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Unstable Homotopy from the Stable Point of View



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Preface

This work had its origins in two projects. The first, undertaken in 1970 with Elmer Rees, was to construct certain low dimensional embeddings of real projective spaces. In order to do this we needed methods for calculating unstable homotopy groups of truncated projective spaces and associate spaces, as well as their images under various Freudenthal suspension homomorphisms. The second was to understand Mahowald's work on the metastable homotopy of spheres.

In 1971 and 1972 my work in surgery made me enlarge the scope of the project and consider an apparently unrelated problem - the stable homotopy of the Eilenberg-MacLane spaces K(Q/Z,n). By means of appropriate fiberings these questions are seen to be merely different faces of the same coin.

Hence, this current work which provides a relatively effective framework for considering such questions. We generalize Mahowald's constructions to allow us to apply Adams' spectral sequence techniques to calculations, and we give detailed calculations for many examples; in particular, those needed for the work with Rees, and those needed in surgery with coefficients.

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Introduction

In recent years, stable homotopy theory has become a standard tool for the working topologist. If \mathscr{X} is a spectrum $\{Y_1, Y_2, \ldots, Y_n, \ldots\}$, one defines the stable homotopy groups of X with coefficients in \mathscr{Y} as

$$H_{i}(X,Y) = \lim_{n \to \infty} \pi_{n+i}(X \land Y_{n}),$$

and these groups, according to G. W. Whitehead, define a generalized homology theory. (This means that they satisfy all the Eilenberg-Steenrod axioms except the dimension axiom.) In particular, if \mathscr{Y} is the sphere spectrum $\mathscr{S} = \{S^1, S^2, \dots, S^n, \dots\}$, then

$$H_{i}(X, \mathscr{G}) = \pi_{i}^{S}(X)$$

defines the stable homotopy groups of the space $\ {\tt X}$.

Since these groups form a homology theory, it is not surprising that homological techniques can be applied in calculations. Indeed, a major tool is the Adams spectral sequence, which has E^2 -term

$$\operatorname{Ext}_{\mathscr{A}(p)}(\operatorname{H}^{*}(X \wedge Y, Z_{p}), Z_{p}),$$

and converges to $H_{\mathbf{x}}(\mathbf{X}, \mathscr{Y} \otimes \mathbb{Z}_{\infty})$. Here $\mathscr{A}(\mathbf{p})$ is the mod (p) Steenrod algebra, and $\mathbb{Z}_{\infty} = \lim_{p \to \infty} \mathbb{Z}_{1}$ (see e.g. [2]). Sometimes there are more efficient methods for calculating these groups, but in a range it is quite efficient, and has led to the calculation of the stable homotopy of spheres and some associated spaces through approximately the first 60 stems. This is a considerable achievement, since it was not that long ago that there was real uncertainty as to the order of even the second stem. These techniques have also yielded spectacular results, such as Adams' first proof of Hopf-invariant 1 and his later solution of the vector field problem for spheres. Moreover, the recent work of Mahowald, Quillen and others on the structure of the stable J-homomorphism promises further profound knowledge in stable homotopy theory.

Summarizing, we can regard the stable groups as reasonably well understood.

For many problems, though, it is unstable groups which are actually needed. For example, the vector field problem for spheres (or its homotopy version) was really a question about how far back a certain element (the Whitehead product [I,I]) in π_4_{n-3} (S²ⁿ⁻¹) desuspends. Its solution was obtained only after it was converted into a problem about the stable homotopy groups of truncated projective spaces, and the fact that this unstable question actually admitted such a reduction is, I suppose, the starting point of this monograph.

In a range of dimensions, there is an exact sequence, discovered in a special case by I. M. James and written down in generality by H. Toda, called the EHP sequence

which relates the unstable homotopy groups of S^n , S^{n+j} to the (stable) groups of a truncated real projective space ($RP_n^m = RP^m/RP^{n-1} = *$).

The existence of an element $\beta \in \pi_{4n-3-i}(S^{2n-i-1})$, which suspends to the Whitehead product [I,I] can be interpreted as equivalent to the existence of an element $\alpha \in \pi_{4n-3-i}(S^{2n-i-2} \wedge P^{2n-1}_{2n-i-1})$, with $P(\alpha) = \beta$ in the EHP sequence, and, on pinching $S^{2n-i-2} \wedge P^{2n-2}_{2n-i-1}$ to a point,

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satisfies $p(\alpha) = e \in \pi_{4n-3-i}(S^{4n-3-i})$ is a generator, provided i is sufficiently small that the EHP sequence is valid. This can be seen at once on considering the commutative diagram of EHP sequences

$$\begin{array}{c} \stackrel{P}{\rightarrow} \pi_{4n-3-i}(s^{2n-i-1}) + \pi_{4n-2}(s^{2n}) \stackrel{H}{\rightarrow} \pi_{4n-3-i}(s^{2n-i-1} \wedge P_{2n-i-1}^{2n-1}) \rightarrow \\ \stackrel{P}{\rightarrow} \pi_{4n-2-i}(s^{2n-i}) \stackrel{E}{\rightarrow} \pi_{4n-2}(s^{2n}) \stackrel{H}{\rightarrow} \pi_{4n-2-i}(s^{2n-i} \wedge P_{2n-i}^{2n-1}) \rightarrow \\ \stackrel{P}{\rightarrow} \pi_{4n-3}(s^{2n-1}) \stackrel{E}{\rightarrow} \pi_{4n-2}(s^{2n}) + \pi_{4n-2}(s^{2n-i} \wedge P_{2n-2}^{2n-1}) \rightarrow \\ \stackrel{P}{\rightarrow} \pi_{4n-3}(s^{2n-1}) \stackrel{E}{\rightarrow} \pi_{4n-2}(s^{2n}) + \pi_{4n-2}(s^{2n-1} \wedge P_{2n-2}^{2n-1}) \rightarrow \end{array}$$

since [I,I] generates the kernel of the bottom suspension map.

There are other problems of a similar nature involving, for example, the number of times a Thom space is really a suspension, which have implications for the geometric dimensions of vector bundles and the immersion dimensions for manifolds ([42]).

Our main object here is to develop machinery which leads to systematic methods for attacking such problems in a range of dimensions. Specifically, we study the problem of passing from the stable to the metastable homotopy groups of a space. In particular, we develop the following generalization of the EHP sequence.

Theorem 1.11. There is a space $\Gamma_L(X) = S^{L-1} \ltimes_T (X \land X)$, which is (2n-1)-connected whenever X is (n-1)-connected and an exact sequence valid in the metastable range (i < 3n-2)

*
$$\dots \stackrel{E}{\rightarrow} \pi_{i+L}(\Sigma^{L}X) \stackrel{H}{\rightarrow} \pi_{i}(\Gamma_{L}(X)) \stackrel{\partial}{\rightarrow} \pi_{i-1}(X) \stackrel{E}{\rightarrow} \pi_{i+L-1}(\Sigma^{L}X) \rightarrow \dots$$

So for L sufficiently large, * determines $\pi_*(X)$ in terms of the stable homotopy of X and $\Gamma_L(X)$ in this range. If X is a sphere S^n ,

then $S^{L-1} \ltimes_T S^n \wedge S^n = \Sigma^n RP_n^{n+1-1}$ ([22]); however, for more complicated X, $\Gamma_L(X)$ becomes considerably more complex. In §§2 and 3, we give $H^*(\Gamma_L(X))$ as a module over the Steenrod algebra $\mathscr{A}(2)$ or $\mathscr{A}(p)$, provided we know the Steenrod algebra structure of $H^*(X)$. Thus, in principle, we can apply the Adams spectral sequence to determine $\pi_*(\Gamma_L(X))$ through a suitable range.

The idea behind the proof of 1.11 is contained in two facts: that $\pi_i(\Omega^n Y) \cong \pi_{n+i}(Y)$, where $\Omega^n Y$ is the nth loop space of Y, and that the natural map (§1)

$$j : X \hookrightarrow \Omega^n \Sigma^n X$$

gives, on passing to homotopy, the map $j_* : \pi_i(X) \to \pi_{n+i}(\Sigma^n X)$, which is the suspension map E in l.ll*. Converting j into a fibration and identifying the fiber with $\Omega(\Gamma_n(X))$ gives l.ll.

In order to make this identification, we need some basic facts about the structure of the space $\Omega^n \Sigma^n X$. When n = 1, I. M. James showed that $\Omega \Sigma X \stackrel{\sim}{=} J_1(X)$, where $J_1(X)$ is the "reduced join"

$$(x \cup_{F_2} x^2) \cup_{F^3} x^3 \cup \dots$$

Here F_2 is defined on $* \times X \cup X \times *$ as the folding map, F_3 is defined on $* \times X^2 \cup X \times * \times X \cup X^2 \times *$ as the folding map, and so on. There is an associative product in $J_1(X)$ defined by juxtaposition, and * is then the identity. In fact, $J_1(X)$ can be described as the free associative H-space generated by X with * as unit. This result was generalized in [18] to give similar constructions for $\Omega^n \Sigma^n X$. We review (and explain) this construction in §0. Specifically, we start almost from first principles and develop the geometric ideas which lead to an understanding of the basic structure of $\Omega^n \Sigma^n X$. These lead to constructions $J_1(X)$, $J_2(X) \dots J_n(X)$, which are, in a sense, minimal models containing all the basic structures just developed. It is then a theorem that, for reasonable X, $J_n(X) \cong \Omega^n \Sigma^n X$. We do not prove this last result (the proof can be found in [18]), but we do explain the considerations which lead to the constructions, and these should make the details in [18] almost unnecessary.

In §4, we consider the problem of looking at $\Omega^n X$ when X is no longer an n-fold suspension. The result is quite intriguing. There is an evaluation map $e : S^n \wedge \Omega^n X \to X$, and looping e n-times gives

 $\tau : \Omega^{n} \Sigma^{n} (\Omega^{n} X) \to \Omega^{n} X$.

The fiber of τ is shown to be $\Gamma_n(X)$ in a range, and we obtain

Theorem 4.4. Suppose X is an (n+m-1)-connected CW-complex. Then in dimensions less than 3m-1, $H^*(\Omega^n X, Z_p)$ depends only on $H^*(X, Z_p)$ for p odd, and on $H^*(X, Z_2)$ as a module over $\mathscr{A}(2)$ for p = 2.

To finish Part 1, we apply the results of §4 to the desuspension problem. The result is

Theorem 5.1. Let X be (n-1)-connected and have dimension less than 3n-2. Then

- 1) if Y is the (2n-L-1)-skeleton of X , there is a unique space Z , so $\Sigma^{\rm L}Z$ = Y ;
- 2) X is itself an L-fold suspension if and only if a certain (universally constructed) map

$$\phi : X/Y \to \Sigma^{L+1} S^{L-1} \ltimes_T Z \land Z$$

is homotopically trivial (as usual, ϕ is a stable map).

The result would be more satisfactory if we knew more about ϕ or even the cofiber of ϕ . In low dimensions, things can be explicitly worked out using unstable higher cohomology operations (see e.g. [36]), but at present the author has no general results.

In this connection, we would like to point out the worked example at the end of §4, $\Omega^{11}(CP_{8}^{12})$, where we show that the mod 2 Steenrod algebra action in $H^{*}(\Omega^{n}X)$ is not determined by its action in X. It is possible to interpret the work of Adém-Gitler on non-immersion theorems ([3]) in terms of examples of this kind, and such analysis could lead to a sharper understanding of ϕ .

In Part 2, we develop means for calculating the maps H and P in 1.11*.

Mahowald showed in [13] how to use Adams spectral sequence techniques to study the ordinary EHP sequence. He constructed a map

$$\mathbb{E}^{2}(\mathbb{H}) : \mathbb{E} \mathfrak{x} \mathfrak{t}^{s, \mathfrak{t}}_{\mathscr{A}(2)}(\mathbb{Z}_{2}, \mathbb{Z}_{2}) \rightarrow \mathbb{E} \mathfrak{x} \mathfrak{t}^{s-1, \mathfrak{t}-n-1}_{\mathscr{A}(2)}(\mathbb{H}^{*}(\mathbb{R}^{p}_{n}), \mathbb{Z}_{2})$$

which commutes with differentials and, at E^{∞} , gives a map associated with H. (Note here that $E^2(H)$ changes the s degrees. It is precisely this change which makes Mahowald's map non-trivial.)

The obvious generalization of $E_2(H)$ to the map in 1.11* fails, however, and $E^2(H)$ does not exist for any space more complicated than a sphere (at the prime 2) !

Our main object in Part 2 is to provide a satisfactory generalization. We first review Adams' method for constructing his spectral sequence, and generalize it slightly so as to define an unstable spectral sequence which <u>approximates</u> the actual homotopy of a space X. Convergence seems difficult in general, but the sequence does — converge in the metastable range. There is a natural suspension map \hat{E} from this spectral sequence to the stable Adams spectral sequence, which is an isomorphism in the stable range and, at E^{∞} , is associated to E. At the E^2 level, \hat{E} is algebraically determined through the metastable range. This situation is quite nice except that the E^2 -term of our sequence for X is very hard to determine above the stable range, so we turn to the maps H and P.

In 6.11, we indicate how to define a spectral sequence for a pair (Y,A), where $A \subset H^*(Y,Z_2)$ is any submodule closed under the action of $\mathscr{A}(2)$. The resulting modified Adams spectral sequence has E^2 -term calculable in terms of $\operatorname{Ext}_{\mathscr{A}(2)}(A,Z_2)$, $\operatorname{Ext}_{\mathscr{A}(2)}(H^*(Y)/A, Z_2)$, and a differential

$$\partial$$
 : Ext^{i,*}(H^{*}(Y)/A, Z₂) \rightarrow Ext^{i+2,*}(A,Z₂).

We denote it $E^2_{**}(Y,A)$.

In $H^*(\Gamma_L(X), Z_2)$, there is a natural $\mathscr{A}(2)$ -submodule A, and in §8 we construct a map

$$\partial^2$$
: Ext^{s,t} $\mathscr{A}(2)$ (H^{*}(X), Z₂) \rightarrow E²_{s-1,t-2}($\Gamma_L(X)$, A),

which provides the desired generalization of Mahowald's map \hat{E}^2 (A is 0 if and only if X is a sphere at 2, in which case $E_{**}^2(\Gamma_L(X), 0) = Ext^{*,*-n-1}(H^*(\Gamma_L(X), Z_2))$. In fact, we are able to prove

<u>Theorem 8.5.</u> Suppose L > 3m, with X (m-1)-connected. Then there is a map

$$J^* : E_{i,j}^*(\Gamma_L(X), A) \neq E_{i+1,j+1}^*(X)$$

and, in the metastable range, the sequence

**
$$\cdots \rightarrow \operatorname{Ext}^{s, t}_{\mathscr{A}(2)}(\operatorname{H}^{*}(X), \mathbb{Z}_{2}) \xrightarrow{\partial^{2}} \operatorname{E}^{2}_{s, t-1}(\operatorname{H}^{*}(\Gamma_{L}(X)), \mathbb{A})$$
$$\xrightarrow{J^{2}}_{\longrightarrow} \operatorname{E}^{2}_{s+1, t}(X) \xrightarrow{\widehat{\operatorname{E}}^{2}} \operatorname{Ext}^{s+1, t}_{\mathscr{A}(2)}(\operatorname{H}^{*}(X), \mathbb{Z}_{2}) \rightarrow \cdots$$

is long exact.

This result makes effective calculations feasible in some cases. To expedite them, we conclude Part 2 with a discussion (§9) of methods for calculating $\operatorname{Ext}^{i,j}_{\mathscr{A}(2)}(\operatorname{H}^{*}(\Gamma_{L}(X)/A, Z_{2}))$. This is highly non-trivial in general, since $\operatorname{H}^{*}(\Gamma_{L}(X)/A)$ is a very complex $\mathscr{A}(2)$ -module; it replaces each Z_{2} -cohomology class of X by the cohomology of a truncated projective space. By an appropriate filtration, we obtain a spectral sequence converging to $\operatorname{Ext}_{\mathscr{A}(2)}(\operatorname{H}^{*}(\Gamma_{L}(X)/A), Z_{2})$, whose E^{1} -term contains a copy of $\operatorname{Ext}_{\mathscr{A}(2)}(\operatorname{H}^{*}(\operatorname{RP}_{n}), Z_{2})$ for each n-dimensional cohomology class in $\operatorname{H}^{*}(X, Z_{2})$. A second spectral sequence is also developed, which makes calculations feasible in case $\operatorname{Ext}_{\mathscr{A}(2)}(\operatorname{H}^{*}(X), Z_{2})$ is sufficiently well-known.

In Part 3, we apply the results of Parts 1 and 2, and give some examples to show that the theorems there cannot be improved too much.

In §§10 and 11, we calculate some of the stable homotopy of the Eilenberg-MacLane spaces K(Z,n), $K(Z_2,n)$, and K(Q/Z, n). The results for K(Q/Z, n) are calculated only so far as we need them in applications ([41])

$$\pi_{2n}^{s}(K(Q/Z, n)) = 0 ,$$

$$\pi_{2n+1}^{s}(K(Q/Z, n)) = \begin{cases} Q/2Z , n-odd \\ 0 , n-even \end{cases}$$

However, in §11, we give the first 10 stable groups for K(Z,8k+1) as an example (Theorem 11.18), and do most of the necessary work to obtain these groups for other values of n as well. In particular, 11.18 corrects some errors in [14].

§12 applies the metastable sequence 8.5** to the case $X = S^n \cup_2 e^{n+1}$, and, as an example, we calculate the homotopy groups of $S^7 \cup_2 e^8$ through the entire metastable range.

Finally, in §13, we give explicit calculations for some truncated projective spaces. In particular, our final calculation is of the unstable resolution for P_2^6 to slightly beyond the metastable range, where we see that wild filtration changes make any reasonable extension of 8.5** impossible. The remaining calculations in §13 provide the homotopy theoretic results needed in [42].

These results were originally obtained in 1969 and 1970. Since then, there has been further work by several authors on some of the questions considered here. B. Drachman has studied the desuspension problem from another point of view, and has obtained geometric criteria for deciding when a space is a suspension. Unfortunately, it seems difficult to iterate his techniques.

Also, an area which has received only partial attention but clearly merits more is the extension of the current results to generalized homology theories, such as unstable MU-theory, which seems ready for serious development in view of W. Steven Wilson's thesis (M.I.T., 1972).

In this connection, it would be interesting to explain Donald Davis' thesis (Stanford, 1971) on the geometric dimensions of bundles over \mathbb{RP}^{n} in terms of obstructions to desuspension of the Thom complexes, since that would probably give insight into the nature of the map ϕ (5.1) in b_0 or b_{II} theory.

PART I

\$0. Iterated loop spaces

We begin by describing the category \mathscr{C}_n of n-fold loop spaces. We can look at $\Omega^n X$ as the space of base point-preserving maps $S^1 \to \Omega^{n-1} X$ or the base point-preserving maps $S^2 \to \Omega^{n-2} X$..., or $S^n \to X$. With respect to the various ways of looking at $\Omega^n X$, there are evaluation maps

$$adj_k^n(1) : S^k \wedge \Omega^n(X) \rightarrow \Omega^{n-k}(X)$$
,

which fit together to give

and the composite

$$\Sigma^{n-s} adj_1^{n-s+1}(1) \dots \Sigma^{n-1} adj_1^n(1) = \Sigma^{n-1} adj_s^n(1)$$
.

The Moore loop space $\Omega_{(M)}(X)$ is the set of maps $f : [0,k_f] \to X$ for variable $k \ge 0$, which satisfy f(0) = f(k) = *. It has an associative product with unit $f * g : [0, k_f + k_g] \to X$, defined by setting

$$f * g(t) = \begin{cases} f(t), & t \leq k_{f} \\ \\ f(t-k_{f}), & t \geq k_{f} \end{cases}$$

The unit of $\Omega_{M}(X)$ is $*:[0] \rightarrow *$. Also, there is a natural homotopy equivalence between $\Omega(X)$ and $\Omega_{M}(X)$ (as, for example, in J. Adams and P. Hilton, "On the chain algebra of a loop space," Comment. Math. Helv. 30 (1956), 305-330), so in the remainder of this paper, we identify them and leave it to the reader to make the necessary modifications to go from $\Omega^{n}(X)$ to $\Omega^{n}_{M}(X)$ or vice versa. 12

If Z and W are n-fold loop spaces, then a map $f : Z \rightarrow W$ is "admissible" if and only if $f = \Omega^{n}(g)$ for some $g : \Omega^{-n}(Z) \rightarrow \Omega^{-n}(W)$.

Lemma 0.2. Let $f : X \rightarrow \Omega^n Y$ be a map. Then there is an admissible map

$$g(f) : \Omega^{n} \Sigma^{n} X \to \Omega^{n} Y$$

and a natural inclusion

$$j : X \rightarrow \Omega^n \Sigma^n X$$
,

so $g(f) \cdot j = f$.

<u>Proof.</u> From the composite $\Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n \Omega^n Y \xrightarrow{\operatorname{adj}(1)} Y$, we can loop down n-times, obtaining the map g(f). Now $j : X \to \Omega^n \Sigma^n X$ is defined by $j(x)(t) = (t,x) \in S^n \land X$, and 0.2 follows.

Remark 0.3.
$$\pi_n(Y) \cong \pi_{n-k}(\Omega^k Y)$$
, $n \ge k$. Also, if $Y = \Sigma^k Z$, then the map
 $j : Z \Rightarrow \Omega^k \Sigma^k Z$

gives $j_{\star}:\pi_{s}(Z) \rightarrow \pi_{k+s}(\Sigma^{k}Z)$, and this is the Freudenthal suspension homomorphism. If Z is an (n-1)-connected CW-complex, then the Freudenthal suspension theorem implies $j: Z \rightarrow \Omega^{k}\Sigma^{k}Z$ is a homotopy equivalence in dimensions less than 2n-1, since Milnor has shown that $\Omega^{k}\Sigma^{k}Z$ also has the homotopy type of a CW-complex ([25]).

<u>Remark</u> 0.4. The universal example for the situation in 0.2 is g(id): $\alpha^{n}\Sigma^{n}(\alpha^{n}Y) \rightarrow \alpha^{n}Y$. Indeed, it has recently been shown by J. P. May that the existence of an H-map $\alpha^{n}\Sigma^{n}Z \rightarrow Z$, with $\pi \cdot j = id$, is essentially equivalent to the associative H-space Z being an n-fold loop space. These observations signal the basic role of the spaces $\Omega^n \Sigma^n Y$ in \mathscr{C}_n . We study the map j in more detail in §1 and $\tau = g(1)$ in §4. Now our object is an explicit description of the homotopy type of the space $\Omega^n \Sigma^n X$.

A model for $\Omega \Sigma X$ was constructed by I. M. James(in "Reduced product spaces," Ann. of Math. 62 (1955), 170-197). It is easily described. Set $J_1(X) = \bigcup_{n=1}^{\infty} X^n/R$, where R is the relation

0.5
$$(x_1...x_i, *, x_{i+2}...x_n) \sim (x_1...x_i, x_{i+2}...x_n)$$

It has an associative product (juxtaposition), a unit *, an obvious topology, and James proved that $J_1(X) \xrightarrow{\sim} \Omega \Sigma X$ for X a CW-complex. The equivalence of $J_1(X)$ with $\Omega \Sigma X$ is obtained by mapping $J_1(X) \neq \Omega \Sigma X$ as the (unique) multiplicative extension of $j : X + \Omega_M(\Sigma X)$.

