Iterated loop spaces

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In this paper, we construct approximations to the iterated loop spaces $\Omega^n\Sigma^n(X)$ where X is a connected CW-complex ($\Sigma^n(X)$ is the n-fold reduced suspension). The approximations are sufficiently good that we can use them to calculate $H_*(\Omega^n\Sigma^n(X), \Gamma)$ for reasonable X and Γ, and we also calculate the cohomology rings $H^*(\Omega^{n-1}\Sigma^n(X), Z_p)$.

In this connection we prove a conjecture of W. Browder [4].

COROLLARY 7.2. *$H_*(\Omega^n\Sigma^n(X), \Gamma)$ depends only on the homology of X and on Γ.*

For $n = 1$ the problem of constructing a reasonable approximation to $\Omega\Sigma(X)$ was resolved by I. M. James in [9], where he introduced the reduced product (here denoted $J_1(X)$, see § 3), and proved its equivalence with $\Omega\Sigma(X)$. We generalize this by proving

THEOREM 5.2. *Let X be a countable, connected* CW-*complex, then there are natural constructions*

$$J_1(X) \subset J_2(X) \subset \cdots \subset J_n(X) \subset \cdots ,$$

which are themselves countable CW-*complexes, and $J_n(X)$ is homotopy equivalent to $\Omega^n\Sigma^n(X)$.* Actually, each of the $J_i(X)$ is a free associative H-space with unit, and the asserted equivalence preserves multiplications. The author believes the restriction that X be countable is unnecessary.[1]

The results in § 8 on the homology and cohomology of these spaces extend the partial results of Kudo and Araki [10], W. Browder [3], [4], and Dyer and Lashof [6]. Also, as we make more use of the geometry of the situation, rather than properties of the Serre spectral sequence, our proofs may be somewhat easier.

In the first part of this paper we will often describe spaces by means of an equivalence relation R in a simpler space X, such a space will be written X/R, and will always have the quotient topology. Also, all loop and path spaces with fixed base point $*$ [written $\Omega(X)$, resp. $P(X)$] will be taken in the

[1] The referee points out that the restriction to countable CW-complexes is, in fact, unnecessary. To avoid it, we need only work in the category of compactly generated spaces, because here the cartesian product of two CW-complexes is again a CW-complex. The reader may easily make the necessary modifications.

sense of J. C. Moore; that is, they will consist of paths of variable length (see for example [2, p. 305–6]). Finally, when we deal with a chain complex A, it will always be graded and free over Γ where Γ is Z or Z_p.

It is a pleasure to acknowledge the aid of Professor W. Browder in preparing this paper. In particular, he suggested that the results might be able to give a proof of 7.2. We would also like to thank the referee for his many helpful comments, especially for pointing out that $J_i(X)$ is actually of the same homotopy type as $\Omega^i\Sigma^i(X)$, instead of weakly homotopic as originally claimed.

1. Free H-spaces

Let X be an associative H-space with unit $*$ and multiplication $M: X \times X \rightarrow X$.

DEFINITION 1.1. *The set of indecomposables* $G(X)$ *are the points of* $X - (M(X' \times X') \cup *)$ *where* $X' = X - *$.

$G(X)$ is, roughly speaking, the set of generators of X, and X is free in case there are no relations among the products of the elements of $G(X)$. More precisely

DEFINITION 1.2. *Let* $G^1(X) = G(X)$, $G^i(X) = M(G(X) \times G^{i-1}(X))$, *then* X *is free if*

(1) $X = * \bigcup_{i=1}^{\infty} G^i(X)$,

(2) $G^i(X) \cap G^j(X) = \varnothing$ *for* $i \neq j$, *and*

(3) $M: G(X) \times G^i(X) \rightarrow X$ *is* $1-1$ *for all* i.

In particular it follows that every point of X may be written uniquely in the form $x_1 \cdots x_n$, $x_i \in G(X)$. We should also note that $\Omega(Y)$ is in general not a free H-space as it fails to satisfy condition 1 above.

DEFINITION 1.3. *If* X *is a countable* CW-*complex, we will call the* CW-*decomposition adapted to* M *if*

(1) *The interior of each cell* e *of* X *is contained in* $G^i(X)$ *for some* i,

(2) *For any two cells* e, f *of* X, M *is a relative homeomorphism of the interior of* $e \times f$ *onto the interior of a cell of* X

(3) $*$ *is a vertex.*

In particular, since M is onto, M is open, closed, and $M^{-1}(C)$ is compact if C is.

From now on all H*-spaces* X *will be assumed free with adapted* CW-*decompositions.*

Let X_k be the k-skeleton of X. The cellular complex $C(X)$ is the free, graded abelian group with

$$C_k(X) = H_k(X_k, X_{k-1}; Z) .$$

The boundary operator is that for the exact sequence of the triple (X_k, X_{k-1}, X_{k-2}) and

$$H_*(C(X) \otimes \Gamma) \cong H_*(X, \Gamma)$$

for any coefficient group Γ (for details see [12]).

The relative Eilenberg-Zilber theorem [7] applied in $X \times X$ gives a natural isomorphism

$$\alpha\colon C(X) \otimes C(X) \to C(X \times X) ,$$

and the following proposition is easily verified.

PROPOSITION 1.4. *$M_*\alpha\colon C(X) \otimes C(X) \to C(X)$ gives $C(X)$ the structure of a* DG. *algebra. Moreover, the generators corresponding to cells with interior contained in $G(X)$ generate $C(X)$ freely as an algebra.*

2. A classifying space for X

Let $h\colon X \to R^+$ be a continuous homomorphism of X into the additive, non-negative reals satisfying $h^{-1}(0) = *$. Set $A = X \times R^+ \times X$, and define an equivalence relation R in A by

$$(x, t, yw) \sim \begin{cases} (x, t, y) & \text{if } t < h(y) \\ (xy, t-h(y), w) & \text{otherwise} . \end{cases}$$

Now, put $E(X) = A/R$, and let $\pi\colon A \to E(X)$ be the identification map.

PROPOSITION 2.1. *$E(X)$ is contractible.*

PROOF. $f\colon I \times A \to A$ is defined by

$$f(t, (x, s, y)) = (*, (1 - t)(s + h(x)), xy) ;$$

it is continuous and preserves R, hence it induces a map $\tilde{f}\colon I \times E(X) \to E(X)$ which gives the desired contraction.

$M \times \text{id}\colon X \times A \to A$ also respects equivalence classes and induces a map

$$\tilde{M}\colon X \times E(X) \to E(X) ,$$

which defines an associative action of X on $E(X)$.

