On the existence of a connection with curvature zero [1])

by JOHN MILNOR

1. Introduction

Let M^2 be a closed oriented C^∞-surface of genus g and let $GL^+(2)$ denote the group of 2×2 real matrices with positive determinant. To each $GL^+(2)$-bundle [2]) over M^2 there corresponds the EULER (or STIEFEL-WHITNEY) class

$$X \in H^2(M^2, Z) \ ,$$

and conversely each class X comes from a unique equivalence class of bundles. Let $X[M^2]$ denote the KRONECKER index of X with the fundamental cycle of M^2.

Theorem 1. *If $|X[M^2]| \geqslant g > 0$ then the $GL^+(2)$-bundle over M^2 with* EULER *class X does not possess a connection [3]) with curvature zero.*

By an *affine connection* on M^2 is meant a connection in the tangent bundle of M^2. BEZECRI [3] has shown that M^2 can possess a symmetric affine connection with curvature zero if and only if $g = 1$. A somewhat stronger result follows from Theorem 1. For the tangent bundle of M^2 the integer $X[M^2]$ is equal to the EULER characteristic $2 - 2g$. Since $|2 - 2g| \geqslant g$ for $g \geqslant 2$ we have:

Corollary. *A surface with genus $g \geqslant 2$ does not possess any (not necessarily symmetric) affine connection with curvature zero.*

(The corresponding assertion for $g = 0$ is easy. In fact a bundle over a simply connected manifold can have a flat connection only if it is a product bundle). A study of the flat affine connections which do exist on the torus has been made by KUIPER [6].

It would be interesting to ask if every manifold with a flat affine connection has EULER characteristic zero, but only partial results have been obtained in higher dimensions. (See AUSLANDER [8].)

The following shows that Theorem 1 was a best possible result.

Theorem 2. *If $|X[M^2]| < g$ then the $GL^+(2)$-bundle over M^2 with* EULER *class X does have a connection with curvature zero.*

Corollary. *If $g \geqslant 2$ then there exist $GL^+(2)$-bundles over M^2 with $X \neq 0$ which have connections with curvature zero.*

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[1]) The author holds an ALFRED P. SLOAN fellowship.

[2]) By a *$GL^+(2)$-bundle* is meant a principal fibre bundle with structural group $GL^+(2)$.

[3]) For the theory of connections see CHERN [4] or AMBROSE and SINGER [1].

This behavior of the EULER class is in contrast with the known behavior of the PONTRJAGIN classes. (See § 4.)

The proofs of Theorems 1 and 2 are outlined in § 2 and completed in § 3. Some related problems are discussed in § 4.

2. Proof of Theorems

Let G be a LIE group and M a C^∞-manifold. The universal covering space of M can be considered as a $\pi_1(M)$-bundle over M. Therefore any homomorphism $\pi_1(M) \to G$ induces a G-bundle over M.

Lemma 1. *A G-bundle over M possesses a connection with curvature zero if and only if it is induced from the universal covering bundle by a homomorphism h: $\pi_1(M) \to G$.*

(Compare STEENROD [7], § 13. The image of h in G is called the holonomy group. See AUSLANDER and MARKUS [2].) More generally for any cell complex K a homomorphism $h: \pi_1(K) \to G$ induces a G-bundle over K. If G is connected, then the first obstruction to the existence of a cross-section in this bundle is an element

$$c(h) \in H^2(K, \pi_1(G)) \ .$$

The next lemma gives an algorithm for computing this obstruction.

Assume that K has only one vertex so that $\pi_1(K)$ has a canonical presentation with generators $\alpha_1, \ldots, \alpha_m$ corresponding to the oriented edges and relations $\varphi_1(\alpha_1, \ldots, \alpha_m) = 1, \ldots, \ \varphi_n(\alpha_1, \ldots, \alpha_m) = 1$ corresponding to the oriented faces. It is important that the relation φ_k should be obtained by listing the edges incident to the face e^2_k (with appropriate signs) in the cyclic order determined by the orientation of e^2_k. Consider the exact sequence

$$1 \to \pi_1(G) \xrightarrow{i} \tilde{G} \xrightarrow{p} G \to 1$$

where $\tilde{G}$ is the universal covering group. For each edge of K define $\gamma_j = h(\alpha_j) \in G$, and choose a representative $\Gamma_j \in p^{-1}(\gamma_j)$. Then for each face we have

$$p\varphi_k(\Gamma_1, \ldots, \Gamma_m) = \varphi_k(\gamma_1, \ldots, \gamma_m) = 1$$

so that $i^{-1}\varphi_k(\Gamma_1, \ldots, \Gamma_m) \in \pi_1(G)$ is defined.

