-2})\big)^4\\ u = u_1 + \phi^1\wedge u_2 + \phi^2\wedge u_3 + \phi^1\wedge\phi^2\wedge u_4.\end{gathered} \tag{9.22}$$

The first 2×2 block is of the form

$$\begin{pmatrix} 0 & t \\ -t & 0\end{pmatrix}$$

corresponding to the eigenvalues $\pm it$ of S. Moreover

$$\sum_{ij=1}^{2k} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j))$$
$$= t\big(\mathrm{clr}(\phi^1)\,\mathrm{clr}(\phi^2) - \mathrm{clr}(\phi^2)\,\mathrm{clr}(\phi^1)\big) + \sum_{ij=1}^{2k-2} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j)\big).$$

The two terms on the right commute, so the exponential becomes
(9.23)

$$\exp\left[\sum_{ij=1}^{2k} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j))\right]$$
$$= \exp\left[t\big(\mathrm{clr}(\phi^1)\,\mathrm{clr}(\phi^2) - \mathrm{clr}(\phi^2)\,\mathrm{clr}(\phi^1)\big)\right]\exp\left[\sum_{ij=1}^{2k-2} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j))\right].$$

The second factor here preserves each of the four spaces in (9.22). Moreover $\mathrm{clr}(\phi^1)\,\mathrm{clr}(\phi^2) - \mathrm{clr}(\phi^2)\,\mathrm{clr}(\phi^1)$ is off-diagonal and has square $-4\,\mathrm{Id}$, so the diagonal part of the first exponential on each of the four components on the right in (9.23) is $1 - 4t^2/2! + 16t^4/4! \cdots = \cos(2t)\,\mathrm{Id}$. The trace of the composite is therefore

$$4\cos(2t)\,\mathrm{tr}\exp\left[\sum_{ij=1}^{2k-2} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j))\right].$$

Proceeding inductively it follows that

$$\mathrm{tr}\exp\left[\sum_{ij=1}^{2k} S_{ij}\,\mathrm{clr}(\phi^i)\,\mathrm{clr}(\phi^j))\right] = 4^k\prod_{j=1}^{k}\cos(t_j), \tag{9.24}$$

where $\pm it_j$ are the eigenvalues of S. Since the eigenvalues of $\cosh(2S)$ are $\cosh(\pm 2it) = \cos(2t_j)$ with multiplicity 2 it follows that the right side of (9.24) reduces to $4^k \det^{\frac{1}{2}}\cosh(2S)$ and this proves (9.21).

Applying (9.21) to the integrand in (9.4) gives

$$\mathrm{str}'\exp(K'_{\mathrm{sign}}) = 4^k\det{}^{\frac{1}{2}}\big(\cosh(R/2)\big).$$

Combining this with the formula for the $\widehat{A}$-genus shows that the integrand (9.4) becomes in this case

$$L_{2k}(X) = \det{}^{\frac{1}{2}}\big(\frac{R/4\pi i}{\tanh R/4\pi i}\big)$$

which is the L-genus of Hirzebruch. This identifies the first term on the right in (9.19).

Next consider the index of the signature operator. If the null space of $\eth^+_{\text{sign}}$ on $x^\epsilon H^\infty_b(X;\Lambda^*)$, i.e. the part of the null space of $d+\delta$ which is in the $+1$ eigenspace of τ, is denoted $\operatorname{null}_\epsilon(\eth^+_{\text{sign}})$ and similarly for $\operatorname{null}_\epsilon(\eth^-_{\text{sign}})$ then by definition

$$\text{(9.25)}\quad \widetilde{\operatorname{ind}}_0(\eth^+_{\text{sign}}) = \frac{1}{2}\big[\dim\operatorname{null}_\epsilon(\eth^+_{\text{sign}}) - \dim\operatorname{null}_{-\epsilon}(\eth^-_{\text{sign}}) \\ + \dim\operatorname{null}_{-\epsilon}(\eth^+_{\text{sign}}) - \dim\operatorname{null}_\epsilon(\eth^-_{\text{sign}})\big]$$

