SOME FREE ACTIONS OF CYCLIC GROUPS ON SPHERES

By J. MILNOR

LET $p \geq 5$ be an integer different from 6 and let $n \geq 5$ be odd. *This note will show that the cyclic group* Π *of order* p *can act differentiably on the* n*-sphere, without fixed points, in infinitely many different ways.* These actions are "different" in the sense that the corresponding quotient manifolds $M = S^n/\Pi$ can be distinguished by their Reidemeister-Franz-de Rham torsion invariants. Hence two such "different" manifolds M, M' cannot have the same simple homotopy type, cannot be piecewise-linearly homeomorphic, and cannot be diffeomorphic. (It is not known whether or not M and M' can be homeomorphic.)

First let me review the basic properties of the torsion invariant, following [3], [4]. Let K be a finite, connected† CW-complex and let Π denote the fundamental group of K. Let

$$f : Z[\Pi] \to \mathbf{C}$$

be a ring homomorphism from the integral group ring to the complex numbers. If the homology groups $H_i(K\,;\mathbf{C}_f)$ are all zero (homology with local coefficients twisted by f) then the torsion invariant $\Delta_f \widetilde{K} \in \mathbf{C}_0/\pm f\Pi$ is defined. (Here $\widetilde{K}$ denotes the universal covering complex, $\mathbf{C}_0$ the multiplicative group of non-zero complex numbers, and $\pm f\Pi$ the subgroup generated by $f(\Pi)$ and ± 1.) To simplify the notation we will henceforth leave off the tilde, and write simply $\Delta_f K$.

Similarly, given a pair K, L with $H_*(K, L\,; \mathbf{C}_f) = 0$ the torsion $\Delta_f(K, L)$ is defined. This satisfies the identity

$$\Delta_f(K, L) = \Delta_f K/\Delta_f L, \qquad (1)$$

providing that the three terms are defined. (If two out of three are defined, then the third is automatically defined.)

† For a complex with several components the torsion can be defined as the product of the torsions of the components.

If W is a triangulated orientable manifold of dimension n with boundary bW, then the following duality theorem holds. We must assume that $|f(t)| = 1$ for $t \in \Pi = \pi_1(W)$. Then

$$\Delta_f(bW) = (\Delta_f W)(\bar{\Delta}_f W)^{\epsilon(n)} \tag{2}$$

where $\bar{\Delta}$ denotes the complex conjugate and $\epsilon(n) = (-1)^n$. We will also need the following variant form. If M is a triangulated manifold without boundary of dimension $n-1$ then

$$\Delta_f M = (\bar{\Delta}_f M)^{\epsilon(n)}. \tag{3}$$

Now consider an h-cobordism $(W; M, M')$. That is, assume that W is a smooth manifold with boundary $M + M'$, and that both M and M' are deformation retracts of W. Choosing a C^1-triangulation of $(W; M, M')$ we will assume that the torsion

$$\Delta_f M \in \mathbf{C}_0 / \pm f\,\Pi$$

is defined.

Lemma 1. *With the above assumptions,* $\Delta_f M'$ *is defined and equal to*

$$(\Delta_f M)\,\Delta_f(W, M)\,(\bar{\Delta}_f(W, M))^{\epsilon(n)}.$$

Proof. Since M is a deformation retract of W it is clear that $\Delta_f(W, M)$ is defined. Thus $\Delta_f W$ is defined, and similarly $\Delta_f M'$ is defined. Consider the duality statement

$$\Delta_f(bW) = (\Delta_f W)(\bar{\Delta}_f W)^{\epsilon(n)}.$$

Since $\Delta_f(bW) = (\Delta_f M)(\Delta_f M')$ and since $\Delta_f W = (\Delta_f M)\,\Delta_f(W, M)$, this can be rewritten as

$$(\Delta_f M)(\Delta_f M') = (\Delta_f M)\,\Delta_f(W, M)\,(\bar{\Delta}_f M)^{\epsilon(n)}\,(\bar{\Delta}_f(W, M))^{\epsilon(n)}.$$

Now dividing through by

$$\Delta_f M = (\bar{\Delta}_f M)^{\epsilon(n)}$$

we obtain the required formula

$$\Delta_f M' = (\Delta_f M)\,\Delta_f(W, M)\,(\bar{\Delta}_f(W, M))^{\epsilon(n)}.$$

Henceforth we will assume that the dimension n of W is even. Thus Lemma 1 can be rewritten in the form

$$\Delta_f M' = (\Delta_f M) \,|\, \Delta_f(W, M)\,|^2. \tag{4}$$

Suppose that we are given the manifold M with fundamental group Π, and wish to construct the h-cobordism $(W; M, M')$.

Lemma 2 (Stallings). *If* $\dim(M) \geqslant 5$ *then the* h*-cobordism* $(W; M, M')$ *can be constructed so that* $\Delta_f(W, M)$ *is equal to the image, in* $\mathbf{C}_0/\pm f\Pi$, *of any unit of the group ring* $Z[\Pi]$.

