Classification of $(n-1)$-connected $2n$-dimensional manifolds and the discovery of exotic spheres

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At Princeton in the fifties I was very much interested in the fundamental problem of understanding the topology of higher dimensional manifolds. In particular, I focussed on the class of $2n$-dimensional manifolds which are $(n-1)$-connected, since these seemed like the simplest examples for which one had a reasonable hope of progress. (Of course the class of manifolds with the homotopy type of a sphere is even simpler. However the generalized Poincaré problem of understanding such manifolds seemed much too difficult: I had no idea how to get started.) For a closed $2n$-dimensional manifold M^{2n} with no homotopy groups below the middle dimension, there was a considerable body of techniques and available results to work with. First, one could easily describe the homotopy type of such a manifold. It can be built up (up to homotopy type) by taking a union of finitely many n-spheres intersecting at a single point, and then attaching a $2n$-cell e^{2n} by a mapping of the boundary ∂e^{2n} to this union of spheres, so that

$$M^{2n} \simeq (S^n \vee \cdots \vee S^n) \cup_f e^{2n} .$$

Here the attaching map f represents a homotopy class in $\pi_{2n-1}(S^n \vee \cdots \vee S^n)$, a homotopy group that one can work with effectively, at least in low dimensions. Thus the homotopy theory of such manifolds is under control. We can understand this even better by looking at cohomology. The cohomology of such an M^{2n}, using integer coefficients, is infinite cyclic in dimension zero, free abelian in the middle dimension with one generator for each of the spheres, and is infinite cyclic in the top dimension where we have a cohomology class corresponding to this top dimensional cell; that is

$$H^0(M^{2n}) \cong \mathbf{Z} , \qquad H^n(M^{2n}) \cong \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} , \qquad H^{2n}(M^{2n}) \cong \mathbf{Z} .$$

Taken from the lecture 'Growing Up in the Old Fine Hall' given on 20th March, 1996, as part of the Princeton 250th Anniversary Conference [9]. For accounts of exotic spheres, see [1]–[4], [6]. The classification problem for $(n-1)$-connected $2n$-dimensional manifolds was finally completed by Wall [10], making use of exotic spheres.

The attaching map f determines a cup product operation: To any two cohomology classes in the middle dimension we associate a top dimensional cohomology class, or in other words (if the manifold is oriented) an integer. This gives a bilinear pairing from $H^n \otimes H^n$ to the integers. This pairing is symmetric if n is even, skew-symmetric if n is odd, and always has determinant ± 1 by Poincaré duality. For n odd this pairing is an extremely simple algebraic object. However for n even such symmetric pairings, or equivalently quadratic forms over the integers, form a difficult subject which has been extensively studied. (See [7], and compare [5].) One basic invariant is the *signature*, computed by diagonalizing the quadratic form over the real numbers, and then taking the number of positive entries minus the number of negative entries.

So far this has been pure homotopy theory, but if the manifold has a differentiable structure, then we also have characteristic classes, in particular the Pontrjagin classes in dimensions divisible by four,

$$p_i \in H^{4i}(M) \ . \qquad (1)$$

This was the setup for the manifolds that I was trying to understand as a long term project during the 50's. Let me try to describe the state of knowledge of topology in this period. A number of basic tools were available. I was very fortunate in learning about cohomology theory and the theory of fiber bundles from Norman Steenrod, who was a leader in this area. These two concepts are combined in the theory of characteristic classes [8], which associates cohomology classes in the base space to certain fiber bundles. Another basic tool is obstruction theory, which gives cohomology classes with coefficients in appropriate homotopy groups. However, this was a big sticking point in the early 50's because although one knew very well how to work with cohomology, no one had any idea how to compute homotopy groups except in special cases: most of them were totally unknown. The first big breakthrough came with Serre's thesis, in which he developed an algebraic machinery for understanding homotopy groups. A typical result of Serre's theory was that the stable homotopy groups of spheres

$$\Pi_n \ = \ \pi_{n+k}(S^k) \qquad (k > n+1)$$

are always finite. Another breakthrough in the early 50's came with Thom's cobordism theory. Here the basic objects were groups whose elements were equivalence classes of manifolds. He showed that these groups could be computed in terms of homotopy groups of appropriate spaces. As an immediate consequence of his work, Hirzebruch was able to prove a formula which he had conjectured relating the characteristic classes of manifolds to the signature. For any closed oriented $4m$-dimensional manifold, we can form the signature of the cup product pairing

$$H^{2m}(M^{4m};\mathbf{R}) \otimes H^{2m}(M^{4m};\mathbf{R}) \;\to\; H^{4m}(M^{4m};\mathbf{R}) \;\cong\; \mathbf{R},$$

using real coefficients. If the manifold is differentiable, then it also has Pontrjagin classes (1). Taking products of Pontrjagin classes going up to the top dimension we build up various *Pontrjagin numbers*. These are integers which depend on the structure of the tangent bundle. Hirzebruch conjectured a formula expressing the signature as a rational linear combination of the Pontrjagin numbers. For example

$$\text{signature}\,(M^4) \;=\; \frac{1}{3}p_1[M^4] \tag{2}$$

and

$$\text{signature}\,(M^8) \;=\; \frac{1}{45}(7p_2 - (p_1)^2)[M^8]\ . \tag{3}$$

Everything needed for the proof was contained in Thom's cobordism paper, which treated these first two cases explicitly, and provided the machinery to prove Hirzebruch's more general formula.