DEFINITION 2.2. *$B(X)$ is the set of maximal orbits in $E(X)$ under the action of X, $\rho\colon E(X) \to B(X)$ is the projection.*

PROPOSITION 2.3. *$E(X)$ is a countable* CW-*complex with cells of the form $e \times *$ or $e \times I \times f$, where e is a cell of X, and the interior of f is contained in $G(X)$. $B(X)$ is also a countable* CW-*complex with a cell $I \times f$ for each f*

as above. $\tilde{M}$ induces the structure of a $C(X)$ module in $C(E(X))$, and the triple $(C(X), C(E(X)), C(B(X)))$ is a construction in the sense of Cartan [5]. (Compare with [2, Th. 2.1, p. 309]).

The proof is direct though tedius. The basic idea is to use the stated properties of M, in particular that $M^{-1}(C)$ is compact if C is, to show that, if T is closed in a cell f with interior contained in $G(X)$, then

$$\pi^{-1}(\pi(e \times [a, b] \times T)) , \qquad (\rho\pi)^{-1}(\rho\pi(e \times [a, b] \times T))$$

are both closed in A, for e any cell of X. We omit the details.

This proposition has the following fundamental corollary.

THEOREM 2.4. *If X is connected, $\rho: E(X) \to B(X)$ is a quasi-fibration with fiber X.* (A map $f: X \to Y$ is a quasi-fibration if, for every point $y \in Y$, $f_*: \pi_i(X, f^{-1}(y)) \to \pi_i(Y, y)$ is an isomorphism for all i.)

This is precisely enough to imply the existence of a homotopy exact sequence

$$\xrightarrow{\partial f_*^{-1}} \pi_i(f^{-1}(y)) \longrightarrow \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \xrightarrow{\partial f_*^{-1}} \pi_{i-1}(f^{-1}(y)) \longrightarrow \cdots$$

and, since $E(X)$ is contractible, this implies

COROLLARY 2.5. *$\partial\rho_*^{-1}$ is an isomorphism $\pi_{i+1}(B(X)) \to \pi_i(X)$.*

PROOF OF 2.4. Restricted to ρ^{-1} of the 0-skeleton, the result is true. Assume the truth of the theorem on ρ^{-1} of the k-skeleton $B(X)_k$. $\rho^{-1}(B(X)_{k+1} - B(X)_k)$ is homeomorphic to $X \times (B(X)_{k+1} - B(X)_k)$. On the other hand, there is a retraction of a neighborhood N of $B(X)_k$ into $B(X)_k$ in $B(X)_{k+1}$ which, because of the way $E(X)$, $B(X)$ were constructed, may be covered by a retraction of $\rho^{-1}(N)$ into $\rho^{-1}(B(X)_k)$ which maps fibers by right translation. Since X is connected, right translation is a homotopy equivalence. Thus by [11, 2.10] $\rho \mid \rho^{-1}(N) \to N$ is a quasi-fibration, and the result now follows from [11, 2.2 and 2.15].

THEOREM 2.6. *Let X be connected, there is a mapping $g: E(X) \to P(B(X))$ so that*

(1) *g restricted to X maps X into $\Omega(B(X))$,*

(2)
$$\begin{array}{ccccc} X \times E(X) & \xrightarrow{\tilde{M}} & E(X) & \searrow^{\rho} & \\ \downarrow{\scriptstyle g \times g} & & \downarrow{\scriptstyle g} & & B(X) \\ \Omega(B(X)) \times P(B(X)) & \xrightarrow{\varphi} & P(B(X)) & \nearrow_{p} & \end{array}$$

is commutative where φ is composition of paths and p is the end point projection.

(3) *$g\colon X \to \Omega(B(X))$ is a homotopy equivalence.*

PROOF. Define g by

$$g\{\pi(*, t, x)\}s = \begin{cases} \rho\pi(*, t, x) & s \geqq t\,, \\ \rho\pi(*, s, x) & s \leqq t\,; \end{cases}$$

g is continuous and clearly satisfies 1, 2. The fact that g induces isomorphisms in homotopy now follows from 2.5. An alternative proof could be given making use of 2.3, the Leray-Serre spectral sequence, and the spectral comparison theorem. It remains to show g restricted to X has a homotopy inverse. This we see as follows:

(1) using the retraction given in [2, p. 306], $\Omega(B(X))$ is retracted onto the space of loops of unit length,

(2) from Milnor's theorem, [13, Th. 3 p. 276], this latter space is homotopy equivalent to a CW-complex, and

(3) from the fact that, if two CW-complexes are weakly equivalent, then they are in fact equivalent, the homotopy inverse is now constructed.

3. The James construction

Let Y be a countable CW-complex, and $*$ a vertex of Y. Let $T(Y)$ be the disjoint union

$$T(Y) = Y + Y \times Y + \cdots + Y^n + \cdots\,.$$

The inclusion map $T(Y) \times T(Y) \to T(Y)$ makes $T(Y)$ into a free associative monoid. The homomorphism $f\colon T(Y) \to J_1(Y)$ (the free associative semi-group with unit $*$ generated by Y) makes $J_1(Y)$ into an identification space of $T(Y)$, $J_1(Y) = T(Y)/R$, and we may specify R by $(x_1 \cdots x_n) \sim (x_1 \cdots \hat{x}_k \cdots x_n)$ if $x_k = *$. $J_1(Y)$ is the reduced product introduced by James in [9]. It is clearly a free H-space with adapted multiplication, and $G(J_1(Y)) = Y - *$.

Let $h\colon Y \to [0, 1]$ be any continuous function on Y with $h^{-1}(0) = *$. Then h extends to a homomorphism $h\colon J_1(Y) \to R^+$ with $h^{-1}(0) = *$, and it is easy to see that $B(J_1(Y))$ is homeomorphic to ΣY. Thus, from 2.6, we have

THEOREM 3.1 (James). *There is an H-map*

$$g\colon J_1(Y) \longrightarrow \Omega\Sigma Y\,,$$

which is a homotopy equivalence.

4. The polygons $C(k)$

(1) Let E^X_{AB} be the set of paths of variable length starting in A and ending in B where A, B are subsets of the space X. If X is the n-cube I^n, and A is the initial point $(0, \cdots, 0)$, B the final point $(1, \cdots, 1)$, $E^X_{A,B}$ will be written P^n.

There is a pairing $M\colon P^n \times P^m \to P^{n+m}$ induced by the inclusions

$$F_n\colon I^n \longrightarrow I^n \times (0, \cdots, 0) \subset I^{n+m} ,$$
$$L_n\colon I^m \longrightarrow (1, \cdots, 1) \times I^m \subset I^{n+m} .$$

In order to generalize the James construction, it is necessary to construct approximations to the P^n satisfying certain conditions imposed by the geometry of the situation. To this end, we introduce some new polygons.

DEFINITION 4.1. *Let* $s = (1, 2, 3, \cdots, n+1) \in R^{n+1}$. *The symmetric group* S_{n+1} *acts on* R^{n+1} *by permutation of coordinates* $[\beta(x_1 \cdots x_{n+1}) = (x_{\beta^{-1}(1)} \cdots x_{\beta^{-1}(n+1)})]$, *and* $C(n)$ *is defined as the convex hull of the points* $\{\beta(s); \beta \in S_{n+1}\}$. (See Figure 1).