for any $\epsilon > 0$ small enough. The space $\operatorname{null}_\epsilon(\eth_{\text{sign}})$ is the image of the relative in the absolute cohomology (see Proposition 6.14). The involution τ maps this space into itself and as in the boundaryless case discussed above it follows that

$$\operatorname{sign}(X) = \dim\operatorname{null}_\epsilon(\eth^+_{\text{sign}}) - \dim\operatorname{null}_\epsilon(\eth^-_{\text{sign}}).$$

Thus the first inequality in (9.19) follows from (9.25) provided

$$\text{(9.26)}\quad \dim\operatorname{null}_{-\epsilon}(\eth^+_{\text{sign}}) - \dim\operatorname{null}_\epsilon(\eth^+_{\text{sign}}) = \dim\operatorname{null}_{-\epsilon}(\eth^-_{\text{sign}}) - \dim\operatorname{null}_\epsilon(\eth^-_{\text{sign}}).$$

To see this, recall Proposition 6.16, and in particular (6.60). Now

$$\operatorname{null}_{-\epsilon}(d+\delta) = \operatorname{null}_{-\epsilon}(\eth^+_{\text{sign}}) \oplus \operatorname{null}_{-\epsilon}(\eth^-_{\text{sign}})$$
$$\operatorname{null}_\epsilon(d+\delta) = \operatorname{null}_\epsilon(\eth^+_{\text{sign}}) \oplus \operatorname{null}_\epsilon(\eth^-_{\text{sign}})$$

and the quotient of these two spaces is identified by BD with a subspace of

$$\text{(9.27)}\quad \mathcal{H}^*_{\text{Ho}}(\partial X) \oplus \mathcal{H}^*_{\text{Ho}}(\partial X).$$

Namely the subspace of those (u,v) such that there exists some element $w \in \operatorname{null}_{-\epsilon}(\eth_{\text{sign}})$ with

$$w \sim u + \frac{dx}{x}\wedge v$$

at ∂X (see (6.56)). Thus τ induces an involution, τ', on the image. It is just

$$\text{(9.28)}\quad \tau'(u,v) = i(\tau R v, \tau R u),$$

as a short calculation shows. Since, from Proposition 6.16, the subspace of (9.27) splits it follows from (9.28) that the ± 1 eigenspaces of τ' acting on it have equal dimension, i.e. (9.26) holds.

Finally it remains to identify the η-invariant term in (9.5). Consider the generalized Dirac operator on the boundary induced by the signature operator $\eth^+_{\mathrm{sign}}$. This is just the boundary operator of $\eth^+_{\mathrm{sign}}$ composed with Clifford multiplication by idx/x. The boundary operator of $d+\delta$ can be computed from (6.49) and the normalizing factors $M_\pm$ show that the η-invariant of the induced generalized Dirac operator is just twice that of $d+\delta$ on the boundary. This completes the proof of Theorem 9.4.

9.4. Flat coefficient bundles.

One furhter generalization of the signature theorem, of considerable geometric interest, arises from the introduction of a flat coefficient bundle. This makes essentially no difference to the analytic structure of the problem. Over a manifold a connection on a vector bundle is said to be flat if its curvature operator vanishes identically. The analogy with the Riemannian case, in Theorem 2.16, easily leads to the conclusion that near every point a flat bundle has a trivialization in which the connection becomes the trivial connection.