These were the tools which I was trying to use in understanding the structure of $(n-1)$-connected manifolds of dimension $2n$. In the simplest case, where the middle Betti number is zero, these constructions are not very helpful. However in the next simplest case, with just one generator in the middle dimension and with $n = 2m$ even, they provide quite a bit of structure. If we try to build up such a manifold, as far as homotopy theory is concerned we must start with a single $2m$-dimensional sphere and then attach a cell of dimension $4m$. The result is supposed to be homotopy equivalent to a manifold of dimension $4m$:

$$S^{2m} \cup e^{4m} \;\simeq\; M^{4m}\ .$$

What can we say about such objects? There are certainly known examples; the simplest is the complex projective plane in dimension four – we can think of that as a 2-sphere (namely the complex projective line) with a 4-cell attached to it. Similarly in dimension eight there is the quaternionic projective plane which we can think of as a 4-sphere with an 8-cell attached, and in dimension sixteen there is the Cayley projective plane which has similar properties. (We have since learned that such manifolds can exist only in these particular dimensions.)

Consider a smooth manifold M^{4m} which is assumed to have a homotopy type which can be described in this way. What can it be? We start with a $2m$-dimensional sphere S^{2m}, which is certainly well understood. According to Whitney, this sphere can be smoothly embedded as a subset $S^{2m} \subset M^{4m}$ generating the middle dimensional homology, at least if $m > 1$. We look at a tubular neighborhood of this embedded sphere, or equivalently at its normal $2m$-disk bundle E^{4m}. In general this must be twisted as we go around the sphere — it can't be simply a product or the manifold wouldn't have the right properties. In terms of fiber bundle theory, we can look at it in the following way: Cut the $2m$-sphere into two hemispheres D^{2m}_+ and D^{2m}_-, intersecting along their common boundary S^{2m-1}. Over each of these hemispheres we must have a product bundle, and we must glue these two products together to form

$$E^{4m} \;=\; (D^{2m}_+ \times D^{2m}) \cup_F (D^{2m}_- \times D^{2m}) \;.$$

Here the gluing map $F(x,y) = (x, f(x)y)$ is determined by a mapping $f : S^{2m-1} \to \mathrm{SO}(2m)$ from the intersection $D^{2m}_+ \cap D^{2m}_-$ to the rotation group of D^{2m}. Thus the most general way of thickening the $2m$-sphere can be described by an element of the homotopy group $\pi_{2m-1}\mathrm{SO}(2m)$. In low dimensions, this group was well understood.

In the simplest case $4m = 4$, we start with a D^2-bundle over S^2 determined by an element of $\pi_1\mathrm{SO}(2) \cong \mathbf{Z}$. It is not hard to check that the only 4-manifold which can be obtained from such a bundle by gluing on a 4-cell is (up to orientation) the standard complex projective plane: This construction does not give anything new. The next case is much more interesting. In dimension eight we have a D^4-bundle over S^4 which is described by an element of $\pi_3(\mathrm{SO}(4))$. Up to a 2-fold covering, the group $\mathrm{SO}(4)$ is just a Cartesian product of two 3-dimensional spheres, so that $\pi_3\mathrm{SO}(4) \cong \mathbf{Z} \oplus \mathbf{Z}$. More explicitly, identify S^3 with the unit 3-sphere in the quaternions. We get one mapping from this 3-sphere to itself by left multiplying by an arbitrarily unit quaternion and another mapping by right multiplying by an arbitrary unit quaternion. Putting these two operations together, the most general $(f) \in \pi_3(\mathrm{SO}(4))$ is represented by the map $f(x)y = x^i y x^j$, where x and y are unit quaternions and where $(i,j) \in \mathbf{Z} \oplus \mathbf{Z}$ is an arbitrary pair of integers.

Thus to each pair of integers (i,j) we associate an explicit 4-disk bundle over the 4-sphere. We want this to be a tubular neighborhood in a closed 8-dimensional manifold, which means that we want to be able to attach a 8-dimensional cell which fits on so as to give a smooth manifold. For that to work, the boundary $M^7 = \partial E^8$ must be a 7-dimensional sphere S^7. The question now becomes this: For which i and j is this boundary isomorphic to S^7? It is not difficult to decide when it has the right homotopy type: In fact M^7 has the homotopy type of S^7 if and only if $i+j$ is equal to ± 1. To fix our ideas, suppose that $i+j=+1$. This still gives infinitely many choices of i. For each choice of i, note that $j=1-i$ is determined, and we get as boundary a manifold $M^7=\partial E^8$ which is an S^3-bundle over S^4 having the homotopy type of S^7. Is this manifold S^7, or not?

