On Spaces Having the Homotopy Type of a CW-Complex

John Milnor

ON SPACES HAVING THE HOMOTOPY TYPE
OF A CW-COMPLEX

BY

JOHN MILNOR[1]

Let $\mathcal{W}$ denote the class of all spaces which have the homotopy type of a CW-complex. The following is intended as propaganda for this class $\mathcal{W}$. Function space constructions such as

$$(A; B, a_0)^{([0,1];[0],[1])}$$

have become important in homotopy theory, and our basic objective is to show that such constructions do not lead outside the class $\mathcal{W}$ (Theorem 3).

The first section is concerned with the smaller class $\mathcal{W}_0$, consisting of all spaces which have the homotopy type of a countable CW-complex. §§1 and 2 are independent of each other.

The basic reference for this paper is J. H. C. Whitehead [15].

1. The class $\mathcal{W}_0$.

THEOREM 1. *The following restrictions on the space A are equivalent*:
(a) *A belongs to the class $\mathcal{W}_0$*;
(b) *A is dominated by a countable CW-complex*;
(c) *A has the homotopy type of a countable locally finite simplicial complex.*
(d) *A has the homotopy type of an absolute neighborhood retract*[2].

Proof. The assertions (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) are due to J. H. C. Whitehead. In fact the implications (c)$\Rightarrow$(a)$\Rightarrow$(b) are trivial; and the implication (b)$\Rightarrow$(c), for a path-connected space, is Theorem 24 of Whitehead [16]. But if the space A is dominated by a countable CW-complex, then each path-component of A is an open set; and the collection of path-components is countable. Therefore it is sufficient to consider the path-connected case.

The assertions (c)$\Rightarrow$(d)$\Rightarrow$(b) are due to O. Hanner [8]. (Hanner's Corollary 3.5 asserts that every countable locally finite simplicial complex is an absolute neighborhood retract; and Theorem 6.1 asserts that every absolute neighborhood retract is dominated by a (countable) locally finite simplicial complex.) This completes the proof of Theorem 1.

As consequences of Theorem 1 we have:

COROLLARY 1. *Every separable manifold belongs to the class $\mathcal{W}_0$.*

Received by the editors April 16, 1957.

[1] The author holds an Alfred P. Sloan fellowship.

[2] Following Kuratowski [11] an absolute neighborhood retract is required to be separable metric, but not necessarily compact.

This follows since every separable manifold is an absolute neighborhood retract. (See Hanner [8, Theorem 3.3].)

COROLLARY 2. *If A belongs to $\mathcal{W}_0$ and C is compact metric, then the function space A^C (in the compact open topology) belongs to $\mathcal{W}_0$.*

Proof. By Theorem 1 we may assume that A is an absolute neighborhood retract. But according to Kuratowski [11, p. 284] this implies that A^C is also an absolute neighborhood retract, which completes the proof.

REMARK. The assumption that C is compact is essential here. For example let A have two elements and let Z be a countable discrete space. Then A^Z is a Cantor set and certainly does not belong to the class $\mathcal{W}$.

It would be possible to generalize these results so as to apply to pairs, triads, etc. (Compare §2.) However we will not try to do this here.

Whitehead has made the following observation [16, Appendix A]:

PROPOSITION 1. *If a compact space A belongs to the class $\mathcal{W}$, then A is dominated by a finite complex.*

REMARK. It would be interesting to ask if every space which is dominated by a finite complex actually has the homotopy type of a finite complex. This is true in the simply connected case, but seems like a difficult problem in general.

The same argument shows the following:

PROPOSITION 2. *If a space A in $\mathcal{W}$ has the Lindelöf property[3], then A belongs to the class $\mathcal{W}_0$.*

Proof. Let $f: A \to K$ be a homotopy equivalence, where K is a CW-complex. Let L be the smallest subcomplex of K which contains $f(A)$. Then A is dominated by L. But if A has the Lindelöf property then $f(A)$ also has the Lindelöf property. Therefore L must be a countable complex. (Compare Whitehead [15, §5(D)].) In view of Theorem 1, this completes the proof.