LEMMA 9.5. *If E is an Hermitian bundle with flat unitary connection over X then each point $p \in X$ has a neighbourhood $U(p)$ over which E has a trivialization*

$$\begin{aligned} E_{\restriction U(p)} &\longrightarrow \mathbb{C}^r \times U(p), \\ e \in \mathcal{C}^\infty(U(p); E) &\longmapsto (e_1, \dots, e_r) \in \mathcal{C}^\infty(U(p)) \\ &\Longrightarrow \nabla_V e \longmapsto (Ve_1, \dots, Ve_r). \end{aligned} \tag{9.29}$$

EXERCISE 9.6. To prove this lemma it suffices to introduce local coordinates in which p is the origin and take a local frame of E which is covariant constant with respect to the radial vector field R. Such a local frame is fixed by its value at p. Since the connection is unitary it is orthonormal near p if it is orthonormal at p. The linear vector fields commute with R, so the frame is covariant constant with respect to these vector fields too. Near (but not at) p the coordinate vector fields are in the span of the linear ones, so the frame is covariant constant and hence (9.29) holds.

Let χ_t, be a continuous one-parameter family of curves, i.e. a continuous map $[0,1]_t \times [0,1]_s \longmapsto \chi_t(s) \in X$ with $\chi_t(0) = p$ and $\chi_t(1) = q$ independent of t. The flatness of the bundle means that transporting a basis from p along χ_t to its endpoint, q, gives a result which is independent of t. If $p = q$, so the curves are closed, this gives a map

$$\pi_1(X) \longrightarrow \hom(E_p), \tag{9.30}$$

where $\pi_1(X)$ is the (pointed) fundamental group of X, the group of homotopy classes of closed curves in X with endpoint p. Clearly (9.30) is a representation of $\pi_1(X)$. If E is Hermitian, with unitary connection, then the representation is unitary. If two bundles with flat connections are identified when they are isomorphic with a flat isomorphism (i.e. one that intertwines the connections) then the representation (9.30) is fixed up to (unitary) equivalence. Conversely any unitary representation of the fundamental group generates an Hermitian bundle with flat unitary connection.

In the case of a manifold with boundary the same results hold, provided the connections are required to be true connections. The fundamental group of the boundary maps into the fundamental group of the manifold

$$\pi_1(\partial X) \hookrightarrow \pi_1(X) \tag{9.31}$$

if the base point is chosen on the boundary. Notice that (9.31) need not be injective. The restriction of a flat bundle to the boundary corresponds to the pull-back of the (unitary) representation under (9.31).

If E is a flat bundle then the exterior differential can be defined on $\Lambda^*(X) \otimes E$ by reference to any locally covariant constant frame. This leads to twisted versions of both the de Rham and the signature complexes. In particular there is a well-defined twisted Laplacian on $\Lambda^*(X) \otimes E$ if E is flat. The representation corresponding to E is said to be acyclic if this Laplacian is invertible, i.e. has no null space.

Exercise 9.7. Show that on a compact manifold with boundary the Laplacian associated to an exact b-metric twisted by a flat coefficient bundle is Fredholm as a map from $H_b^2(X; \Lambda^* \otimes E)$ to $L^2(X; \Lambda^* \otimes E)$ if and only if the induced representation on the boundary is acyclic.

Exercise 9.8. State and prove the twisted signature theorem for a compact manifold with boundary (see [9]).

Exercise 9.9. Recall that a connection on a bundle E over compact manifold with boundary induces a b-connection on the weighted bundle $x^z E$ defined in Lemma 8.6. Show that any bundle, over a compact manifold with boundary, with a flat b-connection (one for which the curvature operator vanishes) is locally isomorphic, with its connection, near each boundary point to $x^{\mathfrak{a}}\mathbb{C}^r$ for some multiweight $\mathfrak{a}$ with its b-connection. Extend the signature theorem to this case.

9.5. Zeta function.

As an indication of the degree to which most of the usual notions from compact Riemann manifolds can be extended to exact b-metrics the zeta function of a non-negative Fredholm elliptic self-adjoint b-differential operator with symbol given by such a metric will be defined. This will be used in the next section to discuss the Ray-Singer analytic torsion associated to any unitary representation of the fundamental group of such a manifold. Although no major results are proved here the existence of these invariants suggests a variety of interesting questions.