Proof. Stallings actually observes that the h-cobordism can be constructed so that the Whitehead torsion invariant $\tau(W, M)$ is any desired element of the Whitehead group

$$\mathrm{Wh}(\Pi) = GL(\infty, Z[\Pi])/(\text{Commutators}, \pm \Pi).$$

(See Stallings [6, §2]. The manifold W is constructed by adjoining handles of index 2 and 3 to $M \times [0, 1]$ along one boundary, in such a way that the matrix of "incidence numbers" between the two types of handles is equal to a given invertible matrix over $Z[\Pi]$.) In particular if u is a unit of $Z[\Pi]$ then W can be chosen so that $\tau(W, M)$ is the element of $\mathrm{Wh}(\Pi)$ corresponding to the matrix

$$\begin{bmatrix} u & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix} \in GL(\infty, Z[\Pi]).$$

It is then clear that $\Delta_f(W, M)$ is equal to the image of u in $\mathbf{C}_0/\pm f\Pi$. (Compare Cockcroft [1], or [3, p. 589].) This completes the proof.

Thus in order to construct examples of h-cobordisms, we need only look for units in $Z[\Pi]$. To be more specific, let us now assume that Π is cyclic of order p with generator t. Define $f: Z[\Pi] \to \mathbf{C}$ by $f(t) = \exp(2\pi i/p)$. The following case is particularly easy.

Lemma 3 (Higman). *If* $p \geqslant 5$ *is an integer of the form* $6k \pm 1$ *then* $Z[\Pi]$ *contains a unit* u *with* $|f(u)| \neq 1$.

Proof. This follows from Higman [2]. Alternatively, here is a direct proof. Let

$$u = t + t^{-1} - 1$$

so that $f(u) = 2\cos(2\pi/p) - 1 \neq \pm 1$. To see that u is a unit it is only necessary for the reader to verify the identity

$$u(1 + t - t^3 - t^4 + t^6 + t^7 - - + + \ldots - - + t^{p-1}) = 1$$

for $p \equiv 1 \pmod 6$; or

$$u(-1 + t^2 + t^3 - t^5 - t^6 + + - - \ldots - - + t^{p-3} + t^{p-2}) = 1$$

for $p \equiv -1 \pmod 6$. This completes the proof. (Some further discussion of this lemma is included as an appendix.)

Now combining the three lemmas we have the following.

Theorem. *Let M be a smooth manifold of odd dimension ≥ 5 whose fundamental group is cyclic of order $p = 6k \pm 1$, $p \geq 5$. Suppose that the torsion $\Delta_f M$ is defined. Then there exist infinitely many manifolds $M_1, M_2, M_3, \ldots$ which are h-cobordant to M, but such that no two have the same simple homotopy type.*

Proof. For each integer m we can choose the h-cobordism $(W_m; M, M_m)$ so that

$$|\Delta_f(W_m, M)| = |f(u^m)|.$$

Then

$$\Delta_f M_m = (\Delta_f M)\,|f(u)|^{2m}.$$

Since $|f(u)| \neq 0, 1$ the real numbers $|\Delta_f M_m|$ are all distinct. This does not yet prove that the M_m all have distinct simple homotopy types, since the invariant $|\Delta_f M_m|$ depends on the choice of f. But there are only finitely many homomorphisms from $Z[\Pi]$ to $\mathbf{C}$, so out of the infinite sequence $M_1, M_2, \ldots$ one can certainly extract an infinite subsequence consisting of pairwise distinct manifolds. This completes the proof.

In particular let us apply this theorem to a lens space

$$L = S^n/\Pi, \; n \text{ odd}.$$

The resulting h-cobordant manifolds $L_1, L_2, \ldots$ will all have universal covering spaces diffeomorphic to the sphere. (See Smale [5].) Thus we have infinitely many distinct free actions of the cyclic group

Π on S^n. But there are only finitely many orthogonal actions of Π on S^n. Thus we have:

COROLLARY. *For n odd $\geqslant 5$ and $p = 6k \pm 1 \geqslant 5$ there exist infinitely many smooth fixed point free actions of the cyclic group of order p on S^n which are not smoothly equivalent to orthogonal actions, and are not smoothly equivalent to each other.*

It would be interesting to know whether any corresponding phenomenon occurs in dimension 3.

APPENDIX: FURTHER DISCUSSION OF LEMMA 3.

Higman's theorem actually applies more generally to any finite abelian group Π which does not have exponent 1, 2, 3, 4 or 6. Hence the theorem also applies in this generality. In fact suppose that $t \in \Pi$ is an element whose order p is different from 1, 2, 3, 4, 6. Then the Euler ϕ-function satisfies $\phi(p) > 2$. Hence there exists an integer a, with $1 < a < p/2$, which is relatively prime to p. Choose b so that $ab \equiv 1 \pmod{p}$, and set

$$x = (t^a - 1)/(t - 1) \quad = 1 + t + t^2 + \dots + t^{a-1},$$

$$y = (t^{ab} - 1)/(t^a - 1) = 1 + t^a + t^{2a} + \dots + t^{(b-1)a}.$$

Then $(t-1)xy = t - 1$, from which it follows easily that $xy - 1$ is a multiple of the element $s = 1 + t + t^2 + \dots + t^{p-1}$. Thus x is a unit modulo s. To obtain an actual unit, choose integers k, l, m so that $a^k = lp + 1$, $b^k = mp + 1$. Then $(x^k - ls)(y^k - ms) = 1$; so that $u = x^k - ls$ is the required unit.

As before we can choose $f : Z[\Pi] \to \mathbf{C}$ so that $f(t) = \exp(2\pi i/p)$. Then $f(s) = 0$, hence $|f(u)| = |f(x)|^k > 1$.

For any integer $p \geqslant 5$, $p \neq 6$, it follows that the cyclic group of order p can act freely on a sphere in infinitely many different ways. *Problem.* Can a cyclic group of order 2, 3, 4 or 6 act freely on a sphere in infinitely many different ways?

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Princeton University
Princeton N. J., U.S.A.