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A COMBINATORIAL COMPUTATION OF THE FIRST PONTRYAGIN CLASS OF THE COMPLEX PROJECTIVE PLANE

ABSTRACT. This paper carries out an explicit computation of the combinatorial formula of Gabrielov, Gel'fand, and Losik for the first Pontryagin class of the complex projective plane with the 9-vertex triangulation discovered by Wolfgang Kühnel. The conditions of the original formula must be modified since the 8-vertex triangulation of the 3-sphere link of each vertex cannot be realized as the complex of faces of a convex polytope in 4-space, but it can be so realized by a star-shaped polytope, and the space of all such realizations is not connected.

0. INTRODUCTION

In a series of papers [24]–[26] in the 1940s, L. S. Pontryagin defined and studied 'characteristic cycles' on smooth manifolds. In the introduction to the first paper of the series, he stressed the importance of having 'a definition of characteristic cycles which would be applicable to combinatorial manifolds' since it would provide an algorithm for their calculation from 'the combinatorial structure of the manifolds'. In the late 1950s, Réne Thom [33], and independently Rohlin and Sarc [28], proved that the rational Pontryagin classes are combinatorial invariants, and in the 1960s. S. P. Novikov [23] proved that these classes are topological invariants.

No combinatorial formula for Pontryagin classes was proved until the mid-1970s, when Gabrielov *et al.* [6], [7] established a formula for the first Pontryagin class $p_1(X)$ of a combinatorial manifold X. The formula expresses $p_1(X)$ in terms of the simplicial structure of X and some additional structure imposed on X, so in this sense the formula is not purely combinatorial. R. McPherson [18], N. Levitt [16], and D. Stone [30] have interpreted this formula from different viewpoints. Gel'fand and MacPherson [9], [19] have recently announced a combinatorial formula for the Pontryagin classes which holds in all dimensions. In none of these papers is the formula computed for any non-trivial example.

In this paper, we carry out an explicit computation of the Gabrielov–Gel'fand–Losik formula for the first Pontryagin class of the complex projective plane with the simplest possible combinatorial structure, the nine-vertex triangulation $\mathbf{CP}^2_9$ discovered by Kühnel [13], [14]. We present an algorithm that can be used to calculate the 'combinatorial part' of the

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formula for an arbitrary combinatorial manifold. The 'non-combinatorial part' of the formula requires a close examination of a particular eight-vertex triangulation M of the 3-sphere which is the link of each of the nine vertices of $\mathbf{CP}^2_9$. This triangulation M, studied by Grünbaum [11] following classical work of Brückner, cannot be realized as the complex of faces of a convex polytope in 4-space. But we show that M can be realized as the complex of faces of a star-shaped polytope in 4-space. Moreover, we will show that the space of all such (orientation-preserving) realizations is not connected, so $\mathbf{CP}^2_9$ violates the connectivity hypothesis made by Gabrielov *et al.* in the proof of their formula (condition (A) of [7]).

The paper is organized as follows. Sections 1–4 contain a brief exposition of the formula for the first Pontryagin class, in the original form presented by Gabrielov *et al.* [7] (cf. also [30]). We set notation and establish some general facts about the required additional structure on X: hypersimplicial sections and configuration data. After stating the formula, we show how work of Kuiper [15] can be used to remove the connectivity hypothesis of Gabrielov *et al.* if the dimension of X is 4.

Roughly speaking, hypersimplicial sections are chains in a cell complex $\Delta(X)$ associated to the combinatorial manifold X. The boundaries of these chains reflect the cell structure of $D(X)$, the dual cell complex of X. In Sections 5–8 we give an algorithm for finding hypersimplicial sections. This algorithm is purely combinatorial and is easy to implement on a computer.

There is no such general algorithm for the configuration data required for the formula. In Section 9–11 we construct these data for the Kühnel triangulation of the complex projective plane. We make considerable use of the simplicial automorphism group of this triangulation. We also investigate the Brückner–Grünbaum triangulation of the 3-sphere.

In Section 12 we apply our algorithm to produce hypersimplicial sections for the Kühnel triangulation, and then we plug these sections and the configuration data into the formula for the first Pontryagin class. A computer-aided computation verifies that the formula gives the expected answer. The final section contains some remarks and questions.

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1. Flattenings and configurations

Let L be a triangulation of a k-sphere, and let cL be a simplicial cone over L.

(1.1) DEFINITION. A *flattening* of L is an embedding

$$\psi: cL \to \mathbf{R}^{k+1}$$

which maps the cone vertex to the origin and which is linear on simplices of cL.

Note that a flattening is uniquely determined by the images of the vertices of L. Using this, it is easy to see that the set of all flattenings of L has a natural (smooth) manifold structure (as an open subset of $(\mathbf{R}^{k+1})^{\{\text{vertices of } L\}}$). We denote this manifold by $F(L)$. The general linear group GL_{k+1} acts on $F(L)$; the quotient space $CF(L)$ is called the *configuration space* of L, and its elements *configurations*.

Not very much is known about $CF(L)$:

(1.2) if $\dim L = 1$ then $CF(L)$ is contractible;
(1.3) if $\dim L = 2$ then $CF(L)$ is path connected;
(1.4) if $\dim L \geqslant 3$ $CF(L)$ can be empty; if nonempty it can be disconnected.

Statement (1.2) is trivial, but (1.3) is not; this is proved by Cairns [4]. The first part of (1.4) is also due to Cairns [2], and we shall verify the second part in Section 10.

Now, suppose that L is a combinatorial manifold, $|L| = S^k$. If p is a vertex of L then its link $L' = \mathrm{Lk}(p, L)$ is a triangulation of a $(k-1)$-sphere. There is a map π_p: $CF(L) \to CF(L')$ sending the configuration of L represented by a flattening

$$\psi: cL \to \mathbf{R}^{k+1}$$

to the configuration $\pi_p(\psi)$ of L' represented by the composition

$$cL' \subseteq cL \xrightarrow{\psi} \mathbf{R}^{k+1} \longrightarrow \mathbf{R}^{k+1}/\psi(p) \cong \mathbf{R}^k. \tag{1.5}$$

Note that, unless $\cong$ is specified, this composition is not well defined. However, flattenings obtained from different choices of $\cong$ are equivalent under the action of GL_k, hence the composition (1.5) represents a well-defined configuration of L'.

Let $F_r(L)$, $r \geqslant 0$, be the subset of $F(L)$ containing all $\psi \in F(L)$ such that the number of subsets S of $\{\text{vertices of } L\}$ with the property

$$|S| = k + 1 \text{ and } \psi(S) \text{ is linearly dependent}$$

is at most r. Let $CF_r(L)$ be $F_r(L) \bmod \mathrm{GL}_{k+1}$. In addition to $CF_0(L)$ (which we call *generic configurations*) we will need $CF_1(L)$ only.

Note that $\pi_p(CF_0(L)) \subseteq CF_0(L')$.

2. Hypersimplices

(2.1) DEFINITION. Let p, q be nonnegative integers. The *standard hypersimplex* $\Delta^{p,q}$ *of type* (p, q) is the convex polyhedron in $\mathbf{R}^{p+q+2}$ defined by

$$\Delta^{p,q} = [0, 1]^{p+q+2} \cap \left\{\sum x_i = q + 1\right\}.$$

Note that:

(a) $\dim \Delta^{p,q} = p + q + 1$,

(b) $\Delta^{p,q}$ and $\Delta^{q,p}$ are isomorphic under $x \mapsto x'$ where $x'_i = 1 - x_i$. If $p + q$ is even this isomorphism preserves orientation; if $p + q$ is odd it reverses orientation.

(2.2) EXAMPLES. (a) $\Delta^{p,0}$ is the standard $(p + 1)$-simplex.

(b) $\Delta^{1,1}$ is a solid octahedron.

(2.3) Let Z and A be disjoint subsets of $\{1, 2, \ldots, m\}$, and let

$$n + 1 - |Z| < |A| < n + 1,\ n < m.$$

Let

$$\Delta_n(Z, A) = [0, 1]^m \cap \{x_i = 0,\ i \notin A \cup Z\} \cap \{x_i = 1,\ i \in A\} \cap \left\{\sum x_i = n + 1\right\}.$$

Then $\Delta_n(Z, A)$ is isomorphic to $\Delta_{n-|A|}(Z, \varnothing)$ and hence is a hypersimplex of the type (p, q) where

$$q = n - |A|,$$
$$p = |Z| - q - 2.$$

An orientation of $\Delta_n(Z, A)$ (i.e. an orientation of $\Delta_{n-|A|}(Z, \varnothing)$) is determined by (the parity of) an ordering of elements of Z. Note that $\Delta_n(Z, A)$ is a face of the standard hypersimplex $[0, 1]^m \cap \{\Sigma x_i = n + 1\}$.

On the other hand, it is easy to see that every face (of positive dimension) of $\Delta = [0, 1]^m \cap \{\Sigma x_i = n + 1\}$ is of the form $\Delta_n(Z, A)$ (each face of Δ is an intersection of a face of $[0, 1]^m$ and the hyperplane $\{\Sigma x_i = n + 1\}$). If $S \subseteq \{1, 2, \ldots, m\}$, $|S| = n + 1$, let $\Delta(S)$ denote the point $x = (x_i)$, where $x_i = 1$ if $i \in S$, $x_i = 0$ otherwise.

(2.4) PROPOSITION. *There are natural correspondences:*

(i) $\{vertices\ of\ \Delta\} = \{\Delta(S) : S \subseteq \{1, 2, \ldots, m\}, |S| = n + 1\}$,
(ii) $\{k\text{-}faces\ of\ \Delta\} = \{\Delta_n(Z, A) : Z, A \subseteq \{1, 2, \ldots, m\}, Z \cap A = \varnothing, |Z| = k + 1, n - k < |A| < n + 1\}$,
(iii) $\{orientations\ of\ \Delta_n(Z, A)\} = \{parity\ classes\ of\ orderings\ of\ Z\}$.

Moreover, $\Delta_n(Z, A)$ *is a face of* $\Delta_n(Z', A')$ *if and only if*

$$A \supseteq A', A \cup Z \subseteq A' \cup Z'.$$

(2.5) PROPOSITION (Gabrielov *et al.* [6, Prop. 5]). *The boundary of an oriented hypersimplex is given by*

(i)
$$\partial\Delta_n(\{z_0, \ldots, z_k\}, A) = \sum_{i=0}^{k} (-1)^i \Delta_n(\{z_0, \ldots, \hat{z}_i, \ldots, z_k\}, A) - \sum_{i=0}^{k} (-1)^i \Delta_n(\{z_0, \ldots, \hat{z}_i, \ldots, z_k\}, A \cup \{z_i\})$$

if $k > 1$, *where, if* $|A| = n - k + 1$ (*resp.* $|A| = n$) *the first sum* (*resp. the second*) *is empty.*

(ii) $\partial\Delta_n(\{z_0, z_1\}, A) = \Delta(A \cup \{z_1\}) - \Delta(A \cup \{z_0\})$.

3. Hypersimplicial sections

Let $\mathbf{X}$ be an n-dimensional combinatorial manifold. Consider the following standard hypersimplex:

$$\Delta = [0, 1]^{v\mathbf{X}} \cap \left\{ \sum_{v \in v\mathbf{X}} x_v = n + 1 \right\}$$

where vK denotes the set of vertices of a simplicial complex K.

(3.1) DEFINITION. $\Delta(\mathbf{X})$ is the subcomplex of Δ consisting of all

$$\Delta(v \operatorname{Lk} \sigma, v\sigma), \sigma \in \mathbf{X}$$

and all their faces. $\Delta(\mathbf{X})$ is a cell complex, with cells of the form $\Delta(Z, A) = \Delta_n(Z, A)$ where $A \supseteq v\sigma$, $A \cup Z \subseteq v \operatorname{St} \sigma$ for some $\sigma \in \mathbf{X}$.

Let

$$C_* = C_*(\Delta(\mathbf{X}); R)$$

be the group of cellular chains with coefficients in R ($R = \mathbf{Z}$ or $R = \mathbf{Q}$).

(3.2) Let $\mathbf{X}'$ be the first barycentric subdivision of $\mathbf{X}$. For every $\sigma \in \mathbf{X}$ define its (closed) dual cell by

$$D\sigma = \{\tau \in \mathbf{X}' : \exists \rho \in \mathbf{X}', \tau < \rho, \sigma \cap \rho = \text{the barycenter of } \sigma\}.$$

Then $D\mathbf{X} = \{D\sigma : \sigma \in \mathbf{X}\}$ is a cell complex and $|D\mathbf{X}| = |\mathbf{X}|$.

Assume that every cell of $D\mathbf{X}$ has some orientation (the vertices of $D\mathbf{X}$, i.e. cells dual to n-simplices of $\mathbf{X}$ are assumed to be positively oriented). Then incidence numbers $\varepsilon_{\sigma\tau}$ of dual cells are defined. For every $\sigma \in \mathbf{X}$, $\partial D\sigma = \sum_{\tau > \sigma} \varepsilon_{\sigma\tau} D\tau$ holds (compare with condition (iv) below).

(3.3) DEFINITION. A collection $\{B\sigma : \sigma \in \mathbf{X}\} \subseteq C_*$ is called a *hypersimplicial section of* $\mathbf{X}$ if it satisfies the following conditions:

(i) $B\sigma \in C_{\operatorname{codim} \sigma}$,
(ii) the support of $B\sigma$ is contained in $\Delta(v \operatorname{Lk} \sigma, v\sigma)$,
(iii) if $\dim \sigma = n$ then $B\sigma = \Delta(v\sigma)$,
(iv) $\partial B\sigma = \sum_{\tau > \sigma} \varepsilon_{\sigma\tau} B\tau$.