In conclusion we mention an interesting example. Borsuk [2] has constructed a locally contractible compact metric space C such that the homology groups $H_n(C, Z)$ are nontrivial for every integer $n \geq 0$. It follows from Proposition 1 that this space C cannot belong to the class $\mathcal{W}$.

2. **The class $\mathcal{W}^n$.** By a CW-n-ad $K = (K; K_1, \cdots, K_{n-1})$ we mean a CW-complex K together with $n-1$ numbered subcomplexes $K_1, \cdots, K_{n-1}$. Let $\mathcal{W}^n$ denote the class of all n-ads which have the homotopy type of a CW-n-ad.

THEOREM 2. *The following restrictions on an n-ad $A = (A; A_1, \cdots, A_{n-1})$ are equivalent:*

(a) *A belongs to the class $\mathcal{W}^n$,*

(b) *A is dominated by a CW-n-ad,*

---

[3] The space A has the *Lindelöf property* if every covering of A by open sets is reducible to a countable sub-covering.

(c) *A has the homotopy type of a simplicial n-ad in the weak topology,*

(d) *A has the homotopy type of a simplicial n-ad in the strong topology.*

(By the *strong topology* on a simplicial complex K we mean the strongest topology such that the barycentric coordinates, considered as functions from K to $[0, 1]$, are continuous. This is the same as the metric topology considered in Eilenberg and Steenrod [5, p. 75].)

For the case $n = 1$ this theorem can be proved as follows. The implications (c)⇒(a)⇒(b) are clear, and the implication (b)⇒(c) follows from Whitehead [16, Theorem 23]. The implications (c)⇔(d) follow from Dowker's result [3] that a "metric complex" has the same homotopy type as the corresponding complex in the weak topology.

Although these proofs could easily be generalized to the case $n > 1$, we will instead give a self-contained proof, based on the following lemma. A map $f: A \rightarrow B$ will be called a *singular homotopy equivalence* if f_* carries $\pi_k(A, a)$ isomorphically onto $\pi_k(B, f(a))$ for each $a \in A$ and each $k \geq 0$. Here π_0 denotes the set of path-components.

Now consider an n-ad $A = (A; A_1, \cdots, A_{n-1})$. For each nonvacuous set S of integers between 1 and $n-1$ define $A_S = \bigcap_{i \in S} A_i$. For the vacuous set ϕ define $A_\phi = A$. Then a map $f: A \rightarrow B$ of n-ads induces 2^{n-1} maps $f_S: A_S \rightarrow B_S$. We will say that f is a *singular homotopy equivalence* if each f_S is a singular homotopy equivalence.

LEMMA 1. *If A and B belong to the class $\mathcal{W}^n$, then every singular homotopy equivalence $f: A \rightarrow B$ is a homotopy equivalence in the ordinary sense.*

For the case $n = 1$ this follows from [15, Theorem 1], and for $n = 2$ it follows from [10, Theorem 3.1].

Proof. We may assume that A and B are CW-n-ads, and that f is cellular. This implies that the mapping cylinder $M = (M; M_1, \cdots, M_{n-1})$ of f is itself a CW-n-ad. (Here M_i is defined as the mapping cylinder of $f_i: A_i \rightarrow B_i$.) Considering A as a sub-n-ad of M, we will prove that A is a strong deformation retract of M. That is we will construct a homotopy $h: M \times [0, 1] \rightarrow M$ satisfying

(*)
$$h(x, 0) = x \qquad \text{for all } x \in M,$$
$$h(x, 1) \in A \qquad \text{for all } x \in M, \text{ and}$$
$$h(a, t) = a \qquad \text{for all } a \in A, t \in [0, 1].$$

This will complete the proof of Lemma 1.

According to the hypothesis each $f_S: A_S \rightarrow B_S$ is a singular homotopy equivalence. This implies that each inclusion $A_S \rightarrow M_S$ is a singular homotopy equivalence; and therefore that

$$\pi_k(M_S, A_S, a) = 0$$

for each $a \in A_S$ and $k \geq 0$. This is equivalent to the following assertion, where E^k denotes a closed k-cell.

(**) *Every map of the pair*

$$((E^k \times [0]) \cup (\dot{E}^k \times [0, 1]), \dot{E}^k \times [1])$$

into (M_S, A_S) *can be extended to a map of* $(E^k \times [0, 1], E^k \times [1])$ *into* (M_S, A_S).