Let us go back to the Hirzebruch-Thom signature formula (3) in dimension 8. It tells us that the signature of this hypothetical 8-manifold can be computed from $(p_1)^2$ and p_2. But the signature has to be ± 1 (remember that the quadratic form always has determinant ± 1), and we can choose the orientation so that it is $+1$. Since the restriction homomorphism maps $H^4(M^8)$ isomorphically onto $H^4(S^4)$, the Pontrjagin class p_1 is completely determined by the tangent bundle in a neighborhood of the 4-sphere, and hence by the integers i and j. In fact it turns out that p_1 is equal to $2(i-j)=2(2i-1)$ times a generator of $H^4(M^8)$, so that $p_1^2[M^8]=4(2i-1)^2$. We have no direct way of computing p_2, which depends on the whole manifold. However, we can solve equation (3) for $p_2[M^8]$, to obtain the formula

$$p_2[M^8] \;=\; \frac{{p_1}^2[M^8]+45}{7} \;=\; \frac{4(2i-1)^2+45}{7}\,. \tag{4}$$

For $i=1$ this yields $p_2[M^8]=7$, which is the correct answer for the quaternion projective plane. But for $i=2$ we get $p_2[M^8]=\frac{81}{7}$, which is impossible! Since p_2 is a cohomology class with integer coefficients, this Pontrjagin number $p_2[M^8]$, whatever it is, must be an integer.

What can be wrong? If we choose p_1 in such a way that (4) does not give an integer value for $p_2[M^8]$, then there can be no such differentiable manifold. The manifold $M^7=\partial E^8$ certainly exists and has the homotopy type of a 7-sphere, yet we cannot glue an 8-cell onto E^8 so as to obtain a smooth manifold. What I believed at this point was that such an M^7 must be a counterexample to the seven dimensional Poincaré hypothesis: I thought that M^7, which has the homotopy type of a 7-sphere, could not be homeomorphic to the standard 7-sphere.

Then I investigated further and looked at the detailed geometry of M^7. This manifold is a fairly simple object: an S^3-bundle over S^4 constructed in an explicit way using quaternionic multiplication. I found that I could actually prove that it was homeomorphic to the standard 7-sphere, which made the situation seem even worse! On M^7, I could find a smooth real-valued function which had just two critical points: a non-degenerate maximum point and a non-degenerate minimum point. The level sets for this function are 6-dimensional spheres, and by deforming in the normal direction we obtain a homeomorphism between this manifold and the standard S^7.

(This is a theorem of Reeb: if a closed k-manifold possesses a Morse function with only two critical points, then it must be homeomorphic to the k-sphere.) At this point it became clear that what I had was not a counterexample to the Poincaré hypothesis as I had thought. This M^7 really was a topological sphere, but with a strange differentiable structure.

There was a further surprising conclusion. Suppose that we cut this manifold open along one of the level sets, so that

$$M^7 \;=\; D^7_+ \cup_f D^7_- \;,$$

where the $D^7_\pm$ are diffeomorphic to 7-disks . These are glued together along their boundaries by some diffeomorphism $g : S^6 \to S^6$. *Thus this manifold M^7 can be constructed by taking two 7-dimensional disks and gluing the boundaries together by a diffeomorphism.* Therefore, at the same time, the proof showed that there is a diffeomorphism from S^6 to itself which is essentially exotic: It cannot be deformed to the identity by a smooth isotopy, because if it could then M^7 would be diffeomorphic to the standard 7-sphere, contradicting the argument above.

References

[1] M. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Ann. Math. 77, 504–537 (1963)
[2] A. Kosinski, "Differential Manifolds", Academic Press (1993)
[3] T. Lance, *Differentiable structures on manifolds*, (in this volume)
[4] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. 64, 399–405 (1956)
[5] —— *On simply connected 4-manifolds*, pp. 122–128 of "Symposium Internacional de Topologia Algebraica", UNAM and UNESCO, Mexico (1958)
[6] —— "Collected Papers 3, Differential Topology", Publish or Perish (in preparation)
[7] —— and D. Husemoller, "Symmetric Bilinear Forms", Springer (1973)
[8] —— and J. Stasheff, "Characteristic Classes", Ann. Math. Stud. 76, Princeton (1974)
[9] H. Rossi (ed.), *Prospects in Mathematics: Invited Talks on the Occasion of the 250th Anniversary of Princeton University*, Amer. Math. Soc., Providence, RI, 1999.
[10] C. T. C. Wall, *Classification of $(n-1)$-connected $2n$-manifolds*, Ann. of Math. 75, 163–189 (1962)

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