If $k \leqslant n$, a collection $\{B\sigma : \sigma \in \mathbf{X}, \operatorname{codim} \sigma \leqslant k\}$ satisfying conditions (i)–(iv) for all σ of $\operatorname{codim} \sigma \leqslant k$ is called a *hypersimplicial k-section of* $\mathbf{X}$.

(3.4) PROPOSITION. *For every* $k = 0, 1, \ldots, n$ *there exists a hypersimplicial k-section of* $\mathbf{X}$.

Proof. First note that if $\operatorname{codim} \sigma > 0$ then $\Delta(v \operatorname{Lk} \sigma, v\sigma)$ is a ball, hence

$$H_{\operatorname{codim} \sigma}(\Delta(v \operatorname{Lk} \sigma, v\sigma); R) = 0. \tag{3.5}$$

The proof is now by induction on k. If $\operatorname{codim} \sigma = 1$ then $B\sigma$ is uniquely determined by (i)–(iv) (and will be of the form $B\sigma = \pm\Delta(v \operatorname{Lk} \sigma, v\sigma)$). If $\operatorname{codim} \sigma > 1$ then the existence of $B\sigma$ follows from (3.5), the assumed existence of $B\tau$ for $\tau > \sigma$ and the fact that

$$\partial\left(\sum_{\tau < \sigma} \varepsilon_{\sigma\tau} B\tau\right) = 0.$$

(3.6) The construction of $B\sigma$ is local: we are using only $B\tau$, $\sigma < \tau$. There are several ways to make it canonical (i.e. to describe an algorithm for constructing $B\sigma$ using the combinatorial structure of $\operatorname{Lk} \sigma$ and nothing else). One such canonical construction is described by Gabrielov *et al.* [7] (see also MacPherson [18, p. 113]), but in Section 5 we will describe another which is easier to implement on a computer (all work with $R = \mathbf{Q}$ only).

4. The formula

Let $\mathbf{X}$ be a combinatorial n-manifold such that the configuration space $CF(\mathrm{Lk}\ \sigma)$ is nonempty for every $\sigma^{n-4} \in \mathbf{X}$.

(4.1) REMARK. There are combinatorial manifolds for which this is not true (e.g. a suspension of Cairns' example [2] of a triangulation L of S^3 with $CF(L)$ empty); however Whitehead [34] proved that every combinatorial manifold has a subdivision with nonempty configuration spaces of all links.

(4.2) DEFINITION. Let $\sigma \in \mathbf{X}$, codim $\sigma = 4$, and let ψ be a generic flattening of $\mathrm{Lk}\,\sigma$. Define a homomorphism

$$C(\psi)\colon C_4(\Delta(\mathbf{X});\ R) \to \mathbf{Q}$$

specifying its values on the generators $\Delta(Z, A)$ of C_4 as follows:

(i) If $\Delta(Z, A)$ is a type (1, 2) face of $\Delta(v\,\mathrm{Lk}\,\sigma,\ v\sigma)$ then

$$Z = \{z_0, \ldots, z_4\} \subseteq v\,\mathrm{Lk}\,\sigma,\ A = v\sigma \cup \{a\},\ a \in v\,\mathrm{Lk}\,\sigma.$$

Consider the following ten bases of $\mathbf{R}^4$:

$$\psi\{z_\alpha, z_\beta, z_\gamma, a\},\quad 0 \leqslant \alpha < \beta < \gamma \leqslant 4.$$

Let $\#$ = number of these having the same orientation, then

$$C(\psi)(\Delta(Z, A)) = \frac{(-1)^{\#}}{48}.$$

(ii) If $\Delta(Z, A)$ is a type (2, 1) face of $\Delta(v\,\mathrm{Lk}\,\sigma,\ v\sigma)$ then

$$Z = \{z_0, \ldots, z_4\} \subseteq v\,\mathrm{Lk}\,\sigma,\ A = v\sigma \cup \{a_0, a_1\},\ a_0, a_1 \in v\,\mathrm{Lk}\,\sigma.$$

Consider the following ten bases of $\mathbf{R}^4$:

$$\psi\{z_\alpha, z_\beta, a_0, a_1\},\quad 0; \leqslant \alpha < \beta \leqslant 4;$$

let $\#$ = number of these having the same orientation, then

$$C(\psi)(\Delta(Z, A)) = \frac{-(-1)^{\#}}{48}.$$

(iii) $C(\psi)(\Delta(Z, A)) = 0$ otherwise.

Note that $C(\psi)$ depends only on the GL_4 orbit of $\psi \in F_0(\mathrm{Lk}\,\sigma)$. If $\tilde{\psi} \in CF_0(\mathrm{Lk}\,\sigma)$ define $C(\tilde{\psi})$ to equal $C(\psi)$ where ψ is any flattening representing the configuration $\tilde{\psi}$.

(4.3) DEFINITION. Let $\tau \in \mathbf{X}$, $\operatorname{codim} \tau = 3$, and let $\theta: [0, 1] \to F_1(\mathrm{Lk}\ \tau)$ be a path with endpoints in $F_0(\mathrm{Lk}\ \tau)$ containing only finite number of points in $F_1 - F_0$. Define a homomorphism

$$C(\theta): C_3(\Delta(\mathbf{X}); R) \to \mathbf{Q}$$

specifying its values on the generators $\Delta(Z, A)$ of C_3 as follows:

(i) If $\Delta(Z, A)$ is a type $(1, 1)$ face of $\Delta(v\,\mathrm{Lk}\,\tau, v\tau)$ then

$$Z = \{z_0, \ldots, z_3\} \subseteq v\,\mathrm{Lk}\,\tau,\ A = v\tau \cup \{a\}, \quad a \in v\,\mathrm{Lk}\,\tau.$$

Consider the cross-ratio $k(t)$ of $\theta(t)\{z_0, z_1, z_2, z_3\} \bmod \theta(t)(a)$: if P_i, $i = 0, \ldots, 4$ are points of $\mathbf{R}^3 - \{0\}$, the cross-ratio k of $P_0P_1P_2P_3 \bmod P_4$ is $k = r_{12}r_{03}/r_{02}r_{13}$ where

$$\begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = \begin{bmatrix} r_{02} & r_{03} & r_{04} \\ r_{12} & r_{13} & r_{14} \end{bmatrix} \begin{bmatrix} P_2 \\ P_3 \\ P_4 \end{bmatrix}.$$

Note that if $k(t) \in \{0, 1, \infty\}$ then $\theta(t) \notin F_0(\mathrm{Lk}\ \tau)$. Let $\mu_+ =$ number of times $k(t)$ passes through one of 0, 1, or ∞ in the direction $\to 0 \to 1 \to \infty \to$, and let $\mu_- =$ number of times it passes through one of these in the opposite direction (both as t increases from 0 to 1). Then

$$C(\theta)(\Delta(Z, A)) = \frac{\mu_+ - \mu_-}{24}.$$

(ii) $C(\theta)(\Delta(Z, A)) = 0$ otherwise.

Similarly as before, if $\tilde{\theta}$ is a path in $CF_1(\mathrm{Lk}\ \tau)$ with endpoints in CF_0 containing only a finite number of points in $CF_1 - CF_0$, set $C(\tilde{\theta}) = C(\theta)$ where $\theta: [0, 1] \to F_1$ is any path such that the diagram

$$\begin{array}{ccc} & & F_1(\mathrm{Lk}\ \tau) \\ & \overset{\theta}{\nearrow} & \downarrow \\ [0, 1] & \xrightarrow{\tilde{\theta}} & CF_1(\mathrm{Lk}\ \tau) \end{array}$$

commutes.

(4.4) For each $\sigma^{n-j} \in \mathbf{X}$, $j = 3, 4$, choose a generic configuration $\tilde{\psi}_\sigma \in CF_0(\mathrm{Lk}\ \sigma)$. Then for each pair $\sigma^{n-4} < \tau^{n-3}$ we have two configurations of $\mathrm{Lk}\,\tau$: $\tilde{\psi}_\tau$ and $\pi\tilde{\psi}_\sigma$ where $\pi: CF_0(\mathrm{Lk}\ \sigma) \to CF_0(\mathrm{Lk}\ \tau)$ is the projection described in (1.5). Choose a path $\tilde{\theta}_{\sigma\tau}: [0, 1] \to CF_1(\mathrm{Lk}\ \tau)$, such that

$\tilde{\theta}_{\sigma\tau}(0) = \tilde{\psi}_\tau$, $\tilde{\theta}_{\sigma\tau}(1) = \pi\tilde{\psi}_\sigma$. Finally, choose a hypersimplicial 4-section $\{B\sigma : \sigma^{n-j} \in \mathbf{X}, j \leqslant 4\}$ of $\mathbf{X}$.

Let

$$(4.5) \qquad P = \sum_{\sigma^{n-4}} \left[C(\tilde{\psi}_\sigma)B\sigma + \sum_{\tau^{n-3} > \sigma} \varepsilon_{\sigma\tau} C(\tilde{\theta}_{\sigma\tau}) B\tau \right] (D\sigma)^*$$

where $(D\sigma)^*$: $C_4(D\mathbf{X}; \mathbf{Q}) \to \mathbf{Q}$ is defined by $(D\sigma)^* D\tau = \delta_{\sigma\tau}(=1$ if $\sigma = \tau$, 0 otherwise). (For a discussion of the formula, see [30].)

(4.6) THEOREM (Gabrielov–Gel'fand–Losik). *Assume*

(A) *for every* $\sigma^{n-4} \in \mathbf{X}$, $CF(\mathrm{Lk}\ \sigma)$ *is connected;*
(B) *for every* $\sigma^{n-3} \in \mathbf{X}$, $CF(\mathrm{Lk}\ \sigma)$ *is simply connected.*

Then

(i) P *is a cocycle in* $C^4(D\mathbf{X}; \mathbf{Q})$;
(ii) *its cohomology class* $[P] \in H^4(D\mathbf{X}; \mathbf{Q})$ *is independent of the choices made in* (4.5);
(iii) *If* $\mathbf{X}$ *is a smooth triangulation of a smooth manifold* X *then* $[P]$ *is the first Pontryagin class of* X.

(4.7) REMARK. We do not know whether $CF(L)$ is simply connected for every triangulation L of S^2. Strong supporting evidence is provided by the following result of Bloch *et al.* [1]: Let L be a triangulation of S^2 and let $\sigma^2 \in L$. Then the diagram space of (L, σ) (= the space of all simplexwise linear homeomorphisms L-int $\sigma \to \Delta^2$) is contractible.

On the other hand, we have an example of a triangulation L of S^3 such that $CF(L)$ is disconnected (the link of a vertex in the combinatorial manifold $\mathbf{X}$ to which we wish to apply the theorem, see Proposition 10.5). But, if $\dim \mathbf{X} = 4$ then assertion (A) can be removed from the theorem. We need the following smoothing result of Kuiper [15].

(4.8) THEOREM (Kuiper). *Let* $\mathbf{X}$ *be a combinatorial n-manifold such that* $CF(\mathrm{LK}\ p) \neq 0$ *for every vertex p of* $\mathbf{X}$. *If*

$$(4.9) \qquad \pi_{k-1}(CF(\mathrm{Lk}\ \sigma)) = 0, \text{ all } \sigma^k \in \mathbf{X}, \text{ all } k > 0$$

then for every choice of $\tilde{\psi}_p \in CF(\mathrm{Lk}\ p)$ *there exists a smooth structure on* $X = |\mathbf{X}|$ *with the following two properties:*

(i) *the triangulation of* X *by* $\mathbf{X}$ *is a smooth triangulation;*
(ii) *for every vertex p of* $\mathbf{X}$ *the natural configuration of* Lk p *(in the tangent space* T_pX*) and the configuration* $\tilde{\psi}_p$ *belong to the same component of* $CF(\mathrm{Lk}\ p)$.

REMARK. Part (i) is a particular case of Kuiper's theorem 4. Part (ii) follows easily from his method of proof: he starts with an arbitrary collection of flattenings (Brouwer charts in his terminology) at vertices of $\mathbf{X}$ and applying carefully chosen homotopies obtains a collection of smoothly compatible charts. The key is his Lemma 3.3 which makes it possible to choose required homotopies f_t so that the first derivatives of f_t depend continuously on t, thus providing a path from ψ_p to the natural flattening of $\mathrm{Lk}\, p$ in T_pX.

(4.10) COROLLARY. *If* $\dim \mathbf{X} = 4$ *then Theorem* 4.6 *holds without assertion* (A); *moreover* (iii) *does not require smoothness of a triangulation.*

Proof. Conclusion (i) is trivially true. The proof of (iii) by Gabrielov *et al.* uses only the following consequence of assertion (A): the natural configuration of $\mathrm{Lk}\,\sigma$ (in $(TX(T\sigma)_b$, $b =$ the barycenter of σ) and the chosen configuration $\tilde{\psi}_\sigma$ belong to the same component of $CF(\mathrm{Lk}\,\sigma)$ for every $\sigma^{n-4} \in \mathbf{X}$.