Lemma 2. *The cocycle in K which assigns to the k-th face of K the element $i^{-1}\varphi_k(\Gamma_1, \ldots, \Gamma_m)$ of $\pi_1(G)$ represents the negative $-c(h)$ of the obstruction cohomology class.*

The proofs of Lemmas 1 and 2 will be given in § 3.

Now let K be the surface M^2 of genus $g > 0$. Using the cell subdivision of

M^2 obtained by matching the edges of a $4g$-gon, we obtain the presentation for $\pi_1(M^2)$ with generators $\alpha_1, \ldots, \alpha_{2g}$ and relation

$$\varphi(\alpha_1, \ldots, \alpha_{2g}) = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1} \ldots \alpha_{2g-1}\alpha_{2g}\alpha_{2g-1}^{-1}\alpha_{2g}^{-1} = 1 \ .$$

The problem of determining which classes $c(h)$ can occur is therefore equivalent to the following: Which elements of $\pi_1(G)$ have the form

$$i^{-1}\varphi(\Gamma_1, \ldots, \Gamma_{2g}) \quad \text{with} \quad \Gamma_1, \ldots, \Gamma_{2g} \in \tilde{G}?$$

Every nonsingular matrix γ can be written uniquely in the form

$$\gamma = r(\gamma)s(\gamma) \tag{1}$$

where $r(\gamma)$ is orthogonal and $s(\gamma)$ is symmetric and positive definite. This defines a retraction

$$r\colon\ GL^+(n) \to SO(n) \ .$$

It is easily verified that r has the following properties:

$$r(\gamma^{-1}) = r(\gamma)^{-1} \ , \tag{2}$$

(3) if γ_1 or γ_2 is orthogonal then

$$r(\gamma_1\gamma_2) = r(\gamma_1)r(\gamma_2) \ .$$

Now consider the special case $n = 2$. For the rest of the section let G denote $GL^+(2)$ and let S denote the circle group $SO(2)$.

The universal covering group $\tilde{S}$ is isomorphic to the real numbers R under the isomorphism $\text{Exp}\colon R \to \tilde{S}$ which covers the homomorphism

$$\alpha \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \ .$$

The retraction r is covered by a retraction $\tilde{r}\colon \tilde{G} \to \tilde{S}$. Define $\theta = \text{Exp}^{-1}\tilde{r}$ so that the following diagram is commutative.

$$\begin{tikzcd} & R \arrow[dr, "\text{Exp}"] & \\ \tilde{G} \arrow[ur, "\theta"] \arrow[rr, "\tilde{r}"] & & \tilde{S} \end{tikzcd}$$

Lemma 3. *The map* $\theta\colon \tilde{G} \to R$ *satisfies*

$$|\,\theta(\Gamma_1\Gamma_2) - \theta(\Gamma_1) - \theta(\Gamma_2)| < \frac{\pi}{2}$$

for all $\Gamma_1, \Gamma_2 \in \tilde{G}$.

Proof. By direct computation we obtain the following explicit formula for r:

$$r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{x^2 + y^2}}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

where $x = a + d$, $y = b - c$. This implies that

(4) if $\gamma \epsilon G$ has positive trace then $r(\gamma) \epsilon S$ also has positive trace.

If σ_1 and σ_2 are two positive definite matrices then the product $\sigma_1 \sigma_2$ has positive trace. (Proof. Choose an orthogonal matrix ϱ so that $\varrho \sigma_1 \varrho^{-1}$ is diagonal. Then

$$\text{Trace}\,(\sigma_1 \sigma_2) = \text{Trace}\,((\varrho \sigma_1 \varrho^{-1})(\varrho \sigma_2 \varrho^{-1}))$$

which is clearly a sum of positive terms.) Together with (4) this implies

(5) if σ_1 and σ_2 are two positive definite elements of G, then $r(\sigma_1 \sigma_2)$ has positive trace.