First consider the b-zeta function, which was introduced by Piazza [74]. We shall only consider the 'geometric case' of a non-negative second order differential operator with diagonal principal symbol acting on sections of some vector bundle (in [74], Piazza considers the b-zeta function for b-pseudodifferential operators). If $\partial X = \emptyset$ then the spectrum of P is discrete and of finite multiplicity. For $\operatorname{Re} s >> 0$ the zeta function is usually defined by

$$\zeta(P,s) = \sum_{\{j;\lambda_j\neq 0\}} \lambda_j^{-s}, \tag{9.32}$$

where the λ_j are the eigenvalues of P repeated according to their (finite) multiplicity. The convergence of the series in (9.32), for $\operatorname{Re} s > \frac{1}{2}\dim X$, follows from the asymptotic properties of the eigenvalues, see for example [47].

Since the heat kernel for $P \in \operatorname{Diff}_b^2(X;E)$ has already been extensively discussed it is convenient to express the zeta function in these terms, avoiding such convergence questions

EXERCISE 9.10. Deduce a bound on the growth rate of the eigenvalues, from the properties of the heat kernel already established, which is good enough to demonstrate the convergence of (9.32) in $\operatorname{Re} s > \frac{1}{2}\dim X$.

If Π_λ is the orthogonal projection onto the eigenspace associated to $\lambda \in \operatorname{spec}(P)$ then the heat kernel can be written

$$\exp(-tP) = \Pi_0 + \sum_{0\neq\lambda\in\operatorname{spec}(P)} e^{-t\lambda}\Pi_\lambda.$$

Thus all terms, except the first, are exponentially decreasing. Since the trace of Π_λ is the dimension of the associate eigenspace, at least formally,

$$\zeta(P,s) = \frac{1}{\Gamma(s)}\int_0^\infty t^s \operatorname{Tr}\exp(-t\underline{P})\frac{dt}{t}. \tag{9.33}$$

Here $\Gamma(s)$ is the gamma function

$$\Gamma(s) = \int_0^\infty t^s e^{-t}\frac{dt}{t}$$

and $\underline{P}$ is P with its null space removed, i.e. (9.33) should really be

$$\zeta(P,s) = \frac{1}{\Gamma(s)}\int_0^\infty t^s \operatorname{Tr}\left[\exp(-tP)\Pi_0^\perp\right]\frac{dt}{t},\ \Pi_0^\perp = \operatorname{Id}-\Pi_0. \tag{9.34}$$

In this form the properties of the zeta function can be analyzed directly (following, at least in general terms, the ideas of Seeley [82]).

LEMMA 9.11. *If $P \in \operatorname{Diff}^2(X;E)$ is an elliptic, non-negative, self-adjoint operator with diagonal principal symbol on a compact manifold X, with $\partial X = \emptyset$, the zeta function, defined by (9.34), is meromorphic as a function of $s \in \mathbb{C}$ with only simple poles at the points $\frac{1}{2}\dim X, \dots, 1$ if $\dim X$ is even and at $\frac{1}{2}\dim X, \dots, \frac{1}{2}$ and $-\frac{1}{2} - \mathbb{N}_0$ if $\dim X$ is odd.*

PROOF: This follows from the properties of the trace of the heat kernel. In particular Lemma 7.18 shows that the integral

$$\int_0^\infty t^s \operatorname{Tr}\exp(-t\underline{P})\frac{dt}{t}$$

extends to a meromorphic function of s, with poles only at the points $s = \frac{1}{2}n - j$, since $\exp(-tP) \in \Psi^{-2}_{H,\text{evn}}(X;E)$. The inverse of the Gamma function, $1/\Gamma(s)$, has zeros at $s \in -\mathbb{N}_0$. So in case n is even the zeta function itself only has poles at $s = \frac{1}{2}n, \frac{1}{2}n-1, \dots, 1$. On the other hand if n is odd there is no such cancellation.