Note that conditions (4.9) are satisfied (cf. (1.2) and (1.3)) so, by the Kuiper theorem, $[P]$ is the first Pontryagin class of some smooth structure on $|\mathbf{X}|$. The remark that a PL 4-manifold has a unique smooth structure (Cerf [5]) finishes the proof.

5. An algorithm for hypersimplicial sections: introduction

We want an algorithm for computing $B\sigma$, $\sigma^{n-k} \in \mathbf{X}$, $k \leqslant 4$ using only the combinatorial structure of $\mathrm{St}\,\sigma$, so all computations are to be done in $\Delta_n(v\,\mathrm{Lk}\,\sigma, v\sigma)$ which is isomorphic to $\Delta_{k-1}(v\,\mathrm{Lk}\,\sigma, \varnothing)$ (cf. (2.3) and (3.3)).

(5.1) LEMMA. *Let* $C_p = C_p(\Delta(v\ \mathrm{Lk}\,\sigma,\ v\sigma);\ \mathbf{Q})$ *be the group of hypersimplicial chains. Then there is a homomorphism* $\Gamma: C_{p-1} \to C_p$ *such that*

(i) $\partial\Gamma\partial = \partial$ *where* $\partial: C_p \to C_{p-1}$ *is the boundary operator,*

(ii) Γ *is invariant under the action of the symmetric group* $S_{v\mathrm{Lk}\sigma}$ (*i.e.* $\Gamma g = g\Gamma$ *for every* $g \in S_{v\mathrm{Lk}\sigma}$).

Proof. C_p and C_{p-1} are vector spaces over $\mathbf{Q}$. Let $\{b_i\}_{i\in I}$ be a basis of C_p. Then $\{\partial b_i\}_{i\in I}$ span ∂C_p. Let $J \subseteq I$ be such that $\{\partial b_j\}_{j\in J}$ is a basis for ∂C_p; define a homomorphism $\Gamma_0: \partial C_p \to C_p$ by $\Gamma_0(\partial b_j) = b_j$, $j \in J$, and extend it arbitrarily to a homomorphism $\Gamma_0: C_{p-1} \to C_p$. It is easy to see that Γ_0 satisfies (i). Then Γ is the average of $\{g\Gamma_0 g^{-1},\ g \in S_{v\mathrm{Lk}\sigma}\}$.

(5.2) PROPOSITION. *Let* $\{B\tau : \tau \in \mathbf{X}, \text{ codim } \tau < k\}$ *be a hypersimplicial* $(k-1)$*-section of* $\mathbf{X}$ *and let* $\Gamma: C_{k-1} \to C_k$ *be as in* (5.1). *Then, if* codim $\sigma = k$

$$B\sigma = \Gamma\left(\sum_{\tau > \sigma} \varepsilon_{\sigma\tau} B\tau\right)$$

satisfies conditions (i)–(iv) *of Definition* 3.3.

Note that in the case $k = 1$, $B\sigma$ is already determined (cf. the proof of Proposition 3.4).

A natural basis for the vector space C_p consists of all p-dimensional faes $\Delta(Z, A)$ of $\Delta_n(v \operatorname{Lk} \sigma, v\sigma)$, so we will determine $\Gamma: C_{k-1} \to C_k$, $k = 2, 3, 4$, in terms of these natural bases. Since Γ is invariant under the action of $S_{v\operatorname{Lk}\sigma}$ it suffices to determine $\Gamma\Delta(Z', A')$ for one representative $\Delta(Z', A')$ of each type of $(k-1)$-dimensional faces of $\Delta_n(v \operatorname{Lk} \sigma, v\sigma)$ (there are $k - 1$ types (p, q),$p + q + 1 = k - 1$).

(5.3) Thus, our algorithm for computing $B\sigma$, codim $\sigma = k$, consist of a list of formulas

$$(5.4) \qquad \Gamma\Delta(Z', A') = \sum_{Z,A} \alpha(Z, A; Z', A)\Delta(Z, A),$$

one formula for each type of $(k-1)$-dimensional hypersimplices. Then, if

$$(5.5) \qquad \sum_{\tau > \sigma s} \varepsilon_{\sigma\tau} B\tau = \sum_{Z', A'} \gamma(Z', A')\Delta(Z', A').$$

$B\sigma$ can be obtained as

$$(5.6) \qquad B\sigma = \sum_{Z', A'} \gamma(Z', A')\Gamma\Delta(Z', A').$$

$\{B\sigma\}$ obtained using this algorithm are simplicial invariants. More precisely

(5.7) PROPOSITION. (i) *$B\sigma$ is independent of the orientations of $D\tau$, $\tau \neq \sigma$; a change in the orientation of $D\sigma$ results in the multiplication of $B\sigma$ by* -1.

(ii) *If $g: v \operatorname{St} \sigma \to v \operatorname{St}(g\sigma)$ is a simplicial isomorphism then $B(g\sigma) = \varepsilon_g g B\sigma$ where $\varepsilon_g = +1$ if orientation of $D(g\sigma) = g($orientation of $D\sigma)$, $\varepsilon_g = -1$ otherwise.*

Proof. (i) A change in the orientation of $D\sigma$ results in the change of the sign of all $\varepsilon_{\sigma\tau}$, $\sigma < \tau$, and all $\varepsilon_{\tau\sigma}$, $\sigma > \tau$. The proof now proceeds by induction on codim σ.

(ii) $B(g\sigma)$ and $B\sigma$ differ only in the names of vertices and (possibly) in the orientations of $Dg\sigma$ and $D\sigma$. The proof is again by induction on codim σ, using the fact that Γ is $S_{v\operatorname{Lk}\sigma}$ invariant.

Before explicitly stating formulas for the Γ's we need to introduce some notation, useful in dealing with actions of finite groups on 'subscripts'. (Think of Z, A as subscripts for α, γ, Δ in (5.4)–(5.6).)

6. Notation: twisted groups and formal sums

(6.1) DEFINITION. A *twisted group* $\tilde{G}$ is a group G together with a homomorphism $\varepsilon: G \to \{1, -1\}$ where $\{1, -1\}$ is considered as a multiplicative group.

(6.2) EXAMPLE. (i) Every group G has the trivial twisting: $\varepsilon(g) = +1$, all $g \in G$.

(ii) $\tilde{S}_X$. the symmetric group on a finite set X, with the usual twisting $\varepsilon_X(g) = (-1)^{\pi(g)}$, $\pi(g)$ the parity of the permutation g.

(iii) Every subgroup of a twisted group $\tilde{G}$ (in particular every subgroup of $\tilde{S}_X$) can be made twisted by restricting the twisting of $\tilde{G}$.

(iv) Let $\Delta(Z, A)$ be a face of $\Delta(v \operatorname{Lk} \sigma, \emptyset)$ and let $G_{Z,A}$ be the subgroup of $S_{v\operatorname{Lk}\sigma}$

$$G_{Z,A} = \{g \in S_{v\operatorname{Lk}\sigma}: gZ = Z, gA = A\}.$$

Then

$$G_{Z,A} = \{gh_1h_2: g \in S_Z,\ h_1 \in S_A,\ h_2 \in S_{v\operatorname{Lk}\sigma - Z - A}\}$$

and we can define $\tilde{G}_{Z,A}$ by

$$\varepsilon_{Z,A}(gh_1h_2) = \varepsilon_Z(g)$$

where ε_Z is the usual twisting of $\tilde{S}_Z$. This means that, for $g \in G_{Z,A}$, $\varepsilon_{Z,A}(g) = +1$ iff g preserves the orientation of $\Delta(Z, A)$, $\varepsilon_{Z,A}(g) = -1$ otherwise (cf. Proposition 2.4(iii)).

(6.3) If M is an element of the rational group ring of $S_{v\operatorname{Lk}\sigma}$ (i.e. if $M = \sum r_i g_i$, $r_i \in \mathbf{Q}$, $g_i \in S_{v\operatorname{Lk}\sigma}$) let

$$M\Delta(Z, A) = \sum r_i \Delta(g_i Z, g_i A)$$

and similarly for anything else indexed by (a collection of) subsets of $v \operatorname{Lk} \sigma$ (e.g. $\alpha(Z, A)$, etc.)

(6.4) DEFINITION. Elements of the rational group ring of $S_{v\mathrm{Lk}\sigma}$ are called *formal sums* (*of permutations*). If M and N are two formal sums, MN will denote their product (in general $MN \neq NM$), so that, for example

$$(MN)\Delta(Z, A) = M(N\Delta(Z, A)).$$

(6.5) EXAMPLES. (i) If $\tilde{G}$ is a twisted group we will use the same symbol $\tilde{G}$ to denote the formal sum $\sum_{g \in G} \varepsilon(g)g$.

(ii) $\tilde{G}_{Z,A} = \tilde{S}_Z S_A S_{v\mathrm{Lk}\sigma - Z - A}$;
$S_X \tilde{S}_X = 0 = \tilde{S}_X S_X$ for every finite set X.

(iii) If $\tilde{H}$ is a subgroup of $\tilde{G}$ and $\varepsilon_H = \varepsilon_{G|H}$ then $\tilde{G}\tilde{H} = \tilde{H}\tilde{G} = n\tilde{G}$ where $n = |\tilde{H}|$ is the size of $\tilde{H}$.

7. Formulas for Γ's

Recall (from (5.4)) that

$$\Gamma\Delta(Z', A') = \sum_{Z,A} \alpha(Z, A; Z', A')\Delta(Z, A)$$

and that the Γ's are invariant under the action of $S_{v\mathrm{Lk}\sigma}$. This means that if $g \in \tilde{G}_{Z',A'}$ then

$$\alpha(gZ, gA; Z', A') = \varepsilon_{Z',A'}(g)\alpha(Z, A; Z', A'),$$

i.e. that $\alpha(Z, A; Z', A')$ depends only on the 'orbit' $\tilde{G}_{Z',A'}\Delta(Z, A)$ so we can

TABLE I

$\Gamma: C_1 \to C_2$

$\Gamma\Delta(12, 3)$	
$2(m-3)!\alpha_i$	Z_i, A_i
$-1/m$	123,0
$(m-3)/m$	134,0
$-(m-3)/m(m-1)$	123,4
$(m-3)/(m-1)$	124,3
$(m-3)/m(m-1)$	134,2
$[(m-3)(m-4)]/m(m-1)$	134,5

TABLE II
$\Gamma: C_2 \to C_3$

(a) $\Gamma\Delta(123,4)$

$3!(m-4)!\alpha_i$	Z_i, A_i
$1/m$	1234,0
$-(m-4)/m$	1245,0
$(m-4)/m(m-1)$	1234,5
$-(m-4)/(m-1)$	1235,4
$-(m-4)/m(m-1)$	1245,3
$-[(m-4)(m-5)]/m(m-1)$	1245,6
$[(m-4)(m-5)]/m(m-1)(m-2)$	1234,56
$-[(m-4)(m-5)]/(m-1)(m-2)$	1235,46
$-[2(m-4)(m-5)]/m(m-1)(m-2)$	1245,36
$-[(m-4)(m-5)(m-6)]/m(m-1)(m-2)$	1245,67

(b) $\Gamma\Delta(123,45)$

$3!(m-4)!\alpha_i$	Z_i, A_i
$-[(m-4)(m-5)]/2(m-2)$	1236,45

write

$$\Gamma\Delta(Z', A') = \sum_{i\in I} \alpha_i \tilde{G}_{Z',A'}\Delta(Z_i, A_i). \tag{7.1}$$

Here I is the set of $G_{Z',A'}$ orbits of hypersimplices $\Delta(Z, A)$, $\Delta(Z_i, A_i)$ is a representative of the orbit i, and

$$\alpha_i = \frac{1}{m_i}\,\alpha(Z_i, A_i; Z', A'),$$

where m_i is the number of elements of $\tilde{G}_{Z',A'}$ leaving $\Delta(Z_i, A_i)$ fixed ($m_i = |\tilde{G}_{Z',A'} \cap \tilde{G}_{Z_i}, A_i|$).

Since $\Delta_n(v\,\mathrm{Lk}\,\sigma, v\sigma)$ is isomorphic to $\Delta_{k-1}(v\,\mathrm{Lk}\,\sigma, \varnothing)$, $\mathrm{codim}\,\sigma = k$ (the isomorphism: $\Delta(Z, A) \leftrightarrow \Delta(Z, A - v\sigma)$, cf. (2.3)) and $\Delta_{k-1}(v\,\mathrm{Lk}\,\sigma, \varnothing)$ is isomorphic to $\Delta_{k-1}(m) = \Delta_{k-1}(12\ldots m, \varnothing)$ where $m = |v\,\mathrm{Lk}\,\sigma|$ we present formulas for

$$\Gamma: C_{k-1}(\Delta_{k-1}(m); \mathbf{Q}) \to C_k(\Delta_{k-1}(m); \mathbf{Q}), \quad k = 2, 3, 4$$

in the form prescribed by (7.1). Note that $m > k$ (since $m = |v\,\mathrm{Lk}\,\sigma| > 1 + \dim \mathrm{Lk}\,\sigma = \mathrm{codim}\,\sigma = k$).