Now let γ_1 and γ_2 be arbitrary elements of G. According to (1)

$$\gamma_1 \gamma_2 = r(\gamma_1) r(\gamma_2) \sigma_1 \sigma_2 \ ,$$

where $\sigma_1 = r(\gamma_2)^{-1} s(\gamma_1) r(\gamma_2)$, $\sigma_2 = s(\gamma_2)$. According to (3) this implies that

$$r(\gamma_1 \gamma_2) = r(\gamma_1) r(\gamma_2) r(\sigma_1 \sigma_2) \ .$$

Therefore according to (5) we have:

(6) for any $\gamma_1, \gamma_2 \epsilon G$ the matrix $r(\gamma_2)^{-1} r(\gamma_1)^{-1} r(\gamma_1 \gamma_2)$ has positive trace.

Given elements Γ_1, Γ_2 of $\tilde{G}$ set $p(\Gamma_i) = \gamma_i$ and define

$$\varDelta(\Gamma_1, \Gamma_2) = \theta(\Gamma_1 \Gamma_2) - \theta(\Gamma_1) - \theta(\Gamma_2) \ .$$

Then $p\,\mathrm{Exp}\,\varDelta(\Gamma_1, \Gamma_2) = r(\gamma_1 \gamma_2) r(\gamma_1)^{-1} r(\gamma_2)^{-1}$ has positive trace by (6). Since $p\,\mathrm{Exp}\,\varDelta = \begin{pmatrix} \cos \varDelta & \sin \varDelta \\ -\sin \varDelta & \cos \varDelta \end{pmatrix}$ by definition, this implies that $\cos \varDelta$ is positive. But since $\varDelta$ is a continuous function of Γ_1 and Γ_2 which vanishes for $\Gamma_1 = 1$, this implies that $-\frac{\pi}{2} < \varDelta < \frac{\pi}{2}$; which completes the proof of Lemma 3.

Proof of Theorem 1. The maps which are used are summarized in the following diagram.

$$\begin{tikzcd}
& 1 \arrow[d] & 0 \arrow[d] & \\
& \pi_1(G) \arrow[r] \arrow[d, "i"] & Z \arrow[d, "2\pi"] & \\
& \tilde{G} \arrow[r, "\theta"] \arrow[d, "p"] & R \arrow[r, "\mathrm{Exp}"] \arrow[d] & \tilde{S} \arrow[dl] \\
\pi_1(M^2) \arrow[r, "h"] & G \arrow[r, "r"] & S &
\end{tikzcd}$$

Here "2π" denotes multiplication by 2π. The above maps are homomorphisms, with the exception of θ and r. The coefficient homomorphism $\pi_1(G) \to Z$

carries the obstruction class $c(h) \in H^2(M^2, \pi_1(G))$ into the Euler class X. The cocycle representing $-c(h)$ carries the unique 2-cell of M^2 into

$$i^{-1}(\Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \ldots \Gamma_{2g}^{-1}) \in \pi_1(G) .$$

Therefore $X[M^2]$ is equal to

$$-\frac{1}{2\pi} \theta(\Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \ldots) \in Z .$$

Applying the previous lemma $4g-1$ times we have

$$|\theta(\Gamma_1 \Gamma_2 \Gamma_1^{-1} \Gamma_2^{-1} \ldots) - \theta(\Gamma_1) - \theta(\Gamma_2) - \theta(\Gamma_1^{-1}) \ldots | < (4g-1)\frac{\pi}{2} .$$

Cancelling the terms $-\theta(\Gamma_j) - \theta(\Gamma_j^{-1})$ and dividing by 2π this gives

$$|X[M^2]| < g - \tfrac{1}{4} < g$$

which completes the proof of Theorem 1.

Let γ_0 denote the matrix $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and let Γ_0 be the matrix in $p^{-1}(\gamma_0)$ satisfying $\theta(\Gamma_0) = 0$. Let K and $\tilde{K}$ denote the conjugate classes of γ_0 and Γ_0 respectively.

Lemma 4. *Any element of the conjugate class* $(\operatorname{Exp} \pi)\tilde{K}$ *can be expressed as the product of two elements in* $\tilde{K}$.

