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A PROCEDURE FOR KILLING HOMOTOPY GROUPS OF DIFFERENTIABLE MANIFOLDS

BY

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A "surgery" on a differentiable manifold W of dimension $n = p + q + 1$ has the effect of removing an imbedded sphere of dimension p from W, and replacing it by an imbedded sphere of dimension q (see §1). This construction is closely related to cobordism theory (§2). The main objective of this paper (§§3–6) is to study the extent to which this construction can be used to simplify the homotopy groups of a given manifold. A typical result is the following.

COROLLARY TO THEOREM 3. *Let W be a triangulated differentiable manifold of dimension $2m$. Suppose that W is compact and that its tangent bundle restricted to the $(m-1)$-skeleton is a trivial bundle. Then by performing a series of surgeries on W, one can obtain a manifold W' which is $(m-1)$-connected.*

CONVENTIONS. All manifolds are to be oriented and differentiable of class C^∞. The letter W is used for a manifold which may have boundaries. The letter V is reserved for a manifold without boundaries.

I am grateful to R. Thom for describing "surgery" to me, and for pointing out the possibility of using it to kill homotopy groups.

Added in proof. Much of the material below has been obtained independently by A. H. Wallace [**14**].

1. The construction. Let D^{p+1} denote the unit disk in the euclidean space R^{p+1}, with boundary S^p and with center 0. The product manifold $S^p \times S^q$ can be considered either

(1) as the boundary of $S^p \times D^{q+1}$,

or

(2) as the boundary of $D^{p+1} \times S^q$.

Given any imbedding of $S^p \times D^{q+1}$ in a manifold W of dimension $n = p + q + 1$, a new manifold W' can be formed by removing the interior of $S^p \times D^{q+1}$ and replacing it by the interior of $D^{p+1} \times S^q$. This procedure will be called surgery. To be more precise:

DEFINITION. Given a differentiable, orientation preserving imbedding

$$f : S^p \times D^{q+1} \to W$$

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[1] The author holds a Sloan fellowship.

with $p + q + 1 = n$, let $\chi(W, f)$ denote the quotient manifold obtained from the disjoint sum

$$(W - f(S^p \times 0)) + (D^{p+1} \times S^q)$$

by identifying $f(u, \theta v)$ with $(\theta u, v)$ for each $u \varepsilon S^p$, $v \varepsilon S^q$, $0 < \theta \leq 1$. Thus $\chi(W, f)$ is an oriented differentiable manifold. The boundary of W (if any) is equal to the boundary of $\chi(W, f)$. If W' denotes any manifold which is diffeomorphic to $\chi(W, f)$ under an orientation preserving diffeomorphism, then we will say that *W' can be obtained from W by surgery of type $(p + 1, q + 1)$.*

This construction clearly makes sense in the range $0 \leq p < n$ (that is the range $p, q \geqq 0$). It will be convenient to extend it to the cases $p = -1$ or $p = n$ by defining

$$D^0 = R^0 = \text{the point } 0; \quad S^{-1} = \text{the vacuous set}.$$

With these conventions, a surgery of type $(0, n + 1)$ replaces W by the disjoint sum $W + S^n$, while a surgery of type $(n + 1, 0)$ replaces $W + S^n$ by W.

It is clear that W and W' play a symmetrical role in this construction. If $W' = \chi(W, f)$ is obtained from W by a surgery of type $(p + 1, q + 1)$, then W can be obtained from W' by a surgery of type $(q + 1, p + 1)$.

Given a sequence $W_1, \cdots, W_r$ of manifolds such that each W_{i+1} can be obtained from W_i by a surgery we will say that W_1 is χ-equivalent to W_r.

2. χ-equivalence and cobordism. Consider manifolds V without boundary which are compact and oriented.

THEOREM 1. *Two such manifolds are χ-equivalent if and only if they belong to the same cobordism class.*

PROOF. Let L denote the locus of points (x, y) in $R^{p+1} \times R^{q+1}$ which satisfy the inequalities

$$-1 \leqq ||x||^2 - ||y||^2 \leqq 1,$$

and

$$||x|| \, ||y|| < (\sinh 1)(\cosh 1).$$

Thus L is a differentiable manifold with two boundaries. The "upper" boundary, $||x||^2 - ||y||^2 = 1$, is diffeomorphic to $S^p \times (\text{Interior } D^{q+1})$ under the correspondence

$$(u, \theta v) \leftrightarrow (u \cosh \theta, v \sinh \theta), \qquad 0 \leqq \theta < 1.$$

The lower boundary, $||x||^2 - ||y||^2 = -1$, is diffeomorphic to $(\text{Interior } D^{p+1}) \times S^q$ under the correspondence

$$(\theta u, v) \leftrightarrow (u \sinh \theta, v \cosh \theta).$$

Consider the orthogonal trajectories of the surfaces $||x||^2 - ||y||^2 = \text{constant}$.