The value of the zeta function at $s = 0$ is of rather special interest since, as can be seen from (9.33), $\zeta(rP,s) = r^{-s}\zeta(P,s)$, so $\zeta(rP,0) = \zeta(P,0)$ for any constant $r > 0$. In fact the simple pole of the gamma function at 0 means that

$$\zeta(P,0) = \begin{cases} a_0 - \dim\operatorname{null}(P) & \dim X \text{ even} \\ -\dim\operatorname{null}(P) & \dim X \text{ odd,} \end{cases} \tag{9.35}$$

where a_0 is the coefficient of t^0 in the expansion of $\operatorname{Tr}\exp(-tP)$ as $t \downarrow 0$ (which vanishes in case $\dim X$ is odd).

Formally the derivative of (9.32), with respect to s, evaluated at $s = 0$ is the (non-convergent) sum $-\sum\limits_{\lambda_j \neq 0} \log\lambda_j$. This makes it reasonable to define the determinant of the operator P by

$$\det(P) = \exp\left(-\zeta'(P,0)\right)$$

which makes sense by Lemma 9.11. Again one can easily see that under multiplication by a positive constant the determinant is transformed to

$$\det(rP) = r^{\zeta(P,0)} \det(P). \tag{9.36}$$

Of course this is not the usual transformation law for a determinant, but shows that the dimension of the space has been 'renormalized' to $\zeta(P, 0)$.

Now consider how these elementary considerations can be extended to the case of a compact manifold with boundary. Suppose that $P \in \mathrm{Diff}_b^2(X; E)$ is elliptic, self-adjoint, non-negative, has diagonal principal symbol corresponding to an exact b-metric and indicial family of the form:

$$I_\nu(P, \lambda) = \lambda^2 + P_\partial. \tag{9.37}$$

Chapter 7 gives a rather complete description of the heat kernel of P. Of course it is not trace class but the trace functional in (9.33) can be replaced by the b-trace functional, with respect to the trivialization of the normal bundle to the boundary given by the exact b-metric. Adding to (9.37) the condition

$$P_\partial \text{ is invertible} \tag{9.38}$$

means that the spectrum of P is discrete near 0 with finite multiplicity and that $\text{b-Tr}_\nu(\exp(-tP)) = \dim \operatorname{null}(P) + e(t)$ with $e(t)$ exponentially decreasing as $t \to \infty$. The direct extension of (9.33):

$$^b\zeta(P, s) = \frac{1}{\Gamma(s)} \int_0^\infty t^s \, \text{b-Tr}_\nu \exp(-t\underline{P}) \frac{dt}{t}$$

therefore behaves essentially as before:

LEMMA 9.12. *If $P \in \mathrm{Diff}_b^2(X; E)$ is elliptic, self-adjoint, non-negative, has diagonal principal symbol corresponding to an exact b-metric on a compact manifold with boundary and has indicial family of the form* (9.37) *satisfying* (9.38) *holds then the conclusions of Lemma 9.11,* (9.35) *and* (9.36) *apply to $^b\zeta(P, s)$.*

Ignoring the proof for the moment, consider what happens if (9.38) does not hold. One then has to be a little more careful in defining the b-zeta function since the operator P has continuous spectrum down to 0. There is another way of looking at the removal of the zero eigenvalue in (9.33) which leads directly to a suitable extension. Namely, choosing any $D > 0$, two 'half zeta functions' can be defined by

$$
{}^b\zeta_0(P,s) = \frac{1}{\Gamma(s)} \int_0^D t^s \,\text{b-Tr}_\nu \exp(-tP) \frac{dt}{t}, \ \operatorname{Re} s > \frac{1}{2} \dim X \tag{9.39}
$$

$$
{}^b\zeta_\infty(P,s) = \frac{1}{\Gamma(s)} \int_D^\infty t^s \,\text{b-Tr}_\nu \exp(-tP) \frac{dt}{t}, \ \operatorname{Re} s < 0. \tag{9.40}
$$