(Note that $\operatorname{Exp} \pi$ is in the center of the group $\tilde{G}$.) Proof. Consider the identity

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2\frac{1}{2} & 4\frac{1}{2} \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ -1\frac{1}{2} & 2\frac{1}{2} \end{pmatrix} .$$

The first two matrices have determinant 1 and trace $2\frac{1}{2}$, and therefore belong to the conjugate class K. Similarly the third belongs to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} K$. Let $\Gamma_1, \Gamma_2 \in \tilde{K}$ be the elements corresponding to the first two matrices. Then $\Gamma_1 \Gamma_2$ must belong to $(\operatorname{Exp} \pi n)\tilde{K}$ for some odd integer n. We will prove that $n = \pm 1$. (Actually $n = +1$.)

Since elements of K have positive trace, assertion (4) implies that $\cos\theta(\Gamma) > 0$ for $\Gamma \in \tilde{K}$. Since $\tilde{K}$ is connected this implies that $|\theta(\Gamma)| < \frac{\pi}{2}$ for $\Gamma \in \tilde{K}$. Therefore $|\theta(\Gamma)| > \frac{5}{2}\pi$ for $\Gamma \in (\operatorname{Exp} \pi n)\tilde{K}$, $n \geqslant 3$.

But since

$$|\theta(\Gamma_1 \Gamma_2)| < |\theta(\Gamma_1)| + |\theta(\Gamma_2)| + \frac{\pi}{2} < \frac{3}{2}\pi ,$$

it follows that $\Gamma_1\Gamma_2$ can only belong to $(\operatorname{Exp}\pi n)\tilde{K}$ for $n = \pm 1$. If $n = +1$ then every element of $(\operatorname{Exp}\pi)\tilde{K}$ can be written in the form

$$\Gamma(\Gamma_1\Gamma_2)\Gamma^{-1} = (\Gamma\Gamma_1\Gamma^{-1})(\Gamma\Gamma_2\Gamma^{-1}) \in \tilde{K}\tilde{K} \ .$$

If $n = -1$ then $\Gamma_2^{-1}\Gamma_1^{-1}$ can be substituted for $\Gamma_1\Gamma_2$ in this formula. This completes the proof of Lemma 4.

One consequence of this lemma is the following

(7) Every element Γ of $(\operatorname{Exp}\pi)\tilde{K}$ has the form $\Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_2^{-1}$.

In fact choose $\Gamma_1, \Gamma_3 \in \tilde{K}$ so that $\Gamma_1\Gamma_3 = \Gamma$. Since Γ_1^{-1} also belongs to $\tilde{K}$, there exists an element $\Gamma_2 \in \tilde{G}$ so that $\Gamma_2\Gamma_1^{-1}\Gamma_2^{-1} = \Gamma_3$, and hence

$$\Gamma = \Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_2^{-1} \ .$$

We are now ready to give the proof of Theorem 2. First recall the statement:

Theorem 2. *If* $|X[M^2]| < g$ *then the G-bundle over* M^2 *with* EULER *class X does have a connection with curvature zero.*

According to Lemma 1 it is sufficient to prove that this bundle is induced by a homomorphism $\pi_1(M^2) \to G$. We will concentrate on the case

$$X[M^2] = 1 - g \ , \quad g > 2 \ ,$$

since the other cases can easily be obtained by the same method.

Applying the previous lemma $g-2$ times there exist elements $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_{g-1} \in \tilde{K}$ so that the product $\bar{\Gamma}_1 \ldots \bar{\Gamma}_{g-1}$ is equal to $(\operatorname{Exp}(g-2)\pi)\Gamma_0$. Setting $\bar{\Gamma}_g = \Gamma_0^{-1} \in \tilde{K}$ this gives

$$\bar{\Gamma}_1 \ldots \bar{\Gamma}_{g-1}\bar{\Gamma}_g = \operatorname{Exp}(g-2)\pi \ .$$

Now choose elements $\Gamma_{2i-1}, \Gamma_{2i}$ in $\tilde{G}$ so that $\Gamma_{2i-1}\Gamma_{2i}\Gamma_{2i-1}^{-1}\Gamma_{2i}^{-1} = (\operatorname{Exp}\pi)\bar{\Gamma}_i$. Then

$$\begin{aligned}&\Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_2^{-1} \ldots \Gamma_{2g-1}\Gamma_{2g}\Gamma_{2g-1}^{-1}\Gamma_{2g}^{-1} = (\operatorname{Exp}\pi)\bar{\Gamma}_i \ldots (\operatorname{Exp}\pi)\bar{\Gamma}_g \\ &= (\operatorname{Exp}\pi)^g \operatorname{Exp}(g-2)\pi = \operatorname{Exp} 2\pi(g-1).\end{aligned}$$

Since this is in the kernel of $p\colon \tilde{G} \to G$, we can define $h\colon \pi_1(M^2) \to G$ by $h(\alpha_i) = p(\Gamma_i)$. The characteristic number of the corresponding G-bundle is

$$-\frac{1}{2\pi}\theta(\operatorname{Exp}(2\pi(g-1))) = 1 - g \ ,$$

which completes the proof.