TABLE III
$\Gamma: C_2 \to C_3$

(a) $\Gamma\Delta(1234,5)$

$4!(m-5)!\alpha_i$	Z_i, A_i
$-1/m$	12345,0
$(m-5)/m$	12356,0
$-(m-5)/m(m-1)$	12345,6
$(m-5)/(m-1)$	12346,5
$(m-5)/m(m-1)$	12356,4
$[(m-5)(m-6)]/m(m-1)$	12356,7
$-[(m-5)(m-6)]/m(m-1)(m-2)$	12345,67
$[(m-5)(m-6)]/(m-1)(m-2)$	12346,57
$[2(m-5)(m-6)]/m(m-1)(m-2)$	12356,47
$[(m-5)(m-6)(m-7)]/m(m-1)(m-2)$	12356,78
$-[(m-5)(m-6)(m,-7)]/m(m-1)(m-2)(m-3)$	12345,678
$[(m-5)(m-6)(m-7)]/(m-1)(m-2)(m-3)$	12346,578
$[3(m-5)(m-6)(m-7)]/m(m-1)(m-2)(m-3)$	12356,478
$[(m-5)(m-6)(m-7)(m-8)]/m(m-1)(m-2)(m-3)$	12356,789

(b) $\Gamma\Delta(1234,56)$

$4!(m-5)!\alpha_i$	Z_i, A_i
$[(m-5)(m-6)]/2(m-2)$	12347,56
$[(m-5)(m-6)(m-7)]/2(m-2)(m-3)$	12347,568

(c) $\Gamma\Delta(1234,567)$

$4!(m-5)!\alpha_i$	Z_i, A_i
$[(m-5)(m-6)(m-7)]/6(m-3)$	12348,567

8. On the proof of the formulas for Γ's

Let $C_p = C_p(\Delta_{k-1}(12\ldots m, \varnothing); \mathbf{Q})$ be the group of hypersimplicial chains. We are looking for $\Gamma: C_{k-1} \to C_k$ satisfying the conditions of Lemma 5.1:

(8.1) $\quad \partial\Gamma\partial = \partial,$

(8.2) $\quad \Gamma g = g\Gamma$ for every $g \in S_{12\ldots m}$.

Condition (8.2) implies that it suffices to determine $\Gamma\Delta(Z, A)$ for one representative of each $S_{12\ldots m}$ orbit of $(k-1)$-dimensional hypersimplices, i.e.

for one representative of each type of $(k-1)$-dimensional hypersimplices. Moreover, it implies that $\Gamma\Delta(Z, A)$ is of the form

$$(8.3)\qquad \Gamma\Delta(Z, A) = \tilde{G}_{Z,A} \sum_{i\in I} \alpha_i \Delta(Z_i, A_i)$$

where $\tilde{G}_{Z,A} = \tilde{S}_Z S_A S_{\{12\dots m\}-Z-A}$, Σ is over I = collection of all nonzero 'orbits' $\tilde{G}_{Z,A}\Delta(Z', A')$ of k-dimensional hypersimplices, and $\Delta(Z_i, A_i)$ is a representative of orbit i.

The idea is now a simple one: plug (8.3) in (8.1) and solve for α_i. We will describe how to do this in the case $k = 2$ and give some comments about the cases $k = 3, 4$.

REMARK. The Γ's listed in Section 7 satisfy (8.2); the computer program A (described briefly in (12.6) can be used to check that they satisfy (8.1) for specific values of m. Here we answer the question how to obtain these Γ's.

There is only one type of 1-face of $\Delta_1(12\dots m,. \varnothing)$, namely (0, 0), so consider its representative $\Delta(12, 3)$. There are eight nonzero 2-dimensional $\tilde{G}_{12,3}\Delta(Z, A)$, whose representatives (Z_i, A_i) are:

$$(8.4)\qquad \begin{cases} i=1, 2, 3: & (123, \varnothing), (124, \varnothing), (134, \varnothing), \\ i=4, 5, 6, 7, 8: & (123, 4), (124, 3), (134, 2), (124, 5), (134, 5). \end{cases}$$

Since it suffices for Γ to satisfy

$$(8.5)\qquad \partial\Gamma\,\partial\Delta(Z, A) = \partial\Delta(Z, A)$$

for only one 2-dimensional $\Delta(Z, A)$ of each type, consider first $\Delta(123, \varnothing)$ and then $\Delta(124, 3)$.

Write $\partial\Delta(123, \varnothing) = -\Delta(23, 1) + \Delta(13, 2) - \Delta(12, 3)$ as

$$\partial\Delta(123, \varnothing) = -\frac{1}{2(m-3)!}\tilde{G}_{123,\varnothing}\Delta(12, 3).$$

Then

$$\begin{aligned} \partial\Gamma\,\partial\Delta(123, \varnothing) &= -\frac{1}{2(m-3)!}\partial\Gamma\,\tilde{G}_{123,\varnothing}\Delta(12, 3) \\ &= -\frac{1}{2(m-3)!}\tilde{G}_{123,\varnothing}\tilde{G}_{12,3}\partial\sum\alpha_i\Delta(Z_i, A_i) \\ &= -\partial\tilde{G}_{123,\varnothing}\sum\alpha_i\Delta(Z_i, A_i) \end{aligned}$$

(since $\tilde{G}_{123,\varnothing}\tilde{G}_{12,3} = 2(m-3)!\,\tilde{G}_{123,\varnothing}$, cf. Example 6.5(iii)), so that (8.5) becomes

$$(8.6)\qquad \sum \alpha_i \partial[Z_i, A_i] = \frac{1}{2(m-3)!}[12, 3],$$

where $[Z, A] = \tilde{G}_{123,\varnothing}\Delta(Z, A)$.

It is easy to compute $\partial[Z_i, A_i]$:

$$i = 1 \quad \partial[123, \varnothing] = -3[12, 3]$$

(since $[23, 1] = -[13, 2] = [12, 3]$; we are always representing $[Z, A]$ by its first representative in lexicographic order).

$i = 2, 3$ $\partial[124, \varnothing] = -[12, 4] + 2[14, 2]$, $\partial[134, \varnothing] = [12, 4] - 2[14, 2]$;

$i = 4, 5$ $\partial[123, 4] = 3[12, 4]$, $\partial[124, 3] = [12, 3] + 2[14, 2]$,

$i = 6, 7$ $\partial[134, 2] = -[12, 3] - 2[14, 2]$, $\partial[124, 5] = [12, 4]$,

$i = 8$ $\partial[134, 5] = -[12, 4]$.

Replace these $\partial[Z_i, A_i]$ in (8.6), and equate the coefficients of $[ij, k]$:

$$(8.7)\qquad -3\alpha_1 + \alpha_5 - \alpha_6 = \frac{1}{2(m-3)!} \qquad ([12, 3])$$

$$-\alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_7 - \alpha_8 = 0 \qquad ([12, 4])$$

$$2\alpha_2 - 2\alpha_3 + 2\alpha_5 - 2\alpha_6 = 0 \qquad ([14, 2])$$

Repeat a similar procedure with $\Delta(124, 3)$:

$$\partial\Delta(124, 3) = \frac{1}{2(m-4)!}\tilde{G}_{124,3}\Delta(12, 3),$$

$$\partial\Gamma\,\partial\Delta(124, 3) = \partial\tilde{G}_{124,3}C_{45\ldots m}\sum \alpha_i\Delta(Z_i, A_i)$$

($C_{45\ldots m}$ is defined by $\tilde{G}_{124,3}\tilde{G}_{12,3} = 2(m-4)!\tilde{G}_{124,3}C_{45\ldots m}$),

$$(8.6')\qquad \sum \alpha_i \partial[C_{45\ldots m}(Z_i, A_i)] = \frac{1}{2(m-4)!}[12, 3]$$

where [ ] denotes $\tilde{G}_{124,3}(\;)$.

Compute $\partial[C_{45\ldots m}(Z_i, A_i)]$, e.g.:

$$\partial[C_{45\ldots m}(124, \varnothing)] = \partial([124, \varnothing] + (m-4)[125, \varnothing])$$

$$= -3[12, 4] + (m-4)[15, 2] - (m-4)[12, 5]$$

and obtain

$$(8.7') \qquad -(m-3)\alpha_1 - \alpha_3 + (m-1)\alpha_5 = \frac{1}{2(m-4)!} \qquad ([12, 3])$$

$$2(m-3)\alpha_1 + 2\alpha_3 + 2\alpha_4 + (m-2)\alpha_6 - \alpha_8 = 0 \qquad ([13, 2])$$

$$-3\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7 = 0 \qquad ([12, 4])$$

$$-(m-4)\alpha_2 + (m-4)\alpha_4 + (m-2)\alpha_7 + \alpha_8 = 0 \qquad ([12, 5])$$

$$2(m-4)\alpha_2 - (m-4)\alpha_6 + 2\alpha_7 + \alpha_8 = 0 \qquad ([15, 2])$$

The general solution of the system (8.7)–(8.7′) is

$$(8.8) \qquad \alpha_1 = -\frac{1}{2m(m-3)!} - x \qquad \alpha_5 = \frac{1}{2(m-1)(m-4)!} - x$$

$$\alpha_2 = x \qquad \alpha_6 = \frac{1}{2m(m-1)(m-4)!} + 2x$$

$$\alpha_3 = \frac{1}{2m(m-4)!} - 2x \qquad \alpha_7 = y$$

$$\alpha_4 = -\frac{1}{2m(m-1)(m-4)!} + x - y \qquad \alpha_8 = \frac{1}{2m(m-1)(m-5)!} - 2y$$

where x and y are arbitrary rational numbers.

REMARK. We must be a little more careful about the cases $m = 3, 4$. If $m = 4$ then $\alpha_7 = \alpha_8 = 0$ (since Z_7, A_7 and Z_8, A_8 do not exist) and, in (8.7′), the equations corresponding to [12, 5] and [15, 2] should be omitted. All this has the same effect as setting $m = 4$, $\alpha_7 = a_8 = 0$ in (8.7)–(8.7′) and (8.8), provided we write $(m-4)/(m-4)!$ instead of $1/(m-5)!$. The discussion of the case $m = 3$ is similar and shows that it is convenient in (8.8) to write $(m-3)/(m-3)!$ instead of $1/(m-3)!$

The formula for $\Gamma\Delta(12, 3)$ given in Table I is the same as (8.8) with $x = y = 0$.

As can be seen from the case $k = 2$, conditions (8.1)–(8.2) do not determine a unique Γ. If $k > 2$ the number of free α_i's increases. In the case $k = 3$ the system corresponding to (8.7)–(8.7′) has 32 equations and 32 variables, 13 of those variables are free. What is remarkable is that a particular solution of this system can be chosen so that $\Gamma\Delta(12, 3)$ and $\Gamma\Delta(123, 4)$ have a similar pattern (cf. Tables I and II in Section 7); eight of the non-free variables in this particular solution are zero. In the case $k = 4$

a conjecture was made that Γ is like the one in Table III and then it was verified that it satisfies (8.1) (finding a general Γ requires a general solution of a system with 80 variables).

9. The Kühnel projective plane CP_9^2

The complex projective plane has a triangulation with nine vertices whose discovery was announced in Kühnel and Banchoff [13] (see also [14]). This triangulation, which we denote by CP_9^2, can be thought of as a subcomplex of $\partial\Delta^8$, the boundary of the 8-simplex Δ^8. The CP_9^2 contains all 1- and 2-simplices of Δ^8 (36 and 84 simplices respectively), and it contains 90 3-simplices and 36 4-simplices (out of 126 3- and 126 4-simplices of Δ^8; in fact, for every 4-simplex σ^4 of Δ^8, either σ^4 or the 3-simplex σ^3 opposite to σ^4 in Δ^8 belongs to CP_9^2). The complete list of 4-simplices of CP_9^2, each given by a 5-tuple of integers from 1 to 9, is given in the Table IV.

TABLE IV
Simplices of CP_9^2

12456	45789	78123
23564	56897	89231
31645	64987	97312
12459	45783	78126
23567	56891	89234
31648	64972	97315
23649	56973	89316
31457	64781	97124
12568	45892	78235
31569	64893	97236
12647	45971	78314
23458	56782	89125

It is indicated in [13] how it can be proved that CP_9^2 is indeed a triangulation of the complex projective plane. A preprint by Morin, Yoshida and Marin [22] contains three different proofs, among them a construction of an explicit homeomorphism

$$h: |CP_9^2| \cong CP^2$$

(§4 of [22]). Unfortunately h is not a smooth triangulation: h is degenerate at the vertices of CP_9^2. This means that configurations cannot be obtained in the usual way for smooth triangulations: as configurations in the tangent space.

The symmetries of $\mathbf{CP}_9^2$ will allow us to reduce the number of configurations to be chosen.

Consider the permutations

$$\alpha = (147)(258)(369),$$

$$\beta = (123)(465),$$

$$\tau = (12)(45)(78),$$

$$\gamma = (\alpha\beta)^{-1}\beta\alpha = (123)(456)(789)$$

and the following subgroups of the symmetric group $S_{12\ldots9}$:

G_{54}, generated by α, β, and τ;

G_{27}, generated by α and β;

G_9, generated by α and γ

(the notation is such that $|G_r| = r$).

(9.1) PROPOSITION. (i) *G_{54} is the full group of simplicial automorphisms of $\mathbf{CP}_9^2$; all elements of G_{54} preserve orientation.*

(ii) *G_9 acts transitively on $\mathbf{CP}_9^2$: for every pair $i, j \in \{1, 2, \ldots, 9\}$ there is a unique $g_{ij} \in G_9$ such that $g_{ij}(j) = i$.*

REMARK. The arrangement of 4-simplices in Table IV reflects the action of G_{54}. Using our notation from Section 6 we can write the complete list of simplices of $\mathbf{CP}^2$ as

$$G_{54}(\tfrac{1}{6}\langle 12456\rangle - \tfrac{1}{2}\langle 12459\rangle)$$

Here 1/6 and 1/2 indicate that $\langle 12456\rangle$ and $\langle 12459\rangle$ are invariant under subgroups of G_{54} of order 6 and 2 respectively and the minus sign indicates that $\langle 12456\rangle$ and $\langle 12459\rangle$ have opposite orientations (since they have a common 3-face $\langle 1245\rangle$).