As already noted, in case (9.38) holds (or $\partial X = \emptyset$), the trace of the heat kernel is, near $t = \infty$, the sum of a constant (equal to the dimension of the null space) plus an exponentially decreasing term. Thus not only do (9.39) and (9.40) make sense but the resulting functions both extend to be meromorphic in $s \in \mathbb{C}$ and

$$
{}^b\zeta(P,s) = {}^b\zeta_0(P,s) + {}^b\zeta_\infty(P,s). \tag{9.41}
$$

To see this simply note that if P is replaced by $\underline{P}$ in (9.40) the resulting function, ${}^b\zeta_\infty(\underline{P},s)$, is entire. Thus ${}^b\zeta(P,s) = {}^b\zeta(\underline{P},s) = {}^b\zeta_0(\underline{P},s) + {}^b\zeta_\infty(\underline{P},s)$. Moreover ${}^b\zeta_\infty(\underline{P},s) = {}^b\zeta_\infty(P,s) - ND^s/s\Gamma(s)$, where $N = \dim \operatorname{null}(P)$, and similarly ${}^b\zeta_0(\underline{P},s) = {}^b\zeta_0(P,s) + ND^s/s\Gamma(s)$ giving (9.41). With this in mind consider (9.39) and (9.40) in the general case.

Proposition 9.13. *If $P \in \operatorname{Diff}_b^2(X;E)$ is elliptic, self-adjoint, non-negative, has diagonal principal symbol corresponding to an exact b-metric on a compact manifold with boundary and satisfies (9.37) then ${}^b\zeta_\infty(P,s)$ extends from $\operatorname{Re} s < 0$, where the integral in (9.40) converges absolutely, to a meromorphic function of s with poles only at $\frac{1}{2}\mathbb{N}$ and ${}^b\zeta_0(P,s)$ extends from $\operatorname{Re} s > \frac{1}{2}\dim X$ to be meromorphic with poles as in Lemma 9.11. If the b-zeta function of P is defined by (9.41) then*

$$
{}^b\zeta(P,0) = \begin{cases} a_0 - \frac{1}{2}(N_1 + N_2) & \dim X \text{ even} \\ -\frac{1}{2}(N_1 + N_2) & \dim X \text{ odd}, \end{cases} \tag{9.42}
$$

where N_1 is the dimension of the null space of P on $H_b^\infty(X;E)$ and N_2 is the dimension of the null space on $\mathcal{C}^\infty(X;E) + L_b^2(X;E)$.

Proof: The meromorphy of ${}^b\zeta_0(P,s)$ proceeds exactly as in the proof of Lemma 9.11 since the behaviour of $h(t) = \text{b-Tr}_\nu \exp(-tP)$ as $t \downarrow 0$ is essentially the same as in the boundaryless case. On the other hand Proposition 7.37 describes the behaviour of $h(t)$ as $t \to \infty$. Thus $h(t)$ has a complete asymptotic expansion in powers of $t^{-\frac{1}{2}}$. Since $1/\Gamma(s)$ vanishes at $s = 0$ the pole there is removed, but otherwise ${}^b\zeta_\infty(P,s)$ can have poles at $\frac{1}{2}\mathbb{N}$ as stated. This proves the proposition, since (9.42) just arises from (7.118).

9.6. Analytic torsion.

From (9.36) the determinant is most interesting in case $\zeta(P,0)=0$. One case in which this occurs, at least in the simplest setting, is in the definition of the analytic torsion of Ray and Singer. If $\varDelta_j$ is the Laplacian acting on j-forms on a compact manifold (initially without boundary) consider the superposition of zeta functions:

$$\zeta_\tau(\varDelta,s)=\sum_{j=0}^{\dim X}(-1)^j j\zeta(\varDelta_j,s).$$

In fact it is worthwhile, following [78] and §9.4, to generalize slightly and to look at twisted versions of the Laplacian, as in §9.4.