Models for $\Omega^{n}\Sigma^{n}X$, n > 1, were constructed in [18]. Several attempts to rework the construction occurred thereafter, culminating with the construction given by J. P. May(in <u>The Geometry of Iterated Loop Spaces</u>, Lecture Notes in Mathematics 271, Springer-Verlag, 1972). In 1.14, we describe the basic germ of his models, but in the remainder of this section, we largely follow [18].

Let us begin by looking at $J_1(\Sigma Y) \stackrel{\sim}{\rightarrow} \Omega \Sigma^2 Y$. Its component building blocks, the $(\Sigma Y)^n$ can be written after shuffling the variables in the form

 $\mathbf{I}^{\mathbf{n}} \times \mathbf{Y}^{\mathbf{n}}$,

where we make the identifications

$$(t_1 \dots \varepsilon_i \dots t_n, y_1 \dots y_n) \sim (t_1 \dots t_n, y_1 \dots, y_i = *, \dots y_n)$$

$$\sim (t_1 \dots t_i \dots t_n, y_1 \dots y_i, \dots y_n), \qquad n > 1,$$

and $(\epsilon_1, y_1) \sim (t_1 \neq) \sim \neq$ for n = 1. Here $\epsilon_i = 0$ or 1.

Let P(n) be the set of variable length paths starting at $(0,0,\ldots,0)$ in Iⁿ and ending at $(1,\ldots,1)$. Crossing with Yⁿ, we map P(n) × Yⁿ into paths on Iⁿ × Yⁿ, starting at $(0,0,\ldots,0) \times Y^n$ and ending at $(1,\ldots,1) \times Y^n$, by defining $(f_{x_1}\ldots y_n)t = (f(t), y_1\ldots y_n)$.

In view of our identification 0.6, we see that $(0,...,0) \times Y^n \sim (1,...,1) \times Y^n \sim *$ in $J_1(\Sigma Y)$. Thus, in $J_1(\Sigma Y)$, each point of $P(n) \times Y^n$ gives rise to a loop; i.e., there is a well-defined and continuous map

$$\phi_{n} : P(n) \times Y^{n} \to \Omega J_{1}(\Sigma Y) \xrightarrow{\sim} \Omega^{2} \Sigma^{2} Y .$$

As a first approximation of $\Omega^2 \Sigma^2 Y$, we could take the free associative H-space generated by the disjoint union of the $P(n) \times Y^n$, and extend the ϕ_n to a multiplicative map in the evident way. However, to do this would be to overlook at least one vital bit of additional structure in the ϕ_n .

Definition 0.7. There is a pairing

$$u_{i,j}$$
; $P(i) \times P(j) \rightarrow P(i+j)$,

defined by

$$u_{i,j}(f,g)t = \begin{cases} (f(t), 0, \dots 0), & t \leq |f| \\ (1, \dots, 1, g(t-|f|)), & t \geq |f| \end{cases}$$

where |f| is the length of the path f.

Clearly, the u_{i,j} are associative in the sense that

$$u_{i+j,k}(u_{i,j} \times 1) = u_{i,j+k}(1 \times u_{j,k})$$
.

Lemma 0.8. The map

$$(P(n) \times Y^{n}) \times (P(m) \times Y^{m}) \xrightarrow{\phi_{n} \times \phi_{m}} \Omega(J_{1}(\Sigma Y)) \times \Omega(J_{1}(\Sigma Y))$$
$$\xrightarrow{u}{\rightarrow} \Omega(J_{1}(\Sigma Y))$$

factors as the composite

$$\begin{array}{c} (P(n) \times Y^{n}) \times (P(m) \times Y^{m}) \rightarrow P(n) \times P(m) \times (Y^{n} \times Y^{m}) \\ \\ & \xrightarrow{u_{m,n} \times 1} P(n+m) \times Y^{n+m} \xrightarrow{\phi_{n+m}} \Omega(J_{1}(\Sigma Y)) \ . \end{array}$$

(The proof is obvious.)

Thus a better model for $\Omega J_1(\Sigma Y)$ would be obtained from the union of the $P(n) \times Y^n$ by making a further identification

$$0.9 (p,y) \cdot (p',y') \sim (u_{n,m}(p,p'), (y,y')) .$$

The resulting model, although better, is still too big. Recall that, in 0.6, if $y_i = *$ in Y^n , we collapse $I^n \times Y^n$ on $I^{n-1} \times Y^{n-1}$ by forgetting the ith coordinates.

Lemma 0.10. The map $\lambda_i : I^n \rightarrow I^{n-1}$, forgetting the ith coordinate, induces a map

$$P(\lambda_{i}) : P(n) \rightarrow P(n-1)$$
,

and if

$$\psi_{n}$$
: P(n) × Yⁿ + $\Omega((\Sigma Y)^{n})$

is the obvious map, $j_n: (\Sigma \Upsilon)^n \to J_1(\Sigma \Upsilon)$ the quotient map, then

 $\phi_n = j_n \psi_n$,

and the diagram

$$\begin{array}{c|c} P(n) \times Y^{n} & \stackrel{\Psi_{n}}{\longrightarrow} \Omega[(\Sigma Y)^{n}] \\ \hline P(\lambda_{i}) \times \lambda_{i} & & & & & \\ P(n-1) \times Y^{n-1} & \stackrel{\Psi_{n-1}}{\longrightarrow} \Omega[(\Sigma Y)^{n-1}] \end{array}$$

commutes.

Thus we can add another relation to our construction:

0.11
$$(p,y) \sim (P(\lambda_i)(p), \lambda_i(y))$$
 if
 $y = (y_1...y_{i-1}, *, y_{i+1}...y_n)$.

Finally, there is one more type of relation which must be taken into account. It is well-known that a second loop space has a <u>homotopy commuta-</u> <u>tive multiplication</u>. We add homotopy commutativity to our model as follows.

Definition 0.12. The symmetric group
$$\mathscr{G}_n$$
 acts on $P(n)$ by $\alpha(p)(\vec{t}) = p(\alpha \vec{t})$, where $\alpha(t_1, \dots, t_n) = (t_{\alpha^{-1}(1)}, \dots, t_{\alpha^{-1}(n)})$.

Lemma 0.13. P(n) is equivariantly contractible with respect to the \mathscr{P}_n -action (i.e., the homotopy of contraction can be chosen so that $H_t(\alpha p) = \alpha H_t(p)$, all $p \in P(n)$).

<u>Proof.</u> We start by defining a contraction of I^n by $\ell_t(t_1...t_n) = (tt_1,...,tt_n)$, and corresponding to ℓ_t , the contraction H_t is given by

$$H_{t}(f)(\tau) = \begin{cases} \ell_{t}f(\tau/t), \quad \tau \leq t | f | \\ (t+\tau-t | f |, \dots, t+\tau-t | f |) & \text{otherwise.} \end{cases}$$



Similarly, we have

Lemma 0.14. The following diagram commutes

where $s_{n,m} \in \mathscr{S}_{n+m}$ is the shuffle of the first n with the last m-coordinates, and T is the interchange.

In particular, this implies the additional identification

0.15
$$(s_{n,m}(u_{n,m}(p,p')), (y,y')) \sim (u_{m,n}(p',p), y', y)$$
.

But $(u_{m,n}(p',p), y', y)$ is equivalent by 0.9 to the product (p',y')(p,y), and since P(n) is connected, $(s_{n,m}(u_{n,m}(p,p')),y,y')$ is homotopic to $(u_{n,m}(p,p'), (y,y')) \sim (p,y) \cdot (p',y')$.

<u>Theorem</u> 0.16. Let $K_2(Y) = \bigcup_n P(n) \times Y^n$ modulo the relations 0.9, 0.11, and 0.15. Then $K_2(Y)$ is an associative H-space with unit, and the natural map $K_2(Y) \rightarrow \Omega J_1(\Sigma Y)$ is a homotopy equivalence if Y is a connected CW-complex.

<u>Proof</u>. Consider the type 0.9 and 0.15 relations on $P(n) \times Y^n$. They imply that the only type of (P(n))-relations occur over the points $\alpha(P(r) \times P(n-r))$, where α runs over all (r,n-r) shuffles. We call these the n-2 "faces" of P(n). This nomenclature is justified by

ii) Let a generator P_{n-1} of $H_{n-1}(P(n), F)$ be given. Then the evaluation map

$$e : (I,\partial) \times (P(n),F) \rightarrow (I^{n}, \partial I^{n})$$

has degree ±1 ; i.e.,

$$e_{*}(e_{1} \otimes p_{n-1}) = \pm e_{n}$$
,

where e_i is the orientation class of (I^i, ∂) . (The proof is by induction.) In J. F. Adams' paper, "On the cobar construction," Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 409-412, it is shown that there is a spectral sequence defined for any space X, and converging to $H_*(\Omega X, Z_p)$. Its E^2 -term has the form $\operatorname{Cotor}_{H_*(X,Z_p)}(Z_p,Z_p)$. A similar spectral sequence can be defined for $K_2(Y)$, and the natural map $K_2(Y) + \Omega(J_1(\Sigma Y))$ induces a map of spectral sequences. From 0.17, it is then an easy calculation to check that, at the E^2 -level, the spectral sequence map is actually an isomorphism. Then if Y is a finite complex, the comparison theorem shows the natural map induces isomorphisms in homology for all coefficients Z_p , and it is known that $\Omega^2 \Sigma^2 Y$ has the homotopy type of a locally finite CW-complex. Hence by the Whitehead theorem (for connected H-spaces), the natural map is a homotopy equivalence. Now, since Y is the limit of its finite subcomplexes, the result follows for general Y.

Notice the role of the "complex" of faces $\alpha(P(r) \times P(k-r))$, $\alpha((\beta P(s) \times P(r-s)) \times P(n-r))$, etc., in the proof of 0.16. Through 0.17, they are the essential things in making the proof work. The complex for P(2) is simply that of an interval; that for P(3) has the form of a hexagon



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As these examples indicate, the complexes above are obtained from cellularly decomposing the boundary of a disc D^{n-1} , and we may simply replace P(n) by D^{n-1} , and make the type 0.9 and 0.15 identifications over the corresponding faces to give a much smaller model for $\Omega^2 \Sigma^2 X$.

The explicit construction follows. Recall first that the faces of the convex hull of a finite point set S in Euclidean space are convex hulls of certain subsets of S.

Definition 0.18. Let C(n) be the convex hull of the translates of $(1,2,3,\ldots,n+1) \in \mathbb{R}^{n+1}$ under the action of \mathscr{P}_{n+1} .

C(n) is easily seen to be the hexagon for n = 2, the figure



whose faces consist of eight hexagons and six squares $C(1) \times C(1)$ for n = 3, and in general C(n) has faces of the form $\alpha(C(r) \times C(n-r-1))$, r = 0, ..., n-1, as α runs over all (r+1, n-r) shuffles. Indeed, let $S' \subset S$ be the orbit of (1, 2, ..., n+1) under the action of $\mathscr{P}_{r+1} \times \mathscr{P}_{n-r}$. Then the convex hull of S' is naturally isomorphic to $C(r) \times C(n-r-1)$. Similarly, $\alpha C(r) \times C(n-r-1)$ is the convex hull of $\alpha(S')$. (See Lemma 4.2, p. 391 of [18] for details.) Let $I^r : C(r) \times C(n-r-1) \rightarrow C(n)$ be the identification above. Note that C(n) is invariant under the action of \mathscr{S}_{n+1} which takes faces to faces. Also, guided by the need for degeneracies in 0.10, 0.11, we can define degeneracies

$$D_i : C(n) \neq C(n-1), i = 1, ..., n+1$$

(as in Lemma 4.5, p. 392 of [18]). Specifically, we have

Lemma 0.19 ([18]). There are maps

$$D_k : C(n) \rightarrow C(n-1), \qquad 1 \le k \le n+1,$$

so that

i)
$$D_1 I^U$$
 is the projection on the second factor;
ii) $D_j I^k = \begin{cases} I^{k-1}(D_j \times id) , & j \leq k \\ I^k (id \times D_{j-k}) & \underline{otherwise}; \end{cases}$
iii) $d_j(\beta)D_{\beta^{-1}(j)} = D_j\beta , & \beta \in n-1;$
iv) $D_i D_j = D_{j-1}D_i & \underline{for} \quad j \geq i.$

(Here d_j : $\mathscr{P}_{n+1} \to \mathscr{P}_n$ is the correspondence which makes the diagram

$$\overset{\lambda}{\beta^{-1}(j)}_{\mathbb{R}^{n}} \overset{R^{n+1}}{\underset{d_{j}(\beta)}{\longrightarrow}} \overset{\beta}{\mathbb{R}^{n}} \overset{R^{n+1}}{\underset{d_{j}(\beta)}{\longrightarrow}} \overset{\lambda_{j}}{\mathbb{R}^{n}}$$

commute.)

Indeed, (i)-(iii) specify D_j on faces, and the map D_j is then defined by extending linearly along rays to the respective centers. The reader is advised to check that (i)-(iv) are forced on us by 0.10. As the remarks preceding 0.18 indicate, the C(n-1) serve as replacements for the P(n), and we have

Corollary 0.20.
$$\Omega^2 \Sigma^2 X \stackrel{\sim}{=} \bigcup_{n=0}^{\infty} C(n) \times X^{n+1}$$

modulo the relations

$$(c, x_1...x_{n+1}) \sim (\alpha^{-1}(c), x_{\alpha^{-1}(1)}, ..., x_{\alpha^{-1}(n+1)})$$

{c,
$$x_1 \dots x_{n+1}$$
} {c', $x_{n+1} \dots x_{n+m+2}$ } = {Iⁿ(c,c'), $x_1 \dots x_{n+m+2}$ },
and with this product, $J_2(x)$ is H-equivalent to $\Omega^2 \Sigma^2 X$.

This provides us with a <u>minimal</u> geometric model for $\Omega^2 \Sigma^2 X$. Basically, the model makes it clear that the fundamental data which go into the statement that a space is a 2-fold loop space are an associative unitary multiplication, together with a series of higher homotopies of commutation.

We now indicate how to extend this construction to obtain models for the higher loop spaces $\,\Omega^n\Sigma^nX$, $\,n\,>\,2$.

Consider, for example, the case n = 3. We have already approximated $\Omega^2 \Sigma^2(\Sigma X)$ as $J_2(\Sigma X) = \cup C(n) \times (\Sigma X)^{n+1}/R = \cup C(n) \times I^{n+1} \times X^{n+1}/R' = \cup I^{n+1} \times C(n) \times X^{n+1}/R''$. Once again we can use the P(n+1) to construct loops by taking elements of P(n+1) × C(n) × X^{n+1} and defining

$$\phi(p,c,x_1...x_{n+1})t = \{p(t),c,x_1...x_{n+1}\}$$
.

By 0.20(i), these paths become loops. Moreover, the obvious analogues of 0.8, 0.10, 0.14 continue to hold. Thus we obtain a model for $\Omega^3 \Sigma^3 X$ as follows.

$$(c,c',x_1...x_{n+1}) \sim (\alpha^{-1}c,\alpha^{-1}c',\alpha(x_1...x_{n+1}));$$

iii) if
$$c \in \alpha I^{k}(C(k) \times C(n-k-1))$$
, then

$$(c,c',x_1,\ldots,x_{n+1}) \sim (\alpha^{-1}c,D^{\alpha}(c'),\alpha(x_1,\ldots,x_{n+1}))$$

Here $D^{\alpha}(c') = [(D_{k+2})^{n-k} \times (D_1)^{k+1}\alpha^{-1}](c')$ is the obvious degeneracy used for paths on $(\Sigma X)^{n+1}$, which are at * on the first k+1 coordinates half the time, and * on the last n-k coordinates the remainder of the time.

To complete the definition, we remark that, after using 0.2(i)-(iii), if the point is equivalent to one of the form $((c_1,c_2), (c_1',c_2')x_1...x_{n+1}x_{n+2}...x_{n+m+2})$, that is, to a product, then we again apply (i)-(iii) separately to $(c_1,c_1'x_1...x_{n+1})$, $(c_2c_2'x_{n+2}...x_{n+m+2})$ and take the product of the results.

It is now possible to prove an analogue of 0.16 for this model, and we have

<u>Theorem</u> 0.22. If X is a connected CW-complex, then $J_3(X)$ has the homotopy type of $\Omega^3 \Sigma^3 X$ as an associative H-space.

It is now clear how to generalize the construction. We obtain

$$(c_1...c_{m-1}, x_1...x_{n+1}) \sim (\alpha^{-1}c_1, ..., \alpha^{-1}c_{j-1}, D^{\alpha}(c_j)...D^{\alpha}(c_{m-1}), \alpha(x, ..., x_{n+1}));$$

iii) the same convention on points equivalent to products as given in 0.21.

Once again we can prove

Theorem 0.24 (Theorem 5.2 of [18], p. 395). Let X be a connected CW-complex. Then there is an H-map

$$j_k : J_k(X) \rightarrow \Omega^k \Sigma^k(X)$$
,

which is a homotopy equivalence.

<u>Remark</u> 0.25. There is a filtration on the points of $J_k(X)$, given by saying y has filtration m if it is in the equivalence class of a point in $(C(m-1)^{k-1} \times X^m)$ under the relations of 0.23. In particular, if * is a vertex of X, and X is (l-1)-connected, then the set of points having filtration $\leq s$, $J_k(X)^{(s)}$, is a subcomplex of $J_k(X)$ and contains all the cells of $J_k(X)$ of dimension < (s+1)l (provided X has no cells of dimension < l except *, which we can assume). Thus if we wish to consider problems dealing with dimensions $\leq (s+1)l-2$, we can replace $J_k(X)$ by $J_k(X)^{(s)}$. The reader is advised to work out the explicit structure of $J_k(X)^{(2)}$ and verify the description given in the proof of 1.11. §1. The Inclusion $X \to \Omega^n \Sigma^n X$

Recall from \$0 that

We convert j into a fibering in the usual way. Thus we first replace $\Omega^n \Sigma^n X$ by the mapping cylinder M(j), and j by the inclusion $X \hookrightarrow M(j)$. Then F_n , the fiber of j, is defined to be the space of paths of unit length $E_X^{M(j)}$ starting in X and ending at *, the base point in $\Omega^n \Sigma^n(X)$. By the result of Milnor ([25]), if X is a CW complex, then F_n is the homotopy type of one also.

In [18], it was

where that $H_*(\Omega^n \Sigma^n(X), Z_p)$ is an explicit functor of $H_*(X, Z_p)$ alone, and the inclusion j_* is injective in homology. Thus, using the Serre spectral sequence of the fibering, it is easily argued that $H_*(F_n)$ <u>depends only</u> on $H_*(X)$ and n <u>in dimensions less than</u> 3m-1 if X is m-1-connected.

Here is an alternate description of F_n . Let G , H be associative H-spaces with units, and f : G \rightarrow H an inclusion which is also a homomorphism. f induces an inclusion of classifying spaces ([19], [32])

$$B_{f} : B_{G} \rightarrow B_{H}$$
,

and we have

<u>Lemma 1.1</u> Let $E_H \xrightarrow{\rho} B_H$ be the universal quasi-fibering ([7], [19], [32]). <u>Then the fiber of</u> B_f is E_H restricted to B_G ; i.e., $\rho^{-1}(B_G)$.

The proof is direct.

In particular, if $G = \Omega(X)$ and $H = \Omega(M(J))$, this provides an explicit and fairly manageable description of F_n . If $X = \Sigma Y$, the classifying space constructions given above can be considerably improved. Indeed, using the constructions introduced in [18], the inclusion $\Omega\Sigma Y \hookrightarrow \Omega^{n+1}\Sigma^{n+1}Y$ is H-equivalent to the inclusion

$$\lambda : J_1(Y) \hookrightarrow J_{n+1}(Y)$$
.

In §2 of [18], an alternate classifying space construction is given: Let X be a (free¹) associative H-space with unit * and homomorphism h : $X \to R^+$ so $h^{-1}(0) = *$; then E_{χ} is defined as $X \times R^+ \times X$ mod the relations

$$(x,t,yz) \sim (xy,t-h(y),z)$$
,
 $(x,0,y) \sim (x,t,*)$.

 ${\rm B}_{\rm X}$ is then defined as $~*\times_{\rm X}{\rm E}_{\rm X}$.

Now, if there is a commutative diagram

(1.2)
$$\begin{array}{c} x \stackrel{g}{\rightarrow} z \\ h \bigvee_{R^+} h' \end{array}$$

with g a homomorphism, then there is an induced map $B_g : B_X \to B_Z$. Moreover, if g is an inclusion, then B_g is an inclusion and $Z \times_X E_X$ is the

 $^{^{\}rm l}$ This is the geometric analogue of unique factorization in algebra.

restriction of E_Z to B_X . Thus the fiber in the map B_g is $Z \times_X E_X$. In particular, 395-396 of [18] shows that (1.2) is true for $X = J_1(Y)$ and $Y = J_{n+1}(Z)$, using the h_1 , h_{n+1} constructed there. Passing to fiberings, we have proved

Lemma 1.3 The fiber F_n in the natural map $\Sigma Y \to \Omega^n \Sigma^{n+1} Y$ is $J_{n+1}(Y) \times_{J_1}(Y) E_{J_1}(X) \cdot$

This space admits a simple description as a CW complex.

There is an algebraic functor ([18], §7) which defines for any chain complex A an associated chain complex $F^n s^n(A)$. It gives $C_{\#}(J_n(Y))$ when applied to $C_{\#}(Y)$, and we have ([18], Theorem 7.2)

<u>Theorem 1.5 Let A, A' be chain complexes over</u> Z_p (for p a prime), and <u>suppose</u> $f : A \to A'$ is an augmentation-preserving chain map inducing isomorphisms in homology. Then $F^n s^n(f) : F^n s^n(A) \to F^n s^n(A')$ also induces isomorin homology.

There are inclusions $\gamma_{i,j} : F^{i}s^{i}(A) \hookrightarrow F^{i+j}s^{i+j}(A)$ which, applied to $C_{\#}(Y)$, are induced from the inclusion $J_{i}(Y) \hookrightarrow J_{i+j}(Y)$, and we have

<u>Corollary</u> 1.6 Let $X = \Sigma Y$ for $Y \ge connected CW$ complex. Then $H_*(F_n, Z_p)$ <u>depends only on</u> $H_*(X, Z_p)$. (Precisely, there are functors $\mathfrak{F}_p(n)$ for each p from the category of graded Abelian groups to graded Abelian groups, and $\mathfrak{F}_p(n)(H_*(\Sigma Y)) \cong H_*(F_n)$.) <u>Proof</u> There is an injection $\gamma : H_*(Y, Z_p) \to C_{\#}(Y) \otimes Z_p$ inducing isomorphisms in homology. Hence we can form the algebraic object corresponding to the complex in 1.3, $G = F^{n+1}s^{n+1}(H_*(Y, Z_p) \otimes \underline{c}(H_*(Y, Z_p)))$ with boundary as in 1.3. γ extends to a chain map $\tilde{\gamma} : G \to C_{\#}(J_{n+1}(Y)) \otimes C_{\#}(\underline{c}Y)$. Now, filtering both sides by the dimension in $\underline{c}(Y)$, we obtain an algebraic Leray-Serre spectral sequence with $E^2 = H_*(J_{n+1}(Y)) \otimes H_*(\Sigma Y)$ in both cases. Moreover, $E_2(\tilde{\gamma})$ is evidently an isomorphism of E^2 terms. 1.6 now follows from the comparison theorem.