(9.2) PROPOSITION. *There are two G_{27} orbits of 1-dimensional simplices of $\mathbf{CP}_9^2$; their representatives are $\langle 69\rangle$ and $\langle 89\rangle$* (see Figure 1).

In order to apply the formula for the first Pontryagin class to the Kühnel triangulation of $\mathbf{CP}^2$ we must make our way around assertion (B) of Theorem 4.6 (and then use Corollary 4.10). The best we can do is repeat the assertion in the special case at hand:

ASSERTION (B′) *If L is either one of triangulation of S^2 indicated in Figure 1, then $CF(L)$ is simply connected.*

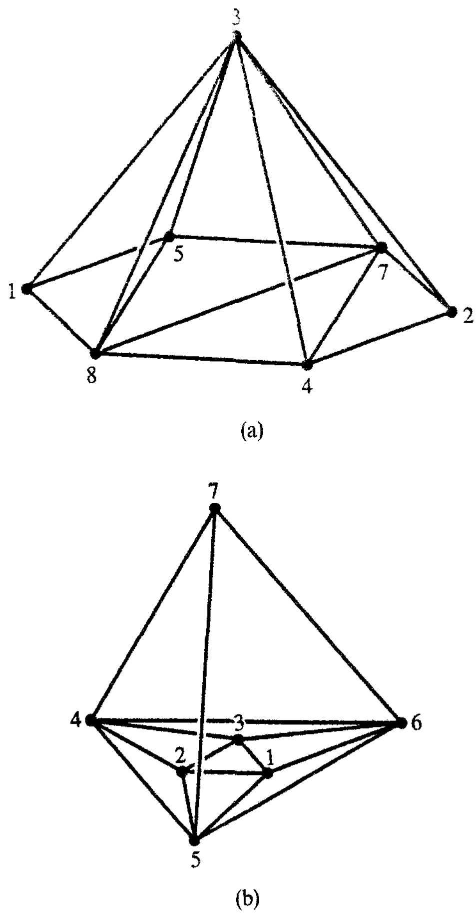

Fig. 1.

By Proposition 9.1(ii) it suffices to choose a configuration $\tilde{\psi}_9$ of Lk⟨9⟩; a configuration $\tilde{\psi}_i$ of Lk⟨i⟩ is then represented by the flattening $\psi_i = \psi_9 g_{9i}$, where ψ_9 is the flattening representing the configuration $\tilde{\psi}_9$. Our choice for the flattening ψ_9 is described in Section 10 and it will be proved there (Proposition 10.4) that ψ_9 and $\psi_9\beta$ belong to the same connected component of F_0(Lk⟨9⟩), the space of generic flattenings of Lk⟨9⟩.

For every 1-simplex ⟨ij⟩ of $\mathbf{CP}^2_9$ there are maps

$$CF_0(\mathrm{Lk}\langle j\rangle) \xrightarrow{\pi_i} CF_0(\mathrm{Lk}\langle ij\rangle) \xleftarrow{\pi_j} CF_0(\mathrm{Lk}\langle i\rangle)$$

Given an unoriented 1-simplex $\{ij\}$ we have two choices for the configuration $\tilde{\psi}_{ij}$ of Lk⟨ij⟩: $\pi_i\tilde{\psi}_j$ and $\pi_j\tilde{\psi}_i$. Since G_{27} contains no element of even order, an orientation chosen on one 1-simplex induces orientations on all 1-simplices in its orbit. Thus define $\tilde{\psi}_{ij}$ as

(9.3) $$\tilde{\psi}_{ij} = \pi_i \tilde{\psi}_j$$

where the oriented simplex $\langle ij \rangle$ is the same G_{27} orbit as the oriented simplex $\langle 69 \rangle$ or $\langle 89 \rangle$ (cf. Proposition 9.2).

(9.4) EXAMPLES. $\tilde{\psi}_{89} = \pi_8 \tilde{\psi}_9$,

$$\tilde{\psi}_{79} = \pi_9 \tilde{\psi}_7,$$

$$\tilde{\psi}_{i9} = \pi_9 \tilde{\psi}_i, \quad i = 1, 2, 3,$$

$$\tilde{\psi}_{i9} = \pi_i \tilde{\psi}_9, \quad i = 4, 5, 6.$$

According to (4.4), for each pair i, $\{i, j\}$ we need a path $\tilde{\theta}_{i,ij}$ between the configurations $\tilde{\psi}_{ij}$ and $\pi_j \tilde{\psi}_i$. Each pair $i, \{i,j\}$ belongs to the G_{27} orbit of exactly one of the following four:

$$8, \{8, 9\};\ 9, \{8, 9\};\ 6, \{6, 9\};\ 9, \{6, 9\}.$$

Moreover, there exists a unique $g \in G_{27}$ such that either

(i) $g(i, j) = 6, 9$ or $9, 6$
or (ii) $g \in G_9$ and $g(i, j) = 8, 9$ or $9, 8$.

(9.5) PROPOSITION. *It suffices to choose*

$$\tilde{\theta}_{6,69},\ \tilde{\theta}_{9,69},\ \tilde{\theta}_{8,89},\ \tilde{\theta}_{9,89}$$

and then define $\tilde{\theta}_{i,ij}$ according to either (i) *or* (ii), *as $\tilde{\theta}_{i,ij} = \tilde{\theta}_{6,69} g$, or $\tilde{\theta}_{9,69} g$, etc. Moreover, $\tilde{\theta}_{9,69}$ and $\tilde{\theta}_{9,89}$ can be chosen to be trivial.*

Proof. If $g(i, j) = 9, 6$ or $9, 8$ then, by the definition of $\tilde{\psi}_{ij}$, $\tilde{\psi}_{ij} = \pi_j \tilde{\psi}_i$, so $\tilde{\theta}_{i,ij}$ has both required endpoints the same.

If $g(i, j) = 8, 9$, $g \in G_9$, then $\tilde{\theta}_{i,ij} = \tilde{\theta}_{8,89} g$ has endpoints $(\pi_8 \tilde{\psi}_9) g = \pi_i \tilde{\psi}_j = \tilde{\psi}_{ij}$ and $(\pi_9 \tilde{\psi}_8) g = \pi_j \tilde{\psi}_i$, which are as required for $\tilde{\theta}_{i,ij}$.

If $g(i, j) = 6, 9$, $g \in G_{27}$, then the endpoints of $\tilde{\theta}_{i,ij} = \tilde{\theta}_{6,69} g$ are $(\pi_6 \tilde{\psi}_9) g$ and $(\pi_9 \tilde{\psi}_6) g$ and the required endpoints for $\tilde{\theta}_{i,ij}$ are $\pi_i \tilde{\psi}_j$ and $\pi_j \tilde{\psi}_i$. Are $(\pi_6 \tilde{\psi}_9) g$ and $\pi_i \tilde{\psi}_j$, $(\pi_9 \tilde{\psi}_6) g$ and $\pi_j \tilde{\psi}_i$ the same?

Recall that $\tilde{\psi}_j = \tilde{\psi}_9 g_{9j}$, $g_{9j} \in G_9$. Since $g g_{9j}^{-1}$ $(9) = 9$ and $g g_{9j}^{-1} \in G_{27}$ it follows that $g g_{9j}^{-1} \in \{\mathrm{id}, \beta, \beta^2\}$. If, say $g = \beta g_{9j}$, then $(\pi_6 \tilde{\psi}_9) g = \pi_i (\tilde{\psi}_9 \beta_{9j})$ and $\pi_i \tilde{\psi}_j = \pi_i (\tilde{\psi}_9 g_{9j})$, if not the same, belong to the same connected component of $CF_0(\mathrm{Lk}\langle ij \rangle)$ (by Proposition 10.4 and the continuity of π_i).

The proof that $(\pi_9 \tilde{\psi}_6) g$ and $\pi_j \tilde{\psi}_i$ belong to the same component of $CF_0(\mathrm{Lk}\langle ij \rangle)$ is analogous.

This is sufficient since all we need to know about the path $\tilde{\theta}_{i,ij}$ is where (and how) it crosses $CF_1 - CF_0$ (cf. (4.3)).

10. A FLATTENING OF Lk⟨9⟩: GRÜNBAUM'S TRIANGULATION OF S^3

(10.1) Lk⟨9⟩ consists of the following 20 3-simplices

1237	7145	7124	1245	
	7264	7236	2364	
	7356	7315	3156	
2138	8254	8215		8457
	8165	8136		8647
	8346	8324		8567

The symmetry group G_6 of Lk⟨9⟩ is generated by $\beta = (123)(465)$ and $\tau = (12)(45)(78)$; all elements of G_6 are orientation preserving. (The above display of 3-simplices of Lk⟨9⟩ reflects the action of G_6; note that orientations of the displayed simplices are consistent, so they determine an orientation of Lk⟨9⟩.)

In fact, Lk⟨9⟩ is simplicially isomorphic to Grünbaum's triangulation **M** of S^3 (see Grünbaum [10, p. 224]; his vertices *ABCDEFGH* correspond to our 78453612, respectively). **M** is a triangulation of S^3 with eight vertices which cannot be realized as the face complex of the boundary of a convex polytope in $\mathbf{R}^4$ (see Grünbaum and Sreedharan [11]). However, this property of **M** does not prevent **M** from having a flattening.

(10.2) PROPOSITION. *Let $\psi: \mathbf{M} \to \mathbf{R}^4$ be a map, linear on simplices, with values on vertices as follows:*

i	$\psi(i)$			
1	-1	$\sqrt{3}$	1	1
2	-1	$-\sqrt{3}$	1	1
3	2	0	1	1
4	0	-2	-1	1
5	$-\sqrt{3}$	1	-1	1
6	$\sqrt{3}$	1	-1	1
7	0	0	0	1
8	0	0	$-a$	-1

(i) *If* $0 < a < 1$, ψ *is a flattening of* $\mathbf{M}$.

(ii) *If* $0 < a < 1$ *and* $a \neq 2 - \sqrt{3}$, ψ *is a generic flattening of* $\mathbf{M}$.

Proof. We will give two proofs.

I. ψ is given by an 8×4 matrix. Let det$(ijkl)$ denote the 4×4 minor (of the matrix ψ) containing the rows $\psi(i)$, $\psi(j)$ $\psi(k)$, $\psi(l)$ in that order. Then

(i) ψ is a flattening of $\mathbf{M}$ if and only if det$(ijkl)$ has the same sign for every 3-simplex $\langle ijkl \rangle$ of $\mathbf{M}$, where the ordering of vertices of all $\langle ijkl \rangle$ is consistent with some orientation or $\mathbf{M}$;

(ii) ψ is a generic flattening if and only if, in addition to condition (i), det$(ijkl) \neq 0$ for every 4-tuple $ijkl$.

Conditions (i) and (ii) are easy to check. In fact, $0 < a < 1$ if and only if det$(ijkl) > 0$ for every 3-simplex $\langle ijkl \rangle$ with the ordering of vertices as in (10.1), and if $a = 2 - \sqrt{3}$, then det(1348) = det(2168) = det(3258) = 0.

II. Let $\mathbf{N} = \mathbf{M} - \mathrm{St}\langle 8 \rangle$. $\mathbf{N}$ is a 3-ball and it is easy to see that $\psi|\mathbf{N}$ is an embedding of $\mathbf{N}$ in $\mathbf{R}^3 \times 1$ (see Figure 2). Hence, $\psi: \mathbf{M} \to \mathbf{R}^4$ embeds $\mathbf{M}$ as $\psi\mathbf{N} \cup$ cone over $\partial(\psi\mathbf{N})$ with vertex $\psi(8)$ (see Figure 3). Next, check that the

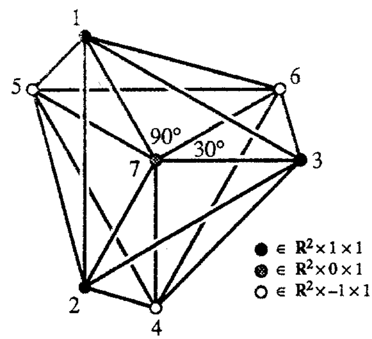

Fig. 2. $\psi|\mathbf{N}$ as seen from above.

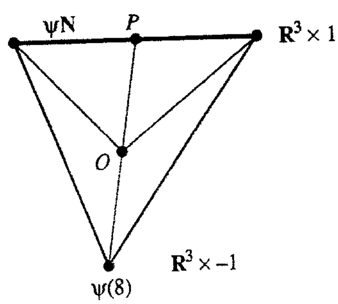

Fig. 3. $\psi(\mathbf{M})$.

origin O (of $\mathbf{R}^4$) is inside $\psi\mathbf{M}$ ('inside' has meaning since $\psi\mathbf{M} = \partial(\psi(8)\,.\,\psi\mathbf{N})$). But, O is inside $\psi\mathbf{M}$ if and only if a point $P(0, 0, a, 1)$, symmetric to $\psi(8)$ with respect to O, is inside $\psi\mathbf{N}$ and P is inside $\psi\mathbf{N}$ iff $0 < a < 1$ since

$$\psi\mathbf{N} \cap 0 \times 0 \times \mathbf{R} \times 1 = 0 \times 0 \times [0, 1] \times 1.$$

All that remains to prove is that $\psi(8)\,.\,\psi\mathbf{N}$ can be realized as $O\,.\,\psi\mathbf{M}$; this follows from the fact that $\psi\mathbf{N}$ can be realized as $P\,.\,\psi(\partial\mathbf{N})$. The verification of this is similar to the verification at the beginning of the proof that $\psi\mathbf{N}$ is an embedding.