The trajectory which passes through the point (x, y) can be parametrized in the form

$$t \rightarrow (tx, t^{-1}y).$$

[Thus the product $||x|| \, ||y||$ is constant along any orthogonal trajectory.] If x or y is zero this trajectory is a straight line segment tending to the origin. For x and y different from zero it is a hyperbola which leads from some well defined point $(u \cosh \theta, v \sinh \theta)$ on the upper boundary of L to the corresponding point $(u \sinh \theta, v \cosh \theta)$ on the lower boundary.

Now let V be a differentiable manifold without boundary of dimension $n = p + q + 1$, and let f: $S^p \times D^{q+1} \rightarrow V$ be an imbedding. Construct an $(n + 1)$-dimensional manifold $\omega(V, f)$ as follows. Start with the disjoint sum

$$(V - f(S^p \times 0)) \times D^1 + L.$$

For each $u \varepsilon S^p$, $v \varepsilon S^q$, $0 < \theta < 1$, and $c \varepsilon D^1$ identify the point $(f(u, \theta v), c)$ in the first summand with the unique point $(x, y) \varepsilon L$ such that

(1) $||x||^2 - ||y||^2 = c$, and

(2) (x, y) lies on the orthogonal trajectory which passes through the point $(u \cosh \theta, v \sinh \theta)$.

It is not difficult to see that this correspondence defines a diffeomorphism

$$f(S^p \times (\text{Interior } D^{q+1} - 0)) \times D^1 \leftrightarrow L \cap (R^{p+1} - 0) \times (R^{q+1} - 0).$$

It follows from this that $\omega(V, f)$ is a well defined differentiable manifold.

This manifold $\omega(V, f)$ has two boundaries, corresponding to the values $c = ||x||^2 - ||y||^2 = +1$, and -1. The upper boundary, $c = +1$, can be identified with V, letting $z \varepsilon V$ correspond to:

$$\begin{cases} (z, 1) \varepsilon (V - f(S^p \times 0)) \times D^1 & \text{for } z \not\varepsilon f(S^p \times 0). \\ (u \cosh \theta, v \sinh \theta) \varepsilon L & \text{for } z = f(u, \theta v). \end{cases}$$

The lower boundary can be indentified with $\chi(V, f)$: letting $z \varepsilon V - f(S^p \times 0)$ correspond to $(z, -1)$; and letting $(\theta u, v) \varepsilon D^{p+1} \times S^q$ correspond to $(u \sinh \theta, v \cosh \theta)$.

It is clear that $\omega(V, f)$ is compact if and only if V is compact. Since orientations can be chosen appropriately, this completes the proof that χ-equivalent manifolds belong to the same cobordism class.

An immediate consequence is the following.

COROLLARY (THOM). *The Stiefel-Whitney numbers, Pontrjagin numbers, and the index of a compact manifold V are invariant under surgery.*

Now suppose that V and V' are compact manifolds which belong to the same cobordism class. Thus V and V' together bound a compact manifold W.

LEMMA 1. *There exists a differentiable map $g : W \rightarrow [0, 1]$ with the following three properties:*

(1) $g^{-1}(0) = V', \quad g^{-1}(1) = V.$

(2) *The gradient of* g *vanishes only at isolated interior points* $w_1, \cdots, w_k \in W$. *Furthermore the matrix of second derivatives of* g *at each* w_i *is non-singular.*
(3) *The values* $c_i = g(w_i)$ *are distinct; say* $0 < c_1 < \cdots < c_k < 1$.

Using a partition of unity, it is easy to construct a function $g_0 : W \to [0, 1]$ which satisfies condition (1), and has non-zero gradient along the boundary of W. Hence g_0 has non-zero gradient along some compact neighborhood K_0 of the boundary.