<u>Remark</u> 1.7 Suppose we consider the inclusion $F_n \rightarrow \Omega^L \Sigma^L F_n$, and study its fiber $F_{n,L}$. If $X = \Sigma^2 Y$, then the model 1.3 for F_n admits a natural description in terms of spaces $C_1^{I} \times (\Sigma Y)^{i}$ where C_1^{I} is a cell, and identifications are made over ∂C_1^{I} or when a coordinate in $(\Sigma Y)^{i}$ is *. There is a natural way to construct loops in these sets; namely, by using the suspension coordinates in $(\Sigma Y)^{i}$. Thus $(\Sigma Y)^{i}$ is an identification space of $I^{i} \times Y^{i}$, and one constructs paths in I^{i} starting at $(0 \cdots 0)$ and ending at $(1, \dots, 1)$. A model for a sufficient number of paths is the Zilchgon C(i-1), introduced in §4 of [18] Replacing I^{i} by C(i-1) in each cell above, performing the appropriate identifications, and forming a universal construction, we obtain natural, minimal, and canonical models for ΩF_n , \dots , $\Omega^{L+1} \Sigma^L F_n$, \dots .

These models also satisfy the property that the inclusions $\Omega^{L}\Sigma^{L-1}F_{n} \hookrightarrow \Omega^{L+1}\Sigma^{L}F_{n}$... are homomorphisms. Now we may apply 1.3 - 1.6 to show that $H_{*}(F_{n,L},Z_{p})$ depends only on $H_{*}(Y,Z_{p})$. Of course, if $X = \Sigma^{3}Y$, we can iterate once more. In general, we have

 $\begin{array}{c|c} \underline{\text{Theorem 1.8}} & \underline{\text{Define}} & F_{n_1} \cdots n_k^{(X)} & \underline{\text{inductively as the fiber in the map}} \\ & & F_{n_1} \cdots n_{k-1}^{n_k} (X) \hookrightarrow \underset{k=1}{\overset{n_k}{\Omega}} \overset{n_k}{\overset{n_k}{\Sigma}} \overset{n_k}{\overset{n_k}{\overset{n_k}{\Gamma}}} n_1 \cdots n_{k-1}^{(X)} (X) \end{array}$

We will have no further need of 1.7 and 1.8 except incidentally in the sequel; it is for this reason that the details are so skimpily sketched.

We now turn to the more limited observations which we can make about ${\rm F}_{\rm L}$ when X is not a suspension.

Lemma 1.9 (Fiber Lemma) Let X, Y be n-1-connected and locally finite CW complexes (n > 2). Suppose $f: X \to Y$ satisfies $f_*: H_t(X,Z_p) \to H_t(Y,Z_p)$ is an isomorphism for t < 2n and a monomorphism for t < 3n. Convert f into a fibering with fiber F. Then through dimensions 3n-2, F is mod p weakly homotopy equivalent to $\Omega(Y/X)$.

<u>Proof</u> F is 2-connected by our hypothesis. Thus, letting C be the class of finite groups having order prime to p, it is enough to show that there is a map g: $F \to \Omega(Y/X)$ so g_* induces a (mod C)-isomorphism $H_*(F) \to$ $H_*(\Omega(Y/X))$ in dimensions less than 3n-1. But our first description gave F as $F = \frac{F}{X} \frac{Y}{*}$, and g is defined as the evident projection

$$E_{X}^{Y} \star \to E_{\star}^{Y/X} \star = \Omega(Y/X) \quad .$$

Note that F and $\Omega(Y/X)$ are both 2n-2-connected. Thus the Serre spectral sequences for the fiberings

$$F \rightarrow X \rightarrow Y$$
,
 $\Omega(Y/X) \rightarrow P \rightarrow Y/X$

are both exact sequences in dimensions $\leq 3n-2$. Moreover, both sequences split, and the fact that g_{χ} is an isomorphism in this range follows.

<u>Remark</u> 1.10 1.9 can obviously be strengthened to give an actual homotopy equivalence in this range if f_* satisfies the hypothesis of 1.9 with the integers as coefficients.

In particular, we can now prove

<u>Theorem 1.11</u> Let X be a locally finite n-1-connected CW complex (n > 1). Then through dimension 3n-2, the fiber \mathbb{F}_n in the inclusion

 $\mathbf{x} \hookrightarrow \mathbf{\Omega}^{\mathbf{n}} \mathbf{\Sigma}^{\mathbf{n}}(\mathbf{x})$

is the space $\Omega(S^{n-1} \ltimes_T X \wedge X)$. (Here $S^{n-1} \ltimes_T X \wedge X$ is given as a quotient space of $S^{n-1} \times (X \wedge X)$ where (x,y,z) is identified with (-x,y,z), and (x,*) is set equal to *.)

<u>Proof</u> From [18], pp. 394-395) we know $\Omega^{n}\Sigma^{n}X \cong X \cup (I^{n-1} \times X \times X)/R$ through dimension 3n-1 where R is a set of relations defined as follows:

1) $(t_1 \dots \varepsilon \dots t_{n-1}, x, y) \sim (T^{\varepsilon}(t_1) \dots T^{\varepsilon}(t_{j-1}), 0 \dots 0)T^{\varepsilon}(x, y))$ where $\int_{j}^{t_h} t_{position}$

T(t) = 1-t and T(x,y) = (y,x) if $\varepsilon = 0$ or 1.

2) $(t_1...t_{n-1},*,y) \sim (t_1...t_{n-1},y,*) \sim y$ for *, the base point of X. Thus, through dimension $\exists n-2$, $\Omega^n \Sigma^n(X)/X \simeq (I^{n-1} \times X \wedge X)/R'$ where R' consists of relations of type (1), and $(t_1...t_{n-1},*) \sim *$. Now, 1.11 follows from

<u>Lemma</u> 1.12 $(I^{n-1} \times X \wedge X)/R' \cong S^{n-1} \ltimes_T X \wedge X$.

<u>Proof</u> Embed $I^{j} \hookrightarrow I^{j+1}$ as the set of points $(\frac{1}{2}, t_{1} \dots t_{j})$. This induces an embedding $(I^{j} \times X \wedge X)/R' \hookrightarrow (I^{j+1} \times X \wedge X)/R$. $(I^{j+1} \times X \wedge X)/R'$ can be given as the equivalence classes of points of the form $(t_{1} \dots t_{j+1}, x, y)$ with $t_1 \leq \frac{1}{2}$, with equivalence relations R' in $(I^j \times X \wedge X)$ together with $(0, t_2 \dots t_{j+1}, x, y) \sim (0, x, y)$, $(t_1, t_2, \dots, t_{j+1}, *) = *$. But $S^j \ltimes_T X \wedge X$ has an identical description in terms of the equitorial embedding $S^j \ltimes_T X \wedge X \rightarrow S^{j+1} \ltimes_T X \wedge X$. l.ll follows by induction.

<u>Theorem</u> 1.13 <u>The equitorial inclusion</u> $S^{n-2} \hookrightarrow S^{n-1}$ <u>induces an inclusion</u> $\gamma_n : S^{n-1} \ltimes_T X \land X \to S^{n-1} \ltimes_T X \land X$, and the diagram

$$\Omega(s^{n-2} \ltimes_{T}^{F_{n-1}} \times x \wedge x) \xrightarrow{\frac{k_{n}}{\gamma_{n}}} \Omega(s^{n-1} \ltimes_{T}^{F_{n}} \times x \wedge x)$$

<u>homotopy commutes</u>. Here k_n is the map of fibers induced from the inclusion $h_{n-1} : \Omega^{n-1}\Sigma^{n-1}X \hookrightarrow \Omega^n\Sigma^nX$.

Proof This follows from the proof of 1.11 when we note that the inclusion

$$(X \cup I^{n-2} \times X \times X)/R \hookrightarrow (X \cup I^{n-1} \times X \times X)/R$$

defined on points by

$$(t_1,\ldots,t_{n-2},x,y)\mapsto (t_1,\ldots,t_{n-2},0,x,y)$$

induces a map H-homotopic to the map h_{n-1} (eg. see [18], §5).

<u>Remark</u> 1.14 The "little cubes" category of Boardman and Voit ([37]) provides an easy way of including the space $X \cup (S^{L-1} \ltimes_T X \times X)/R$ in $\Omega^L \Sigma^L X$, where R is the relation

$$(x,y,*) \sim (x,*,y) \sim y$$
.

Specifically, the space $C^2(L)$ is defined as the set of all disjoint
embeddings

$$I_1^{L} \cup I_2^{L} \rightarrow I_{(3)}^{L}$$

which are linear and take faces into sets parallel to the corresponding faces in $\mbox{ I}_{\{\mbox{ 3}\}}$.



Z₂ acts (freely!) on C²(L) by interchanging I₁ and I₂. An arbitrary point x of $\Omega^{L}Y$ can be regarded as a map $f_{x}(I^{L},\partial I^{L}) \rightarrow$ (Y,*). Given two maps f_{x} , f_{y} and a point z in C²(L), there is a map (I^L, ∂) → (Y,*) defined as



This provides a pairing $C^2(L) \ltimes_T \Omega^L(Y) \times \Omega(Y) \to \Omega^L(Y)$.

Lemma 1.14 (P. May) $C^2(L)$ is equivariantly homotopic to S^{L-1} with the antipodal action.

(Indeed, the equivariant inclusion $S^{L-1} \to C^2(L)$ takes x to the embedding of a cube of length $\frac{1}{4}$ with center at the point x , and the

second with center at -x. It is now an easy geometric argument to prove 1.14.)

Now, using the pairing above in $\Omega^{L}\Sigma^{L}X$ and restricting to X , the desired inclusion of X \cup (S^{L-1} $\ltimes_{T} X \times X$) in $\Omega^{L}\Sigma^{L}X$ is readily obtained.

§2. The Map
$$F_n(X) \rightarrow \Omega F_{n-1}(\Sigma X)$$

 $\underline{\text{Lemma}} \text{ 2.1 } \Sigma J_{1}(X) = \Sigma X \vee \Sigma(X \wedge X) \vee \Sigma(X \wedge X \wedge X) \vee \dots$

Proof This follows from the well-known ([38]) splittings

$$\Sigma X^{n} = \Sigma (X \land \dots \land X) \lor (n) \Sigma X \land \dots \land X, \lor \dots \lor (\binom{n}{j}) \Sigma \underbrace{X \land \dots \land X}_{n-j} \lor \dots ,$$

writing $\Sigma J_1(X)$ as an identification space of $\bigcup \Sigma(X^n)$. n <u>Corollary</u> 2.2 <u>There are H-maps</u> $H_r : J_1(X) \to J_1(X \land \dots \land X)$ so r times

 $(\otimes \operatorname{H}_{r})_{*} \operatorname{H}_{*} \operatorname{J}(\operatorname{X})) \to \operatorname{H}_{*}(\otimes \operatorname{J}_{1} \Sigma(\operatorname{X} \wedge \ \dots \ \wedge \ \operatorname{X})) \quad \underline{\operatorname{is}} \text{ injective}.$

H_r is called the rth Hopf-invariant map of X. Clearly, if an element α in $\pi_*(\Omega\Sigma X)$ comes from $\pi_*(X)$, then $H_{r*}(\alpha) = 0$ for all r.

Presumably there is a similar splitting for $\Sigma^n J_n(X)$. Thus it seems reasonable, in particular, to conjecture a splitting

$$\Sigma^{2} J_{2}(X) = \Sigma^{2} X \vee \Sigma^{2} S^{1} \ltimes_{T} X \wedge X \vee \Sigma^{2} C(2) \ltimes_{R} (X \wedge X \wedge X) \vee \cdots$$

Recent results of D. S. Kahn have shown that these splittings exist for $Q(X) = \lim_{n \to \infty} \alpha^n \Sigma^n X$; however, at present the splitting theorem for the J_n has not been proved.

Thus, we adopt an alternate "Hopf invariant" for the purposes of this section. The Hopf invariant of a class $\alpha \in \pi_m(\Sigma X)$ is defined by taking $\sigma^{-1}(\alpha) \in \pi_{m-1}(\Omega \Sigma X) = \pi_{m-1}(J_1(X))$, and projecting onto $\pi_{m-1}(J(X)/X)$. We denote its image by $H(\alpha)$, and, in the metastable range, the results of §1 show that $H(\alpha) = 0$ is both necessary and sufficient in order that α be in the image of σ .

In the metastable range, $H(\alpha) = H_2(\alpha)$ defined above. Now we consider the problem of when an element $\beta \in \pi_m(\Sigma^L X)$ desuspends L-l times but not L times.

Consider the diagram

$$\begin{split} \mathbf{F}_{\mathbf{L}}(\mathbf{X}) &\to \mathbf{X} \to \boldsymbol{\Omega}^{\mathbf{L}}\boldsymbol{\Sigma}^{\mathbf{L}}(\mathbf{X}) \\ & \downarrow^{\tau} \mathbf{t} & \downarrow^{\pi} \mathbf{t} & \downarrow^{\text{id}} \\ \boldsymbol{\Omega}^{\mathsf{t}} \mathbf{F}_{\mathbf{L}-\mathbf{t}}(\boldsymbol{\Sigma}^{\mathsf{t}} \mathbf{X}) \to \boldsymbol{\Omega}^{\mathsf{t}} \boldsymbol{\Sigma}^{\mathsf{t}} \mathbf{X} \to \boldsymbol{\Omega}^{\mathsf{L}} \boldsymbol{\Sigma}^{\mathsf{L}} \mathbf{X} \end{split}$$

Clearly, β does not desuspend L times if and only if $\partial \beta \neq 0$ in $\pi_*(F_L(X))$. On the other hand, it desuspends L-t times if and only if $\tau_{t*}(\partial \beta) = 0$. Thus the key step (outside of analyzing ∂ , which we defer for the moment) is to study the map τ_t . We can reduce this to the study of τ_1 since τ_t can clearly be decomposed as

$$\mathbf{F}_{\mathbf{L}}(\mathbf{X}) \xrightarrow{\tau_{1}} \Omega \mathbf{F}_{\mathbf{L}-1}(\Sigma \mathbf{X}) \xrightarrow{\Omega \tau_{1}} \Omega^{2} \mathbf{F}_{\mathbf{L}-2}(\Sigma^{2} \mathbf{X}) \rightarrow \dots \xrightarrow{\Omega^{t-1}(\tau_{1})} \Omega^{t} \mathbf{F}_{\mathbf{L}-t}(\Sigma^{t}(\mathbf{X})) \quad .$$

We have

<u>Theorem</u> 2.2 Let $f: \Sigma[S^{n-2} \ltimes_T (X \land X)] \to S^{n-2} \ltimes_T (\Sigma X \land \Sigma X)$ be given on points by

$$f(t, (x,y,z)) = (x (t,y), (t,z))$$
.

Then the diagram

$$S^{n-2} \ltimes_{T} (X \land X) \xrightarrow{\text{adj}(f)} S^{n-2} \ltimes_{T} (\Sigma X \land \Sigma X)$$

$$F_{n}(X) \xrightarrow{\tau_{1}} S^{n-2} (\Sigma X)$$

homotopy commutes in the metastable range.

<u>Proof</u> The map $\Omega(\Sigma X) \to \Omega[\Omega^{n-1}\Sigma^{n-1}(\Sigma X)]$ is given by the inclusion $J_1(X) \hookrightarrow J_n(X)$ ([18]). This inclusion satisfies the conditions of 1.9, so the diagram

$$\begin{array}{ccc} X \rightarrow J_{n}(X) \\ \downarrow \\ J_{1}(X) \rightarrow J_{n}(X) \end{array}$$

induces the inclusion of cofibers in dimensions less than 3n-1,

$$s^{n-1} \ltimes_{_{\mathrm{T}}} \mathtt{X} \wedge \mathtt{X} \to (s^{n-1} \ltimes_{_{\mathrm{T}}} \mathtt{X} \wedge \mathtt{X}) \big/ \mathtt{X} \wedge \mathtt{X} \ .$$

In the proof of Theorem 5 of [18], a map (adjoint to the identity) $\Sigma J_n(X) \rightarrow J_{n-1}(\Sigma X)$ is constructed. Precisely, there is a map $\phi : I \times I \rightarrow I \times I$ defined by

$$\varphi(t,\tau) = \begin{cases} (2t,0), t < \frac{1}{2} - \tau \\ (1-2\tau, 2(t+\tau-\frac{1}{2})), \frac{1}{2} - \tau \le t \le 1-2\tau \\ (t,t), 1-2\tau \le 2t \end{cases}$$

for $\tau < \frac{1}{2}$, and $\phi(t\,,\frac{1}{2}\!+\!\tau)$ = $\mathbb{T}[\phi(t,\tau)]$. This then defines a map

$$r : \Sigma[(X \cup I^{n-1} \times X^2)/R] \rightarrow (\Sigma X \cup I^{n-2} \times \Sigma X \times \Sigma X)/R$$

by $r(t,t_1,\ldots,t_{n-1},x,y) = \{t_2,\ldots,t_{n-1}(\phi(t,t_1)x,y\} \text{ (identifying } I^2 \times X^2 \text{ with } \Sigma X \times \Sigma X \text{). In particular, } r(t,\varepsilon_1,t_2,\ldots,t_n,x,y) \to \Sigma X \text{ . Thus, factoring } X \text{ to a point, the induced map factors through}$

$$\Sigma[s^{n-1} \ltimes_T (x \land x) / x \land x]$$

Finally, note that, for $t_1 = \frac{1}{2}$,

$$\varphi(t, \frac{1}{2}, t_2, \dots, t_{n-1}, x, y) = \{t_2, \dots, t_{n-1}(t, x, t, y)\}$$

Thus 2.2 follows.

Actually, we have proved more than 2.2

<u>Corollary</u> 2.3 $\Sigma[S^{n-1} \ltimes_T (X \land X)/X \land X]$ <u>is homotopy-equivalent to</u> $S^{n-2} \ltimes_T \Sigma X \land \Sigma X$ for X <u>a connected</u> CW <u>complex</u>.

<u>Proof</u> Note that, using the cell decomposition of these spaces by the $[I^{j} \times X_{r} \times X_{r}]$, ϕ constructed above is cellular and induces an isomorphism of cellular chain complexes.

§3. The Cohomology of the F_n

We start by examining the cohomology of the $S^{n-1} \ltimes_T X \wedge X$. After that we give the structure of $\#^*(F_n(\Sigma X))$.

There are maps

$$\begin{split} J \; : \; X \; \wedge \; X \; \to \; S^n \; \kappa_{_{\rm T}} \; X \; \wedge \; X \; , \\ {\rm K} \; : \; S^n \; \kappa_{_{\rm T}} \; X \; \wedge \; X \; \to \; \Sigma^n \! X \; \wedge \; X \; . \end{split}$$

Proposition 3.1 (a) J^* is surjective onto the invariant subalgebra under (T)^{*} of $H^*(X \wedge X, Z_p)$ for p an odd prime. Moreover, kernel $J^* = im(K^*)$, and the following sequence is exact:

 $(3.2) \qquad \operatorname{H}^{*}(\Sigma^{n}X \wedge X, Z_{p}) \xrightarrow{1+(-1)^{n}(\Sigma^{n}T)^{*}} \operatorname{H}^{*}(\Sigma^{n}X \wedge X, Z_{p}) \rightarrow \operatorname{im} (K^{*}) \rightarrow 0 .$

(b) Mod (2) J^* is surjective as in (a), 3.2 is again exact, but there are additional elements $e^i \cup (\theta \otimes \theta)$ for $1 \le i \le n$ where $\theta \in \widetilde{H}^*(X, Z_2)$, and these completely describe $H^*(\Gamma^n(X), Z_2)$.

<u>Proof</u> Consider the filtration of $\Gamma^{n}(X)$, $X \wedge X \hookrightarrow \Gamma^{2}(X) \hookrightarrow \ldots \hookrightarrow \Gamma^{n}(X)$, obtained by embedding successive spheres equitorially. The resulting quotient spaces are the $\Sigma^{i}[X \wedge X]$. Moreover, the d_{1} differential on $H^{*}(\Sigma^{i}X \wedge X, Z_{p})$ is exactly $[1+(-1)^{i}(\Sigma^{i}T)^{*}]$.

<u>Lemma</u> 3.3 <u>A chain complex</u> for $s^n \ltimes_{\pi} X \land X$ is obtained as

where C is any chain complex homotopy equivalent to $C_{\#}(X)$, and W is any free resolution of Z_2 (e.g. see [18], [27]). (This is immediate from the geometry.)

Now, to show $E_{p} = E_{\infty}$ in our special sequence, note, for example,

 $\partial e_i \otimes (x \otimes x) = [(1+(-1)^i T)e_{i-1}] \otimes x \otimes x$

if x is a cycle in \widetilde{C} . But $(Te_{i-1}) \otimes x \otimes x = (-1)^{\dim x} e_{i-1} \otimes (x \otimes x)$ due to the action of T in $S^n \ltimes X \land X$. Hence

$$\partial e_{i} \otimes x \otimes x = \begin{cases} 0, & i \neq \dim (x)(2) \\ 2e_{i-1} \otimes x \otimes x, & i \equiv \dim x(2) \end{cases}$$

in $W \otimes_{\underline{r}} \widetilde{C \otimes C}$. Thus cycles in \underline{E}_2 are represented by cycles in $W \otimes_{\underline{r}} \widetilde{C \otimes C}$, and 3.1 follows.

It is also fairly easy to verify that cup products (mod 2) are given by the formulae

$$[e^{i} \cup (\theta \otimes \theta)] \cup [e^{j} \cup (\tau \otimes \tau)] = e^{i+j} \cup (\theta\tau \otimes \theta\tau) ,$$

$$(3.4)$$

$$[e^{i} \cup (\theta \otimes \theta)] \cup \langle a,b \rangle = \begin{cases} 0 , i > 0 , \\ \langle a\theta, b\theta \rangle , i = 0 . \end{cases}$$

Here, $\langle a,b \rangle$ is an appropriate choice of generator, so $J^*\langle a,b \rangle = a \otimes b + b \otimes a$, and $e^{\circ} \cup \theta \otimes \theta$ is an element for which $J^*(e^{\circ} \cup \theta \otimes \theta) = \theta \otimes \theta$. Indeed, 3.4 follows directly from:

<u>Lemma</u> 3.5 \triangle : $\Gamma^{\mathbf{n}}(\mathbf{X}) \to \Gamma^{\mathbf{n}}(\mathbf{X}) \times \Gamma^{\mathbf{n}}(\mathbf{X})$ <u>admits a chain approximation</u> φ : $(\mathbf{W} \otimes_{\mathbf{T}} \underbrace{\mathsf{C} \otimes \mathsf{C}}) \xrightarrow{\Delta_1 \otimes (\Delta_2 \otimes \Delta_2)} \mathbf{W} \otimes \mathbf{W} \otimes_{(\mathbf{T} \times \mathbf{T})} (\underbrace{\mathsf{C}^2 \otimes \mathsf{C}^2})$

$$\xrightarrow{\text{shuff}} (W \otimes_{Z_2} \widetilde{c \otimes c}) \otimes (W \otimes_{Z_2} \widetilde{c \otimes c})$$

where \triangle_1 is a (T \otimes T, T) equivariant diagonal map for W, and \triangle_2 is any chain approximation to the diagonal in X. (This is again immediate from the geometry; eg. see [27].) We turn now to the question of higher Bocksteins. Mod (p) for p odd, these Bocksteins are determined by 3.1(a); however, their structure mod 2 is somewhat more involved.