ψ is generic unless P belongs to some plane determined by $\psi(i)$, $\psi(j)$, $\psi(k)$, $i, j, k \leqslant 6$. It is easy to see (in Figure 2 for example) that the only plane intersecting $0 \times 0 \times [0, 1] \in P$ are 134, 126, and 235 and then to compute that the intersection point is $(0, 0, 2 - \sqrt{3})$.

(10.3) DEFINITION. Define $\tilde{\psi}_i \in CF_0(\mathrm{Lk}\langle i \rangle)$ to be the configuration given by the flattening ψ with $a = \frac{1}{2}$ if $i = 9$; if $i \neq 9$ then set $\tilde{\psi}_i = \tilde{\psi}_9 g_{9i}$ where $g_{9i} \in G_9$, $g_{9i}(i) = 9$.

(19.4) PROPOSITION. *ψ and $\psi\beta$ ($\beta = (123)(465)$) belong to the same connected component of $F_0(\mathbf{M})$, the space of generic flattenings of* $\mathbf{M}$.

Proof. β is rotation through 120° in the first two coordinates.

The next proposition is not needed in the sequel, but is rather interesting in view of assertion (A) of Theorem 4.6.

(10.5) PROPOSITION. *ψ and $\psi\tau$ ($\tau = (12)(45)(78)$) belong to distinct connected components of $F(\mathbf{M})$, the space of all flattenings of* $\mathbf{M}$.

(10.6) COROLLARY. *The configuration space $CF(\mathbf{M})$ has at least two connected components.*

Proof of (10.5). Suppose that φ_t is a path in $F(\mathbf{M})$ such that $\varphi_0 = \psi$, $\varphi_1 = \psi\tau$. Let $\det_t(ijkl)$ be the corresponding minor of the 8×4 matrix φ_t.

Note that, if $\langle ijkl \rangle$ is a 3-simplex of $\mathbf{M}$ (as in (10.1)) then

$$\det_t(ijkl) > 0, \quad \text{all } t,$$

but if $ijkl \in \{1234, 1235, 1236\}$ then

$$\det_0(ijkl) > 0, \ \det_1(ijkl) < 0$$

(since $\det_1(ijkl) = \det_0(\tau(ijkl))$). Also note that since the codimension of $F - F_1$ (in F) is at least 2, without loss of generality we can assume that $\varphi_t \in F_1(\mathbf{M})$, all t (i.e. that for every t, at most one quadruple $ijkl$ has $\det_t(ijkl) = 0$).

Now, if for some t $\det_t(1234) = 0$ then there are $\lambda_i \neq 0$ such that

$$\varphi_t(4) = \lambda_1\varphi_t(1) + \lambda_2\varphi_t(2) + \lambda_3\varphi_t(3).$$

Since

$$\det_t(7124) = \Sigma\lambda_i \det_t(712i) = \lambda_3 \det_t(7123) = -\lambda_3 \det_t(1237)$$

and both $\langle 7124\rangle$ and $\langle 1237\rangle$ are (properly oriented) simplices of $\mathbf{M}$, it follows that $\lambda_3 < 0$. But, since $\langle 1245\rangle \in \mathbf{M}$,

$$0 < \det_t(1245) = \lambda_3 \det_t(1235)$$

so that $\det_t(1235) < 0$.

Thus, 1234 cannot be the first element of $\{1234, 1235, 1236\}$ for which $\det_t$ changes sign (we proved that $\det_t(1235)$ must change sign before $\det_t(1234)$).

Proofs that neither $\det_t(1235)$ nor $\det_t(1236)$ can be first to change sign are similar to (and in fact equal to β^2 and β multiples of) the proof for 1234.

11. $\theta_{8,89}$ AND $\theta_{6,69}$

According to Proposition 9.5 it suffices to find paths

$$\tilde{\theta}_{8,89}:[0, 1] \to CF_1(\mathrm{Lk}\langle 89\rangle)$$

and

$$\tilde{\theta}_{6,69}:[0, 1] \to CF_1(\mathrm{Lk}\langle 69\rangle)$$

with required endpoints: $\pi_8\tilde{\psi}_9$ and $\pi_9\tilde{\psi}_8$, $\pi_6\tilde{\psi}_9$ and $\pi_9\tilde{\psi}_6$ respectively. The additional requirement is that both paths have only a finite number of points in $CF_1 - CF_0$ (see (4.3)). The paths $[0,1] \to F_1$ we will find are paths in the flattening spaces and $\tilde{\theta}_{8,89}$ and $\tilde{\theta}_{6,69}$ are their projections into the corresponding configuration spaces.

(11.1) Given φ_0, $\varphi_1 \in F_0(\mathbf{L})$, $\mathbf{L}$ a triangulation of the 2-sphere, our strategy for finding a path $\{\varphi_t\} \subseteq F_1(\mathbf{L})$ between φ_0, φ_1 is as follows: First, find $\bar{\varphi}_0$, $\bar{\varphi}_1 \in F_0(\mathbf{L})$ such that

(i) $\bar{\varphi}_0$, φ_1 and $\bar{\varphi}_1$ are in the same connected component of $F_0(\mathbf{L})$;
(ii) for every vertex v of $\mathbf{L}$, $\bar{\varphi}_i v \in \mathbf{R}^2 \times \{-1, 1\}$, $i = 0, 1$;
(iii) for at least four vertices $\varphi_0 = \bar{\varphi}_1$.

Then we have to move each of the remaining vertices from position $\bar{\varphi}_0$ to position $\bar{\varphi}_1$, so that no more than one triple of vertices is linearly dependent at any moment, and certain triples are not allowed to be linearly dependent at all (those triples are vertices of 2-simplices of **L**). These conditions can be easily visualized if we plot $-\bar{\varphi}_i v$ in $\mathbf{R}^2 \times 1$ for each $\bar{\varphi}_i v \in \mathbf{R}^2 \times (-1)$; then everything is going on in $\mathbf{R}^2 \times 1$ and there linear dependency = colinearity.

Next, for each case (i, j = 8, 9 or 6, 9), we list, in Tables V and VI, the

TABLE V

$\theta_{8,89}$

(i) Lk⟨89⟩

123	125	136	156	234
245	346	457	467	567

(ii)

$\varphi_0 = \pi_8\psi_9$				$\varphi_1 = \pi_9\psi_8 = (\pi_7\psi_9)\gamma$, $\gamma = (123)(456)(789)$			
1	-1	$\sqrt{3}$	$1/2$	1	-1	$-\sqrt{3}$	1
2	-1	$-\sqrt{3}$	$1/2$	2	2	0	1
3	2	0	$1/2$	3	-1	$\sqrt{3}$	1
4	0	-2	$-3/2$	4	$-\sqrt{3}$	1	-1
5	$-\sqrt{3}$	1	$-3/2$	5	$\sqrt{3}$	1	-1
6	$\sqrt{3}$	1	$-3/2$	6	0	-2	-1
7	0	0	$-1/2$	7	0	0	$-1/2$

$\pi_8\psi_9$ and $\pi_7\psi_9$ are obtained by expressing ψ_9 in terms of the basis (1, 0, 0, 0), (0, 1, 0, 0), $-2\psi_9(7)$, $-2\psi_9(8)$ and then omitting the last and third coordinates respectively.

(iii)

$\bar{\varphi}_0$				$\bar{\varphi}_1$			
1	-2	6	1	1	-2	6	1
2	-2	-6	1	2	-2	-6	1
3	4	0	1	3	4	0	1
4	0	-2	-1	4	3	3	-1
5	-1	1	-1	5	0	-6	-1
6	1	1	-1	6	-3	3	-1
7	0	0	-1	7	0	0	-1

$\bar{\varphi}$: normalize each $\varphi_0(\mathrm{i})$ so that the 3rd coordinate is ± 1; rounding and scaling in the first two columns of φ_0.

$\bar{\varphi}_1$: rotation through $-120°$ in the first two coordinates then similarly as in φ_0.

(iv)

$\theta_{8,89}$										
Step	4			5			6			Linearly dependent triple
$\bar{\varphi}_0$	0	−2	−1	−1	1	−1	1	1	−1	
1	0.3	−1.5	−1							145
2							0.6	1.2	−1	246
3				−0.9	0.3	−1				356
4	0.6	−1	−1							147
5				−0.8	−0.4	−1				357
6							0.2	1.4	−1	267
7							0	1.5	−1	256
8				−0.7	−1.1	−1				345
9	0.9	−0.5	−1							146
10	1.5	0.5	−1							347
11				−0.5	−2.5	−1				257
12							−1	2	−1	167
13	3	3	−1							124
14				0	−6	−1				135
15							−3	3	−1	236
$\bar{\varphi}_1$	3	3	−1	0	−6	−1	−3	3	−1	

Read, for example row 8, as: Step 8 is to move vertex 5 along the straight line from previous position (−0.8, −0.4, −1.0) to (−0.7, −1.1, −1.0); the only linearly dependent triple occurring during this move is 345

TABLE VI

$\theta_{6,69}$

(i)

Lk⟨69⟩				
135	138	158	234	237
247	348	357	478	578

(ii)

$\varphi_0 = \pi_6\psi_9$				$\varphi_1 = \pi_9\psi_6 = (\pi_3\psi_9)\alpha$, $\alpha = (147)(258)(369)$			
1	$1-\sqrt{3}$	$1-\sqrt{3}$	$2\sqrt{3}$	1	$1-\sqrt{3}$	$2\sqrt{3}$	$1-\sqrt{3}$
2	1	0	0	2	1	0	0
3	$1+\sqrt{3}$	−1	$-2\sqrt{3}$	3	$1+\sqrt{3}$	$-2\sqrt{3}$	−1
4	0	1	0	4	0	1	0
5	−3	$-1+\sqrt{3}$	6	5	1/2	3/2	$-\sqrt{3}/6$
7	0	0	1	7	0	0	1
8	−1/2	$\sqrt{3}/6$	−1/2	8	−3	6	$-1+\sqrt{3}$

φ_0: express ψ_9 in terms of the basis $\psi_9(2)$, $\psi_9(4)$, $\psi_9(7)$, $\psi_9(6)$ and then omit the last coordinate.
φ_1: express ψ_6 in terms of the basis $\psi_6(2)$, $\psi_6(4)$, $\psi_4(7)$, $\psi_4(6)$ and then omit the last coordinate.

(iii)

	$\bar{\varphi}_0$				$\bar{\varphi}_1$		
1	−1	−6	8	1	−1	8	−6
2	1	0	0	2	1	0	0
3	1	−8	−8	3	1	−8	−8
4	1	3	0	4	1	3	0
5	−1	6	38	5	−1	−4	−2
7	1	0	3	7	1	0	3
8	−1	2	−2	8	−1	38	16

Multiply both φ_0 and φ_1 by

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \in GL_3^+$$

and get first columns with corresponding entries having the same sign (GL_3^+ is connected!). Then proceed similarly as in the case 89.

(iv)

	$\theta_{6,69}$									
Step	1			5			8			Linearly dependent triple
$\bar{\varphi}_0$	−1	−6	8	−1	6	38	−1	2	−2	
1	−1	−6	5.5							128
2	−1	−6	2.1							147
3	−1	−6	0.6							148
4	−1	−6	−1.1							124
5	−1	−6	−4.1							134
6				−1	−1.4	8.4				257
7				−1	−3.1	1.6				258
8				−1	−3.43	0.28				457
9				−1	−3.48	0.08				458
10				−1	−3.6	−0.4				245
11	−1	−4.9	−5.2							123
12	−1	−4.1	−6							178
13	−1	−1.3	−6							137
14				−1	−4	−2				345
15							−1	38	16	248
16	−1	1.5	−6							127
17	−1	8	−6							147
$\bar{\varphi}_1$	−1	8	−6	−1	−4	−2	−1	38	16	

following:

(i) 2-simplices of $\mathbf{L} = \mathrm{Lk}\langle ij\rangle$ (see Figure 1).
(ii) $\varphi_0 = \psi_{ij} = \pi_i\psi_j$ and $\varphi_1 = \pi_j\psi_i$ (note that π_i is not well defined on flattenings unless an isomorphism $\mathbf{R}^4/\psi_i(j) \cong \mathbf{R}^3$ is specified (see (1.5)), so we have to do this also).
(iii) $\bar{\varphi}_0$ and $\bar{\varphi}_1$ satisfying conditions 11.1(i)–(iii) above. (We do not know whether such $\bar{\varphi}_0$ and $\bar{\varphi}_1$ exist for every pair of flattenings φ_0 and $\varphi_1 \in F_0(\mathbf{L})$ of an arbitrary triangulation of S^2, so we give a brief description of how they are obtained in the cases at hand.)
(iv) Paths between $\bar{\varphi}_0$ and $\bar{\varphi}_1$ (described in steps, one step changes position of only one vertex) together with all linearly dependent triples along the way.