DEFINITION 9.14. For any unitary representation of the fundamental group of a compact manifold with boundary and exact b-metric the analytic torsion is

$$\tau(g,\rho)=-{}^b\zeta_\tau'(g,\rho,0).$$

EXERCISE 9.15. Extend the results of [78] to exact b-metrics.

9.7. The b-eta invariant.

In the same spirit as the discussion of the analytic torsion, the definition of the eta invariant of the Dirac operator can be extended to the case of a graded Clifford module for an exact b-metric. This b-eta invariant appears in the generalization of the Atiyah-Patodi-Singer index theorem to exact b-metrics on manifolds with corners.

In case $\partial X=\emptyset$ the eta invariant, defined by (In.28), can be obtained as the value at $s=0$ of the eta function:

$$\eta(\eth,s)=\frac{1}{\Gamma(s+\frac12)}\int_0^\infty t^{s+\frac12}\operatorname{Tr}\left(\eth e^{-t\eth^2}\right)\frac{dt}{t}.$$

Since the null space is annihilated by the factor of $\eth$, the integral is absolutely convergent near $t=\infty$; the convergence near $t=0$ is discussed in Chapter 8. The latter discussion extends directly to the case of an exact b-metric, as does the former if (9.38) is assumed for the Dirac operator and the definition is taken to be:

$$(9.43)\quad {}^b\eta(\eth,s)=\frac{1}{\Gamma(s+\frac12)}\int_0^\infty t^{s+\frac12}\,\text{b-Tr}_\nu\left(\eth e^{-t\eth^2}\right)\frac{dt}{t},\ {}^b\eta(\eth)={}^b\eta(\eth,0).$$

Even without the assumption that $I_\nu(\eth,0)$ is invertible the same formula can be used.

PROPOSITION 9.16. *If X is an odd-dimensional exact b-spin manifold then the integral in* (9.43) *converges absolutely for $-\frac{1}{2} < \operatorname{Re} s < \frac{1}{2}$ and extends to a meromorphic function of $s \in \mathbb{C}$.*

PROOF: The convergence near $t = 0$ follows from the discussion in §8.13. Similarly Proposition 7.37 can be generalized to show that

$$e(t) = \text{b-Tr}_\nu\left(\eth e^{-t\eth^2}\right) \sim \sum_{j=0}^{\infty} b_j t^{-1-\frac{1}{2}j} \text{ as } t \to \infty. \tag{9.44}$$

To see this we start from (7.119), which gives the long-time component of the heat kernel and apply $\eth$:

$$e(t) = \frac{1}{2\pi i} \int\limits_{\gamma_A} \text{b-Tr}_\nu \, e^{-t\tau}\left(\eth G'(\tau)\right) d\tau. \tag{9.45}$$

The analytic extension of $G'(\tau)$ in the variable z, $\tau = z^2$, has a double pole at $z = 0$, as discussed in Proposition 6.28. Since the leading term is the null space of $P = \eth^2$ on L^2, this is annihilated by $\eth$. Similarly the residue has range in the null space of $\eth^2$ on $\mathcal{C}^\infty(X; E) + L^2(X; E)$ and this is just the null space of $\eth$ on $x^{-\epsilon} H_b^\infty(X; E)$, for $\epsilon > 0$ small, by the properties of the boundary pairing. Thus the analytic continuation of $\eth G'(\tau)$ has no pole at $z = 0$. Following through the proof of Proposition 7.37, the leading terms, as $t \to \infty$, arising from both H_1' and $H_2'(t)$ vanish. Taking into account the parity of the terms, it follows that the leading term in $e(t)$ is of order t^{-1}, i.e. (9.44) holds. This completes the proof of the proposition.

In particular if Tr is replaced by b-Tr_ν then the formula (In.28) defines the b-eta invariant. The transition, by 'analytic surgery,' from the eta invariant on a compact manifold without boundary to the b-eta invariant is discussed in [57].

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