<u>Proposition</u> 3.6 <u>Suppose</u> $\beta_i(a) = b$ and:

- a) dimension a is even; then $\beta_j(a \otimes a) = 0, j \le i$ and $\beta_i(a \otimes a) = \langle a, b \rangle$, while $Sq^{1}(b \otimes b) = e^{1} \cup b \otimes b$;
- b) dimension a odd implies that $Sq^{1}(a \otimes a) = e^{1} \cup a \otimes a$

+
$$\begin{cases} 0, i > 1 \\ \langle a, b \rangle, i = 1 \end{cases}$$

while $\beta_{i+1}(\langle a,b\rangle + \theta e^1 \cup a \otimes a) = b \otimes b$;

c) for $i \ge 0$ and dim a even, $Sq^{l}(e^{2i+l} \cup (a \otimes a)) = e^{2i+2} \cup a \otimes a$, and for dim (a) odd, $Sq^{l}(e^{2i} \cup a \otimes a) = e^{2i+l} \cup a \otimes a$.

(The proof is a routine exercise using the explicit chain complex for $\Gamma^{n}(X)$ in 3.3; e.g., as in [20], [27].)

Finally, it remains to evaluate the action of G(2) and G(p) in $H^*(\Gamma^n(X))$. For p odd, this is immediate from 3.1 (modulo an extension problem, but that is handled in the next section; it turns out that the extension is trivial). Here is the result for p = 2.

$$\begin{array}{l} \underline{\text{Theorem }} 3.7 & \underline{\text{Assume}} \quad \theta \in \text{H}^{n}(\text{X}, \mathbb{Z}_{2}) \text{ } \cdot \underline{\text{Then}}:\\ \text{a)} & \text{Sq}^{i}[e^{k} \cup (\theta \otimes \theta)] = \sum_{r,j} \binom{k}{r} \binom{n-j}{i-r-2j} e^{k+r-2j} \cup (\text{Sq}^{j}\theta) \otimes \text{Sq}^{j}\theta \text{ ,}\\ \text{b)} & \text{Sq}^{i}(e^{0} \cup \theta \otimes \theta) = \sum_{r < i}^{r} \langle \text{Sq}^{r}\theta, \text{Sq}^{i-r}\theta \rangle + \sum_{i-2j}^{n-j} e^{i-2j} \cup \text{Sq}^{j}(\theta) \otimes \text{Sq}^{j}(\theta) \text{ ,}\\ \text{c)} & \text{Sq}^{i}(a,b) = \sum_{r=0}^{i} \langle \text{Sq}^{r}a, \text{Sq}^{i-r}b \rangle \text{ ,} \end{array}$$

at least modulo terms in im (K).

<u>Proof</u> The functor $\theta \to e^{\circ} \cup \theta \otimes \theta$ is natural and representable, so it suffices to establish a, b for $\iota \otimes \iota$ where ι is the fundamental class of a $K(\mathbb{Z}_2,n)$. Moreover, there is the map $\lambda : \mathbb{RP}^{\circ} \times \ldots \times \mathbb{RP}^{\circ} \to K(\mathbb{Z}_2,n)$, taking ι to $e_1 \otimes \ldots \otimes e_n$ which induces a monomorphism in cohomology for dimensions $\leq 2n$. Reference to 3.1 (and the obvious naturality) shows that $\Gamma^n(\lambda) : \Gamma^n(\mathbb{RP}^{\circ} \times \ldots \times \mathbb{RP}^{\circ}) \to \Gamma^n(K(\mathbb{Z}_2,n))$ also introduces a monomorphism in dimensions $\leq 4n$. Moreover, $\Gamma^n(\lambda)^*[\iota \otimes \iota] = (e^1 \otimes e^1) \cup \ldots \cup (e^n \otimes e^n)$. Thus we can apply the Cartan formula (noting by 3.4(b) that $\operatorname{Sq}^1(e \otimes e) = \beta_1(e \otimes e) = e^1 \cup (e \otimes e) + \langle e^2, e \rangle$ and $\operatorname{Sq}^2(e \otimes e) = (e \otimes e)^2 = e^2 \otimes e^2$). Now a, b follow directly.

Finally, to prove (c), consider the map

S:
$$\mathbb{RP}^{n} \times \mathbb{K}(\mathbb{Z}_{p}, \mathbb{m}) \times \mathbb{K}(\mathbb{Z}_{p}, \ell) \to \Gamma^{n}(\mathbb{K}(\mathbb{m}) \times \mathbb{K}(\ell))$$

defined by $S(x,y,z) = \{x(y,z)(y,z)\}$. It is easy to show that $S^* \langle \iota_m, \iota_\ell \rangle = 0$, but $S^* [e^i \cup (Sq^I \iota_m \cup Sq^J \iota_\ell)^{(2)} = e^i \otimes (Sq^I \iota_m \cup Sq^J \iota_\ell)^2 + W$ is non-zero. Hence $Sq^i \langle \iota_m, \iota_\ell \rangle$ can only involve terms $\langle Sq^i \iota_m, Sq^{j-r} \iota_\ell \rangle$ and perhaps a term in im (K^*) .

Finally, we will need the evaluation of the suspension map τ_1 (2.1) in cohomology.

Theorem 3.8
$$(\sigma\tau_1)^* \langle a, b \rangle = 0$$
. $(\sigma\tau_1)^* (e^i \cup \sigma(a) \otimes \sigma(a)) = e^{i+1} \cup (a \otimes a)$.

The proof is direct from 2.3.

We now consider the structure of $\text{H}^{*}(\text{F}_{n}(\Sigma X))$. By use of the Eilenberg-Moore spectral sequence ([26], [30], [31]), there is a spectral sequence converging to $H_*(F_n(\Sigma X))$ having E^2 term $\operatorname{Tor}^*_{T[\widetilde{H}_*(X,Z_p)]}(H_*(\Omega^{n+1}\Sigma^{n+1}(X), Z_p), Z_p)$. Here T(A) is the tensor algebra on A , and $H^*(\Omega^{n+1}\Sigma^{n+1}(X), Z_p)$ is a module over $T[\widetilde{H}_*(X,Z_p)]$ from the inclusion of H-spaces $J_1(X) \hookrightarrow J_{n+1}(X)$.

Moreover, the arguments of 1.6 show that $E^2 = E^{\infty}$. Thus, to calculate $H_*(F_n(\Sigma X))$, it suffices to calculate these Tor groups.

Note first that $H_*(J_{n+1}(X), Z_p) = P_n \otimes R$ where P is a polynomial algebra $P(\ldots \gamma_I(x) \ldots)$, with x running over $\widetilde{H}_*(X, Z_p)$ and the γ_I over some basis for the universal loop homology operations, and R is $A(\widetilde{H}_*(X, Z_p))$, the universal commutative algebra generated by $\widetilde{H}_*(X, Z_p)$. The action of T on $P \otimes R$ is then obtained by projecting T on $I \otimes R$.

Also, T is free; hence a resolution of T has the form $0 \rightarrow T \otimes s(\widetilde{H}_*(X,Z_p)) \rightarrow T \rightarrow Z_p$. Now, tensoring with P \otimes R and taking homology, we see that a basis for $\operatorname{Tor}_T^1(P \otimes R, Z_p)$ as a P \otimes R module is given by the cycles

$$\langle a,b \rangle = as(b)-(-1)^{|a|+|b|}bs(a)$$
 for $a \neq b$ in $\widetilde{H}_{*}(X,Z_{n})$.

We have thus calculated

<u>Theorem</u> 3.9 $H_*(F_n(\Sigma X), Z_p) \cong P_n \otimes L$ where L is a module over R with generators (a,b) of degree |a|+|b|+1, where a, b are non-equal basis elements in $\tilde{H}_*(X, Z_p)$. L is completely determined as a module by the relations $a(b,c)-(-1)^{|a|} \cdot |b|_{b(a,c)}+(-1)^{|b|} \cdot |c|+|a| \cdot |c|_{c(b,a)} = 0$.

We now turn to the cohomology structure of $\mbox{\bf F}_n$. First note the fibering $\Omega^{n+1}\Sigma^{n+1}X\xrightarrow{J}\mbox{\bf F}_n\to\Sigma X\to\Omega^n\Sigma^{n+1}X$.

This proof was suggested by J. C. Moore.

Lemma 3.10 There is a map φ : $\Sigma F_n \rightarrow \Omega^n \Sigma^{n+1} X / \Sigma X$, so the diagram

$$\begin{array}{c|c} \Sigma & \Omega^{n+1} & \Sigma^{n+1} X \xrightarrow{\text{adj} (1)} & \Omega^n \Sigma^{n+1} X \\ & & \downarrow^{\Sigma(J)} & & \downarrow^{\pi} \\ & \Sigma & F_n & \xrightarrow{\phi} & \Omega^n \Sigma^{n+1}(X) / \Sigma X \end{array}$$

<u>homotopy commutes</u>. (π is the evident projection.)

<u>Proof</u> We can write $F_n = E_{\Sigma X} \alpha^{n_{\Sigma}n+1}(X)_{*}$ (as in the remarks preceding 1.1), and take φ^{*} as the adjoint of the obvious map $F_n \to E_* \alpha^{n_{\Sigma}n+1}(X)/\Sigma(X)_{*} = \alpha[\alpha^{n_{\Sigma}n+1}(X)/\Sigma(X)]$. The inclusion $J: \alpha^{n+1}\Sigma^{n+1} \to F_n$ is given in this notation by the natural inclusion $E_* \alpha^{n_{\Sigma}n+1}(X)_{*} \hookrightarrow E_{\Sigma X} \alpha^{n_{\Sigma}n+1}(X)_{*}$, and, by suspending and adjointing, 3.9 follows.

By tracing through the map φ , we find $\varphi_*(a,b) = \pi_*(a \circ b)$ (the Pontrjagin Product) and $\varphi_*\alpha_\circ(a,b) = 0$. This gives us some information on passing to cohomology. However, to obtain more complete information, it is now necessary to use the action map

$$\Omega^{n+1}\Sigma^{n+1}(X) \times F_n \xrightarrow{\lambda} F_n$$

and our knowledge of the structure of this latter space as a module over the Steenrod algebra $\Omega(p)$ (e.g., see [21], [28]). Also note that λ_* is given by 3.8.

 $\S4$. The Structure Of Iterated Loop Spaces In The Metastable Range

We start with the following basic result:

Lemma 4.1 Let X be the nth loop space of Y. Then the fibering

$$G_n \to \Omega^n \Sigma^n X \xrightarrow{\tau} X$$

has a cross-section. Hence, up to weak homotopy equivalence,

$$\Omega^{n}\Sigma^{n}X = G_{n} \times X .$$

Indeed,

$$\Omega G_n = F_n$$
,

the fiber in the inclusion

$$\pi : X \hookrightarrow \Omega^{n} \Sigma^{n} X .$$

<u>Proof</u> The cross-section of τ is exactly the inclusion $X \hookrightarrow \Omega^n \Sigma^n X$. Hence, since $\Omega^n \Sigma^n X$ is an H-space, there is a map

$$G_n \times X \xrightarrow{W} \Omega^n \Sigma^n X$$
 ,

and $\tau \circ W$ is projection on the first factor. Moreover, if we let G_n be given explicitly as $\Omega^n(H)$, where H is the fiber in the adjoint map $\Sigma^n X \to Y$, then the diagram

$$\begin{array}{c} \mathbf{G}_{n} \times \mathbf{X} \xrightarrow{\mathbf{W}} \Omega^{n} \Sigma^{n} \mathbf{X} \\ \mathbf{\rho}_{2} \end{array} \xrightarrow{\mathbf{V}} \mathbf{I}^{\tau} \end{array}$$

commutes up to reparametrization of paths. Now both maps are fiberings with fiber G_n . Thus, by the five-lemma, W is a weak equivalence.

<u>Corollary</u> 4.2 $H_*(X,Z_p) \stackrel{\sim}{=} H_*(\Omega^{n_{\sum}n}X,Z_p)//_{H_*(G_n,Z_p)} \xrightarrow{as Hopf algebras. In particular, if X is m-1-connected, then <math>H_*(X,Z_p)$ determines $H_*(G_n,Z_p)$ completely in dimensions < 4m-1, and G_n is 2m-1-connected. (Indeed, a Hopf algebra basis for $H_*(G_n,Z_p)$ may be given with generators $Q_I[\pi_*(X)] - \pi_*[Q_I(X)]$, $\pi_*(X_1) \circ \cdots \circ \pi_*(X_n) - \pi_*(X_1 \circ \cdots \circ X_n)$ where $x_1 \cdots x_n$ run over a suitable basis for $H_*(X,Z_p)$, and the Q_I over a basis for the loop homology operations.)

<u>Corollary</u> 4.3 <u>Suppose again that</u> X is m-l-connected. Then in dimensions less than 3m-l, G_n is homotopic to $S^{n-1} \ltimes_T X \wedge X$ for X, the homotopy type of a CW complex. Moreover, this equivalence is natural in the same range (from 1.11 and 4.1).

Corollary 4.4 Under the assumptions of 4.3, $H^{i}(X,Z_{p})$ depends only on $H_{*}(Y,Z_{p})$ for p odd, and on $H^{*}(Y,Z_{2})$ as an G(2)-module for i < 3m-1.

<u>Proof</u> Since $G_n = \Omega^n H$, where H is the fiber in the map $\Sigma^n X \to Y$, 4.3 implies that $H \cong \Sigma^n (S^{n-1} \ltimes_T X \land X)$ in dimensions less than n+3m-1. Also, in this range of dimensions, the Serre spectral sequence of this fibration becomes a long exact sequence

(4.5)
$$\rightarrow \operatorname{H}^{i}(\operatorname{H}) \xrightarrow{\delta} \operatorname{H}^{i+1}(\operatorname{Y}) \xrightarrow{\sigma} \operatorname{H}^{i+1}(\Sigma^{n} \operatorname{X}) \xrightarrow{j} \operatorname{H}^{i+1}(\operatorname{H}) \rightarrow ,$$

and, to obtain $H^*(X,Z_p)$ for p a prime, it suffices to evaluate the map δ . For example, for p = 2, to evaluate $\delta(e^1 \cup a \otimes a)$, consider the map

 $Y \xrightarrow{\sigma(a)} K(Z_{2}, \ell)$.

By 4.3, this then induces a map of exact sequences

Moreover, $\Gamma^{n}(\sigma(\alpha))(e^{i} \cup \iota \otimes \iota) = e^{i} \cup \alpha \otimes \alpha$, and δ is determined by its behavior in the universal model. 4.4 follows.

In considering the proof of 4.4, it becomes clear that we need to know the map

$$\delta : \operatorname{H}^{i}(\operatorname{H}_{\ell,n}) \to \operatorname{H}^{i+1}(\operatorname{K}(\operatorname{Z}_{2},\ell), \operatorname{Z}_{2})$$

in order to determine the explicit form of the functor occurring in 4.4. <u>Theorem</u> 4.6 $\delta\sigma^{n}[e^{i} \cup (Sq^{I}(\iota) \otimes Sq^{I}(\iota)] = Sq^{\ell} + deg(I) + i+1(Sq^{I}(\iota))$, <u>while</u>

$$\mathbf{j}^* \sigma^n(\mathrm{Sq}^{\mathrm{I}}(\iota) \cup \mathrm{Sq}^{\mathrm{J}}(\iota)) = \sigma^n(\mathrm{Sq}^{\mathrm{I}}(\iota), \mathrm{Sq}^{\mathrm{J}}(\iota)) \quad (\mathrm{in } 4.5).$$

<u>Proof</u> The second statement is obvious. To prove the first assertion, it suffices, by naturality and the known behavior $(\S\S2,3)$ under suspension, to check that $\delta[\sigma(\iota_{\ell} \otimes \iota_{\ell})] = \mathrm{Sq}^{\ell+1}(\iota_{\ell+1}) = (\iota_{\ell+1})^2$ in the fibering $\Sigma K(\mathbb{Z}_2, \ell) \rightarrow K(\mathbb{Z}_2, \ell+1)$. Indeed, by the known results ([6]) on $\operatorname{H}^*(K(\mathbb{Z}_2, \ell+1))$, the kernel of σ^* in dimension $2\ell+2$ is exactly $(\iota)^2$. On the other hand, $\operatorname{H}^{2\ell+1}(\Sigma(K(\mathbb{Z}_2, \ell) \wedge K(\mathbb{Z}_2, \ell)), \mathbb{Z}_2) = \mathbb{Z}_2$, and has generator $\sigma(\iota \otimes \iota)$. 4.6 follows.

<u>Remark</u> 4.7 The results 4.4, 4.6 give the structure of $H^*(\Omega^n Y, Z_p)$ completely in the metastable range as a module over Z_p . In particular, we have <u>Corollary</u> 4.8 <u>Suppose</u> Y is m+n-1-connected, and $\operatorname{H}^{*}(Y, Z_{2})$ satisfies $\operatorname{Sq}^{I}(a) = 0$ for excess (I) > [dim(a)-n]. Then in dimensions less than 3m-1, $\operatorname{H}^{*}(\operatorname{\Omega}^{n}Y, Z_{2}) = \operatorname{s}^{-n}\operatorname{H}^{*}(Y) \oplus \operatorname{H}^{*}(G_{n})$.

However, this splitting need not be valid over G(2). For example, CP_{-8}^{12} certainly satisfies the hypothesis of 4.8 if m = 5. However, the secondary operation Φ_8 , associated with the relation

$$sq^{1}sq^{8} + sq^{2}sq^{1}(sq^{6}) + sq^{8}sq^{1} = 0$$
,

§5. The Obstructions To Desuspension In The Metastable Range

We conclude the first part of this paper by considering a basic example. In the metastable range, we reduce the question of desuspension to the determination of when a certain map φ of known spaces is homotopy-trivial. Berstein and Ganea independently have obtained related results ([36]).

Theorem 5.1 Let X be n-1-connected, and have dimension less than 3n-2. Then:

- 1) if Y is the 2n-L-l skeleton of X, there is a unique space Z, so $\Sigma^{L}Z = Y$;
- 2) X itself is an L-fold suspension if and only if a certain map

$$\varphi \; : \; X/Y \to \Sigma^{L+1} S^{L-1} \ltimes_{\pi} Z \land Z$$

is homotopy-trivial;

3) if φ is trivial, the number of distinct L-fold desuspensions of X is equivalent to the set of homotopy classes of maps

$$[X/Y, \Sigma^{L}S^{2-1} \ltimes_{T} Z \land Z]$$
.

(To avoid low dimensional complications, we also require $\ L$ to be less than n-3 .)

<u>Remark</u> 5.2 In 5.1.3, the equivalence classes comprise: (a) distinct homotopy types of desuspensions, and (b) maps

h :
$$W \rightarrow W$$

where $\Sigma^{L}W = X$, and where h is a homotopy equivalence $\simeq 1$ on the 2n-2L-1 skeleton of W .

<u>Remark</u> 5.3 By the dimensional restrictions, φ is actually a stable map since $S^{L-1} \ltimes_T Z \wedge Z$ is 2(n-L)-l-connected. Similarly, the set occurring in 5.1.3 is stable. Thus stable techniques are sufficient to determine them.

<u>Proof</u> $\Omega^L Y$ is a CW complex. The dimensional restrictions imply that the adjoint map $\Sigma^L \Omega^L Y \xrightarrow{\bullet} Y$ induces an isomorphism in homology in dimensions < 2n-L. Now taking the associated cross-section of $\Sigma^L \Omega^L Y$, we can assume that Y is actually the 2n-L-1 dimensional skeleton of $\Sigma^L \Omega^L Y$. Thus Y

is a suspension $\Sigma^{L}Z$. Moreover, the attaching map of every cell of Z is stable! Hence Z is indeed unique. This proves 5.1.1.

From 5.1.1, $X/Y=\Sigma^{L+1}W$ for a unique W. Moreover, there is a map τ : $\Sigma^LW\to\Sigma^LZ$, so the following is a cofiber sequence:

(5.4)
$$\Sigma^{\mathbf{L}} \mathbb{W} \xrightarrow{\tau} \Sigma^{\mathbf{L}} \mathbb{Z} \to \mathbb{X} \to \Sigma^{\mathbf{L}+1} \mathbb{W} \to \dots$$

Then X is a suspension $\Sigma^{\bf L}(M)$ if and only if $\tau=\Sigma^{\bf L}\tau'$ for some τ' : W $\to Z$. Consider the diagram

(5.5)
$$\begin{array}{c} \Omega^{\mathbf{L}}\Sigma^{\mathbf{L}}W \xrightarrow{\Omega^{\mathbf{L}}\tau} \Omega^{\mathbf{L}}\Sigma^{\mathbf{L}}Z \\ P & & & \downarrow \\ W \xrightarrow{\tau'} Z . \end{array}$$

We assert that τ' exists if and only if the composite

(5.6)
$$W \stackrel{\Omega}{\to} \Omega^{L} \Sigma^{L} W \stackrel{\Omega^{L} \tau}{\longrightarrow} \Omega^{L} \Sigma^{L} Z \stackrel{\pi}{\to} \Omega^{L} \Sigma^{L} Z / Z$$

is homotopy trivial. This follows from

Lemma 5.7 Let **3** be the fiber in the map π : $\Omega^{L}\Sigma^{L}Z \rightarrow \Omega^{L}\Sigma^{L}Z/Z$. Then through dimension 3(n-L)-2, $\mathfrak{F} \cong \mathbb{Z}$, and the inclusion $\mathfrak{F} \rightarrow \Omega^{L}\Sigma^{L}Z$ factors the inclusion $\mathbb{Z} \rightarrow \Omega^{L}\Sigma^{L}Z$. (This is immediate from the fiber lemma, §1.)

On the other hand, from the proof of l.ll, it follows that, in the range of dimensions which concern us, $\Omega^L \Sigma^L Z/Z \cong S^{L-1} \ltimes_T Z \wedge Z$, and this concludes the proof of 5.l.2 when we note (as in 5.3) that $\pi \circ \Omega^L \tau \circ \rho$ is a stable map, hence is homotopic to zero if and only if $\Sigma^{L+1}(\pi \circ \Omega^L \tau \circ \rho) = \phi : X/Y \to \Sigma^{L+1} S^{L-1} \ltimes_T Z \wedge Z$ is.