REMARK. The procedure described in (11.1) is needed only for finding the paths. Verifying that these are indeed paths with all the required properties and that the list of linearly dependent triples (i.e. the list of points of $F_1 - F_0$ on the path) is complete is much easier; a computer can do it.

(11.2) DEFINITION. $\tilde{\theta}_{8,89}: [0,1] \to CF_1(\mathrm{Lk}\langle 89\rangle)$ is the path in the configuration space represented by the path $\theta_{8,89}$ in the flattening space; $\tilde{\theta}_{6,69}$ is represented by $\theta_{6,69}$.

Note that, although (11.2) does not determine $\tilde{\theta}_{8,89}$ and $\tilde{\theta}_{6,69}$ completely (parametrizations are missing), it does determine $C(\tilde{\theta})$ (see Definition 4.3).

12. $P_1(\mathbf{CP}_9^2)$

The formula for $P_1(\mathbf{CP}_9^2)$ is of the form

$$P_1 = \sum_{i=1}^{9} p_i D^*\langle i\rangle$$

where $D^*\langle i\rangle$ is a cocycle in $C^4(D(\mathbf{CP}_9^2); \mathbf{Q})$ defined by $D^*\langle i\rangle(D\langle j\rangle) = \delta_{ij}$ ($D\langle k\rangle$ = the cell dual to the vertex k of $\mathbf{CP}_9^2$), and

$$p_i = C(\tilde{\psi}_i)B\langle i\rangle + \sum_{i \neq j} \varepsilon_{i,ij} C(\tilde{\theta}_{i,ij})B\langle ij\rangle$$

where ψ_i is the configuration of $\mathrm{Lk}\langle i\rangle$ (chosen according to 10.3), $\tilde{\theta}_{i,ij}$ is a path in $F_1(\mathrm{Lk}\langle ij\rangle)$ (9.5, 11.2), $B\langle i\rangle$, $B\langle ij\rangle$ are elements of the hypersimplical section determined using the algorithm of Section 5, and $\varepsilon_{i,ij}$ is the incidence number of dual cells $D\langle i\rangle$ and $D\langle ij\rangle$.

Note that, in order to apply the algorithm of Section 5, we must specify orientations of dual cells (and these orientations will determine the incidence numbers $\varepsilon_{i,ij}$).

Since $\mathbf{CP}^2$ is orientable we can make use of the following

(12.1) PROPOSITION. *If* $\mathbf{X}$ *is an oriented n-dimensional combinatorial manifold then there exists a natural correspondence orientations of* $\sigma \leftrightarrow$ *orientations of* $D\sigma$.

Namely, an orientation of $\sigma \in \mathbf{X}$ (given as a parity of an ordering of its vertices, say $\langle v_0 \ldots v_k \rangle$) determines an orientation for every $(n-k)$ simplex τ of $\mathbf{X}'$ belonging to $D\sigma$: the vertices of τ are $b_0, \ldots, b_l$, the barycenters of simplices

$$\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_l \; (k + l = n)$$

respectively; each σ_i, $i \geqslant 1$ has an additional vertex w_i (in addition to vertices of σ_{i-1}), and we say that the orientation of τ determined by $\sigma = \langle v_0 \ldots v_k \rangle$ is

$$\text{either } \langle b_0 b_1 \ldots b_l \rangle \quad \text{or} \quad -\langle b_0 b_1 \ldots b_k \rangle$$

depending on whether the orientation $\langle v_0 \ldots v_k w_1 \ldots w_l \rangle$ agrees with the orientation of $\mathbf{X}$ or not.

It is easy to see that orientations of top dimensional simplices of $D\sigma$ determined by the orientation $\langle v_0 \ldots v_k \rangle$ of σ are compatible; therefore they determine an orientation of $D\sigma$.

(12.2) COROLLARY. *If* σ *denotes an oriented simplex of* $\mathbf{X}$ *and* $D\sigma$ *the correspondingly oriented dual cell, then*

(i) $D(-\sigma) = -D\sigma$;

(ii) $(-1)^{\dim \tau} \varepsilon^{\mathrm{S}}_{\sigma\tau} = \varepsilon^{\mathrm{D}}_{\sigma\tau}$, where ε^{S}, ε^{D} are incidence numbers of simplices $\sigma < \tau$, dual cells $D\sigma > D\tau$ respectively;

(iii) $\varepsilon^{\mathrm{D}}_{\langle i_0 \ldots i_{k-1} \rangle, \langle i_0 \ldots i_k \rangle} = +1$.

We will use the above correspondence with the orientation of $\mathbf{CP}^2_9$ determined by $\langle 12456 \rangle$.

(12.3) PROPOSITION. (i) *For any two vertices i, j of* $\mathbf{CP}^2_9$

$$C(\tilde{\psi}_i)B\langle i \rangle = C(\tilde{\psi}_j)B\langle j \rangle.$$

(ii) *If i, ij and k, kl are in the same* G_{27} *orbit, then*

$$C(\tilde{\theta}_{i,ij})B\langle ij \rangle = C(\tilde{\theta}_{k,kl})B\langle kl \rangle.$$

Proof. (i) Let $g \in G_9$, $g(i) = j$. Then $\tilde{\psi}_i = \tilde{\psi}_j g$ and $B\langle j \rangle = gB\langle i \rangle$. It easily follows from the definition of $C(\tilde{\psi})$ in (4.2) that $C(\tilde{\psi}_j g)B\langle i\rangle = C(\tilde{\psi}_j)gB\langle i\rangle$.

(ii) Analogous to (i).

(12.4) COROLLARY. *For any two vertices i, j of* $\mathbf{CP}_9^2$

$$p_i = p_j.$$

(12.5) PROPOSITION.

$$p_9 = C(\tilde{\psi}_9)B\langle 9\rangle + C(\tilde{\theta}_{8,89})B\langle 89\rangle + 3C(\tilde{\theta}_{6,69})B\langle 69\rangle.$$

Proof. $\tilde{\theta}_{9,98}$ and $\tilde{\theta}_{9,9i}$, $i = 4, 5, 6$ are trivial paths (9.5). 9,97 and 8,89 are in the same G_{27} orbit; 9,91, 9,92, 9,93 and 6,69 are in the same G_{27} orbit.

(12.6) To compute p_9, we use three computer programs: A, B, and C.

Program A is designed to compute Tv, $v \in V_1$, where $T: V_1 \to V_2$ is a linear map and V_i, $i = 1, 2$ is a $\mathbf{Q}$-vector space spanned by all $\Delta(Z, A)$ (of a certain dimension). Various examples of T for which this program is used are $T = \Gamma$, $T = \partial$, and $T =$ formal sum of elements of a twisted group acting on $V_1 = V_2$.

Programs B and C are used to compute $C(\tilde{\psi}_\sigma)B\sigma$ and $C(\tilde{\theta}_{\sigma,\sigma\tau})B\tau$.

All three programs can be used for computing $P_1(\mathbf{X})$, $\mathbf{X}$ any combinatorial manifold, provided configuration data $\{\tilde{\psi}_\sigma, \sigma^{n-4} \in \mathbf{X}\}$ and $\{\tilde{\theta}_{\sigma,\sigma\tau}, \sigma^{n-4} R < \tau^{n-3}\}$ are given. (I know of no algorithm for finding configuration data, say on a computer.)

The procedure for computing $B\langle 9\rangle$, $B\langle 69\rangle$, and $B\langle 89\rangle$ has several steps.

STEP I (Preparation). The Γ's must be transformed from the form (7.1) (given in Tables I, II, and III) to the form as in (5.4). We need the following Γ's: $k = 2$, $m = 3, 4, 5, 6$; $k = 3$, $m = 7$; $k = 4$, $m = 8$. Program A can be used; moreover, using the same program it can be checked that these Γ's satisfy condition 5.1(i).

STEP II. If $\dim \sigma = 4$ then, by Definition 3.3(iii), $B\sigma = \Delta(v\sigma)$. If $\dim \sigma = 3$, say $\sigma = \langle ijkl\rangle$, then it follows from the proof of Proposition 3.4 that $B\sigma = \Delta(mn, ijkl)$ where $\langle ijkln\rangle$ has orientation agreeing with that of $\mathbf{CP}_9^2$ (and $\langle ijklm\rangle$ has the opposite orientation).

STEP III. For each 2-simplex $\langle ijk\rangle \in \mathbf{CP}_9^2$, compute

$$C\langle ijk\rangle = \Sigma\{B\langle ijkl\rangle : l \in v\,\mathrm{Lk}\langle ijk\rangle\}.$$

Note that, since the $B\sigma$'s are invariant under the action of G_{54}, $C\sigma$ will be invariant too, so it suffices to find $C\langle ijk\rangle$ for only one representative of each G_{54} orbit. There are five distinct orbits of 2-simplices of $\mathbf{CP}_9^2$; their representatives are (see Figure 1):

$$\langle 269\rangle,\ \mathrm{Lk}\langle 269\rangle = 347;$$

$$\langle 369\rangle,\ \mathrm{Lk}\langle 369\rangle = 157248;$$

$$\langle 389\rangle,\ \mathrm{Lk}\langle 389\rangle = 1246;$$

$$\langle 689\rangle,\ \mathrm{Lk}\langle 689\rangle = 13475;$$

$$\langle 789\rangle,\ \mathrm{Lk}\langle 789\rangle = 456.$$

Using Program A, compute $B\langle ijk\rangle = \Gamma(C\langle ijk\rangle)$ in each of the five cases.

STEP IV. There are only two G_{54} orbits of 1-simplices of CP_9^2, represented by $\langle 69\rangle$ and $\langle 89\rangle$ (for their links see Figure 1). When computing

$$C\langle ij\rangle = \Sigma\{B\langle ijk\rangle : k \in v\,\mathrm{Lk}\langle ij\rangle\}$$

($ij = 69$ or 89) we use Proposition 5.7 and (orbit representatives of) $B\langle ijk\rangle$ computed in Step III so that

$$\begin{aligned} C\langle 69\rangle &= \Sigma\{B\langle k69\rangle : k \neq 6, 9\} \\ &= [(12)(45)(78) + \mathrm{id}]B\langle 269\rangle + B\langle 369\rangle \\ &\quad + [(184275)(396) + (174)(285)(396)]B\langle 389\rangle \\ &\quad - [(12)(45)(78) - \mathrm{id}]B\langle 689\rangle \end{aligned}$$

and similarly

$$\begin{aligned} C\langle 89\rangle &= [(123)(465) + (132)(456) + \mathrm{id}]B\langle 389\rangle \\ &\quad + [(132)(456) + (123)(465) + \mathrm{id}]B\langle 689\rangle + B\langle 789\rangle. \end{aligned}$$

Program A is again used. Finally, compute $B\langle 69\rangle = \Gamma(C\langle 69\rangle$ and $B\langle 89\rangle = \Gamma(C\langle 89\rangle)$.

STEP V. Similarly, as in Step IV, first compute

$$\begin{aligned} C\langle 9\rangle &= \Sigma\{B\langle 9i\rangle : i \neq 9\} \\ &= [(169)(247)(358) + (158)(269)(347) + (147)(258)(369) \\ &\quad - (132)(456) - (123)(465) - \mathrm{id}]B\langle 69\rangle \\ &\quad + [-(12)(45)(78) - \mathrm{id}]B\langle 89\rangle, \end{aligned}$$

and then compute $B\langle 9\rangle = \Gamma(C\langle 9\rangle)$.

Now, using $\tilde{\psi}_9$, $\tilde{\theta}_{8,89}$, and $\tilde{\theta}_{6,69}$ obtained in Sections 10 and 11 and programs B and C, we compute

$$C(\tilde{\psi}_9)B\langle 9\rangle = -1/42,$$

$$C(\tilde{\theta}_{8,89})B\langle 89\rangle = 3/28$$

$$C(\tilde{\theta}_{6,69})B\langle 69\rangle = 1/12.$$

Thus, by Proposition 12.5,

$$p_9 = -\frac{1}{42} + \frac{3}{28} + 3\cdot\frac{1}{12} = \frac{1}{3}.$$

Therefore

$$P_1(\mathrm{CP}_9^2) = \frac{1}{3}\,\Sigma\{D^*\langle i\rangle : i = 1,\ldots,9\}.$$

REMARKS. (i) Since $9(1/3) = 3$, $P_1(\mathrm{CP}_9^2)$ agrees with the smooth $P_1(\mathrm{CP}^2)$ (compare [21, Example 15.6]).

(ii) Had we chosen the opposite orientation on $\mathrm{CP}_9^2(-\langle 12456\rangle$ instead of $\langle 12456\rangle)$, $B\langle ijkl\rangle$ in Step II would have the opposite sign, so that all subsequent $B\sigma$ would change sign, thus resulting in $p_9 = -1/3$. But a change in orientation of CP_9^2 will cause a change in the orientation of $D\langle i\rangle$, so that $P_1(\mathrm{CP}^{29})$ remains unchanged.