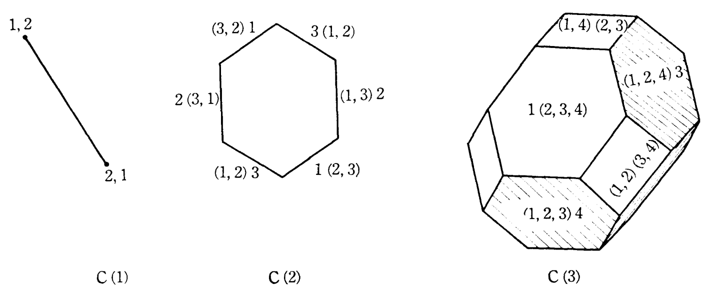

FIGURE 1
The first few polygons and faces.

S_{n+1} obviously acts as a group of homeomorphisms $C(n) \to C(n)$. There are also homeomorphisms $I^j\colon C(j-1) \times C(n-j) \to \partial C(n)$ defined by $I^j(x_1 \cdots x_j, y_1 \cdots y_{n-j+1}) = (x_1 \cdots x_j, y_1 + j, \cdots, y_{n-j+1} + j)$. Let $\alpha, \alpha' \in S_{n+1}$ be $(j, n+1-j)$ shuffles. Then $\operatorname{im}(\alpha I^j) \cap (\operatorname{im} \alpha' I^j)$ are points common only to the boundaries of the two sets, moreover the same is true if α' is a $(k, n+1-k)$ shuffle for $\operatorname{im}(\alpha I^j) \cap \operatorname{im}(\alpha' I^k)$. In fact we have

LEMMA 4.2. *$C(n)$ is topologically a closed disk with a regular* CW-*decomposition with one n-cell and the lower dimensional cells exactly the images of the cells of* $C(j-1) \times C(n-j)$ *under the maps* αI^j, *where* $1 \leqq j \leqq n$, *and α runs over all* $(j, n+1-j)$ *shuffles.*

PROOF. $C(n)$ is convex, compact, and n-dimensional, thus it is homeomorphic to D^n. On the other hand, the convex hull of a finite point set in R^n is a CW-complex having 1-top dimensional cell, and faces which are convex hulls of certain subsets of the generating sets. Such a set is contained in a hyperplane $\Sigma b_i x_i = 0$ (the plane through the origin which intersects $\Sigma x_i = (n+1)(n+2)/2$ in the plane containing the $n-1$ dimensional face $H(s_1 \cdots s_r)$).

Let $b_\alpha \cdots b_\gamma$, $(b^*_{m_1} \cdots b^*_{m_r})$ be the minimal (respectively maximal) b's in this equation, and suppose $H(s_1 \cdots s_r)$ is not $\alpha I^j C(j) \times C(n-j-1)$ for any j or α. It follows that, for some $s_k \in (s_1 \cdots s_r)$, $(s_k)_\tau < (s_k)_m$ for $\tau = \alpha \cdots \gamma$, and for some s'_k, $(s'_k)_n < (s'_k)_{m_l}$, $1 \leqq l \leqq r$. Let $\bar{s} = s_k$ with the α^{th} and m^{th} positions interchanged, and $\bar{\bar{s}} = s'_k$ with the n^{th} and $m_1{}^{\text{th}}$ positions interchanged, then $\bar{s}$ and $\bar{\bar{s}}$ lie on opposite sides of the hyperplane, a contradiction.

Passing to the cellular chain algebra of $C(n)$, we can easily verify

$$\partial\{C(n)\} = \textstyle\sum_{\alpha, j} (-1)^{\alpha+j}\{\alpha I^j C(j-1) \times C(n-j)\}\ , \tag{4.3}$$

where α runs over all $(j, n-j+1)$ shuffles, and $1 \leqq j \leqq n$.

(2) Let $\hat{j}\colon R^n \to R^{n-1}$ be the map which omits the j^{th} coordinate. $\hat{j}$ induces a map $d_j\colon S_n \to S_{n-1}$ where $d_j(\beta)$ is the permutation which makes the following diagram commutative

$$\begin{array}{ccc} R^n & \xrightarrow{\ \beta\ } & R^n \\ {\scriptstyle \widehat{\beta^{-1}(j)}}\big\downarrow & & \big\downarrow{\scriptstyle \hat{j}} \\ R^{n-1} & \xrightarrow{\ d_j(\beta)\ } & R^{n-1}\ . \end{array}$$

From the definition, we obtain

$$d_j(\alpha\beta) = d_{\beta^{-1}(j)}\alpha d_j(\beta)\ . \tag{4.4}$$

LEMMA 4.5. *There are maps*

$$D_k\colon C(n) \longrightarrow C(n-1)\ , \qquad 1 \leqq k \leqq n+1$$

so that

(i) $D_1 \circ I^0$ *is projection onto the second factor*

(ii) $D_j I^k = \begin{cases} I^{k-1}(D_j \times \mathrm{id}) & j \leqq k \\ I^k(\mathrm{id} \times D_{j-k}) & \textit{otherwise} \end{cases}$

(iii) $d_j(\beta) D_{\beta^{-1}(j)} = D_j \beta$

(iv) $D_i D_j = D_{j-1} D_i$ for $j \geqq i$.

PROOF. $D_1 = D_2\colon C(1) \to C(0)$, now (i), (ii), (iii), serve to define D_j by induction on $\partial C(n)$, (consistency follows from 4.4). Let $y_n = ((n+2)/2, \cdots, (n+2)/2) \in C(n)$, each point of $C(n)$ can be uniquely written $(t(y_n) + (1-t)z)$ where $z \in \partial C(n)$. On such a point define

$$D_j\big(ty_n + (1-t)z\big) = ty_{n-1} + (1-t)D_j(z)\ .$$

Lemma 4.5 (iii), (iv) continue to hold as they are true on the boundary. $\hat{j}$ restricts to $\hat{j}\colon I^n \to I^{n-1}$, which induces a map $\hat{j}_\#\colon P^n \to P^{n-1}$. Similarly $\beta \in S_n$ restricts to a map $\beta\colon I^n \to I^n$ which induces a map $\beta_\#\colon P^n \to P^n$, so that P^n also admits S_n as a group of homeomorphisms.

LEMMA 4.6. *There are mappings*

$$J_k\colon C(k) \longrightarrow P^{k+1}$$

so that

(i) $J_k\beta = \beta_{\#}J_k$, $\beta \in S_{k+1}$.
(ii) $\hat{j}_{\#}J_k = J_{k-1}D_j$ *(modulo re-parametrization of paths).*
(iii) $J_kI^r = M(J_{r-1} \times J_{k-r})$
(iv) *The evaluation map*

$$E\colon I \times C(k) \longrightarrow I^{k+1}$$

is cellular and of degree plus or minus one. (See Figure 2.)