Choose open sets $B_1, \cdots, B_r \subset$ Interior W which are diffeomorphic to open subsets of euclidean space, and which cover the compact set Closure$(W - K_0)$. Then g_0 restricted to B_1 can be approximated by a function g_0' which satisfies condition (2) throughout B_1. (See Morse [**6**, Theorem 16], or Whitney [**13**, Theorem 12A].)

Let $A_1, \cdots, A_r$ be an open covering of Closure$(W - K_0)$ such that Closure $A_i \subset B_i$. Let $\lambda_i : W \to [0, 1]$ be a differentiable function which takes the value of 1 on A_i, and has carrier contained in B_i. Now define

$$g_1(w) = (1 - \lambda_1(w))g_0(w) + \lambda_1(w)g_0'(w) \quad \text{for} \quad w \in B_1,$$

$$g_1(w) = g_0(w) \quad \text{for} \quad w \notin B_1.$$

Then $g_1 : W \to R$ will satisfy condition (2), at least throughout Closure A_1. Furthermore, if g_0' is sufficiently close to g_0, then
(a) the modified function g_1 will still have values between 0 and 1; and
(b) g_1 restricted to K_0 will still satisfy condition (2).
In other words g_1 will satisfy condition (2) throughout the compact set

$$K_1 = K_0 \cup \text{ Closure } A_1.$$

Now continue by induction, constructing functions $g_i : W \to [0, 1]$ which satisfy condition (1), and which satisfy (2) throughout the set

$$K_i = K_{i-1} \cup \text{ Closure } A_i.$$

Since K_r is equal to W, it follows that g_r satisfies condition (2) everywhere.

If g_r takes on the same value at two of its critical points, then by adding a function which is equal to ϵ in a small neighborhood of one point, and is zero outside a larger neighborhood, we can separate the two values. This completes the proof of Lemma 1.

The proof of Theorem 1 continues as follows. Define $c_0 = 0$, $c_{k+1} = 1$. If both c' and c'' lie between c_i and c_{i+1} note that the manifolds $g^{-1}(c')$ and $g^{-1}(c'')$ are diffeomorphic. This is proved by taking any Riemannian metric on W, and using the orthogonal trajectories of the surfaces $g^{-1}(c)$ to define the diffeomorphism. Call any one of these surfaces V_i. Thus V_0 is diffeomorphic to V' and V_k is diffeomorphic to V. We will prove that each V_{i-1} can be obtained from V_i by surgery.

Let $B(r)$ denote the open ball of radius r in R^{n+1}. According to Morse [**7**, p. 172] there exists a neighborhood U of w_i in W, and a coordinate diffeomorphism

$$\phi : B(2\epsilon) \to U$$

so that the function $g\phi(z_0, \cdots, z_n)$ takes the form

$$c_i + z_0^2 + \cdots + z_p^2 - z_{p+1}^2 - \cdots - z_n^2,$$

for some $-1 \leq p \leq n$, and some $\epsilon > 0$. We will abbreviate this as

$$g\phi(x, y) - c_i = ||x||^2 - ||y||^2,$$

where $x \in R^{p+1}$, $y \in R^{q+1}$, $p + q + 1 = n$. Note that $\phi(0, 0) = w_i$.

We may assume that $g^{-1}(c_i + \epsilon^2) = V_i$ and that $g^{-1}(c_i - \epsilon^2) = V_{i-1}$. Define an imbedding

$$f : S^p \times D^{q+1} \to V_i$$

by $f(u, \theta v) = \phi(\epsilon u \cosh \theta, \epsilon v \sinh \theta)$. It will be shown that V_{i-1} is diffeomorphic to $\chi(V_i, f)$.

An imbedding $h: L \to W$ is defined by $h(x, y) = \phi(\epsilon x, \epsilon y)$. Choose a Riemannian metric on W which reduces to the euclidean metric

$$dx_0^2 + \cdots + dx_p^2 + dy_0^2 + \cdots + dy_q^2$$

throughout the closure of $h(L)$. The imbedding h carries the surfaces $||x||^2 - ||y||^2 =$ constant into surfaces $g^{-1}(c)$. Hence it carries orthogonal trajectories into orthogonal trajectories.