The homotopy h can now be constructed as follows, by induction on the skeletons M^k of M. Define X^k as the subset

$$(M \times [0]) \cup (A \times [0, 1]) \cup (M^{k-1} \times [0, 1])$$

of $M \times [0, 1]$. The conditions (*) define h uniquely on the set X_0. Suppose that h has been defined on the set X^k. Let e^k be any k-cell of $M - A$, and let S be the set of integers i such that $e^k \subset M_i$. Then h has already been defined on

$$(\bar{e}^k \times [0]) \cup (\dot{e}^k \times [0, 1]),$$

and maps this set into M_S. Furthermore it maps $\dot{e}^k \times [1]$ into $A \cap M_S = A_S$. Therefore, according to (**), h can be extended over $\bar{e}^k \times [0, 1]$ so as to map this set into M_S, and so as to map $\bar{e}^k \times [1]$ into A_S. Continuing by induction, this completes the proof.

Proof of Theorem 2. The implications (c)$\Rightarrow$(a)$\Rightarrow$(b) are clear. To prove that (b)$\Rightarrow$(c) let $|S(A)|$ denote the geometric realization[4] of the singular complex of A. M. Barratt [1] has recently proved that the realization of any semi-simplicial n-ad can be triangulated. In other words $|S(A)|$ can be considered as a simplicial n-ad in the weak topology.

If K is the CW-n-ad which dominates A, then there exist maps $A \xrightarrow{f} K \xrightarrow{g} A$ such that gf is homotopic to the identity. Consider the commutative diagram

$$
\begin{array}{ccccc}
|S(A)| & \xrightarrow{|f|} & |S(K)| & \xrightarrow{|g|} & |S(A)| \\
{\scriptstyle j}\downarrow & & {\scriptstyle j'}\updownarrow\,{\scriptstyle k} & & \downarrow{\scriptstyle j} \\
A & \xrightarrow{\quad f \quad} & K & \xrightarrow{\quad g \quad} & A
\end{array}
$$

Note that $|g| \, |f|$ is homotopic to the identity. (See [12, Corollary to Theorem 2].) Since K is a CW-n-ad, Lemma 1, together with [12, Theorem 4], implies that j' is a homotopy equivalence. Let k be a homotopy inverse to j'; and consider

$$|g| \, kf : A \to |S(A)|.$$

It follows from the above diagram that this map is a homotopy inverse to j; which completes the proof that (b)$\Rightarrow$(c).

----

[4] We will use realizations in the sense of the author's paper [12] (making an obvious generalization to n-ads), although Giever [7] or Whitehead [16] could equally well be used.

Proof that (c)⟺(d). Let K_s denote a simplicial n-ad in the strong topology, and let K_w denote the same n-ad in the weak topology. We will show that the natural map

$$i: K_w \to K_s$$

is a homotopy equivalence.

Choose a locally finite open covering $\{U_\beta\}$ of K_s, indexed by the collection $\{\beta\}$ of vertices; so that each U_β is contained in the open star neighborhood of the vertex β. For example if ξ_β denotes the βth barycentric coordinate of the point x in K_s, then the sets

$$U_\beta = \left\{ x \mid \xi_\beta > \frac{1}{2} \operatorname*{Max}_{\gamma} \xi_\gamma \right\}$$

form such a covering. Choose a partition of unity $\{p_\beta\}$ on K_s so that $p_\beta(K_s - U_\beta) = 0$, and define $p: K_s \to K_w$ by letting $p(x)$ be the point in K_w with barycentric coordinates $p_\beta(x)$. It is clear that p is continuous, and maps each $(K_j)_s$ into $(K_j)_w$. Since the composition $ip: K_s \to K_s$ maps each simplex into itself, it is homotopic to the identity. Similarly $pi: K_w \to K_w$ is homotopic to the identity. This completes the proof of Theorem 3.

Given n-ads A and C let A^C denote the subspace of the function space A^C consisting of all maps $f: C \to A$.