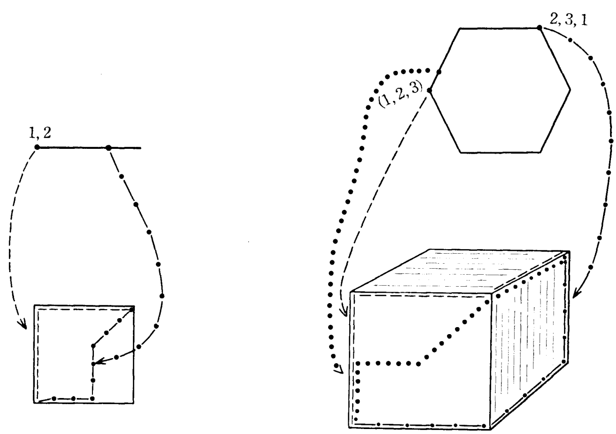

FIGURE 2
Some typical paths.

PROOF. Set $[J_0(1)]t = t$, then (i), (iii) serve to define J_k on $\partial C(k)$ for $k > 0$ (again consistency follows from (4.4)), and J_k is extended to the interior by

$$[J_k((1-t)y_k + tz)]\tau = \begin{cases} f_t\Big([J_k(z)]\Big(\dfrac{\tau}{t}\Big)\Big) & \tau \leqq t\,|\,J_k(z)\,| \\ (t \cdots t) + (\tau - t\,|\,J_k(z)\,|)(1, \cdots, 1) & \text{otherwise.} \end{cases}$$

Here $f_t(x_1 \cdots x_k) = (tx_1 \cdots tx_k)$, and $|\,J_k(z)\,|$ is the length of $J_k(z)$. Note that the length of $J_k((1-t)y_k + tz) = 1 + t(|\,J_k(z)\,| - 1)$. (i), (ii), (iii) continue to hold, and the truth of (iv) is a consequence of the fact that the evaluation map is of degree 1 or -1 from $\partial(I \times C(k))$ into $\partial(I^{k+1})$ from (iii).

5. The spaces $J_i(X)$

For any space X, $T(X)$ is the disjoint union $X + X^2 + X^3 + \cdots + X^n + \cdots$. It has the structure of an associative monoid with multiplication defined by the inclusion $M\colon T(X) \times T(X) \to T(X)$.

If X is graded, then $T(X)$ has a grading defined by $\text{degree}(x_1 \cdots x_r) = \sum_{j=1}^{r} \text{degree } x_j$. We will denote the set of points of $T(X)$ of degree k by $T^k(X)$.

As we have seen in § 3, the reduced product $J_1(X)$ is defined by means of an equivalence relation in $T(X)$. We now define certain higher functors $T_i(X)$ and equivalence relations which allow us to construct spaces $J_i(X)$ equivalent to $\Omega^i\Sigma^i X$, for $i \geqq 1$.

X is graded by $\deg(x) = 1$ for all $x \in X$. $T_1(X) = T(X)$, and suppose inductively $T_i(X)$ is defined and graded. We set

$$Y_i = C(0) \times T_i^1(X) + C(1) \times T_i^2(X) + \cdots + C(n) \times T_i^{n+1}(X) + \cdots ,$$

and define a grading in Y by degree $(x) = n + 1$ if $x \in C(n) \times T_i^{n+1}(X)$. Then $T_{i+1}(X) = T(Y)$. Thus $T_i(X)$ is a graded associative monoid. We can write its points in the form

$$(y_1 \cdots y_{i-1}, x_1 \cdots x_n) ,$$

where each y_j is contained in a product $C(i_1) \times \cdots \times C(i_r)$ with $i_1 + \cdots + i_r = n - r$. In this notation the multiplication is specified by

$$\begin{aligned} M[(y_1 \cdots y_{i-1}, x_1 \cdots x_r), (y'_1 \cdots y'_{i-1}, x_{n+1} \cdots x_{n+m})] \\ = \big((y_1, y'_1), \cdots, (y_{i-1}, y'_{i-1}), x_1 \cdots x_{n+m}\big) . \end{aligned}$$

There is an obvious identification map

$$K: \{I \times T'_i(X) + \cdots + I^n \times T_i^n(X) + \cdots\} \rightarrow T_i(\Sigma X)$$

given on points by

$$K(t_1 \cdots t_n, y_1 \cdots y_{i-1}, x_1 \cdots x_n) = \big(y_1 \cdots y_{i-1}, (t_1, y_1) \cdots (t_n, y_n)\big) ,$$

and 4.6 implies there are maps

$$J_k \times \text{id}: C(k) \times T_i^{k+1}(X) \longrightarrow P^{k+1} \times T_i^{k+1}(X) .$$

These induce a map $K_\# T$: $T_{i+1}(X) \rightarrow P(T_i(\Sigma X))$. Let π be the identification map $T_i(X) \rightarrow J_i(X)$. We assume $\pi_\# K_\# T$ maps $T_{i+1}(X)$ into $\Omega J_i(\Sigma X)$. This being so, we will (roughly speaking) define $J_{i+1}(X)$ to be the image under this map of $T_{i+1}(X)$.

We will have to be more precise! $J_{i+1}(X)$ is an identification space of $T_{i+1}(X)$, and our task now is to give the equivalence relation explicitly.

To this end it is convenient to introduce some new notation. First we extend the mappings D_k to products by induction: suppose D_k defined on all products of length $< r$, then on $C(s) \times A$, $D_k = D_k \times \text{id}$ if $k \leqq s + 1$ or $\text{id} \times D_{k-s+1}$ otherwise (here $D_1 \mid C(0)$ is the forgetful functor). Now, let α be an $(i + 1, j + 1)$ shuffle. We define a mapping D^α: $C(i + j + 1) \rightarrow C(i) \times C(j)$ by

$$D^\alpha = (D_{i+2}^{j+1} \times D_1^{i+1})\alpha^{-1} .$$

Then D^α, restricted to the face $\alpha C(i) \times C(j)$, is just the projection onto $C(i) \times C(j)$, and unless a cell $\beta(C(i_1) \times \cdots \times C(i_r))$ is a face of $\alpha C(i) \times C(j)$, D^α maps it onto a lower dimensional cell. Also, using (4.4) we have

$$D_k D^\alpha = D^{d_{\alpha^{-1}(k)}\alpha} D_{\alpha^{-1}(k)} . \tag{5.1}$$

$S_{(i_1+1)} \times \cdots \times S_{(i_r+1)}$ acts as a group of homeomorphisms on the product $C(i_1) \times \cdots \times C(i_r)$. Let now α be an $(l+1, i_j - l + 1)$ shuffle. Then $D^{1\times\cdots\times 1\times\alpha\times 1\cdots\times 1} = \mathrm{id} \times \cdots \times D^\alpha \times \mathrm{id} \times \cdots \times \mathrm{id}$. Where no confusion is possible, we will also denote this last map as D^α.