13. Remarks

13.1. *On our Version of the Formula*

A careful reader will note a few minor differences between our statement of the formula (in Section 4) and the original version in Gabrielov *et al.* [6]. Notational differences aside, these are:

(a) Our P is a cochain in $C^4(D\mathbf{X};\mathbf{Q})$ while theirs is a chain in $\hat{C}_{n-4}(\mathbf{X};\mathbf{Q})$, the chain complex on the simplices of $\mathbf{X}$ with the boundary operator $\hat{\partial}$ defined using incidence numbers of dual cells:

$$\hat{\partial}\sigma = \Sigma\{{}^{\mathrm{D}}_{\varepsilon\sigma}\tau : \tau < \sigma\};$$

it is obvious that $C^*(D\mathbf{X})$ and $\hat{C}_{n-*}(\mathbf{X})$ are isomorphic. This isomorphism, together with the natural correspondence of incidence numbers of simplices and incidence numbers of dual cells (see 12.1), is, in fact, the simplicial form of Poincaré duality.

(b) When defining $C(\psi)$ (see 4.2), Gabrielov *et al.* [6] use $-(-1)^{\#}$ for both types (1.2) and (2.1) of hypersimplices; we follow Stone [3] in using $(-1)^{\#}$ for type (1,2) and $-(-1)^{\#}$ for type (2,1). A possible cause for the discrepancy is in a typographic error (our computation would give an incorrect result if the signs from [6] were used).

(c) Finally, [6] does not have assertion (B) that the configuration spaces of triangulations of S^2 are simply connected. Instead, there is a claim that assertion (B) easily follows from a result of Chung-Wu Ho [12].

13.2. *Flattenings vs. Diagrams*

If K is a triangulation of a sphere S^k then, for every $\sigma^k \in K$, $K - \sigma$ is simplicially isomorphic to a subdivision of the standard n-simplex Δ^k (with no new vertices on the boundary $\partial\Delta^k$ of Δ^k). A homeomorphism $h: |K - \sigma| \cong \Delta^k$ which is linear on simplices of $K - \sigma$ is called a (Schlegel) diagram of K (based on σ) (compare Grünbaum [10, p. 42]). The space of all diagrams of K (based on the same σ) is an open subset of $(\operatorname{int} \Delta^k)^{vK - v\sigma}$ (cf. Section 1); denote it by $L(K, \sigma)$.

Diagram spaces have been studied by several people, sometimes with the motivation that these spaces are used by Cairns [3], Whitehead [35], Thom [33] and Kuiper [15] in their studies of the problem of smoothing a combinatorial manifold. In the first non-trivial case, $k=2$, $L(K, \sigma)$ turns out to be contractible (Bloch *et al.* [1]; earlier Cairns [4] proved its connectedness; and the already mentioned result of Ho [12] is that $L(K, \sigma)$ is simply connected), but almost nothing is known in the next case $k = 3$. There are sporadic examples showing that

(a) $L(K, \sigma)$ depends on the base simplex $\sigma^3 \in k$. In fact,

$$L(\mathbf{M}, \langle 1245 \rangle) = \emptyset \neq L(\mathbf{M}, \langle 4578 \rangle)$$

where $\mathbf{M}$ is Grünbaum's triangulation of S^3 with vertices as in Section 10 (see pp. 222–226 of [9]).

(b) The diagram space can be disconnected: Starbird [29] describes a subdivision K of Δ^3 with no new vertices on $\partial\Delta^3$ (and at least 17 vertices altogether) such that $L(K)$ is disconnected.

However, the smoothing theory of Cairns and others uses configuration spaces, not diagram spaces: a typical theorem shows that obstructions for smoothing a combinatorial manifold $\mathbf{X}$ have coefficients in homotopy groups of configuration spaces of links of simplices of $\mathbf{X}$ (e.g. [15], theorem 4.8). There are some variations in the spaces used (orientation-preserving flattenings, configurations whose vertices are on the unit sphere, geodesic triangulations of a sphere, etc.) but these spaces are always based on flattenings. The only exception is Thom's lecture at the 1958 International Congress in Edinburgh in which he goes from a geodesic triangulation of a sphere to a diagram ('par une inversion on est ramené à considérer...') without further explanation and then makes a conjecture that every diagram space is aspherical [32, p. 251].

The problem here is that the connection between flattening and diagrams requires that the flattening is convex (i.e. that a flattening realizes K as the boundary of a convex polytope; one then uses a central projection to obtain a diagram) and flattenings are not *a priori* convex. For example, if for some σ, $L(K, \sigma) = \varnothing$ then K has no convex flattening (thus $\mathbf{M}$ has no convex flattening, but, as we now know, it has a flattening). Note that a generic flattening of K realizes K as the boundary merely of a star-shaped polytope.

It is interesting to note that Cairns in [4] proves that for a triangulation K of S^2, both the flattening space $F(K)$ and the diagram space $L(K, \sigma)$ are connected, but he gives two similar but independent proofs (and does not derive one as the corollary of the other). There is a similar situation with the non-connectedness result for $F(\mathbf{M})$ (10.5): McCrory [20] has proved that the diagram space $L(\mathbf{M}, \langle 4578 \rangle)$ is also disconnected (thus providing an example much simpler than Starbird's [29]). It is worth mentioning here that Kuiper [15] has proved that the direct limit (under the directed system of compatible subdivisions) of configuration spaces of S^3 is connected; the proof depends on a deep (and hard) result of Cerf [5].

The only conclusion which can be drawn from all this is in the form of QUESTIONS. (1) What are the homotopy properties of configuration spaces? What is the relationship between diagram spaces and configuration spaces?

(2) Does the existence of convex flattenings of K imply some nice homotopy properties of $CF(K)$?

(3) If $|K| = S^2$, is $CF(K)$ contractible? simply connected? (In particular, is our assertion (B′) true?)

(4) MacPherson's variant of the formula [18] requires only $\pi_1(CF(\mathrm{Lk}\,\sigma^{n-3})) = 0$; Levitt's variant [16] does not require it. Is the formula (Theorem 4.6) valid without the assumptions (A) $\pi_0(CF(\mathrm{Lk}\,\sigma^{n-4})) = 0$ and (B) $\pi_1(CF(\mathrm{Lk}\,\sigma^{n-3})) = 0$? (In general?, for 4-manifolds?)

13.3. *The Role of Flattenings*

Levitt [16] has an interpretation of the formula which does not require hypersimplicial data. He uses transverse plane fields instead of flattenings, but these two are more or less equivalent (see Whitehead [35]). On the other hand, the smoothing theory of Cairns, Whitehead, and Kuiper shows that any generalization of the formula to higher Pontryagin classes, requiring flattening data, might, in fact, require the existence of some kind of smooth structure (say with singularities contained in some skeleton of the combinatorial manifold in question). A similar message is provided by Stone [31] where he constructs a 'combinatorial Gauss map' for a C^1 submanifold of Euclidean space. Recent work on the combinatorics of flattenings has been done by MacPherson [19]. The work of Levitt and Rourke [17] shows that local combinatorial formulas for characteristic classes could be obtained from explicit cocycles representing the cohomology of the classifying space for piecewise-linear bundles.

QUESTIONS. (1) How much information about **X** is retained in a hypersimplicial section of **X**? What minimum additional information is needed to recover characteristic classes of **X**? (Note that Gel'fand and MacPherson [8] showed that there are natural hypersimplices inside the Grassmannian G_n^k.) In the case of rational classes, does this additional information restrict the scope of the formula?

(2) Note that the conditions 5.1(i)–(ii) do not determine a unique hypersimplicial section Γ. Is there some canonical way to choose Γ, which implies a meaningful property of $\{B\sigma\}$ (compare Proposition 5.7)? It is reasonable to expect that by requiring more from Γ in 5.1, the amount of information carried by $\{B\sigma\}$ can be increased.

REFERENCES

1. Bloch, E. D., Connelly, R. and Henderson, D. W., 'The space of simplexwise linear homeomorphisms of a convex 2-disc', *Topology* **23** (1984), 161–175.
2. Cairns, S., 'Triangulated manifolds which are not Brouwer manifolds', *Ann. Math.* **41** (1940), 792–795.
3. Cairns, S., 'Homeomorphism between topological manifolds and analytic manifolds', *Ann. Math.* **41** (1940), 796–808.
4. Cairns, S., 'Isotopic deformations of geodesic complexes on the 2-sphere and on the plane', *Ann. Math.* **45** (1944), 207–217.
5. Cerf, J., 'Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$)', *Lecture Notes in Math.*, No. 53, Springer, Berlin, 1968.
6. Gabrielov, I. M., Gel'fand, I. M. and Losik, M. V., 'Combinatorial calculation of characteristic classes', *Funktsional. Anal. Prilozhen.* **9** (1975): No. 1, 54–55; No. 2, 12–28; No. 3, 5–26.

7. Gabrielov, I. M., Gel'fand, I. M. and Losik, M. V., 'A local combinatorial formula for the first Pontryagin class', *Funktsional. Anal. Prilozhen.* **10** (1976), No. 1, 14–17.
8. Gel'fand, I. M. and MacPherson, R. D., 'Geometry in Grassmannians and a generalization of the dilogarithm', *Adv. Math.* **44** (1982), 279–312.
9. Gel'fand, I. M. and MacPherson, R. D., 'A combinatorial formula for the Pontryagin classes', *Bull. Amer. Math. Soc.* **26** (1992), 304–309.
10. Grünbaum, B., *Convex Polytopes*, Wiley, New York, 1967.
11. Grünbaum, B. and Sreedharan, V. P., 'An enumeration of simplicial 4-polytopes with 8 vertices', *J. Combin. Theory* **2** (1967), 437–465.
12. Ho, Chung-Wu, 'On certain homotopy properties of some spaces of linear and piecewise linear homeomorphisms. I', *Trans. Amer. Math. Soc.* **181** (1973), 213–243.
13. Kühnel, W. and Banchoff, T. F., 'The 9-vertex triangulation of the complex projective plane', *Math. Intelligencer* **5** (1983), No. 3, 11–22.
14. Kühnel, W. and Lassmann, G., 'The unique 3-neighborly 4-manifold with few vertices', *J. Combin. Theory A* **35** (1983), 173–184.
15. Kuiper, N. H., 'On the smoothings of triangulated and combinatorial manifolds', *Differential and Combinatorial Topology* (*A Symposium in Honor of Marston Morse*), Princeton University Press, Princeton, 1965, pp. 3–22.
16. Levitt, N., 'Some remarks on local formulae for p_1', *Algebraic and Geometric Topology, Proceedings* (*Rutgers 1983*). *Lecture Notes in Math.* **1026**, Springer, 1985, pp. 96–126.
17. Levitt, N. and Rourke, C., 'The existence of combinatorial formulae for characteristic classes', *Trans. Amer. Math. Soc.* **239** (1978), 391–397.
18. MacPherson, R., 'The combinatorial formula of Gabrielov, Gel'fand and Losik for the first Pontryagin class', *Séminare Bourbaki* **29**, No. 497 (1976–77).
19. MacPherson, R. D., 'Combinatorial differential manifolds', preprint MIT, February 1992.
20. McCrory, C., 'Configuration spaces of simplicial spheres', Convegno Nazionale di Topologia, Sorrento 1988, *Rend. Circ. Mat. Palermo* (2), *Suppl.*, No. 24 (1990), 159–168.
21. Milnor, J. and Stasheff, J. D., *Characteristic Classes*, Princeton University Press, Princeton, 1974.
22. Morin, B. and Yoshida, M., 'The Kühnel triangulation of the complex projective plane', Max Planck Institut, Bonn, 1986 (preprint).
23. Novikov, S. P., 'On manifolds with free abelian fundamental group and their applications', *Izv. Akad. Nauk SSSR* **30** (1966), 207–246 (Russian).
24. Pontryagin, L. S., 'Characteristic cycles on differentiable manifolds', *Mat. Sb.* (*N.S.*) **21** (63) (1947), 233–284 (Russian).
25. Pontryagin, L. S., 'Vector fields on manifolds', *Mat. Sb.* (*N.S.*) **24** (66) (1949), 129–162 (Russian).
26. Pontryagin, L. S., 'Some topological invariants of closed Riemannian manifolds', *Izv. Akad. Nauk SSSR, Ser. Mat.* **13** (1949), 125–162 (Russian).
27. Pontryagin, L. S., English translations of [24]–[26] in *Translations Amer. Math. Soc.* **32** (1950), **13** (1949), **49** (1951); also in *Algebraic Topology, Translations, Ser. I*, vol. 7, Amer. Math. Soc., Providence (1962), pp. 149–219, 220–278, 279–331.
28. Rohlin, V. A. and Svarc, A. S., 'The combinatorial invariance of Pontryagin classes', *Dokl. Akad. Nauk SSSR* (N.S.) **114** (1957), 490–493 (Russian).
29. Starbird, M., 'A complex which cannot be pushed around in E^3', *Proc. Amer. Math. Soc.* **63** (1977), 363–367.
30. Stone, D., 'Notes on "A combinatorial formulae for $p_1(x)$" ', *Adv. Math.* **32** (1979), 36–97.
31. Stone, D., 'On the combinatorial Gauss map for C^1 submanifolds of Euclidean space', *Topology* **20** (1981), 247–272.
32. Thom, R., 'Des variétés trianguleés aux variétés différentiables', *Internat. Congr. Math., Edinburgh*, 1958, pp. 248–255.

33. Thom, R., 'Les classes caractéristiques de Pontryagin des variétés triangulées', *Symp. Internacional de Topologia Algebraica, Mexico*, 1958, pp. 54–67.
34. Whitehead, J. H. C., Note on manifolds, *Quart. J. Math.* **12** (1941), 26–29.
35. Whitehead, J. H. C., Manifolds with transverse fields in Euclidean space, *Ann. Math.* **73** (1961), 154–212.

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