THEOREM 3. *If A belongs to the class $\mathcal{W}^n$ and if C is a compact n-ad, then the function space A^C belongs to $\mathcal{W}$. In fact the n-ad*

$$(A^C; (A, A_1)^{(C,C_1)}, \cdots, (A, A_{n-1})^{(C,C_{n-1})})$$

belongs to the class $\mathcal{W}^n$.

The first assertion follows from the second since A^C is equal to the intersection

$$(A, A_1)^{(C,C_1)} \cap \cdots \cap (A, A_{n-1})^{(C,C_{n-1})}.$$

As illustration of the way this theorem can be applied, let Ω denote the space of loops in A based at a_0, and let ω_0 denote the constant loop at a_0.

COROLLARY 3. *If the pair (A, a_0) belongs to $\mathcal{W}^2$, then the pair (Ω, ω_0) also belongs to $\mathcal{W}^2$.*

Proof. Set $C = (I; \dot{I}, I)$, where I denotes the unit interval, and set $A = (A; a_0, a_0)$. Then Theorem 4 asserts that the triad $(A^I; \Omega, \omega_0)$ has the homotopy type of a CW-triad $(K; K_1, K_2)$. Therefore (Ω, ω_0) has the homotopy type of the pair $(K_1, K_1 \cap K_2)$.

The proof of Theorem 3 will be based on the following considerations[5].

---

[5] An alternative proof would be based on the concept of "NES (metric)" in the sense of Hanner [9]. Compare [9, Theorems 2.11, 26.1, 28.6], as well as Dugundji [4, Theorem 7.5].

A space A has been called "locally equiconnected" by Fox[6] [6] and "ULC" by Serre [13] if there exists a neighborhood U of the diagonal in $A \times A$ and a map

$$\lambda: U \times [0, 1] \to A$$

satisfying

(1) $$\lambda(a, b, 0) = a, \qquad \lambda(a, b, 1) = b$$

for all $(a, b) \in U$, and

(2) $$\lambda(a, a, t) = a$$

for all $a \in A$, $t \in [0, 1]$. We will call the space ELCX (equi locally convex) if it also satisfies:

(3) there exists an open covering of A by sets V_β which are *convex*, in the sense that $V_\beta \times V_\beta \subset U$ and $\lambda(V_\beta \times V_\beta \times [0, 1]) = V_\beta$.

Similarly we will say that an n-ad $(A; A_1, \cdots, A_{n-1})$ is ELCX if the A_i are closed subsets of A, if the above conditions are satisfied for the space A, and if the following condition is satisfied:

(4) if $a, b \in A_i$ and $(a, b) \in U$ then $\lambda(a, b, t) \in A_i$ for every $t \in [0, 1]$.

We will prove the following lemmas.

LEMMA 2. *Every simplicial n-ad in the strong topology is ELCX.*

LEMMA 3. *If A is ELCX and C is compact then the n-ad*

$$(A^C; (A, A_1)^{(C,C_1)}, \cdots, (A, A_{n-1})^{(C,C_{n-1})}) \text{ is ECLX.}$$

LEMMA 4. *Every paracompact ELCX n-ad belongs to the class $\mathfrak{W}^n$.*

Proof of Theorem 3, assuming Lemmas 2, 3, 4. If A belongs to $\mathfrak{W}^n$ then it has the homotopy type of a simplicial n-ad K in the strong topology. According to Lemmas 2 and 3 the n-ad $F = (K^C; (K, K_1)^{(C,C_1)}, \cdots, (K, K_{n-1})^{(C,C_{n-1})})$ is ELCX. Since K is metrizable and C is compact, the function space K^C is metrizable, and hence paracompact. Therefore, by Lemma 4, F belongs to the class $\mathfrak{W}^n$. But it is easily verified that F has the same homotopy type as the n-ad

$$(A^C; (A, A_1)^{(C,C_1)}, \cdots, (A, A_{n-1})^{(C,C_{n-1})}).$$

Hence this n-ad belongs to $\mathfrak{W}^n$, as asserted.

Another consequence of the same argument is the following. By the product of the n-ad A with the m-ad B we mean the $(n+m-1)$-ad

$$(A \times B; A_1 \times B, \cdots, A_{n-1} \times B, A \times B_1, \cdots, A \times B_{m-1}).$$

---

[6] Fox requires an additional "uniformity" condition which we do not need.