The equivalence relation R is now defined by specifying generators of two kinds:

(1) $(y_1 \cdots y_{i-1} x_1 \cdots x_n) \sim \big(\alpha^{-1}y_1 \cdots \alpha^{-1}y_s, D^\alpha y_{s+1} \cdots D^\alpha y_{i-1}, \alpha^{-1}(x_1 \cdots x_n)\big)$
if y_{s+1} belongs to a face $C(i_1) \times \cdots \times (\alpha C(m) \times C(i_j - m - 1)) \times \cdots \times C(i_j)$ of $C(i_1) \times \cdots \times C(i_j)$,

(2) $(y_1 \cdots y_{i-1}, x_1 \cdots x_n) \sim \big(D_k(y_1) \cdots D_k(y_{i-1}) x_1 \cdots \hat{x}_k \cdots x_n\big)$
if $x_k = *$. We now set $J_i(X) = T_i(X)/R$.

From 5.1, it follows that these two types of relations commute. Hence, every point is uniquely equivalent to one of the form $(y_1 \cdots y_{i-1}, x_1 \cdots x_n)$ with each y_j an interior point of its cell, and no $x_j = *$. Thus, if X is a countable CW-complex with $*$ a vertex, so is $J_i(X)$. It contains X as a subcomplex and otherwise it has cells of the form $C^1 \times \cdots \times C^{i-1} \times e_1 \times \cdots \times e_n$, where each C^j is a product of $C(k)$'s, and no $e_j = *$. This cell complex will be studied in more detail in the next three sections. For now we content ourselves with observing that $J_i(X)$ is an associative H-space with unit $*$, and the cell complex is adapted to the multiplication. Its generators are the cells which contain the interior points of Y_{i-1}.

THEOREM 5.2. *Let X be a connected countable* CW-*complex with $*$ a vertex. Then there is an H-map*

$$j_k \colon J_k(X) \longrightarrow \Omega^k \Sigma^k(X) ,$$

which is a homotopy equivalence.

PROOF. It suffices to find an H-map $T\colon J_k(X) \to \Omega J_{k-1}(\Sigma X)$ which is a homotopy equivalence.

The map $K_\# T$ induces a map $\mathrm{Exp}\colon R^+ \times T_k(X) \to T_{k-1}(\Sigma X)$, where $\mathrm{Exp}(t, z) = K_\# T(z)t$. Define $L\colon R^+ \times T_k(X) \to R^+$ by

$$L(t, y_1 y_2 \cdots y_{i-1}, x_1 \cdots x_n) = \int_0^t \Big(\sum_{j=1}^n \Big(\frac{dJ(y)}{dt_j}\Big)^2 h(x_j)^2\Big)^{1/2} dt ,$$

where $h\colon X \to [0, 1]$ is some function so that $h^{-1}(0) = *$. To put it another way, define $H(x_1 \cdots x_n)\colon R^n \to R^n$ by $H(x_1 \cdots x_n)(t_1 \cdots t_n) = (h(x_1)t_1 \cdots h(x_n)t_n)$. Then $L(t, y_1 y_2 \cdots y_{i-1}, x_1 \cdots x_n)$ is the distance traveled on the path $H(x \cdots x_n)J(y)$ as τ runs between zero and t.

That L may be assumed continuous, follows from a close examination of the proof of 4.6.

Define $h\colon T_k(X) \to R^+$ by $h(z) = \max_t (L(t, z))$. h is continuous, $hM = hp_1 + hp_2$; and, if $z \sim z'$, then $h(z) = h(z')$. This is true since, if the two points are related by a relation of type one or two, then $h(z) = h(z')$. Thus h extends to a multiplicative map $\tilde{h}\colon J_k(X) \to R^+$ with $\tilde{h}^{-1}(0) = *$; and from § 2, we may use $\tilde{h}$ to define a classifying space $B_k(X)$ for $J_k(X)$.

There are maps φ, ψ which will make the following diagram commutative

$$
\begin{array}{ccc}
R^+ \times T_k(X) & \xrightarrow{\text{Exp}} & T_{k-1}(\Sigma X) \\
\Big\downarrow {\scriptstyle L \times \pi p_2} & & \Big\downarrow {\scriptstyle \pi} \\
R^+ \times J_k(X) & \xrightarrow{\varphi} & J_{k-1}(\Sigma X) \\
& {\scriptstyle \pi\rho} \searrow \quad \nearrow {\scriptstyle \psi} & \\
& B_k(X) . &
\end{array}
\tag{5.3}
$$

This follows from the fact that

$$
L(t, M(z, z')) = \begin{cases} L(t, z) & t \leqq h(z) \\ h(z) + L(t - h(z), z') & \text{otherwise} , \end{cases}
$$

and, if $z \sim z'$ while $L(t, z) = L(t', z')$, then $\text{Exp}\,(t, z) \sim \text{Exp}\,(t', z')$.

As a consequence of 2.6, the theorem will be proved if we can show ψ is a homotopy equivalence.

Both $B_k(X)$ and $J_{k-1}(\Sigma X)$ are simply connected, so the result will follow from the fact that $\psi_*\colon C_\#(B_k(X)) \to C_\#(J_{k-1}(\Sigma X))$ is an isomorphism. But this is a consequence of 4.6 (iv) and the diagram. q.e.d.

6. The cellular chain groups $C_\#(J_i(X))$

Let e be a cell of $J_i(X)$. If e contains an interior point in the equivalence class of $(y_1, \cdots, y_i, x_1, \cdots, x_j)$ with no x_k equal to $*$, then e has degree j, and it follows that $\deg M(e \times f) = \deg(e) + \deg(f)$.

Let $[C_\#(J_i(X))]^j$ be the submodule of $C_\#(J_i(X))$ generated by the cells of degree j. *We assume* $[C_\# J_i(X)]^j$ *is a subcomplex of* $C_\# J_i(X)$. For this to happen, it is sufficient that $*$ be the only vertex of X; and since X is connected, X is homotopy equivalent to a CW-complex with a single vertex. Thus our assumption is always satisfied, at least up to homotopy equivalence.

Let $P_1 \cdots P_n \cdots$ be the generating cells of $C_\# J_{i-1}(X)\,(G^1 J_{i-1}(X))$. Their products $M(P_1 \times \cdots \times P_n)$ will be written $P_1 \cdots P_n$. Thus, the generating cells of $C_\# J_i(X)$ will be of the form

$$\{C(r-1) \times (P_1 \cdots P_n)\}$$

with $r = \sum_{k=1}^{n} \deg P_k$. For convenience we will use the symbol $|P_1 \cdots P_n|$ for such a cell.