To prove 5.1.3, note that different suspensions satisfying the relations imposed by 5.2 are given by different homotopy classes of liftings in (5.5). But by 5.7, these are given by maps of W into the fiber. $\alpha(\alpha^{\rm L}\Sigma^{\rm L}Z/Z) \simeq \alpha(S^{\rm L-1} \ltimes_{\rm T} Z \wedge Z) \quad \text{in our range. Dimensional considerations}$ show that these are again stable, and 5.1.3 follows.

<u>Remark</u> 5.8 Combining 5.1 with 4.6 and the structure of $S^{L-1} \ltimes_T Z \wedge Z$, it is direct to calculate the first few obstructions explicitly in terms of higher order cohomology operations in X. For the first two obstructions, see [36] in particular.

<u>Remark</u> 5.9 Recent work of D. Anderson ([35]) makes it also possible to give analogues of 4.6 for certain exotic cohomology theories, eg. K-theory. This in turn makes it possible to carry through a program analogous to 5.8 in these theories as well. This remark will be considerably amplified in a forthcoming paper.

PART II

§6. An Unstable Adams Spectral Sequence

In this section, we introduce a version of the Adams spectral sequence which gives information about the unstable homotopy of a space X. It is invariantly defined from E^2 on; however: (1) little is known about its convergence properties, and (2) in general, E^2 is not just a functor of $H^*(X,Z_p)$ over G(p) but actually depends on the space itself.

The construction we use is similar to the one given in [17]. However, due to the special cohomological properties of the spaces they considered, it there turns out that E^2 is a functor of $H^*(X,Z_p)$ over G(p).

In §8, we will show that, in the metastable range, E^2 is algebraically determined (explicitly) from $H^*(X,Z_2)$ over G(2). Thus we reduce many of the problems involved in metastable calculations to formal algebra and the determination of differentials in this sequence.

Definition 6.1 An Adams (p,q)-resolution of a space X for p a prime and q a positive integer is a sequence of fiberings

$$\begin{array}{c} x \xrightarrow{\rho_{1}} E_{1} \xrightarrow{\rho_{2}} E_{2} \xrightarrow{\rho_{3}} E_{3} \\ \downarrow \pi_{1} \qquad \downarrow \pi_{2} \qquad \downarrow \pi_{3} \qquad \downarrow \pi_{4} \\ B_{H_{1}} \qquad B_{H_{2}} \qquad B_{H_{3}} \qquad B_{H_{4}} \end{array}$$

where:

i) E_{i} is the fiber in the map π_{i} ;

ii) $B_{H_{\frac{1}{2}}}$ is a generalized Eilenberg-MacLane space $K(Z_p, \vec{n})$ for the prime p, and

It is clear that, if X has the homotopy type of a CW complex, then (p,q)-resolutions of X exist for all (p,q). They also satisfy the naturality properties:

Lemma 6.2 Let $f: X \rightarrow Y$ be a map of CW complexes, and suppose given sequences

a)
$$\begin{array}{c} Y & \longleftarrow E'_{1} & \longleftarrow E'_{2} & \longleftarrow E'_{3} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ B'_{H_{1}} & B'_{H_{2}} & B'_{H_{3}} & B'_{H_{4}} \end{array}$$

where (a) satisfies (i), (ii) of 6.1, and (iii) with q-l in place
of q;

b)

$$x \leftarrow e_1 \leftarrow e_2 \leftarrow e_3 \leftarrow$$

where (b) satisfies all of 6.1. Then there are maps $f_i : E_i \to E'_i$, so the diagram

Х	< E ₁	< E ₂	< E3	<
	f	f	f ₂	f _z
Ŷ	< E'1	< E'₂	<- E'3	<i>~</i>

commutes.

Lemma 6.3 Under the assumptions of 6.2, suppose $f \simeq g : X \rightarrow Y$ and

$$\{ \mathbf{f}_{i} : \mathbf{E}_{i} \to \mathbf{E}_{i}^{\prime} \} ,$$
$$\{ \mathbf{g}_{i} : \mathbf{E}_{i} \to \mathbf{E}_{i}^{\prime} \}$$

are given. Then there are homotopies

$$\textbf{K}_{i}$$
 : $\textbf{I} \times \textbf{E}_{i} \rightarrow \textbf{E}_{i-1}'$,

so

i)
$$K_{i}(0,E_{i}) = \rho_{i}f_{i}$$
,

ii) $K_{i}(l,E_{i}) = \rho_{i}g_{i}$,

iii) the diagram



commutes.

As usual, taking the homotopy exact couple of the (p,q)-resolution in 6.1 gives a spectral sequence. Since $E_i \rightarrow E_{i-1} \rightarrow B_{H_i}$ is a fibering, it follows that $\pi_*(E_{i-1}, E_i) \cong \pi_*(B_{H_i})$. Moreover, the d_1 differential is obtained by passing to homotopy in the composition

$$\mathtt{H}_{i} \xrightarrow{\mathtt{j}} \mathtt{E}_{i} \xrightarrow{\pi_{i}} \mathtt{B}_{\mathtt{H}_{i+1}}$$

where $j : H_i \rightarrow E_i$ represents H_i as the fiber in the map ρ_i . By using 6.2, 6.3, we define our desired spectral sequence by passing to inverse limits over q when we note

<u>Corollary</u> 6.4 $E_{i,j}^{2}(X)[(p,q)]$ depends only on X for i-j < q-1, and is isomorphic to $E_{i,j}^{2}(X)$. In particular, $E_{i,j}^{2}(X)$ depends only on X and not the resolutions used to take the inverse limit.

Note that, when X is a (single) suspension, the convergence proof given in [34] carries over without change. (It depends only on 6.2 and the fact that it is possible to map a suspension onto a space Y_x , which satisfies 6.5(1) below for $x \in \pi_*(X)$, so the image of x in $\pi_*(Y_x)$ is non-zero.) Thus we have

<u>Corollary</u> 6.5(1) If X satisfies the condition that, for each j, there is an $r_j < \infty$ so $p \overset{r_j}{\pi_j}(X) \otimes Z = 0$, then the spectral sequence converges to $\pi_{\star}(X) \otimes Z$.

(ii) If X is a suspension, then the spectral sequence always converges to $\pi_*(X) \otimes Z$.

For general X, the difficulty in extending 6.5 is in the elements of infinite order in $\pi_*(X)$. If X is simply connected, has the homotopy type of a locally finite CW complex, and is also an associative unitary H-space, then: (a) the only elements of $\pi_*(X)$ of infinite order are contained in Z-direct summands, and (b) have non-trivial images under the Hurewicz map ([43]).

Put another way, this says that the Postnikov invariants are all finite for X . An easy argument now shows:

Corollary 6.6 If X is a simply connected, locally finite unitary H-space, then the spectral sequence converges to $\pi_*(X) \otimes Z$.

6.5 and 6.6 establish convergence insofar as we need it. It is probable that more extensive results in this direction can be obtained from [44]. <u>Example</u> 6.7 Let $X = K(Z_{p},n)$. Then $B_{H_{1}} = K(Z_{p},n) \times K(Z_{p},n+1)$, and the k-invariants are (ι) and $\beta(\iota)$. It is easily verified that $E_{1} = K(Z_{p},n)$, and the map ρ_{1*} is multiplication by p. Thus $E_{2} = \ldots = E_{n} = \ldots = K(Z_{p},n)$, and ρ_{n*} is always multiplication by p. In particular, $E_{1,j}^{1} = E_{1,j}^{2} = 0$ unless j-i = n,n+1 when it is Z_{p} . The differentials are all d_{3} 's and are all non-trivial.

A map of spectral sequences $\sigma : \mathbf{E}_{i,j}^*(X) \to \mathbf{E}_{i,j+1}(\Sigma X)$ is defined from 6.2, 6.3, the map $\pi : X \hookrightarrow \Omega \Sigma X$, and the sequence

(6.8)
$$\begin{array}{c} & & & & & \\ & & & & \\ &$$

associated to a (p,q)-resolution of ΣX .

<u>Corollary 6.9</u> If X is n-l-connected, the sequence of (6.8) satisfies 6.1(iv) in dimensions $\leq 2n-2$. Thus

$$\sigma : E_{i,j}^{*}(X) \to E_{i,j+1}^{*}(\Sigma X)$$

is an isomorphism in dimensions j-i \leq 2n-2 .

Corollary 6.10 If X is n-1-connected, then

$$\mathbf{E}_{i,j}^{2}(\mathbf{X}) \stackrel{\sim}{=} \operatorname{Ext}^{i,j}_{\mathbf{G}(\mathbf{p})}(\mathbf{H}^{*}(\mathbf{X}), \mathbf{Z}_{\mathbf{p}})$$

for $j-i \leq 2n-2$.

<u>Remark</u> 6.11 It is possible to generalize somewhat the above construction. Let $A \hookrightarrow H^*(X, Z_p)$ be an unstable sub-G(p)-module. Then we can resolve X by a sequence of fiberings



so im $(\pi_1)^*$ is exactly A, while the sequence $E_1 \leftarrow E_2 \leftarrow E_3 \leftarrow$ is a (p,q)-resolution of E_1 . Analogs of 6.2 through 6.6 continue to hold. Thus we again have an Adams-type spectral sequence with invariantly-defined E^2 -term. We denote it $E^2(X,A)$. Clearly, in the stable range, there is an exact sequence $\rightarrow \operatorname{Ext}^{i+1,*}_{G(p)}(A,Z_p) \rightarrow E^2_{i+1,*}(X,A) \rightarrow \operatorname{Ext}^{i,*}_{G(p)}(H^*(X)/A, Z_p) \xrightarrow{\partial} \operatorname{Ext}^{i+2,*}_{G(p)}(A,Z_p) \rightarrow \dots$, though, as we will see in the examples, ∂ may well be non-trivial.

7. The Loop Space Functor For Resolutions

Let X be m-l-connected, and suppose that

(7.1)
$$\Sigma^n X \stackrel{\rho_1}{\longleftarrow} E_1 \stackrel{\rho_2}{\longleftarrow} E_2 \stackrel{\rho_3}{\longleftarrow} E_3 \stackrel{\rho_4}{\longleftarrow} \cdots$$

is a (2,q)-resolution with q < 3m+n. In this section, we wish to study the behavior of (7.1) under the operation of taking loop spaces. It will appear that the sequence

(7.2)
$$\Omega^{n}\Sigma^{n}X \stackrel{\Omega^{n}\rho_{1}}{\longleftrightarrow} \Omega^{n}E_{1} \stackrel{\Omega^{n}\rho_{2}}{\longleftrightarrow} \Omega^{n}E_{2} \stackrel{\Omega^{n}E_{3}}{\longleftrightarrow} \cdots$$

is not a (2,q-n)-resolution, as $(\Omega^n \rho_1)^*$ is not zero in general. However, this is the only point at which the sequence fails to satisfy the definition for a (2,q-n)-resolution. Moreover, we will be able to calculate exactly im $(\Omega^n \rho_1)^*$.

We begin by proving

Theorem 7.3 Under the above assumptions, the sequence

$$\Omega^{n}\mathbf{E}_{1} \xleftarrow{\Omega^{n}\rho_{2}}{ \Omega^{n}\mathbf{E}_{2}} \xleftarrow{\Omega^{n}\rho_{3}}{ \Omega^{n}\mathbf{E}_{3}} \xleftarrow{\Omega^{n}\mathbf{E}_{3}}{ \cdots} \cdots$$

is a (2,q-n)-resolution of $\Omega^n \mathbb{E}_{1}$ for q < 2n.

<u>Proof</u> We proceed in two steps. First, consider the diagram of (vertical) fibrations

(7.4)
$$\begin{array}{c} G_1 & \stackrel{f_1}{\longrightarrow} & G_2 & \stackrel{f_2}{\longrightarrow} & G_3 & \stackrel{\cdots}{\longrightarrow} & \cdots \\ \downarrow_{\tau} & \downarrow_{\tau} & \downarrow_{\tau} & \downarrow_{\tau} & \downarrow_{\tau} \\ \Sigma^n \Omega^n E_1 & \stackrel{}{\longrightarrow} \Sigma^n \Omega^n E_2 & \stackrel{}{\longrightarrow} \Sigma^n \Omega^n E_3 & \stackrel{\cdots}{\longrightarrow} & \cdots \\ \downarrow_{\pi} & \stackrel{}{\longrightarrow} & I_{\pi} & \stackrel{}{\longrightarrow} & I_{\pi} & \stackrel{}{\longrightarrow} & I_{\pi} & \stackrel{}{\longrightarrow} & \cdots \end{array}$$

The key observation about (7.4) which we need is

Lemma 7.5 Through dimension 3m+n-2, f^{*}_i is the zero map in mod p cohomology.

<u>Proof</u> From 4.3, $G_i = S^{n-1} \ltimes_T (\Omega^n E_i \wedge \Omega^n E_i)$ in our range. Moreover, by 3.1 and naturality, f_i^* is determined in our range by $(\Omega^n \rho_i)^*$ restricted to $H^r(\Omega^n E_i)$ with r < 2m. But by suspension, $H^r(\Omega^n E_i) \cong H^{n+r}(E_i)$ in this range. Hence $f_i^* = 0$, and 7.5 now follows.

Next we must consider the fibering

$$\mathbf{H}_{\mathbf{i}} \xrightarrow{\mathbf{j}} \mathbf{E}_{\mathbf{i}} \rightarrow \mathbf{E}_{\mathbf{i-l}} \ .$$

Since q < 2n, we know that $j^* : H^r(E_j) \to H^r(H_j)$ is injective for r < q. Now consider the functor $\Sigma^n \Omega^n$ applied to the map j. We obtain the diagram of fibrations

(7.6)
$$\begin{array}{ccc} G_{1}^{i} & \stackrel{\sigma}{\to} & G_{1}^{i} \\ & \downarrow & \downarrow^{\tau} \\ \Sigma^{n} \Omega^{n} H_{1} & \rightarrow \Sigma^{n} \Omega^{n} E_{1}^{i} \\ & \downarrow & \downarrow \\ H_{1} & \stackrel{j}{\to} & E_{1}^{i} \end{array}$$

From 3.1, it now follows that θ_* is injective in our range. In particular, given $x \in H^r(E_1)$, we can suppose that $j^*(x) = \alpha$ and $\alpha \neq 0$; then the class $j^*(\operatorname{Sq}^{\dim(\alpha)+k+1-n}(x)) = \operatorname{Sq}^{\dim(\alpha)+k+1-n}(\alpha) \neq 0$. Thus $\operatorname{Sq}^{\dim(\alpha)+k+1-n}(x) \neq 0$, provided, of course, dimension $(\alpha) < n+m$. From the exact cohomology sequence for the left-hand fibering in (7.6), we find

$$\delta(e^k \cup \alpha \otimes \alpha) = Sq^{\dim(\alpha)+k+1-n}(\alpha)$$

Hence $\delta(e^k \cup x \otimes x) \neq 0$ in $H^*(E_i)$. Hence the only possible elements in $H^*(G_i)$ which are in the kernel of δ are the $\langle x, y \rangle^*$ with $x \neq y$ in $H^*(E_i)$. (Of course, x and y must each have dimension less than n+m.) Now note (as a consequence of 4.6) that

(7.7)
$$\langle \mathbf{x}, \mathbf{y} \rangle^* = \tau^* \sigma^n(\Omega^n(\mathbf{x}) \cup \Omega^n(\mathbf{y})) ;$$

hence $\langle x,y \rangle^{\star} \in \ker \delta$, and these elements give the entire kernel.

To complete the proof of 7.3, we suppose there is a $\lambda \in \operatorname{H}^{*}(\Sigma^{n}\Omega^{n}E_{i},Z_{2})$ with $(\Sigma^{n}\Omega^{n}\rho_{i})^{*}(\lambda) \neq 0$. In the diagram (7.4), it is certainly true that λ is not in the image of π^{*} .

Hence $\tau^*(\lambda) \neq 0$.

Hence ((7.6), (7.7)), it follows that

$$\lambda = \sigma^{n}(\Omega^{n}(\mathbf{x}) \cup \Omega^{n}(\mathbf{y})) + \upsilon$$

for some υ in the image of π .

On the other hand, $(\Omega^n \rho_1)^* (\Omega^n(x) \cup \Omega^n(y)) = (\Omega^n \rho_1)^* (\Omega^n x) \cup (\Omega^n \rho_1)^* \Omega^n(y) = 0$ Also, trivially, $(\Sigma^n \Omega^n \rho_1)^* \pi^*(x) = 0$. But this implies

$$(\Sigma^n \Omega^n \rho_i)^* (\lambda) = 0$$

Hence $\lambda = 0$, and 7.3 is proved.

We now turn to the first map in (7.2), $\Omega^n \rho_{\perp} : \Omega^n E_{\perp} \to \Omega^n \Sigma^n X$. As in the proof of 7.3, we suspend and consider the diagram of (vertical) fibrations

(7.8)
$$\begin{array}{c} \mathbf{G} < \overset{\mu}{-} & \mathbf{G}_{1} < \overset{\theta}{-} & \mathbf{G}_{1}' \\ \downarrow^{\tau} & \downarrow^{\tau}_{1} & \downarrow^{\tau}_{1} & \downarrow \\ \boldsymbol{\Sigma}^{n}(\boldsymbol{\Omega}^{n}\boldsymbol{\Sigma}^{n}\mathbf{X}) < \overset{\boldsymbol{\Sigma}^{n}\boldsymbol{\Omega}^{n}\boldsymbol{\rho}_{1}}{-} \boldsymbol{\Sigma}^{n}\boldsymbol{\Omega}^{n}\mathbf{E}_{1} < \overset{\boldsymbol{\Gamma}}{-} \boldsymbol{\Sigma}^{n}\boldsymbol{\Omega}^{n}\mathbf{H}_{1} \\ \downarrow^{\pi} & \downarrow^{\pi}_{1} & \downarrow \\ \boldsymbol{\Sigma}^{n}\mathbf{X} < \overset{\boldsymbol{\rho}_{1}}{-} & \mathbf{E}_{1} < \overset{\boldsymbol{J}}{-} & \mathbf{H}_{1} \end{array}$$

Note that π has a homotopy inverse. Hence, in the metastable range,

(7.9)
$$\Sigma^{n} \Omega^{n} \Sigma^{n} X \simeq \Sigma^{n} X \vee G$$

Before proceeding further, we need to consider the bottom (horizontal) fibration in (7.8). For $\sigma^n(\alpha)$ in $\operatorname{H}^*(\Sigma^n X, Z_2)$, let $\operatorname{t}(\alpha)$ be any element in $\operatorname{H}^*(\operatorname{H}_1)$ which satisfies

(7.10)
$$\delta(\underline{t}(\alpha)) = \sigma^{n}(\alpha)$$

in the Serre exact sequence of the fibration. Note that, for all $\,k\geq 0$, we have

(7.11)
$$\operatorname{Sq}^{\dim(\alpha)+k+1}(\mathfrak{t}(\alpha)) \in \operatorname{im}(\mathfrak{j}^*)$$
.

(In our range, this is due to exactness and the fact that $Sq^{\dim(\alpha)+l+k}(\alpha) \equiv 0$ for $k \geq 0$. For general $k \geq 0$, it follows from the Borel transgression theorem.) Hence, for each $k \geq 0$, we can choose

$$\beta_{k}(\alpha) \in H^{*}(E_{1})$$

satisfying

(7.12)
$$j^*(\beta_k(\alpha)) = \operatorname{Sq}^{\dim(\alpha)+k+1}(\mathfrak{t}(\alpha))$$
.

<u>Theorem</u> 7.13 <u>Under the splitting</u> (7.9), <u>let</u> $q_k(\alpha)$ <u>be the cohomology class</u> corresponding to $e^k \cup (\alpha \otimes \alpha)$ in $H^*(\Sigma^n \Omega^n \Sigma^n X)$. <u>Then for some choice of</u> $\beta_k(\alpha)$, <u>we have</u>

$$(\Sigma^{n}\Omega^{n}\rho_{1})^{*}q_{k}(\alpha) = \beta_{k}(\alpha)$$

<u>Proof</u> It suffices to verify 7.13 in the universal situation. The space which is universal for $\sigma^{n}(\alpha)$ is $\Sigma^{n}(K(\mathbb{Z}_{2}, \dim(\alpha)))$. Thus, in place of the bottom line of (7.8), we consider the fibering

(7.14)
$$K(Z_2, n-1 + \dim (\alpha)) \xrightarrow{\phi} \mathcal{U} \to \Sigma^n(K(Z_2, \dim (\alpha)))$$
.

Of course,

$$t(\sigma\iota_{\dim(\alpha)}) = \iota_{n+1+\dim(\alpha)}$$
.

Now consider the fibering

(7.15)
$$\Omega^{n}(\mathcal{U}) \xrightarrow{\theta} \Omega^{n} \Sigma^{n} K(\mathbb{Z}_{2}, \dim (\alpha)) \to K(\mathbb{Z}_{2}, \dim (\alpha)) .$$

Clearly, $\Omega^n \mathcal{U}$ is 2 dim (α)-1-connected, and $\theta^*(\iota * \iota) = \nu$, the first non-zero element in $H^*(\mathcal{U})$. (Here, $\iota * \iota$ is the class dual to the Pontrjagin

product.) By 4.3, $\Omega^{n} \mathcal{U} = S^{n-1} \ltimes_{T} K(Z_{2}, \dim(\alpha)) \wedge K(Z_{2}, \dim(\alpha))$ in the metastable range, and 7.13 follows by naturality under suspension. (Explicitly, the class corresponding to $e^{k} \cup \iota \otimes \iota$ in $H^{*}(U, Z_{2})$ pulls back under φ^{*} in (7.14) to $Sq^{k+1+\dim(\alpha)}\iota$.)

<u>Remark</u> 7.15 Finally, it should be noted that 7.3 is valid whether or not the space Y with which we start is $\Sigma^n X$. Indeed, the only crucial condition is that Y be n+m-l-connected.

8. The Metastable Exact Sequence

In this section, we assume X is m-l-connected, and restrict our attention to the 3m-2 skeletons of all the spaces under consideration.

Consider the fibering $F_L \to X \to \Omega^L \Sigma^L X$; then, as we have observed, F_L has the homotopy type of

$$\Omega(S^{L-1} \ltimes_{\pi} X \land X)$$

in our range. Consequently, $H^*(F_L,Z_2)$ has a natural sub-G(2)-module $A = \{\sigma^*(a,b) ; a,b \in H^*(X)\}$.

A plays a fundamental role in the sequel. Before proceeding to our main result, 8.5, we illustrate why this must be so.

Lemma 8.1 Consider the inclusion

$$\theta : \Omega^{L+1} \Sigma^L X \to F$$

<u>Then</u> ker (θ^*) is exactly A. (This follows immediately from the results of Part I (notably 3.10).)

Now suppose we start to construct (2,q)-resolutions. Observe that the composite

(8.2)
$$X \to \Omega^{L} \Sigma^{L} X \to \Omega^{L} \Sigma^{L} X \to \Omega^{L} B_{H_{L}}$$

is onto in cohomology. Hence the first stage in the resolution of X can be assumed to be the fiber in the map of (8.2). The following diagram of (vertical and horizontal) fibrations is thus obtained.



(since E'_L can be regarded as the pull-back via $\Omega^L(\rho_L)$ of the fibering $\pi\,:\,X\to\Omega^L\Sigma^LX)$.