If A and B are metrizable and ELCX, then it is clear that $A \times B$ is metrizable and ELCX. Therefore Lemmas 2 and 4 imply:

PROPOSITION 3. *If A belongs to $\mathcal{W}^n$ and B belongs to $\mathcal{W}^m$ then $A \times B$ belongs to $\mathcal{W}^{n+m-1}$.*

(An alternative proof of this proposition would be based on Dowker's result that the product of two "metric complexes" is a "metric complex.") The rest of the paper will be devoted to proofs.

Proof of Lemma 2. Let K be a simplicial n-ad in the strong topology. Let V_β denote the open star neighborhood of the vertex β; and let U denote the union over all vertices β of $V_\beta \times V_\beta$. Given any pair $(x, y) \in U$ with barycentric coordinates $\{\xi_\beta\}$ and $\{\eta_\beta\}$ respectively, define the "average" $\mu(x, y)$ as the point with barycentric coordinates

$$\zeta_\beta = \mathrm{Min}\,(\xi_\beta, \eta_\beta) \Big/ \sum_\gamma \mathrm{Min}\,(\xi_\gamma, \eta_\gamma).$$

The denominator is nonzero, since x and y belong to some common set V_γ.

The resulting map

$$\mu \colon U \to K$$

is clearly continuous (since we are using the strong topology!).

Note that $\mu(x, y)$ lies in the intersection of the smallest simplex containing x and the smallest simplex containing y. Hence we can define

$$\lambda\left(x, y, \frac{1}{2}t\right) = (1 - t)x + t\mu(x, y)$$

$$\lambda\left(x, y, \frac{1}{2} + \frac{1}{2}t\right) = (1 - t)\mu(x, y) + ty$$

for $0 \leq t \leq 1$. The resulting map

$$\lambda \colon U \times [0, 1] \to K$$

clearly satisfies Conditions 1, 2, 3, 4. In particular the star neighborhoods V_β are convex sets covering K.

Proof of Lemma 3. Define $U' \subset A^C \times A^C$ as the set of pairs (f, g) with $(f(c), g(c)) \in U$ for all $c \in C$. Define the map $\lambda' \colon U' \times [0, 1] \to A^C$ by

$$\lambda'(f, g, t)(c) = \lambda(f(c), g(c), t).$$

Every point in A^C has a convex open neighborhood of the form

$$(A; V_{\beta_1}, \cdots, V_{\beta_k})^{(C; D_1, \cdots, D_k)},$$

where $D_1, \cdots, D_k$ are compact sets covering C. Since the necessary identities 1, 2, 4 are easily verified, this completes the proof.

Proof of Lemma 4. We will first consider the case $n=1$. That is we will prove that any paracompact ELCX space A is dominated by a CW-complex.

Choose a locally finite covering $\{W_\gamma\}$ of A which is sufficiently fine so that the star of any point a of A (that is the union of all sets W_γ which contain a) is contained in some convex set V_β. This is possible since every paracompact space is fully normal. (See Stone [14, Theorem 2].)

Let N denote the nerve of the covering $\{W_\gamma\}$, considered as a simplicial complex in the weak topology. (To avoid confusion, assume that the sets W_γ are nonvacuous.) Choose a partition of unity $\{p_\gamma\}$ on A so that $p_\gamma(A-W_\gamma)$ $=0$. Then for each $a \in A$, the numbers $\{p_\gamma(a)\}$ can be considered as the barycentric coordinates of a point $p(a)$ in N. The resulting function $p: A \to N$ is clearly continuous.

Choose a representative point w_γ in each set W_γ; and choose an ordering of the simplicial complex N. Then a map $q: N \to A$ is defined as follows, by induction on the skeletons of N. For each vertex γ set $q(\gamma) = w_\gamma$. Consider any k-simplex of N with vertices $\gamma_0 < \cdots < \gamma_k$. Each point x in this simplex can be written uniquely in the form $x = (1-t)\gamma_0 + ty$ where y lies in the face spanned by $\gamma_1, \cdots, \gamma_k$. Now if q has been defined on the $(k-1)$-skeleton then the formula

$$q(x) = \lambda(w_{\gamma_0}, q(y), t)$$

defines an extension of q over the k-skeleton. It is easy to see that this extension is well defined and continuous; which completes the induction.