THEOREM 6.1. *There is a map* Δ *of* DGA *algebras*

$$\Delta\colon C_\# J_i(\Sigma(X)) \longrightarrow C_\# J_i(\Sigma X) \otimes C_\#(J_i(\Sigma X)) ,$$

which is a diagonal approximation, and is defined on generators by

$$\Delta |P_1 \cdots P_n| = |P_1 \cdots P_n| \otimes 1 + 1 \otimes |P_1 \cdots P_n| .$$

Stated in other words, Δ *makes* $C_\# J_i(\Sigma X)$ *into a co-commutative, primitively generated, Hopf algebra.*

PROOF. The identification map

$$K\colon \{\textstyle\sum_{j=1}^{\infty} I^j \times T_i^j(X)\} = Z \longrightarrow T_i(\Sigma X)$$

is multiplicative and onto. We will construct a diagonal approximation in Z by means of the compositions

$$I^j \times T_i^j(X) \xrightarrow{F \times j} I^j \times I^j \times T_i^j(X) \cdot T_i^j(X) \xrightarrow{\text{shuff}} (I^j \times T_i^j(X)) \times (I^j \times T_i^j) ,$$

where F is the usual cellular approximation to the diagonal

$$(t_1 \cdots t_n) \longrightarrow (\overline{2t_1} \cdots \overline{2t_n}; \underline{2t_1 - 1}, \cdots, \underline{2t_n - 1})$$

($\bar{\alpha} = \min(\alpha, 1)$, $\underline{\beta} = \max(0, \beta)$). The homotopy between F and d is given by

$$H(t, t_1 \cdots t_n) = (\overline{(1+t)}t_1, \cdots \overline{(1+t)}t_n; \underline{(1+t)t_1 - t}, \cdots \underline{(t+1)t_n - 1}) ,$$

and it is seen that this homotopy induces a homotopy h so that

$$\begin{array}{ccc} I \times Z & \xrightarrow{H} & Z \times Z \\ {\scriptstyle \mathrm{id} \times \pi K}\downarrow & & \downarrow{\scriptstyle \pi K \times \pi K} \\ I \times J_i(\Sigma X) & \xrightarrow{h} & J_i(\Sigma X) \times J_i(\Sigma X) \end{array}$$

is commutative. Now it is clear that h_1 is cellular and commutes with M. Moreover, on a cell $A = I^j \times C(j-1) \times P_1 \cdots P_n$, $h_1 \pi K A = \pi K(A) \times 1 + 1 \times \pi K(A) + \Sigma B_i \times B_i'$ where the $B_i \times B_i'$ are of lower dimensions. q.e.d.

If A is a co-associative, co-augmented co-algebra with co-unit, then the cobar construction $F(A)$ is defined (see [1] for definition). We now can prove the main result of this section.

THEOREM 6.2. $F[C_\# J_i(\Sigma X)]$ *is isomorphic to* $C_\# J_{i+1}(X)$.

PROOF. It suffices to show this for an arbitrary generator $|P_1 \cdots P_r|$ of

$C_{\#}J_{i+1}(X)$. Such a generator corresponds to a cell $C(m) \times P_1 \times \cdots \times P_r$. In turn, each cell P_i is of the form $C(m_i) \times M_i \times e_{1,i} \times \cdots \times e_{m_i+1,i}$, where M_i is a certain product of $C(j)$'s. The map $\psi: B_{i+1} \to J_i(\Sigma X)$ maps the cell $I \times C(m) \times P_1 \times \cdots \times P_r$ into

$$(-1)^\beta \prod_{j=1}^r C(m_j) \times M_j \times \sigma e_{1,j} \times \cdots \times \sigma e_{m_j+1,j} ,$$

where β is the appropriate sign and σe represents the suspension of the cell e of X. Now using 4.3 and 6.1 we can easily check that

$$\partial | P_1 \cdots P_r | = -| \partial W | + \Sigma(-1)^{a(j)} | W_j | W_j' | ,$$

where $W = \psi(I \times C(m) \times P_1 \times \cdots \times P_r)$ and $\Sigma W_j \otimes W_j' + W \otimes 1 + 1 \otimes W = \Delta W$, with $a(j)$ equal to the dimension of W_j.

7. Some invariance theorems for the cobar construction

In this section all chain complexes A will be assumed connected and augmented over the ground ring Γ (Z or Z_p for p a prime).

DEFINITION 7.1. *$s(A)$ is the complex*

$$s(A)_{i+1} \underset{J}{\overset{\cong}{\longleftrightarrow}} A_i , \qquad i > 0 ,$$
$$s(A)_0 = A_0 ,$$
$$s(A)_1 = 0$$

with boundary operator induced by the isomorphism J.

The operation s may be iterated, and we write

$$ss(A) = s^2(A), \cdots, s(s^{n-1}(A)) = s^n(A) .$$

Moreover, a diagonal approximation is given in $s(A)$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$. Thus if A is free over Γ, $F(s^n(A))$ is defined, and a diagonal approximation on generators is given by $\Delta | x | = | x | \otimes 1 + 1 \otimes | x |$. It is extended to F by requiring that it commute with the multiplication. If $F(s^n(A))$ is simply connected, i.e., if $n \geqq 2$, we can define $F(F(s^n(A))) = F^2(s^n(A))$, and this procedure may be iterated until we obtain $F^n(s^n(A))$.

THEOREM 7.2. *Let A, A' be free chain complexes over Γ, and suppose $f: A \to A'$ is an augmentation preserving chain map inducing isomorphisms in homology, then for $i \leqq n$,*

$$F^i(s^n f): F^i(s^n A) \longrightarrow F^i(s^n A')$$

also induces isomorphisms in homology.

COROLLARY 7.2. *$H_*(\Omega^n \Sigma^n(X), \Gamma)$ depends only on n, and the homology of*

X, if X has the homotopy type of a connected, countable CW-*complex.*[2]

PROOF OF 7.2. Let A be a minimal complex for the homology of X, and suppose $f\colon A \to C(X)$ induces isomorphisms in homology, the result now follows from 7.1 and the results of § 6. Similarly we have

COROLLARY 7.3. *The cohomology ring* $H^*(\Omega^{n-1}\Sigma^n(X), \Gamma)$ *depends only on n and the homology of X.*

PROOF OF 7.1. Let $\bar{F}(A) = F(A) - (\Gamma)$, then if we set

$$F^i(A) = \operatorname{im} M(\underbrace{\bar{F}(A) \otimes \cdots \otimes \bar{F}(A)}_{i \text{ times}}) \qquad i > 1\ ,$$

we have

$$F(A) = F^1(A) \supset F^2(A) \supset F^3(A) \supset \cdots \supset F^n(A) \supset \cdots ;$$

and if x has dimension i, then x does not belong to $F^{i+1}(A)$. Thus, the spectral sequence induced by the filtration converges and

$$s^iE_i^0 = s^i(F^i/F^{i+1}) \cong \underbrace{A \otimes \cdots \otimes A}_{i \text{ times}}\ .$$

Hence

$$E_i^1F(f)\colon E_i^1(F(A)) \longrightarrow E_i^1(F(A'))$$

is an isomorphism, thus the same is true for $E_i^rF(f)$ $r \geqq 2$, and thus also for $F(f)_*$.