Lemma 8.4 (a)
$$\operatorname{im}(\overline{j})^* = A$$
,
(b) $\operatorname{H}^*(\mathbf{E}'_1) \cong A \oplus B$
where $B = \operatorname{H}^*(\Omega^{\mathrm{L}}(\mathbf{E}_1)) / \operatorname{im} \Omega^{\mathrm{L}}(\rho_1)^*$.

<u>Proof</u> In our range, the Leray-Serre spectral sequence becomes a long exact sequence. Note that, in the sequence of the left-hand fibering in (8.3), $\delta(x,y) = x \cup y$ and $\Omega^{L}(\rho_{1})^{*} x \cup y = \Omega^{L}(\rho_{1})^{*}(x) \cup \Omega^{L}(\rho_{1})^{*}(y) = 0$. On the other hand, $\delta e^{i} \cup (x \otimes x) = (Q_{i}(x))^{*}$, and the results of §7 show that $\Omega^{L}(\rho_{1})^{*}(Q_{i}(x))^{*} \neq 0$. Indeed, the $\Omega^{L}(\rho_{1})^{*}(Q_{i}(x))^{*}$ span im $\Omega^{L}(\rho_{1})^{*}$. 8.4 follows. We are now ready to state the main result of Part II.

<u>Theorem</u> 8.5 <u>Suppose</u> L > 3m. <u>Then there are maps of spectral sequences</u>

$$\begin{split} \partial^{i}_{*} : & E^{*}_{i,j}(\Sigma^{L}X) \to E^{*}_{i,j-L-1}(F_{L},A) , \\ J^{i}_{*} : & E^{*}_{i,j}(F_{L},A) \to E^{*}_{i+1,j+1}(X) , \end{split}$$

and the resulting sequences

$$\xrightarrow{J_2} \mathbf{E}_{i,j+1}^2(\mathbf{X}) \xrightarrow{\sigma^{\mathbf{L}}} \mathbf{E}_{i,j+L+1}^2(\Sigma^{\mathbf{L}}\mathbf{X}) \xrightarrow{\partial_2} \mathbf{E}_{i,j}^2(\mathbf{F}_{\mathbf{L}},\mathbf{A}) \xrightarrow{J_2} \dots$$

are exact for j-i $\leq 3m-2$, and converge to the generalized E.H.P. sequence. <u>Remark</u> 8.6 The mechanism for determining ∂_{*}^{i} will be made clear during the course of the proof.

<u>Proof</u> We begin by constructing the maps ∂_* , J_* . Consider the first stage of an (F_L , A)-resolution of F_L :

$$(\mathbf{F}_{\mathbf{L}})_{1} \xrightarrow{\boldsymbol{\rho}} \mathbf{F}_{\mathbf{L}} \xrightarrow{\boldsymbol{\phi}} \mathbf{B}_{(\mathbf{A})}$$

where im ϕ^{\star} = A . From Lemma 8.1, there is no obstruction to extending θ to a map θ_{1} so

$$\begin{array}{c} \Omega^{L+1} E_{L} \xrightarrow{\theta_{1}} (F_{L})_{L} \\ \Omega^{L+1} \rho_{1} \\ \Omega^{L+1} \Sigma^{L} X \xrightarrow{\theta} F_{L} \end{array}$$

commutes. The map θ_1 may now be continued to a map of resolutions. This gives ∂_* on passing to spectral sequences. Moreover, by 6.3, ∂_* is invariantly defined from \mathbf{E}^2 on.

<u>Remark</u> 8.7 From 7.13, it follows that $\theta_1^{*}(e^i \cup x \otimes x) = (\Omega^{L+1}\rho_1)^{*}(Q_i(x_*))^{*}$. Thus the map θ_1^{*} is completely determined except on secondary elements associated to the resolution of A. We shall discuss this situation more carefully later in this section.

Now, consider the map

$$F_L \xrightarrow{j} X$$
.

Since π^* : $H^*(\Omega^L \Sigma^L X) \to H^*(X)$ is surjective, it follows that $j^* = 0$. Let E'_1 be the first stage in a resolution of X. It follows that j lifts to

$$\overline{j} : F_{L} \to E_{1}'$$
.

From 8.4(a), im $(\overline{j})^* = A$; thus if E'_2 is the next stage in a resolution of X, it follows that \overline{j} lifts to

$$\overline{j}_1$$
 : $\left(\mathbf{F}_{\mathrm{L}}\right)_1 \rightarrow \mathbf{E}_2'$.

We then continue \overline{j}_1 to a map of resolutions

This defines J_* . Of course, J_* is not, as it stands, well-defined since it depends on the choice of lifting \overline{j} . However, anticipating the fact that the sequence 7.5 is exact, it is certain that the image of J_2 is invariant under the different choices of liftings.

In order to demonstrate this exactness, we construct a very special resolution of X , one which tries to be the direct sum of the resolution

for $\Omega^{L}E_{l}$ and (F_{L},A) . Only the first two steps, due to the influence of A , must be handled with special care.

To begin, take E'_1 as given in (8.3). Consider the situation in (8.3). We have the fibering

$$\boldsymbol{\Omega^{L}E_{2}} \rightarrow \boldsymbol{\Omega^{L}E_{1}} \rightarrow \boldsymbol{\Omega^{L}B_{H_{2}}} \ ,$$

and we can induce a fibering over $\Omega^{L}E_{2}$ from the fibering over $\Omega^{L}E_{1}$ on the left in (8.3). This gives the diagram of (vertical) fiberings

(8.9)
$$\begin{array}{c} \mathbf{F}_{\mathbf{L}} & \stackrel{=}{\longrightarrow} & \mathbf{F}_{\mathbf{L}} \\ \mathbf{\psi} & \stackrel{\overline{\rho}}{\longrightarrow} & \mathbf{\psi} \\ \mathcal{E}_{2} & \stackrel{\overline{\rho}}{\longrightarrow} & \mathbf{E}_{1}^{\mathsf{L}} \\ \mathbf{\mu} & \stackrel{\mathbf{\mu}}{\sum} & \stackrel{\mathbf{\mu}}{\sum} & \stackrel{\mathbf{\mu}}{\sum} \\ \mathbf{\mu} & \stackrel{\mathbf{\mu}}{\sum} & \stackrel{\mathbf{\mu}}{\sum} \\ \mathbf{\mu} & \stackrel{\mathbf{\mu}}{\sum} & \stackrel{\mathbf{\mu}}{\sum} \\ \mathbf{\mu} & \stackrel{\mathbf{\mu}}{$$

From 8.4(b), im $(\overline{\rho})^* = A$ since the map $(\Omega^L \rho_2)^* \equiv 0$ in our range. Thus we can kill A in $H^*(\mathcal{E}_2)$. This gives us the diagram

Lemma 8.11 In (8.10), the composite map ρ_2' has as fiber a generalized Eilenberg-MacLane space.

Proof Consider the diagram



Here the 45° lines are fiberings, as well as the horizontal line starting with Θ . Clearly, there are liftings $\overline{\rho_1}$, $\overline{\rho_2}$ to make the triangles in (8.12) commute. But \mathbf{E}_2' is universal for such pairs of maps. Hence there is a λ : $\Theta \rightarrow \mathbf{E}_2'$ making 8.11 commutative in its entirety. Thus we have a map of fibrations

Note $\pi_*(\mathbf{F}) \cong \pi_*(\mathbf{H}(\mathbf{A}) \times \Omega^{\mathbf{L}}\mathbf{H}_1)$, and reference to (8.13) shows that the isomorphism is actually induced by $\lambda|_* \cdot \Theta$, \mathbf{E}'_2 have the homotopy type of CW complexes. Thus, since the five-lemma shows $\lambda_* : \pi_*(\Theta) \to \pi_*(\mathbf{E}'_2)$ is an isomorphism, it follows that $\Theta \simeq \mathbf{E}'_2 \cdot \mathbf{S}$.ll follows.

Again in view of 8.4(b), $(\rho'_2)^* = 0$ in our range. Hence, from 8.11, it follows that ρ'_2 is the second stage in our resolution.

To continue the process, consider the diagram

Here \mathcal{E}_{3} is the fibering over \mathbf{E}_{2}' induced by $(\pi \cdot \omega)$ from $(\Omega^{L}_{\rho_{3}})$. Since $(\Omega^{L}_{\rho_{3}})^{*} = 0$, it follows as in 8.4 that

(8.15)
$$\operatorname{H}^{*}(\mathcal{E}_{\mathfrak{Z}}) \cong \operatorname{H}^{*}((\operatorname{F}_{L})_{1}) \oplus \operatorname{H}^{*}(\Omega^{L} \mathbf{E}_{\mathfrak{Z}})$$

in our range. We construct E'_{3} by now killing $H^{*}((F_{L})_{1})$ in $H^{*}(\mathcal{E}_{3})$. As in the proof of 8.11, we find that the composite

$$E_3 \rightarrow E_3 \rightarrow E_2'$$

has as fiber a generalized Eilenberg-MacLane space. Moreover, by construction, the map is zero in cohomology.

This process evidently continues giving us a resolution of X , with fiber at each stage the product of a fiber in a resolution of F_L with a fiber in the resolution of $\Omega^L E_1$.

Passing to E¹-terms gives

(8.16)
$$E_{i,j}^{1}(X) \cong E_{l,j+L}(\Sigma^{L}X) \oplus E_{i-l,j+1}^{1}(F_{L},A)$$
.

Now we need

Lemma 8.17 The d₁ operator for the resolution given in (8.15) is the pair

$$(\Omega^{L}\partial_{1}, \underline{\partial}_{1} \oplus \partial_{1}(F_{L}, A))$$

where \hat{g}_1 is the boundary map constructed in the first step of our proof from the resolution of $\Sigma^L X$ to that of $(F_{\tau,r}A)$.

Thus, algebraically, $E_{**}^2(X)$ is the homology of the mapping cone, and the exactness of 8.5 follows.

Proof We find immediately that the diagram



commutes. The fiber of the map

$$E'_{i+1} \rightarrow E'_i$$

is $\Omega^{L+1}B_{H_i} \times \Omega B_{(F_L)_{i-1}}$, and the map iou embeds $\Omega^{L+1}B_{H_i}$ as the first factor in this fiber. On the other hand, E'_{i+2} is the fiber in the map

$$\pi : \mathbf{E}'_{i+1} \rightarrow \mathbf{B}_{(\mathbf{F}_{L})_{i}} \times \mathbf{\Omega}^{L} \mathbf{B}_{\mathbf{H}_{i+1}} ,$$

and $\gamma : E'_{i+1} \to B_{(F_L)_i}$ is exactly $p_1 \circ \pi$, the projection on the first factor. The differential d_1 is determined by the map in homotopy in the

composite

(8.18)
and now 8.17 follows.

This completes the proof of 8.5.

We now turn to the problem of determining the map ∂_{\star} as explicitly as possible.

In 6.11, an exact sequence was exhibited which determines $E^2(F_L,A,Z_2)$ in terms of an exact sequence. Part of this sequence is a map

(8.20)
$$\mu : \mathbb{E}^{2}_{i,j}(\mathbb{F}_{L},\mathbb{A},\mathbb{Z}_{2}) \to \mathbb{E} \times t^{i-l,j+l}_{G(2)}(\mathbb{H}^{*}(\mathbb{F}_{L})/\mathbb{A},\mathbb{Z}_{2}) .$$

In terms of the map, we have

Theorem 8.21 The composite

$$\mu \circ \partial_{2} : \operatorname{Ext}^{i,j}_{\mathbb{G}(2)}(\operatorname{H}^{*}(X), \operatorname{Z}_{2}) \to \operatorname{Ext}^{i-1,j+2m-2}_{\mathbb{G}(2)}(\operatorname{H}^{*}(\operatorname{F}_{L})/A, \operatorname{Z}_{2})$$

is algebraically determined, and commutes with the action of $\operatorname{Ext}_{G(2)}(Z_2, Z_2)$ on each of these modules. (This is an immediate consequence of 8.7 and the fact that the map (8.20) is determined by the algebraic inclusion of $\operatorname{H}^*(F_L)/A$ in $\operatorname{H}^*((F_L)_1)$.)

<u>Remark</u> 8.22 In the special case that X is the sphere S^{m} , note that $A = \emptyset$ Thus (8.20) is an isomorphism, and we obtain as a corollary the main result of [13], on the existence of 8.5 for spheres.

<u>Remark</u> 8.23 In particular, this gives us many examples where ∂_2 is non-trivial. For example, if $X = S^7$, then, in the first non-trivial dimension, we have

$$\operatorname{Ext}^{1,8}_{0(2)}(Z_2,Z_2)$$

has generator h_3 , and ${\rm Ext}^{0,0}_{~~G(2)}(\texttt{H}^*(P_7^{~~\infty}),\, Z_2)=Z_2$ with generator e_7 , while

$$\partial_2(h_3) = e_7$$
.

9. Calculating The Groups
$$Ext^{i,j}_{G(2)}(H^*(F_L)/A, Z_2)$$

In \$8, we saw how the composite

$$(\mathbf{u} \cdot \partial_*) : \operatorname{Ext}^{i,j}_{G(2)}(\operatorname{H}^*(X), \operatorname{Z}_2) \to \operatorname{Ext}^{i-1,j-2L-2}_{G(2)}(\operatorname{H}^*(\operatorname{F}_L)/A, \operatorname{Z}_2)$$

is algebraically determined. In the context of the Adams spectral sequence approach to metastable homotopy theory, it replaces the classical ∂ -invariant in the E.H.P. sequence.

In this section, we review the most effective techniques for calculating this Ext group for L greater than the connectivity of X .

 $H^{\star}(F_{L})/A$ has generators $e^{i} \cup x \otimes x$ for $x \in H^{\star}(X)$. There are two obvious filtrations for these elements:

(9.1)
$$e^{i} \cup x \otimes x \in \mathfrak{F}_{i}$$
 if $\dim x \geq j$,

(9.2) $e^r \cup x \otimes x \in G_i$ if $r \ge i$.

The results of §3 show that each of these \mathfrak{F}_j , \mathfrak{G}_i are closed under the action of the Steenrod algebra G(2). Hence they give rise to spectral sequences ${}_1\mathbf{E}$, ${}_2\mathbf{E}$ with

(9.3)
$$1^{E_{j}^{1} \cong Ext_{G(2)}(\mathfrak{Z}_{j}/\mathfrak{Z}_{j+1}, \mathbb{Z}_{2})}$$
,

(9.4)
$$_{2}E_{j}^{1} \cong Ext_{G(2)}(G_{i}/G_{i+1}, Z_{2})$$
.

In order to identify these terms further, we note

Theorem 9.5
$$\mathbf{\mathfrak{F}}_{j}/\mathbf{\mathfrak{F}}_{j+1} \cong \mathrm{H}^{*}(\mathrm{P}^{\circ}_{j}) \otimes \mathrm{H}^{j}(\mathrm{X})$$

<u>Proof</u> This is immediate from 3.1, or see [22, §2]. These groups have been tabulated in a range ([13]), and hence may be regarded as available for calculations.

Now, turning to the second filtration, let $S^{2}(H^{*}(X))$ be the T-invariant subalgebra of $H^{*}(X) \otimes H^{*}(X)$, and let

```
B(H^{*}(X))
```

be the quotient of $S^2(H^*(X))$ by the sub-G(2)-module image (1+T) . We evidently have

<u>Theorem</u> 9.6 The $2^{E^1} * k^{-term}$ of the spectral sequence defined by filtration 2 is a copy of

$$\operatorname{Ext}_{G(2)}^{**}(s^{k}B(H^{*}(X)), Z_{2})$$
.

There are cases, for example: when X has very few cells, when the first spectral is more convenient. However, when the structure of $H^*(X)$ is very intricate, but $Ext_{\mathbb{C}(2)}(H^*(X), Z_2)$ is completely known in a range, then the second spectral sequence is usually much easier to work with. (For example, in §11 we will study the case when X = K(Z,m).)

In order to calculate the functor

(9.7)
$$\operatorname{Ext}_{G(2)}(B(H^{*}(X)), Z_{2})$$
,

we proceed by exploiting the "doubling" homomorphism. Let $D: G(2) \to G(2)$ be the doubling map $(D(xy) = D(x)D(y), D(Sq^{2j}) = Sq^{j})$. It is a map of algebras, and makes G(2) into an algebra over itself. We denote G(2) in this context as $G(2)_{D}$. Similarly, we can "double" modules over G(2). Thus if M is a graded G(2)-module, via the doubling map M becomes an ungraded G(2)-module, $([Sq^{i}](m) = D(Sq^{i})m)$.

Lemma 9.8 There is a unique (graded) G(2)-module D(M) and ungraded isomorphism f : $D(M) \rightarrow M$ so that the diagram

$$\begin{array}{ccc} G(2) \otimes D(M) & \underline{D \otimes f} \\ & & \downarrow \\ & & \downarrow \\ D(M) & \underline{f} \\ & & M \end{array} \end{array}$$

<u>commutes</u>. (Obvious: $D(M)_{2i} \cong (M)_i$, $D(M)_{2i+1} = 0$.

We call D(M) the double of M . Clearly, $B(H^{\star}(X)) \stackrel{\sim}{=} D(H^{\star}(X))$. We can expand the diagram of 9.5 as

$$\begin{array}{c|c} G(2) \otimes D(G) \otimes D(M) & \underline{u' \otimes 1} \\ D(u) & & \downarrow D(u) \\ G(2) \otimes D(M) & \underline{u' \otimes 1} \\ D(u) & & \downarrow D(u) \\ D(u) & & \downarrow D(u) \\ \end{array}$$

In particular, D(M) is actually a module over D(G), and we can use the change of rings spectral sequence ([12]) converging to (9.7). Thus we have

<u>Lemma 9.10 There is a spectral sequence converging to</u> $Ext_{G(2)}(BH^{*}(X), Z_{2})$ with E₂ term

$$\operatorname{Ext}_{G(2)}(M, \operatorname{Ext}_{G(2)}(DG(2), \mathbb{Z}_2))$$

(Here the tilde over the internal Ext denotes the fact that this is a spectral sequence with <u>twisted coefficients</u>.)

However, 9.10 is still very useful since $Ext_{G(2)}(DG(2), Z_2)$ has a very simple form; e.g., see [23, §2]. Specifically, we have

<u>Theorem</u> 9.11 Ext^{**}_{G(2)}(D(G), Z_2) \cong P(q₀,q₁,...,q₁...) where q₁ has bidegree (1, 2ⁱ⁺¹-1).

To illustrate methods, we outline the proof. The dual G^* of G(2) is a polynomial algebra on generators ξ_i of degree 2^i -1. Clearly, the map $d: \xi_i \rightarrow {\xi_1}^2$ extends to a map of Hopf algebras. Its dual is D! Thus, dual to the sequence of Hopf algebras

(9.12)
$$G(2)^* \stackrel{d}{\to} G(2)^* \to G(2)^* //dG(2)^*$$
,

there is the sequence

$$(9.13) \qquad (\text{Ker D}) \to G(2) \xrightarrow{D} G(2) //\text{Ker (D)} \cong D(G(2))$$

Of course, $G(2)^*//dG(2)^* \xrightarrow{\sim} E(\xi_1 \cdots \xi_i \cdots) \xrightarrow{\sim} Ker(D)$. Now the sequence (9.13) becomes an exact sequence of G(2)-modules as

$$(9.14) \qquad \qquad 0 \to \mathbb{Q}(2) \cdot \overline{\mathrm{Ker}} (D) \to \mathbb{Q}(2) \to \mathbb{D}(2) \to 0 \ .$$

(9.14) gives rise to a long exact sequence of Ext groups, and, since G(2) is G(2)-free, this implies

$$\operatorname{Ext}^{i}_{G(2)}(\operatorname{DG}(2), \operatorname{Z}_{2}) \cong \operatorname{Ext}^{i-1}_{G(2)}(G(2) \cdot \overline{\operatorname{Ker}} (D), \operatorname{Z}_{2}) \ .$$

On the other hand, $\operatorname{Ext}^{i-1}_{G(2)}(G(2) \cdot \overline{B}, Z_2) \cong \operatorname{Ext}^{i}_{B}(Z_2, Z_2)$ for any sub-Hopf algebra of G(2) ([23, §2]). Thus the result follows from the routine calculation of the Ext groups for a graded exterior algebra.

<u>Remark</u> 9.15 The difficulty with using 9.10 and 9.11 for the spectral sequence (9.4) is that the twisting in 9.10 is almost always non-trivial.

<u>Example</u> 9.16 In case $X = K(Z_2,m)$, we find the spectral sequence (9.4) by far the more convenient. Its E_r^1 term in our range is a copy of $P(q_0 \cdots)$; however, the differentials are very involved. Considerable work on this case has been done by J. Harper ([10], [11]).

PART III

APPLICATIONS AND EXAMPLES

10. Calculations Of The Stable Homotopy Of $K(\pi,n)$'s

In [4], [5], and [14], methods were given for studying the stable homotopy groups $\pi^{s}_{*}(K(\pi,n))$. These techniques give only fragmentary information in actual practice. More recently, work of Browder and Brown ([38], [39]) and Brumfiel, Madsen, and the author ([40]) has shown the importance of these groups for problems relating to the classification of manifolds. Also, and rather tautologically, these groups give the classification of higher order one-variable cohomology operations which vanish for dimensional reasons, [3], [8], [10], [11] universally on appropriate cohomology classes.

In this section, we apply the methods of Sections 4 and 9 to the problem, obtaining somewhat more efficient tools for the calculations. We begin by considering the first few groups when $\pi = Z_2$ or Q/Z, the cases of interest in [38], [39], [40]. We then continue in the next section by giving more extensive calculations for the case $\pi = Z$.

In all cases, we begin by considering the fibering

(10.1)
$$G_{L} \to \Sigma^{L} K(\pi, n) \to K(\pi, n+L)$$

This is of the type considered in 4.1; hence in our range

$$G_{\mathbf{L}} = \Sigma^{\mathbf{L}} S^{\mathbf{L}-1} \ltimes_{\mathbf{T}} (K(\pi,n) \wedge K(\pi,n))$$

Moreover, since $\pi_i(K(\pi,n+L)) = 0$ for i > n+L, it follows from the homotopy exact sequence of (10.1) that $\pi_i^{S}(K(\pi,n)) \cong \pi_i^{S}(G_L)$ in our range.

Thus to compute $\ \pi_1^{\ \ S}K(\textbf{Z}_2,n)$, we must calculate the homotopy of

$$s^{L-1} \ltimes_T K(Z_2,n) \wedge K(Z_2,n)$$
 .