3. Proof of Lemmas

Suppose that $q\colon E \to M$ is the projection map of a G-bundle with a flat connection. Then each C^∞-path $\lambda\colon [0, 1] \to M$ induces an isomorphism of $q^{-1}\lambda(1)$ onto $q^{-1}\lambda(0)$, invariant under C^∞-homotopies of the path which keep the end points fixed. Identify G with the fibre over a base point. If λ is a closed path representing the element α of $\pi(M)$, then the corresponding isomorphism of the fiber G onto itself is left-translation by an element $h(\alpha)$ of G. It is easily verified that

$$h\colon \pi_1(M) \to G$$

is a homomorphism.

Define a map $\eta\colon \tilde{M} \times G \to E$ as follows, where $\tilde{M}$ denotes the universal covering space. Given $(x,\gamma) \in \tilde{M} \times G$ choose a C^∞-path from x to the base point in $\tilde{M}$, and project onto a path λ in M. Then the isomorphism along λ carries $\gamma \in G$ into a point $\eta(x,\gamma)$ in the fibre corresponding to x. The identity

$$\eta(x\cdot\alpha, \gamma) = \eta(x, h(\alpha)\cdot\gamma)$$

which is easily verified, shows that E is the G-bundle over M induced by h. This proves the first half of Lemma 1.

It will be convenient to let $\pi_1(M)$ act on the left of $\tilde{M}$ by the rule

$$\alpha\cdot x = x\cdot\alpha^{-1}$$

for $\alpha \in \pi_1(M)$, $x \in \tilde{M}$; and on the left of G by the rule

$$\alpha\cdot\gamma = h(\alpha)\gamma \ .$$

Then $\pi_1(M)$ acts on the left of $\tilde{M} \times G$, and the collapsed space E is the total space of the G-bundle over M induced by h. The product bundle $\tilde{M} \times G$ has a canonical flat connection. Since this connection is invariant under the action of $\pi_1(M)$, it induces a flat connection in E. This completes the proof of Lemma 1.

The proof of Lemma 2 will be based on the following. Since $\pi_1(K)$ acts on the left of $\tilde{K}$ without fixed points, and also acts on the left of G, we can ask if there exists a $\pi_1(K)$-equivariant map $\tilde{K} \to G$. Let

$$c'(h) \in H^2(K, \pi_1(G))$$

be the first obstruction to the existence of such a map. We will first compute this class $c'(h)$.

For each edge e^1_j of K let $\lambda_j\colon [0, 1] \to K$ denote a characteristic map, and let $\Lambda_j\colon [0, 1] \to \tilde{K}$ be the unique lifting which carries 0 into the base point $\tilde{e}^0$. Then $\Lambda_j(1)$ will be equal to $\alpha_j\cdot\tilde{e}^0$.

Let μ_j: $[0, 1] \to G$ be any path from 1 to γ_j. An equivariant map f from the 1-skeleton $(\tilde{K})^1$ to G is defined by

$$\alpha \cdot \Lambda_j(t) \to \alpha \cdot \mu_j(t)$$

for each $\alpha \in \pi_1(K)$, $t \in [0, 1]$, and $1 \leqslant j \leqslant m$.

Consider the boundary of the k-th 2-cell e_k^2 in K. This closed path, which represents the element

$$\varphi_k(\alpha_1, \ldots, \alpha_m) = \alpha_{l_1}^{\varepsilon_1} \ldots \alpha_{l_r}^{\varepsilon_r}$$

of $\pi_1(K^1)$, lifts into a path in $\tilde{K}$ which goes from $\tilde{e}^0$ to $\alpha_{l_1}^{\varepsilon_1} \cdot \tilde{e}^0$ to $\alpha_{l_1}^{\varepsilon_1} \alpha_{l_2}^{\varepsilon_2} \cdot \tilde{e}^0$ to $\ldots$ to $\varphi_k(\alpha_1, \ldots, \alpha_m) \cdot \tilde{e}^0 = \tilde{e}^0$. Mapping this closed path into G we obtain a representative $f \partial \tilde{e}_k^2$ of the required homotopy class