Now, suppose A, A' are simply connected, co-associative, co-algebras which are free over Γ. A co-multiplication is defined in $A \otimes A'$ by the composition

$$A \otimes A' \xrightarrow{\Delta \otimes \Delta} A \otimes A \otimes A' \otimes A' \xrightarrow{1 \otimes T \otimes 1} (A \otimes A') \otimes (A \otimes A')\ ,$$

and the cobar constructions $F(A)$, $F(A')$, $F(A \otimes A')$ are all defined.

THEOREM 7.4. *The map of algebras*

$$\rho\colon F(A \otimes A') \longrightarrow F(A) \otimes F(A')$$

defined on generators by

$$\rho\,|\,a \otimes 1\,| = |\,a\,| \otimes 1$$
$$\rho\,|\,1 \otimes a'\,| = 1 \otimes |\,a'\,|$$

with ρ *identically* 0 *on other generators, induces isomorphisms in homology.*

PROOF. $F(C) \otimes C$ is made into a construction [5] over $F(C)$ by defining

[2] The referee points out that the restriction to countable CW-complexes is unnecessary, as is any restriction involving CW-complexes at all, since the singular complex of a space is the direct limit of its countable subcomplexes, and homology preserves direct limits both for the original complex and for the induced limits in the loop space.

the twisted boundary

$$\partial 1 \otimes c = |c| \otimes 1 + \textstyle\sum_j |c_j| \otimes c'_j + 1 \otimes \partial c \,, \tag{7.5}$$

where $\Delta(c) = \Sigma c_j \otimes c'_j$. Using this formula it is easy to check that ρ extends to a map of constructions

$$\rho \otimes \mathrm{id}\colon F(A \otimes A') \otimes A \otimes A' \longrightarrow F(A) \otimes F(A') \otimes (A \otimes A') \,.$$

Define filtrations by

$$\begin{aligned} H^i &= F(A \otimes A') \otimes (A \otimes A')_i \\ G^i &= F(A) \otimes F(A') \otimes (A \otimes A')_i \,, \end{aligned}$$

where $(A \otimes A')_i$ is the i-skeleton of A. Then

$$\begin{aligned} E^1_j(H^i) &= H_*(F(A \otimes A')) \otimes (A \otimes A')_j/(A \otimes A')_{j-1} \\ E^1_j(G^i) &= H_*(F(A) \otimes F(A')) \otimes (A \otimes A')_j/(A \otimes A')_{j-1} \\ E^1(\rho \otimes \mathrm{id}) &= \rho_* \otimes \mathrm{id} \,, \end{aligned}$$

and

$$\begin{aligned} E^2_j(H^i) &= H_j(A \otimes A'; H_*(F(A \otimes A'))) \\ E^2_j(G^i) &= H_j(A \otimes A'; H_*(F(A) \otimes F(A'))) \,. \end{aligned}$$

Since A, A' are simply connected, $F(A \otimes A')$, $F(A) \otimes F(A')$ are connected, and we may apply the spectral comparison theorem, from which it follows that ρ_* is an isomorphism.

8. The homology of $J_i(X)$

In this section we obtain some information on the homology and cohomology of $J_i(X)$ if X is of finite type. In particular, we assume $C(X)_j$ is finitely generated for every j.

Let A be a connected free Γ complex with trivial boundary. Then $F(sA)$ is just the tensor algebra generated by A, $T(A)$. It now follows from the Poincaré-Birkhoff-Witt theorem (see for example [8]) that we can write $T(A) = \bigotimes_i P(x_i)$ as a Γ module, where the $\{x_i\}$ form a basis for the graded Lie algebra generated by A, and $P(x)$ is the polynomial algebra generated by x.

In the case of interest here, we assume A is itself a suspension sA', then we can make $T(A)$ into a Hopf algebra by setting $\Delta(a) = a \otimes 1 + 1 \otimes a$ for $a \in A$. (If $T(A)$ represents $\Omega(\Sigma^2 X)$, it will follow that $F(T(A))$ represents $\Omega^2(\Sigma^2 X)$.) But if a, b, are primitives in a Hopf algebra, so is $[a, b]$. Hence the graded Lie algebra generated by A is primitive in $T(A)$, and $T(A)$ is thus represented as a tensor product of primitively generated co-algebras, and we may apply 7.4 when we iterate.

We may simplify $P(x)$ still further. For example, if x is odd dimensional, then $P(x) \cong E(x) \otimes P(x^2)$ as a co-algebra $\Delta x^2 = x^2 \otimes 1 + 1 \otimes x^2$; and if $\Gamma = Z_p$, and x is even dimensional,

$$P(x) \cong T(x) \otimes T(x^p) \otimes \cdots \otimes T(x^{p^j}) \otimes \cdots ,$$

where $T(a)$ is the polynomial algebra on a truncated by the relation $a^p = 0$, and $\Delta(x^{p^j}) = x^{p^j} \otimes 1 + 1 \otimes x^{p^j}$.

LEMMA 8.1. *If* $\Gamma = Z_p$, $n \geqq 1$, *then*

$$F\big(E(2n+1)\big) \cong T(2n) \otimes T(2np) \otimes \cdots \otimes T(2np^j) \otimes \cdots$$

as a co-algebra.

The proof is obvious.

LEMMA 8.2. *If* $\Gamma = Z_p$, $n \geqq 1$, *there is a chain map of co-algebras*

$$\rho\colon F\big(T(2n)\big) \longrightarrow E(2n-1) \otimes T(2np-2) \otimes \cdots \otimes T\big(p^j(2np-2)\big) \otimes \cdots$$

inducing isomorphisms in homology.

PROOF. We define a twisted boundary operator in $F(T(2n)) \otimes T(2n)$ by 7.5. We claim that the diagonal

$$\bar{\Delta} = (1 \otimes T \otimes 1)\Delta \otimes \Delta\colon F(T) \otimes T \longrightarrow \big(F(T) \otimes T\big)^2$$

is a chain map. It is clearly sufficient to verify this on an element $1 \otimes x^j$. We have $\bar{\Delta}(1 \otimes x^j) = \sum \binom{j}{k} 1 \otimes x^k \otimes 1 \otimes x^{j-k}$, while

$$\begin{aligned} \bar{\Delta}\partial(1 \otimes x^j) &= \bar{\Delta}\Big(|\, x^j \,| \otimes 1 + \textstyle\sum_{k=1}^{j-1} \binom{j}{k} |\, x^k \,| \otimes x^{j-k}\Big) \\ &= (|\, x^j \,| \otimes 1) \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (|\, x^j \,| \otimes 1) \\ &\quad + \textstyle\sum_{k=1}^{j-1} \binom{j}{k} [(|\, x^k \,| \otimes x^{j-k}) \otimes 1 \otimes 1 + |\, x^k \,| \otimes 1 \otimes 1 \otimes x^{j-k} \\ &\quad + 1 \otimes x^{j-k} \otimes |\, x^k \,| \otimes 1 + 1 \otimes 1 \otimes |\, x^k \,| \otimes x^{j-k}] \end{aligned}$$

and this is $\partial\bar{\Delta}(1 \otimes x^j)$.