In the first few dimensions, this has cells as follows:

$$e_{3} \cup \iota \otimes \iota \quad e_{1} \cup Sq^{1}\iota \otimes Sq^{1}\iota \langle Sq^{1}\iota, Sq^{2}\iota \rangle \quad \langle \iota, Sq^{2}Sq^{1}\iota \rangle \quad \langle \iota, Sq^{3}\iota \rangle$$

$$e_{2} \cup \iota \otimes \iota \qquad Sq^{1}\iota \otimes Sq^{1}\iota \quad \langle \iota, Sq^{2}\iota \rangle$$

$$(10.2)$$

$$e_{1} \cup \iota \otimes \iota \qquad \langle \iota, Sq^{1}\iota \rangle$$

$$\iota \otimes \iota .$$

The action of the Steenrod algebra G(2) on the elements enumerated above depends, of course, on n . Thus for $n \equiv 2(4)$, we find from 3.7 that

$$\begin{split} & \mathrm{Sq}^{1}(\iota \otimes \iota) = \langle \iota, \mathrm{Sq}^{1}\iota \rangle , \\ & \mathrm{Sq}^{2}(\iota \otimes \iota) = \mathrm{Sq}^{1}\iota \otimes \mathrm{Sq}^{1}\iota + \langle \iota, \mathrm{Sq}^{2}\iota \rangle + \mathrm{e}_{2} \cup \iota \otimes \iota \\ & \mathrm{Sq}^{3}\iota \otimes \iota = \langle \mathrm{Sq}^{2}\iota, \mathrm{Sq}^{1}\iota \rangle + \langle \mathrm{Sq}^{3}\iota, \iota \rangle , \\ & \mathrm{Sq}^{1}\mathrm{e}_{1} \otimes \iota \otimes \iota = \mathrm{e}_{2} \cup \iota \otimes \iota , \\ & \mathrm{Sq}^{2}\mathrm{e}_{1} \otimes \iota \otimes \iota = \mathrm{e}_{3} \cup \iota \otimes \iota + \mathrm{e}_{1} \cup \mathrm{Sq}^{1}\iota \otimes \mathrm{Sq}^{1}\iota , \\ & \mathrm{Sq}^{1}(\mathrm{Sq}^{1}\iota \otimes \mathrm{Sq}^{1}\iota) = \mathrm{e}_{1} \cup \mathrm{Sq}^{1}\iota \otimes \mathrm{Sq}^{1}\iota . \end{split}$$

In this range, there are at least four non-trivial Z_2 's in homotopy; the first dual to $\iota \otimes \iota$, the second dual to $e_1 \cup \iota \otimes \iota$, the third dual to $\operatorname{Sq}^1 \iota \otimes \operatorname{Sq}^1 \iota$, and the fourth dual to $\langle \operatorname{Sq}^1 \iota, \operatorname{Sq}^2 \iota \rangle$. Applying 3.7 in the remaining cases, we obtain the table

$$n \equiv 0(2) \qquad \frac{i}{\pi_{2n+i}^{s}(K(Z_{2},n))} \begin{vmatrix} Z_{2} & Z_{2} \\ generator \\ n \equiv 1(2) & \pi_{2n+i}^{s}(K(Z_{2},n)) \begin{vmatrix} Z_{2} & Z_{2} \\ (\iota \otimes \iota)_{*} & (e_{1} \cup \iota \otimes \iota)_{*} \end{vmatrix} \begin{pmatrix} Z_{2} \\ (sq^{1}\iota \otimes sq^{1}\iota)_{*} \\ Z_{2} \\ generator \end{vmatrix} \begin{pmatrix} z_{2} & Z_{2} \\ (sq^{1}\iota \otimes sq^{1}\iota)_{*} \\ Z_{2} \\ (e_{1} \cup \iota \otimes \iota)_{*} \end{vmatrix} \begin{pmatrix} z_{2} & Z_{2} \\ (sq^{1}\iota \otimes sq^{1}\iota)_{*} \\ Z_{2} \\ (e_{2} \cup \iota \otimes \iota)_{*} \end{vmatrix}$$

Now we consider the group Q/Z. Suprisingly, this involves us in some algebraic problems. The first is a suitable description of Q/Z. To this end, let $Z = \lim_{\to \infty} (Z_{p^i})$ for each prime p.

<u>Lemma</u> 10.4 $Q/Z \cong \bigoplus (Z)$ as p runs over all primes.

<u>Proof</u> Injections $\theta_p : Z_{(p)}^{\infty} \to Q/Z$ are defined as the direct limits of maps

$$\rho_i : Z_i \to Q/Z$$

where $\rho_i(n) = n/p^i$. These in turn induce a morphism

$$\theta : \bigoplus \hat{Z}_{(p)^{\infty}} \to Q/Z$$

given on elements by

$$\theta(n_p \cdots n_q) = \theta_p(n_p) + \cdots + \theta_q(n_q)$$

To see that ker $(\theta) = 0$, note that

$$n_p/p^i + \ldots + n_q/q^j \in Z$$

if and only if $n_p(\cdots q^j) + \ldots + (p^j \cdots)n_q \equiv O(p^j \cdots q^j)$. In particular,

$$n_{p}(\cdots q^{j}) + \ldots + (p^{j} \cdots)n_{q} \equiv O(p^{j})$$
$$\vdots$$
$$\equiv O(q^{j}),$$

and these congruences imply $n_p \equiv 0(p^i)$, ..., $n_q \equiv 0(q^j)$. Thus ker $(\theta) = 0$. To see the converse, suppose the fraction $\frac{n}{2^i j^j \dots q^k}$ is given in Q/Z. Since the ideals $(p^j) + (q^k) = Z$ in the integers, for p a prime and q relatively prime to p, there are integers m_p , n_p so

$$n_2^{2^i} + m_2(3^j \cdots q^k) = n$$
.

Similarly, there are integers m_3 , n_3 so

$$m_3(5^s \cdots q^k) + n_3 3^j = n_2$$
,

etc. Hence we can write

$$n/2^{i}\cdots q^{k} = m_{2}^{2}/2^{i} + \ldots + m_{q}^{2}/q^{k}$$
,

and this shows that θ is onto. 10.4 follows.

Now note that the inclusion $Z_{p^{i}} \hookrightarrow Z_{p^{i+1}}$ induces an inclusion $K(Z_{p^{i}},n) \hookrightarrow K(Z_{p^{i+1}},n)$ (e.g., see the models for $K(\pi,n)$ introduced in [19]). Clearly, we have

Lemma 10.5 $K(Z_{(p)}^{\infty}, n) \xrightarrow{\lim_{p \to \infty}} (\text{weak}) K(Z_{pi}, n) \xrightarrow{\text{under these inclusions.}} (Here, (p)^{\infty})$ the topology in $K(Z_{(p)}^{\infty}, n)$ is given by specifying its compact sets [which are (p)^{\infty}, n) is given by specifying its compact sets [which are p^{1}, n) for some i, so $X \cap K(Z_{pi}, n)$ is compact], and letting Y be open if and only if $Y \cap X$ is open for each compact set X.)

<u>Corollary</u> 10.6 $K(Q/Z,n) = \prod_{p \text{ prime}}^{W} K(Z_{p},n)$ (weak limit of the finite Cartesian products).

Corollary 10.7 Consider the fibering

$$G_{L} \rightarrow \Sigma^{L} K(Q/Z, n) \rightarrow K(Q/Z, n+L)$$
.

 $\underline{\text{Then}} \quad \textbf{G}_{L} = \prod_{p \text{ prime}} \overset{W}{\underset{\rightarrow}{\text{ lim}}} (\text{weak}) \Sigma^{L}[\textbf{S}^{L-1} \ltimes_{T} \textbf{K}(\textbf{Z}_{p^{i}}, \textbf{n}) \land \textbf{K}(\textbf{Z}_{p^{i}}, \textbf{n})] \quad \text{in our range.}$

Thus we can calculate the stable homotopy groups of K(Q/Z,n) from the observation

10.8
$$\pi_*$$
 (lim (weak) X_i) $\stackrel{\sim}{=}$ lim $\pi_*(X_i)$.

Using 3.1 and 3.7, we now find

<u>Theorem</u> 10.9 (a) $\pi_{2n}^{s}(K(Q/Z,n) = 0$,

(b)
$$\pi^{s}_{2n+1}(K(Q/Z,n)) \cong \begin{cases} Q/2Z \cong Q/Z , & n \text{ odd,} \\ 0 & , & n \text{ even} \end{cases}$$

$$\underline{Proof} \ \pi_{2n+L}(G_{L}(K(Z_{i},n)) = \begin{cases} Z_{i}, n \text{ even} \\ p \text{ } \end{cases}, n \text{ odd, } p \neq 2 \\ Z_{2}, n \text{ odd, } p = 2. \end{cases}$$

In homology, $(\iota_{p^{i}})_{*} \rightarrow p(\iota_{p^{i+1}})_{*}$ under our injection. Hence $(\iota_{p^{i}})_{*} \otimes (\iota_{p^{i}})_{*}$ $\rightarrow p^{2}((\iota_{p^{i+1}})_{*})$, and the kernel of the map is generated by

$$p^{i-1}(\iota_{p^{i}})_{*} \otimes (\iota_{p^{i}})_{*}$$

Moreover, the image of $p^{i-2}[(\iota_p)_* \otimes (\iota_p)_*]$ is in the kernel of the next map, and, in general, $p^{i-j}[(\iota_p)_* \otimes (\iota_p)_*]$ is in the kernel of the jth iterate. Thus the direct limit is indeed zero, and 10.9(a) follows.

To prove (b), note that

(10.10)
$$\pi_{2n+1+L}(G_{L}(K(Z_{p^{i}},n))) = \begin{cases} Z_{p^{i}}, n \text{ odd, } p \neq 2 \\ Z_{p^{i}}, n \text{ odd, } p = 2 \\ 2^{i+1}, n \text{ odd, } p = 2 \\ 0, n \text{ even, } p \neq 2 \\ Z_{2}, n \text{ even, } p = 2 \end{cases}$$

Moreover, from examining the action in homology of the Hurewicz images of the generators, we conclude that, in each case, the inclusion of the one in the next is given by multiplication by p. Hence

$$\lim_{\rightarrow} (Z_{4} \subset , Z_{8} \subset \ldots) = Z_{(2)}^{\infty}$$

is the two-local component of the limit. Clearly, there is a surjection of this Z on Z with kernel Z_2 . Thus it is natural to embed the $\binom{2^{\infty}}{2}$ resulting group in $\mathbb{Q}/2\mathbb{Z}$. This completes the proof of (b).

<u>Remark</u> 10.11 Theorem 10.9 is exactly the homotopy theoretic result which is needed to do simply-connected surgery ([41]), and also plays a key role in [40]. The results of Section 2 also allow us to know exactly at which suspension the various homotopy elements first appear. For example, the reader is asked to verify that, for n odd, a Q/Z appears in $\pi_{2n+1}(K(Q/Z,n) \wedge KQ/Z,n)$, and hence in $\pi_{2n+2}(\Sigma(KQ/Z,n))$. However, while the map

$$\pi_{2\mathbf{n}+2}(\Sigma(\mathtt{K}(\mathtt{Q}/\mathtt{Z},\mathtt{n}))) \rightarrow \pi_{2\mathbf{n}+1}^{s}(\mathtt{K}(\mathtt{Q}/\mathtt{Z},\mathtt{n}))$$

is an isomorphism, the approximating maps

$$\begin{array}{c} z \\ 2^{i} \xrightarrow{Z} 2^{i+1} \end{array}$$

for $K(Z_{2^i},n)$ always have cokernel Z_2 .

11. Some Calculations Of The Stable Homotopy Groups For The K(Z,n)

In this section, we apply the techniques of §9 to study $\pi_{*}^{S}(K(Z,n))$ in some special cases. The calculational results illustrate the structure of the exact sequence 6.11.

(10.1) shows that we need study only $\pi_*(S^{L-1} \ltimes_T K(Z,n) \wedge K(Z,n))$ for L > n. To do this, we shall calculate the groups

(11.1)

$$Ext_{G(2)}^{**}(H^{*}(\Gamma_{L}K(Z,n)), Z_{2}),$$

 $Ext_{G(2)}^{**}(H^{*}(\Gamma_{L}), A, Z_{2})$

in three steps. First, we calculate $\operatorname{Ext}_{G(2)}^{**}(\operatorname{H}^{*}(G_{L})/A, Z_{2})$; then we calculate $\operatorname{Ext}_{G(2)}^{**}(A, Z_{2})$; finally, we put them together to obtain the desired groups (ll.1).

We begin with the calculation of ${\rm Ext}_{C(2)}^{**}({\rm H}^*({\rm G}_L)/{\rm A},\,{\rm Z}_2)$, using the second spectral sequence in §9.

Lemma 11.2
$$\operatorname{Ext}^{**}_{G(2)}(D[H^*(K(Z,n))], Z_2) \cong$$

 $\cong \operatorname{Ext}^{**}_{G_1}(Z_2, Z_2) \otimes \operatorname{Ext}^{**}_{\Lambda(Q_3 Q_4, \dots)}(Z_2, Z_2)$

in total degrees less than 4n+1 . Here G_1 is the subalgebra of G(2) generated by Sq^1 and Sq^2 .

<u>Proof</u> We imitate the main idea in the proof of 9.11. The map u $H^*(K(Z,n))$ is onto in dimensions less than or equal to 2n and has kernel $G(2)Sq^1$. Thus, the kernel of the doubling homomorphism

 $G(2) \xrightarrow{D} DG(2) \xrightarrow{D(u)} DH^*(K(Z,n))$

is $D^{-1}(G(2)Sq^1) = Ker(D) \oplus G(2)(Sq^2)$, but this is $G(2)\overline{E}(Sq^1, Q_2, Q_3^{-\cdots}) \oplus G(2)(Sq^2)$, but this ideal is clearly isomorphic to

$$G(2) \cdot \overline{(C_1 \otimes E(Q_3 Q_4 \cdots)))}$$
,

and 11.2 follows.

Now we recall the well-known

<u>Lemma</u> 11.3 Ext^{**} $(Z_2, Z_2) = P(h_0, h_1 Q, P)/R$ where R is the set of relations

$$Q^{2} = h_{0}^{2}P,$$
$$h_{0}h_{1} = 0,$$
$$h_{1}^{3} = 0,$$
$$h_{1}Q = 0.$$

<u>Here</u> h_0 <u>has bidegree</u> (1,1), h_1 <u>has bidegree</u> (1,2), $Q \in Ext^{3,7}$ and $P \in Ext^{4,12}$. (See e.g. [9, §6.1] for a proof.)

We mention in passing that Q has Massey product representation $\langle h_1^2, h_1, h_0 \rangle$. Also, P has representation $\langle h_1, h_1^2, h_1, h_1^2 \rangle$.

ll.2 and ll.3 together determine the E^1 term of our spectral sequence. We now turn to the evaluation of the differentials. I know of no way to do this without going to the chain level. First, we need some notation. In E^1 , we denote an element in the kth copy of $Ext_{G_1} \otimes E(Q_3 \cdots)^{(Z_2, Z_2)}$ with a (k) on the right. Thus $\theta \otimes q_3^{i_1} \cdots q_r^{i_r}$ in the kth copy will be written $\theta \otimes q_3^{i_1} \cdots q_r^{i_r}(k)$.

The generators of \mathbf{C}^0 are thus the $\{(k)_{\star}\}$, and those in \mathbf{C}^1 are the $h_0(k)_{\star}$, $h_1(k)_{\star}$, $q_3(k)_{\star}$. In principal, we can completely describe the differentials once we have specified the boundaries of the generators in $\mathbf{C}^{(1)}$.

In the associated graded gadget, $\partial h_0(k) = Sq^1(k)_*$, $\partial h_1(k)_* = Sq^2(k)_*$, $\partial q_i(k) = Q_i(k)_*$. Of course, these are not in the kernel of the surjection $C^0 \to H^*(G_L)/A$ in general, so there are higher filtration correction terms which must be added to these (presumptive) boundaries. In particular,

(11.4)

$$Sq^{2}(k)_{*} + {\binom{k-1}{2}}(k+2)_{*}$$
,
 $Sq^{1}(k)_{*} + {\binom{k-1}{1}}(k+1)_{*}$

are in the kernel, and hence are appropriate choices for the actual differentials. Now, using standard calculational techniques, we have

(11.5)
$$Q_{1}Sq^{k} = Sq^{(0\cdots 0^{2})}Sq^{k+1} + Sq^{(0\cdots 0^{4})}Sq^{k+3} + \theta$$

where θ vanishes on elements of dimension k+4 . Hence if we are concerned only with the first four or five differentials, we can assume

(11.6)
$$\partial q_{i}(k)_{*} = Q_{i}(k)_{*} + Sq^{(0\cdots 0^{2})}(k+1)_{*} + Sq^{(0\cdots 0^{4})}(k+3)_{*}$$
.

Thus we obtain

Lemma 11.7
$$\delta^{1}(k) = {\binom{k}{1}}h_{0}(k-1)$$
,
 $\delta^{1}(q_{3}(k)) = {\binom{k-1}{1}}h_{0}q_{3}(k-1)$,
 $\delta^{2}(k) = {\binom{k-3}{2}}h_{1}(k-2)$,
 $\delta^{2}(q_{3}(k)) = {\binom{k-2}{2}}h_{1}q_{3}(k-2)$.

These differentials and some routine algebra allow the complete determination of $E_{s,r,t}^{4}$ in dimensions r-s < 12 . For example, when n = 4L+1 , we find



Our conventions for writing the Ext groups are the usual ones (e.g., see [16]). However, for the reader's convenience, we list them here. A vertical line connecting two dots represents the fact that the upper is h_0 times the lower. A 45° line connecting two dots says that the upper is h_1 times the lower. Finally, a dashed vertical line implies again that the upper is h_0 times the lower, but the extension is not obvious in E^1 . (Actually, the precise significance is δ (lower) = $(Sq^1)^*$ (upper) + θ where θ has filtration at least 4 higher.)

Examining (11.8), we see that the only possible differential is $\delta^{7}(10) = \begin{cases} q_{3}(3) \\ 0 \end{cases}$ In fact, we have

 $\underline{\text{Lemma}} \text{ ll.9 } \delta^{7}(10) = \begin{cases} q_{3}(3) , & n \equiv 1(8) \\ 0 , & n \equiv 5(8) \end{cases}$ (For the proof, it is necessary 0 , $n \equiv 5(8)$ only to expand (ll.6) two further stages in these cases.)

Thus, in view of the dashed vertical lines, this determines the remaining differentials, and (ll.8) represents $\operatorname{Ext}_{G(2)}(\operatorname{H}^{*}(\Gamma_{n})/\operatorname{A}, \mathbb{Z}_{2})$ for $n \equiv 5(8)$, while

$$\operatorname{Ext}_{G(2)}(\operatorname{H}^{*}(\Gamma_{n})/A, \mathbb{Z}_{2})$$

for $n \equiv 1(8)$ has the form



Now, following our program, we turn to the calculation of $\operatorname{Ext}_{G(2)}(A, Z_2)$ Unfortunately, I know no indirect way of studying this group, so I found it necessary to write down an explicit resolution in our range of interest. We suppress the calculations (in the interest of mercy to all concerned) and record the result:



with, perhaps, some missing terms in r-s = 11. Combining this with (11.10), and calculating some obvious differentials in the exact sequence

(11.12) $\dots \operatorname{Ext}_{G(2)}(\operatorname{H}^{*}(\operatorname{F}_{L}), \operatorname{Z}_{2}) \to \operatorname{Ext}_{G(2)}(\operatorname{A}, \operatorname{Z}_{2}) \xrightarrow{\delta} \operatorname{Ext}_{G(2)}(\operatorname{H}^{*}(\operatorname{F}_{L})/\operatorname{A}, \operatorname{Z}_{2}) \to \dots,$ we find that $\operatorname{Ext}_{G(2)}(\operatorname{H}^{*}(\operatorname{F}_{8k+1}), \operatorname{Z}_{2})$ has the form



There are a number of "suprising" facts concealed in (11.13). For one, note that δ in (11.12) is highly non-trivial. For another, note e.g. that, for h_0D , H, there are extensions of the $\operatorname{Ext}_{G(2)}(\mathbb{Z}_2,\mathbb{Z}_2)$ module structure connecting the remaining terms from (11.10) and (11.11). The original interpretation of elements in $\operatorname{Ext}_{G(2)}(\operatorname{H}^*(\operatorname{F}_n)/\operatorname{A}, \mathbb{Z}_2)$ was as "stable" higher order cohomology operations which vanish universally on integral cohomology classes of dimension n (e.g. see [11]). However, it is clear that that hypothesis can no longer be supported since the terms from $\operatorname{Ext}_{G(2)}(\operatorname{A},\mathbb{Z}_2)$ play a very definite role in determining the actual Ext groups of F_n , and the differential depends on n. Thus the general defining system for an operation represented in $\operatorname{Ext}_{G(2)}(\operatorname{H}^*(\operatorname{F}_n)/\operatorname{A}, \mathbb{Z}_2)$ may well have been built on relations which depend on lower order elements in $\operatorname{Ext}_{G(2)}(\operatorname{A},\mathbb{Z}_2)$ and n.

We now calculate the differentials in (11.13), and conclude this section by evaluating

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$$E^{2}_{**}(F_{n}, A, Z_{2})$$

in this range.

We first observe that the $\rm Z_{l_4}$ Bockstein (Proposition 3.6) implies some $\rm \partial_2$ differentials. Thus we have

Lemma 11.14 In (11.13), we have

$$\partial_2 g_3(0) = h_0 B$$
,
 $\partial_2(H) = h_0 D$.

Somewhat less immediately, we have

<u>Lemma</u> 11.15 (i) $\partial_2(g_3(1)) = 0$,

(ii) $\partial_{\rho}(E) = 0$.

Proof (i) Consider the inclusion

$$u : S^{1} \ltimes_{T} K \land K \to S^{n-1} \ltimes_{T} K \land K .$$

It is easy to verify that $g_3(1)$ is present in im (u_*) on Ext groups. However, $h_0^{-3}D$ is not in this image. Hence $\partial_2(g_3(1)) = u_*\partial_2(g_3(1)) = 0$, and (i) follows. To prove (ii), note that E^* has representative cycle $\langle \dot{s}q^3, sq^4sq^2 \rangle$. Consider the inclusion

$$\mathtt{I} : \mathtt{K} \land \mathtt{K} \to \mathtt{S}^{n-1} \ltimes_{_{\mathrm{T}}} \mathtt{K} \land \mathtt{K} \ .$$

 $K \wedge K$ is a wedge of $K(Z_2,n)$'s and K(Z,n) 's in low dimensions. A direct calculation shows that $Sq^4Sq^2\iota \otimes Sq^3\iota$ is a generator in $H^*(K \wedge K)$ over

G(2) . Hence it is dual to an element b in $\pi_{\ast}(K \wedge K)$. Clearly, $I_{\ast}(b)$ is detected by E^{\ast} , and (ii) follows.

<u>Remark</u> 11.16 The splitting of $K \wedge K$ used in the proof of 11.15(ii), together with the cofibrations

(11.17)
$$S^r \ltimes_T K \land K \to S^{r+1} \ltimes_T K \land K \to \Sigma^{r+1} K \land K \to \Sigma S^r \ltimes_T K \land K \to \dots$$

provide us with a good method for constructing homotopy classes in our range, while the global calculations of Ext groups provide an effective way to limit the number of elements whose study is required. It would seem to the author that this remark provides the reader with effective tools for analyzing the stable homotopy of Eilenberg-MacLane spaces so far as desired in our range!

Returning to our calculation, we easily see that 11.14 and 11.15 give all the ∂_2 differentials. Moreover, $E^3 \cong E^{\infty}$, and we have <u>Theorem 11.18 The first ten stable homotopy groups of K(Z,8k+1) are given</u> by the table

For j = 10, $\pi_{16k+11}^{s}(K(Z, 8k+1))$ has order at most 8.