$$c'(h)(e_k^2) \in \pi_1(G) \ .$$

To evaluate this homotopy class we will lift the path to $\tilde{G}$. Suppose that the path μ_j, when lifted to $\tilde{G}$, goes from 1 to Γ_j. Then the path $f \partial \tilde{e}_k^2$, lifted to $\tilde{G}$, goes from 1 to $\Gamma_{l_1}^{\varepsilon_1}$ to $\Gamma_{l_1}^{\varepsilon_1} \Gamma_{l_2}^{\varepsilon_2}$ to $\ldots$ to $\varphi_k(\Gamma_1, \ldots, \Gamma_m)$. Therefore this path represents the element $-i^{-1} \varphi_k(\Gamma_1, \ldots, \Gamma_m)$ of $\pi_1(G)$.

To complete the proof of Lemma 2 the following is needed.

Lemma 5. *The obstruction class $c'(h)$ is equal to the obstruction class $c(h)$.*

Proof. A cross section $K^1 \to E = \eta(\tilde{K} \times G)$ over the 1-skeleton is defined by the rule

$$\lambda_j(t) \to \eta(\Lambda_j(t), \mu_j(t)) \ ,$$

for each $t \in [0, 1]$ and each j. The obstruction to extending this over the 2 cell e_k^2 is defined as follows. Lifting e_k^2 to the 2-cell $\tilde{e}_k^2$, over which the bundle is a product, the given cross section on the 1-skeleton can be considered as a map $\partial \tilde{e}_k^2 \to \partial \tilde{e}_k^2 \times G$. Projecting into G we obtain a representative of the required class $c(h)(e_k^2) \in \pi_1(G)$. But evidently the class obtained in this way is the same as the class $c'(h)(e_k^2)$ studied previously. This completes the proof of Lemma 2.

4. Further remarks

Although the following two results are topological in nature, the author only knows how to prove them by methods of differential geometry, using theorems of CHERN and WEIL.

Theorem 3. *Let K be a finite complex.*

(a) *The $SO(n)$-bundle over K induced by any homomorphism $\pi_1(K) \to SO(n)$ has trivial EULER class with rational coefficients.*

(b) *The $GL(n)$-bundle over K induced by any homomorphism $\pi_1(K) \to GL(n)$ has trivial* Pontrjagin *classes with rational coefficients.*

These results are in contrast with Theorem 2, which showed that a homomorphism $\pi_1(K) \to GL^+(n)$ may induce a bundle with non-trivial Euler class.

Both results would be false with integer coefficients. For example consider the Lens space $K = S^5/Z_p$ and a non-trivial homomorphism $h: \pi_1(K) \to SO(2)$. The Euler class $X \in H^2(K, Z) \approx Z_p$ can be computed by Lemma 2, and turns out to be non-zero. This implies that the Pontrjagin class $p_1 = X^2$ is also non-zero.

Proof of 3a. Imbed K in some euclidean space as a deformation retract of a neighborhood U. Then we may replace K by the C^∞-manifold U for the rest of the proof. The following assertion follows easily from Chern [4]. (See also [5].) For any $SO(n)$-bundle over U choose a connection with curvature forms Ω_{ij}. Then the form

$$\Sigma \varepsilon_{i_1 \ldots i_n} \Omega_{i_1 i_2} \ldots \Omega_{i_{n-1} i_n}$$

represents a cohomology class which is a non-zero multiple of the Euler class.

If this $SO(n)$-bundle is induced by a homomorphism $\pi_1(U) \to SO(n)$, then according to Lemma 1 there exists a connection with $\Omega_{ij} = 0$. Therefore the Euler class must be zero.

The proof of 3b is similar, being based on the following assertion which also follows easily from [4]. For any $GL(n)$-bundle over U choose a connection with curvature forms Ω^i_j. Then the form

$$\Sigma \delta^{j_1 \ldots j_{2k}}_{i_1 \ldots i_{2k}} \Omega^{i_1}_{j_1} \ldots \Omega^{i_{2k}}_{j_{2k}}$$

represents a cohomology class which is a non-zero multiple of the k-th Pontrjagin class. The rest of the proof is clear.

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