Thus the dual complex Hom $(F(T) \otimes T, Z_p)$ becomes a twisted tensor product of *commutative* DGA-algebras. Let $Y = (E(2n-1) \otimes T(2np-2) \otimes \cdots)$. We define an inclusion of algebras $\rho^*\colon Y \subset F^*(T) = \text{Hom}\,(F(T), Z_p)$ on generators by

$$\begin{aligned} \rho^*\big(e(2n-1)\big) &= |\, x \,|^* \\ \rho^*\big(f(2np-2)\big) &= (|\, x \,|\, x^{p-1} \,|)^* \\ \rho^*\big(f(2np-2)p^j)\big) &= (\underbrace{|\, x \,|\, x^{p-1} \,|\, \cdots \,|\, x \,|\, x^{p-1} \,|}_{p^j \text{ times}})^* \end{aligned}$$

etc. We define a twisted boundary in $Y \otimes T(x)$ by

$$\delta f\big((2np-2)p^j\big) = \alpha_j\big(f(2np-2)p^{j-1}\big)^{p-1} \cdots \big(f(2np-2)\big)^{p-1} e(2n-1) \otimes x^{p-1} ,$$

$\delta e(2n-1) = x$ where α_j is the appropriate non-zero coefficient, defined so as to make the inclusion $\rho^* \otimes \mathrm{id}\colon Y \otimes T^* \to F^*(T) \otimes T^*$ into a chain map. If we now define a spectral sequence by $F^i(F(T)^* \otimes T^*) = \sum_{j \geq i} F(T)^* \otimes x^j$ and, by restriction, give a sequence for $Y \otimes T^*$, an easy induction shows that $E^2(\rho^* \otimes \mathrm{id})$ is an isomorphism of spectral sequences, and this proves the lemma.

Because of 7.4 and the preliminary remarks, 8.1 and 8.2 allow us to give an essentially complete description of the cohomology rings $H^*(J_i(\Sigma X), Z_p)$ for all i, since in this case there is always an A with trivial differential, and a map $f\colon A \to C(X) \otimes Z_p$ inducing isomorphisms in homology. For example $H^*(\Omega^k S^n, Z_p)$ $k < n$ is a tensor product of exterior algebras on odd dimensional generators, and twisted polynomial algebras $T(2n)$ on even generators. Note that

$$H^*(\Omega S^{2n}, Z_p) \cong H^*(S^{2n-1}, Z_p) \otimes H^*(\Omega S^{4n-1}, Z_p) ;$$

thus $H^*(\Omega^k S^{2n}, Z_p) \cong H^*(\Omega^{k-1} S^{2n-1}; Z_p) \otimes H^*(\Omega^k S^{4n-1}, Z_p)$, and it suffices to give the generators for odd dimensional spheres only. Given any sequence of positive integers $(i_1 \cdots i_r)$ with $r \leqq k$, we have the long sequences

$$\begin{aligned}\{(n, pn, \cdots, p^{i_1-1}n), 1, &\big((np^{i_1}-1), \cdots, (np^{i_1}-1)p^{i_2-1}\big), 1,\\ &\big([(np^{i_1}-1)p^{i_2}-1 \cdots] \cdots\big), 1,\\ &\big(\cdots [(np^{i_1}-1)p^{i_2}-1 \cdots p^{i_{r-1}}-1]p^{i_r-1}\big), \varepsilon\} ,\end{aligned}$$

where ε may equal 0 or 1 if $r < k$, and $\varepsilon = 0$ if $r = k$. To such a long sequence there corresponds a generator of dimension

$$\begin{aligned}(2n-k) + 2n(p-1) + \cdots + \varepsilon n p^{i_1-1}(p-1) - 1 + 2(np^{i_1}-1)(p-1)\\ + \cdots + 2\big[\big([(np^{i_1}-1)p^{i_2}-1] \cdots p^{i_{r-1}}-1\big)p^{i_r-1}\big](p-1) - \varepsilon ,\end{aligned}$$

and these are precisely the generators of $H^*(\Omega^k S^{2n+1}, Z_p)$.

One may note the rough duality that occurs between these results and the results of Cartan for Eilenberg-MacLane spaces $K(Z, n)$. This duality may be extended to the case where $\Gamma = Z$ also. By methods similar to those used by Cartan, or to those used in the proof of 8.2, we may prove

LEMMA 8.3. *There is a map $\rho^*\colon D(2n) \to F^*(E(2n+1))$ inducing isomorphisms in cohomology. There is a map*

$$\begin{aligned}\bar{\rho}^*\colon E(2n-1) \otimes E_2(4n-1) \otimes \cdots \otimes E_2(2^{j+1}n-1) \otimes \cdots\\ \otimes E_p(p^j(2n)-1) \otimes \cdots \longrightarrow F^*\big(D(2n)\big) \cdots\end{aligned}$$

inducing isomorphisms in cohomology. Similarly there are maps

$$\bar{\bar{\rho}}^*\colon D_p(2n) \otimes E_p(2np-1) \otimes \cdots \otimes E_p(2np^j-1) \otimes \cdots \longrightarrow F^*\big(E_p(2n+1)\big)$$
$$\bar{\bar{\bar{\rho}}}^*\colon E_p(2n-1) \otimes E_p(2np-1) \otimes \cdots \otimes E_p(2np^j-1) \otimes \cdots \longrightarrow F^*\big(D_p(2n)\big) ,$$

which induce isomorphisms in homology.

Here $D(2n)$ is the divided polynomial algebra on a generator of dimension $2n$. $E_p(2n-1) = E(2n-1) \otimes D(2n-2)$ with $\delta w = pe$ if w is the generator of $D(2n-2)$, and $D_p(2n)$ is an enormously complex algebra with integral cohomology generated by elements V_i in dimension $2ni$, and the order of V_i is ph where h is the largest power of p dividing i.

Thus, it would seem that, for X more general than spheres, the problems of calculating the integral cohomology rings of $\Omega^i\Sigma^{i+j}(X)$ reduces to finding a reasonable representation of $T(S^{i+j}(A))$ where A is a minimal complex for $C(X)$. The author, however, has not been able to achieve this.

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