<u>Remark</u> 11.19 The groups obtained in 11.18 for j = 6, 7 do not agree with those obtained by Mahowald and Williams in [14]. They apparently missed the generator C corresponding to the cycle $\langle Sq^2, Sq^4 \rangle$.

Finally, we apply 11.18 to calculate the differential

 ∂ : Extⁱ_{G(2)}(H^{*}(F_n)/A, Z₂) \rightarrow Extⁱ⁺²_{G(2)}(A,Z₂) defined in 6.11. We easily find that

$$E^{2}_{*}(F_{8k+1}, A, Z_{2})$$

has the form



Again using 11.18, the reader can easily calculate the higher differentials in (11.20).

12. An Example For The Metastable Exact Sequence

In [13], Mahowald used his special case of 8.5 to give extensive calculations of the metastable homotopy groups of spheres. The most elementary spaces after the spheres are the various two-cell complexes; in particular, the four such non-trivial ones distinguished by their structure as G(2)-modules.

In this section, we apply 8.5 to calculate some of the unstable groups of the space $X_7 = s^7 U_2 e^8$ as an example of the techniques involved.

The stable homotopy of $X_{7}^{}$ is easily studied. We have an exact sequence

$$(12.1) \xrightarrow{\partial} \operatorname{Ext}_{\mathbb{G}(2)}(\mathbb{Z}_{2},\mathbb{Z}_{2}) \xrightarrow{\mathrm{I}} \operatorname{Ext}_{\mathbb{G}(2)}(\mathbb{H}^{*}(\mathbb{X}_{7}),\mathbb{Z}_{2}) \xrightarrow{\mathrm{J}} \operatorname{Ext}_{\mathbb{G}(2)}(\mathbb{Z}_{2},\mathbb{Z}_{2}) \xrightarrow{\mathrm{J}}$$

and ∂ is exactly multiplication by h_0 , $\partial x = h_0 X$ (eg., see [2]). Hence $Ext_{\Omega(2)}(H^*(X_7), Z_2)$ consists of elements of two kinds:

(12.2) i) elements
$$\alpha$$
 in the image of I,
ii) elements β for which $J(\beta) \neq 0$ and $h_J(\beta) = 0$.

Elements of this second kind may be written explicitly as Massey products in $Ext_{G(2)}(H^*(X_7), Z_2)$. Exactly, we have

(12.3)
$$\beta = \langle J(\beta), h_0, \underline{\iota} \rangle$$

where $t = I(\iota)$ is the unique non-zero element in $Ext^{0,7}_{C(2)}(H^*(X_7), Z_2)$. Consequently, we obtain <u>Proposition 12.4</u> The following extensions occur in $\text{Ext}_{G(2)}(\text{H}^{*}(X_{7}), Z_{2})$:

- i) $h_0(\gamma,h_0\mu) = h_1I(\gamma)\mu;$
- ii) If $\alpha\gamma = 0$ in $\operatorname{Ext}_{\mathbb{G}(2)}(\mathbb{Z}_2,\mathbb{Z}_2)$, then $\alpha\langle\gamma,h_{\circ},\downarrow\rangle = \langle\alpha,\gamma,h_{\circ}\rangle_{\downarrow}$.

Proof By the slide formula for Massey products,

$$\alpha(\beta,\gamma,\delta) = \langle \alpha,\beta,\gamma\rangle\delta$$
,

we obtain (ii). To show (i), note that

$$\mathbf{h}_{o}\langle\gamma,\mathbf{h}_{o},\iota\rangle = \langle\mathbf{h}_{o},\gamma,\mathbf{h}_{o}\rangle_{\mathbf{w}}$$

from 12.2(ii) and 12.4(ii). On the other hand, the "Hirsch" formula ([17], [24]) $\langle a,b,a \rangle = (a \cup_1 a)b$ implies $\langle h_0,\gamma,h_0 \rangle = h_1\gamma$ since $h_0 \cup_1 h_0 = h_1$, and (i) now follows.

Using the tabulated results ([16], [33]) on $Ext_{G(2)}(Z_2,Z_2)$, we obtain $Ext_{G(2)}(H^*(X_7), Z_2)$ for t-s < 20 as



Proposition 12.6 In the range of (12.5), there are only two non-zero differentials:

$$\partial_2(e_0) = h_1^2 d_0$$
,
 $\partial_2 T = h_1^2 d_0 L$.

Moreover, there is a non-trivial extension

$$2\{h_{h}\} = Q$$
.

<u>Proof</u> These differentials are non-trivial since the sequence (12.1) is actually a long exact sequence of Adams spectral sequences, being induced from the obvious cofiber map of spaces, and the corresponding differentials are non-zero for spheres.

The next possibility for a non-zero differential is $h_{l_{L}}L$. In fact, $\partial_{2}(h_{l_{L}}L) = (2h_{3}^{2})L = \{2\sigma^{2}\}L$ by naturality, but $2\sigma^{2}L = \sigma^{2}\eta^{2}L = 0$ since $\sigma^{2}\eta = 0$. Thus $h_{l_{L}}L$ represents the Toda bracket

$$\left\{\sigma^{2}, (2,\eta^{2}), \left(L_{l}, L_{l}\right)\right\}$$

in $\pi^{s}_{\star}(X_{7})$, and is indeed an infinite cycle.

There remains the possiblity of a differential on f. In the Adams spectral sequence for the sphere, $\partial_2(f_0) = 2h_2(d_0)$. Hence f_0 represents the Toda bracket

in $\pi^{\rm S}_{}({\rm X}_7)$. This completes the first part of 12.6.

For the second assertion, note that ${\rm h}_{\rm l_{\rm l}}$ is a permanent cycle representing the Toda bracket

{\sigma², 2, <u>1</u>}.

Hence

(12.7)
$$2\{\sigma^2, 2, 1\} = \{\sigma^2, 2, 1\} = \{\sigma^2, 2, 2\} = \{\sigma^2, 4, 1\}$$
.

But this last represents $\langle h_3^2, h_0^2, \iota \rangle = 0$, and 12.6 follows.

<u>Remark</u> 12.8 The reader might wonder why we did not use the slide formula in (12.7), obtaining $2\{\sigma^2, 2, \iota\} \subset \{2\sigma^2, 2\}_{\iota} = \eta\sigma^2_{\iota} = 0$. But these equations are valid only modulo the total indeterminacy which in this case is $2\pi^{s}_{22}(X_{7})$.

Now we turn to the calculation of $\pi_*(S^7 \cup_2 e^8)$ in the metastable range. The fiber in the map

$$\mathbf{F}_{\mathbf{L}} \rightarrow \mathbf{S}^{7} \cup_{2} \mathbf{e}^{8} \rightarrow \boldsymbol{\Omega}^{\mathbf{L}} (\mathbf{S}^{\mathbf{L}+7} \cup_{2} \mathbf{e}^{\mathbf{L}+8})$$

is given as $\Omega S^{L-1} \ltimes_{T} [S^7 \cup_{2} e^8) \land (S^7 \cup_{2} e^8)]$. The calculation of $Ext \overset{**}{}_{C(2)}(H^*(F_L),A,Z_2)$ is routine using the first spectral sequence in §9 (in this case an exact sequence), and we find $Ext \overset{s,r}{}_{C(2)}(H^*(F_L),A,Z_2)$ has the form



In particular, the reader should note that, since $\beta_{l_1} \langle e_7, e_8 \rangle = \langle e_8 \otimes e_8 \rangle$, it follows that the element γ , in the space $E_1(F_L, A)$, corresponding to $\operatorname{Sq}^1(\iota_0)$ satisfies $\operatorname{Sq}^1(\gamma) = e_8 \otimes e_8$. This is an example of the type of twisting referred to in §6, particularly 6.11. It is easy to check that there are no differentials in this range in the resulting Adams sequence, so $E^2 = E^{\infty}$. We now evaluate the map ∂^i_2 of 8.5.

<u>Proposition</u> 12.10 $\partial_2(h_{jk}) = I_0$, and there are no further non-trivial images for r-s < 8.

<u>Proof</u> Since the k-invariant of h_{3L} is $Sq^{8}(\iota_{7})$, it is evident that $\partial_{2}(h_{3L}) = I_{0}$. The only other possibility is $\partial(h_{3L}) = \epsilon h_{1}\langle e_{7}, e_{8}\rangle$, but this is part of the tower associated purely to A. Hence $\epsilon = 0$, and 12.10 follows.

<u>Corollary</u> 12.11 The unstable resolution of $S^7 \cup_2 e^8$ in the metastable range has the form



There are clearly no further differentials, so $E^2 = E^{\infty}$ in this range.

13. Further Calculations For Some Truncated Projective Spaces

In this section, we study the groups

(13.1)
$$\pi_{i+n}(\Sigma^{n}(P_{2}^{6})) , i \leq 6,$$
$$\pi_{i+n}(\Sigma^{n}(P_{2}^{6})) , i \leq 6, n = 2.$$

They are of importance in the joint work of the author and E. Rees on embedding projective spaces ([42]), and provide examples illustrating much of the theory of Part I not already explored in §§9-12.

We being by recording the stable homotopy of the spaces in (13.1) and in our range.

Lemma 13.2 $\operatorname{Ext}_{\mathbb{Q}(2)}(\operatorname{H}^{*}(\operatorname{p}_{2}^{6}), \operatorname{Z}_{2})$ is given by the table



in our range.



in <u>our range</u>. (Both 13.2 and 13.3 are evaluated by using the exact sequence of cofiberings

(13.4)
$$P_2^{\ 6} \xrightarrow{j} P_2^{\ \infty} \rightarrow P_7^{\ \infty} ,$$
$$P_1^{\ 6} \rightarrow P_1^{\ \infty} \rightarrow P_7^{\ \infty} ,$$

taking the resulting long exact sequences of Ext groups, and using the tables in [13], pp. 54, 55, and 60 to evaluate $\operatorname{Ext}_{G(2)}(\operatorname{H}^{*}(\operatorname{P}_{i}^{\infty}), \operatorname{Z}_{2})$ in our range.)

From 13.2, 13.3, and the fact that, in $\pi_8(P_2^{\infty})$, the class corresponding to 4L(5) is non-zero ([13], p. 55, and the observation $j_*(L(5)) = 2\{e_5\}$), it follows that $\partial_2 h_2(5) = 0$ in the Adams spectral sequences for which 13.2 and 13.3 are the E^2 terms. On the other hand, by inspection we see that $\partial_2(h_2(5))$ is the only possible non-zero differential. Thus $E^2 = E^{\infty}$ in both cases, and we have the table (localized at 2)

Now we consider the unstable homotopy groups (13.1) for $\Sigma^2 P_1^6$, $\Sigma^2 P_2^6$. Since these are suspensions, the fibers

(13.6)

$$F_{L}^{1} \rightarrow \Sigma^{2} P_{1}^{6} \rightarrow \Omega^{L} \Sigma^{L+2} P_{1}^{6}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$F_{L}^{2} \rightarrow \Sigma^{2} P_{2}^{6} \rightarrow \Omega^{L} \Sigma^{L+2} P_{2}^{6}$$

are given explicitly in §1 and their homology calculated in §3 (particularly 3.9, 3.10). There is no difficulty in constructing resolutions to evaluate $\pi_*(F_L^{\epsilon})$ for $* \leq 8$. (For the fiber F_L^{1} , we are one dimension beyond the point where 1.11 is valid; thus the fact that we are dealing with suspensions is essential for effective calculation.)

Lemma 13.7
$$\pi_7(F_L^2) = Z$$
 generator A,
 $\pi_8(F_L^2) = Z_4 \oplus Z_2$ generators B, C

with relation $\eta A = 2B$.

 \underline{Proof} We are in the range in which 1.11 applies. The fiber ${\rm F}_{\rm L}^{\ 2}$ has homology generators

(13.8)

$$7$$
 8
 9
 $\sigma(e_{4} * e_{4})$
 $\sigma Q_{1}(e_{4})$
 $\sigma Q_{2}(e_{4})$
 $\sigma(e_{4} * e_{5})$
 $\sigma(e_{5} * e_{5})$
 $\sigma(e_{4} * e_{6})$

Passing to cohomology over G(2) , we apply 3.7 to show that

$$Sq^{1}(\sigma Q_{1}(e_{4})^{*}) = \sigma(Q_{2}(e_{4}))^{*}$$
,
 $Sq^{1}(\sigma(e_{4} * e_{5}))^{*} = Sq^{2}(\sigma(e_{4} * e_{4}))$,

and there are no further non-zero operations in (13.8). 13.7 now follows on taking $\text{Ext}_{G(2)}(\text{H}^{*}(\text{F}_{L}^{2}), \text{Z}_{2})$ in this range.

The situation for \mathbb{F}_{L}^{1} is similar. However, we must also take into account the classes $\Sigma^{2}e_{1}\langle\Sigma^{2}e_{1}, \Sigma^{2}e_{2}\rangle$, $\Sigma^{2}e_{2}\langle\Sigma^{2}e_{1}, \Sigma^{2}e_{2}\rangle$ (in the notation of 3.9).

$$\underbrace{\text{Lemma }}_{13.8} 13.8 \quad \pi_{5}(F_{L}^{1}) = Z_{2} \qquad \underbrace{\text{generator}}_{\text{generator}} D = (\Sigma^{2}e_{1} * \Sigma^{2}e_{1})$$

$$\pi_{6}(F_{L}^{1}) = Z_{4} \qquad \underbrace{\text{generator}}_{\pi_{7}(F_{L}^{1})} = Z_{2} \qquad \underbrace{\text{generator}}_{\pi_{7}(F_{L}^{1})} F = \langle \Sigma^{2}e_{1}, \Sigma^{2}e_{2} \rangle$$

$$\pi_{8}(F_{L}^{1}) = Z_{2} \qquad \underbrace{\text{generator}}_{\text{generator}} F = \langle \Sigma^{2}e_{1}, \Sigma^{2}e_{2} \rangle$$

$$\pi_{8}(F_{L}^{1}) = Z_{2} \qquad \underbrace{\text{generator}}_{\text{generator}} \int_{D}^{G} = \Sigma^{2}e_{1} \cdot \langle \Sigma^{2}e_{1}, \Sigma^{2}e_{2} \rangle$$

$$\pi_{8}(F_{L}^{1}) = Z_{2} \qquad \underbrace{\text{generator}}_{\text{generator}} \int_{D}^{H} = \{\eta, 4, D\} .$$

Also, in the map φ : $F_L^{\ l} \rightarrow F_L^{\ 2}$, we find $\varphi_*(D) = \varphi_*(E) = \varphi_*^*(F) = \varphi_*(G) = 0$, but

 $\phi_*(H) = \eta A$.

<u>Proof</u> Through dimension 8, $\pi_*(F_L^{\ l}) = \pi_*^{\ s}(F_L^{\ l})$ since $F_L^{\ l}$ is 4-connected. We thus take a stable resolution. $H^*(F_L^{\ l})$ has generators

in dimensions less than or equal to 9. A basis over G(2) is easily seen to be $(e_3 * e_3)^*$, $Q_1(e_3)^*$, $(e_4 * e_4)^*$, $\langle e_3, e_5 \rangle^*$, $e_3 \langle e_3, e_4 \rangle^*$, $Q_4(e_3)^*$, $(e_5 * e_5)^*$ in this range. Relations are $Sq^1(Q_1(e_3))^* = Sq^1(e_4 * e_4)^* = 0$,

(13.10)
$$sq^{*}(e_{3} * e_{3})^{2} = sq^{2}(Q_{1}(e_{3}))^{2},$$
$$(sq^{4}+sq^{3}sq^{1})(e_{3} * e_{3})^{*} = sq^{2}(e_{4} * e_{4})^{*}$$

Next, since $\beta_{i_1}(e_{i_1} * e_{i_2}) = \langle e_3, e_{i_2} \rangle + Q_1(e_3)$, we see that, in the Adams spectral sequence for F_L^{-1} , $\partial_2(e_{i_1} * e_{i_2}) = h_0^{-2}(Q_1(e_3))$. Moreover, there are no further differentials in our range, and $E^3 = E^{\infty}$.

Finally, noting the fact that

$$\varphi^{*}(e_{i_{4}}^{*} * e_{i_{4}}^{*})^{*} = (e_{i_{4}}^{*} * e_{i_{4}})^{*},$$

we see that the filtration 2 class, due to the second relation in (13.10), maps to ηA in $\pi_{\star}({F_L}^2)$.

Corollary 13.11 The E^2 term of our spectral sequence 6.4 for $\Sigma^2 P_2^{-6}$ has the form



up to extensions (which are marked with dotted lines).

<u>Remark</u> 13.12 By using the techniques of §§1 and 8, we note that $j_*(G)$, $j_*(H)$ generate the unstable part of the homotopy group $\pi_8(\Sigma^2 P^6)$. Moreover, a simple argument with Whitehead products shows that $j_*(G)$ cannot be a suspension element. Thus $j_*(H)$ is the only candidate for an element in $\sigma^2 \pi_6(P^6)$ in $\pi_8(\Sigma^2 P^6)$. In particular, the generator of $\pi_6(P^6)$ is the attaching map of the cell building P^7 :

(13.13)
$$s^6 \xrightarrow{\lambda} p^6 \rightarrow p^7 \rightarrow s^7$$
,

and it must be clear from (13.13) that the top class in $\Sigma^2 P^7$ is spherical if and only if $\sigma^2(\lambda) = 0$ in $\pi_8(\Sigma^2 P^6)$. But the only non-zero candidate is $j_*(H)$; and in $(\Sigma^2 P_2^6)$, $i_* j_*(H) \neq 0$ by 13.8. This proves the key result needed in [43]:

Theorem 13.13 $\Sigma^2 P^7$ has top-class spherical if and only if $\Sigma^2 P_2^7$ does.

We now complete the calculations of this paper by calculating $\pi_{2+i}(\Sigma P_2^6)$ and $\pi_{1+i}(P_2^6)$ in the range of (13.1).

Without difficulty, we find that the homotopy of the fiber $\,F_{L}^{\,\,\,5}\,$ in the map

$$\Sigma P_{2}^{6} \rightarrow \Omega^{L} \Sigma^{L+1} P_{2}^{6}$$

is given by

(13.14)
$$\frac{j}{\pi_{j}(F_{L}^{3})} = \frac{5}{Z_{2}} = \frac{6}{Z_{2}} = \frac{7}{Z_{2}}$$
Generator I J K, $\{\eta, 2, I\}$

Clearly, $\partial_2(v_2) = I$, and no other non-trivial boundary is possible. A schematic representation of $\pi_*(\Sigma P_2^6)$ in our range can now be given as

(13.15)

$$\begin{array}{c|c}
 & 4 \\
 & 3 \\
 & 2 \\
 & 1 \\
 & s \\
 & t-s \rightarrow 3 4 5 6 7
\end{array}$$

$$\begin{array}{c|c}
 & 4 \\
 & 3 \\
 & 5 \\
 & 1 \\
 & s \\
 & t-s \rightarrow 3 4 5 6 7
\end{array}$$

$$\begin{array}{c|c}
 & 4 \\
 & 5 \\
 & 1 \\
 & s \\
 & t-s \rightarrow 3 4 5 6 7
\end{array}$$

The extensions $2v(3) = j_*(K)$, $\eta[h_0h_2(2)] = j_*(\eta,2,I)$ must be verified. (Of course, the second extension follows directly from the fact that $\partial[h_2(2)] = I$.) To obtain the remaining extension, note that we are outside the range where 8.5 holds, so the extension is not necessarily surprising. (The point is, if 8.5 were true in this dimension, then in the resolution both K and $h_2(3)$ would occur in filtration 1, and the extension $2v(3) = j_*K$ would clearly be impossible.)

Specifically, the difficulty with extending 8.5 occurs here because $Sq^{4}(\iota_{4}) = \iota_{4}^{2}$ is a Z_{4} class, and $\beta_{4}(\iota_{4})^{2} = [Sq^{1}\iota_{4} \cup \iota_{4} + Sq^{4}Sq^{1}\iota_{4}]$ in $K(Z_{2}, 4)$. Using the fact that we can take an unstable "resolution" of ΣP_{2}^{6} with first K-invariants given as
(13.16)
$$\Sigma P_{2}^{6} \xrightarrow{(\sigma e_{2}, \sigma e_{3})} \kappa(z, 3) \times \kappa(z_{2}, 4) \xrightarrow{Sq^{1}\iota_{4}+Sq^{2}\iota_{3}, \iota_{4}^{2}} \kappa(z_{2}, 5) \times \kappa(z_{4}, 8)$$

we easily justify (13.15), and this provides a good example of why the range of dimensions in which 8.5 holds cannot be extended.

<u>Remark</u> 13.17 We note in passing that, in the map $\mu : \mathbb{F}_{L}^{3} \to \Omega \mathbb{F}_{L}^{1}$, we have $\mu_{*}(K) = 0$, $\mu_{*}(\eta, 2, I) = \eta A$; and in the map $\mu : \Sigma \mathbb{F}_{2}^{6} \to \Omega \Sigma^{2} \mathbb{P}_{2}^{6}$, we have $\mu_{*}(h_{\Omega}h_{2}(2)) = \mathbf{j}_{*}(A) + 2\upsilon(2)$.

Now we conclude the discussion by studying briefly the homotopy of P_2^6 itself. It is a routine calculation with the Serre spectral sequence to give the homology of the fiber in the map

$$\mathbf{P_2}^6 \to \mathbf{\Omega}^{\mathbf{L}} \mathbf{\Sigma}^{\mathbf{L}} \mathbf{P}^{\mathbf{6}}_2$$

in dimensions less than 8. The fact that there are non-trivial cup products in P_2^6 produces some minor unpleasantries such as: (i) the class which should have transgressed to $(e_3 \circ e_4)^*$ is identified with the class transgressing to $(e_2 \circ e_5)^*$, and (ii) in $H^7(F)$, the class which transgresses to $(e_2 \circ e_2 \circ e_2 \circ e_2)^*$ is a Z₈-Bockstein,

$$\beta_8 \{\sigma^{-1}(\langle e_3, e_4 \rangle^* + e_1 \cup e_3 \otimes e_3)\} .$$

This illustrates the way in which the results of §l fail when X is not a suspension.

The homotopy of F is given by

Further, we have relations $\eta^2 \bar{K} = 2M$, $\eta[\iota_2, \iota_2] = 2\bar{K}$, and $\Sigma(\bar{K}) = I$, $\Sigma(M) = \{\eta, 2, I\}$ while $\Sigma P = K$ on suspending, i.e., taking the map

$$F \rightarrow \Omega F_L^3$$

Clearly, $\partial v(2) = \overline{K} \oplus L$, $\partial v(3) = [\iota_3, \iota_3]$. We may also verify $\partial(L(5)) = 2P$. This is not trivial; it involves the construction of an unstable resolution of P_2^6 , and makes essential use of the fact ([29]) that $\Sigma^2 P_2^7$ is reducible. Here is what the resolution looks like through dimension 6:



The k-invariants may be easily calculated. Notice that M has filtration 2 here but σM has filtration 3 in the resolution of ΣP_2^6 . This change of filtration degree would seem to merit further study (e.g., see [15] for an example of what can happen).

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