For each point $a \in A$ we assert that the pair $(a, qp(a))$ belongs to the neighborhood U of the diagonal. In fact let V_β be a convex set in A which contains the star of a. Then $qp(a)$ is a convex combination of points in V_β, which implies that $(a, qp(a)) \in V_\beta \times V_\beta \subset U$.

Therefore the formula

$$(a, t) \to \lambda(a, qp(a), t)$$

defines a homotopy between qp and the identity map of A. This shows that A is dominated by N, which completes the proof for the case $n=1$.

This proof extends without essential change to the case $n=2$. However for $n>2$ it is necessary to be more careful in choosing the sets W_γ.

We will say that an open subset W of A is *admissible* with respect to A if, whenever W intersects sets $A_{i_1}, \cdots$, and A_{i_k} of A, it also intersects the intersection $A_{i_1} \cap \cdots \cap A_{i_k}$.

ASSERTION. *If $A_1, \cdots, A_{n-1}$ are closed subsets of A, then every locally finite open covering $\{W_\gamma\}$ of A has a locally finite open refinement $\{W_\delta'\}$ consisting of admissible sets.*

Proof. Let $\{W_\delta'\}$ consist of all sets of the form

$$W_\gamma - A_{i_1} - \cdots - A_{i_k}$$

which happen to be admissible. It is easily verified that this is a locally finite open covering of A.

Proof of Lemma 4 for $n > 1$. Let N' denote the nerve of the covering $\{ W'_\delta \}$, and define subcomplexes N'_i as follows. The vertices $\delta_0, \cdots, \delta_k$ span a simplex of N'_i if and only if $W'_{\delta_0} \cap \cdots \cap W'_{\delta_k}$ intersects A_i. We can map $\boldsymbol{A}$ into the resulting simplicial n-ad $\boldsymbol{N'}$ just as before.

Choose representative points $w'_\delta \in W'_\delta$ so that if W'_δ intersects A_i then $w'_\delta \in A_i$. This is possible since each W'_δ is admissible. Now the map $q : \boldsymbol{N'} \to \boldsymbol{A}$ and the homotopy $A \times [0, 1] \to A$ are defined just as before. This completes the proof of Lemma 4 and Theorem 3.

REFERENCES

1. M. Barratt, *Simplicial and semisimplicial complexes* (mimeographed), Princeton University, 1956.

2. K. Borsuk, *Sur un espace compact localement contractile qui n'est pas un rétracte absolu de voisinage*, Fund. Math. vol. 35 (1948) pp. 175–180.

3. C. H. Dowker, *Topology of metric complexes*, Amer. J. Math. vol. 74 (1952) pp. 555–577.

4. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. vol. 1 (1951) pp. 353–367.

5. S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton, 1952.

6. R. H. Fox, *On fibre spaces* II, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 733–735.

7. J. B. Giever, *On the equivalence of two singular homology theories*, Ann. of Math. vol. 51 (1950) pp. 178–191.

8. O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. vol. 1 (1950) pp. 389–408.

9. ———, *Retraction and extension of mappings of metric and nonmetric spaces*, Ark. Mat. vol. 2 (1952) pp. 315–359.

10. I. James and J. H. C. Whitehead, *Note on fibre spaces*, Proc. London Math. Soc. vol. 4 (1954) pp. 121–137.

11. C. Kuratowski, *Sur les espaces localement connexes et péaniens en dimension n*, Fund. Math. vol. 24 (1935) pp. 269–287.

12. J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. of Math. (1957).

13. J. P. Serre, *Homologie singulière des espaces fibrés*, Ann. of Math. vol. 54 (1951) pp. 425–505. (See page 490.)

14. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 977–982.

15. J. H. C. Whitehead, *Combinatorial homotopy* I, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 213–245.

16. ———, *A certain exact sequence*, Ann. of Math. vol. 52 (1950) pp. 51–110.

PRINCETON UNIVERSITY,
    PRINCETON, N. J.