Define a diffeomorphism $k: \chi(V_i, f) \to V_{i-1}$ as follows. Starting at any point z of $V_i - f(S^p \times 0)$ the orthogonal trajectory is a non-singular curve which leads from z to some well defined point $k(z)$ in V_{i-1}. In the case $z = f(u, \theta v) = h(u \cosh \theta, v \sinh \theta)$ this orthogonal trajectory leads from z to the point $h(u \sinh \theta, v \cosh \theta)$ of V_{i-1}. Hence defining

$$k(\theta u, v) = \phi(\epsilon u \sinh \theta, \epsilon v \cosh \theta)$$

for all $(\theta u, v) \in D^{p+1} \times S^q$, we obtain a well defined diffeomorphism from $\chi(V_i, f)$ to V_{i-1}. This completes the proof that cobording manifolds are χ-equivalent.

Remark. A similar proof shows that the region $g^{-1}[c_i - \epsilon^2, c_i + \epsilon^2]$ between V_i and V_{i-1} is diffeomorphic to $\omega(V_i, f)$.

3. Killing homotopy classes. Let W be a manifold of dimension n and let $\lambda \in \pi_p(W)$ be a homotopy class which we wish to kill. To simplify the discussion, assume that W is connected.

Definition. An imbedding $f : S^p \times D^{n-p} \to W$ *represents* the homotopy class λ if $\lambda = f_*(\iota)$, where ι is a generator of the infinite cyclic group $\pi_p(S^p \times D^{n-p})$. [To be more precise one should specify base points for these homotopy groups, but this will be left to the reader.]

Suppose that λ is represented by such an imbedding $f : S^p \times D^{n-p} \to W$. Suppose further that $n \geq 2p + 2$. Let $W' = \chi(W, f)$.

LEMMA 2. *Under these conditions, the homotopy groups $\pi_i(W')$ are isomorphic to $\pi_i(W)$ for $i < p$; and $\pi_p(W')$ is isomorphic to $\pi_p(W)$ modulo a subgroup which contains λ.*

Briefly: the effect of the surgery is to kill the homotopy class λ.

PROOF. As usual set $q = n - p - 1$. Note that $p < q$. Let X denote the space which is formed from the topological sum $W + (D^{p+1} \times D^{q+1})$ by identifying (u, y) with $f(u, y)$ for each $(u, y) \in S^p \times D^{q+1}$. The subset $W \cup (D^{p+1} \times 0)$ is clearly a deformation retract of X. This subset is formed from W by attaching a $(p + 1)$-cell; using the map $u \to f(u, 0)$ as attaching map. It follows immediately that the inclusion homomorphism $\pi_i(W) \to \pi_i(X)$ is an isomorphism for $i < p$, and is onto for $i = p$. Furthermore the homotopy class λ of the attaching map lies in the kernel of this homomorphism.

But W' is also imbedded topologically in X, and a similar argument shows that the homomorphism $\pi_i(W') \to \pi_i(X)$ is an isomorphism for $i < q$. In particular it is an isomorphism for $i \leq p$. Together with the preceding paragraph, this completes the proof.

REMARK. The manifold $\omega(W, f)$ could be used in place of the space X in this argument.

The above argument can also be applied to the case $p = 0$. In this case the appropriate statement is:

LEMMA 2′. *Let W be a manifold of dimension $n \geq 2$ with k components, $k \geq 2$. Let $f : S^0 \times D^n \to W$ be an orientation preserving*[2] *imbedding which carries the two components of $S^0 \times D^n$ into distinct components of W. Then $\chi(W, f)$ has only $k - 1$ components.*

The proof will be left to the reader.[3]

In order to make use of Lemma 2 it is necessary to answer the following question. For which homotopy classes $\lambda \in \pi_p(W)$ does there exist an imbedding $S^p \times D^{n-p} \to W$ which represents λ?

Let τ^n denote the tangent bundle of W and let $f_0 : S^p \to W$ denote any map in the homotopy class of λ. Let $f_0^*\tau^n$ denote the induced bundle over S^p.

LEMMA 3. *Assume that $n \geq 2p + 1$. Then there exists an imbedding $S^p \times D^{n-p} \to W$ which represents λ if and only if the induced bundle $f_0^*\tau^n$ is trivial.*

The proof of Lemma 3 will be based on the following.

LEMMA 4. *Let ξ^m be an m-dimensional vector space bundle over a complex K of dimension $p < m$. Let o^1 denote the trivial line bundle over K. Then ξ^m is a trivial bundle if and only if the Whitney sum $\xi^m \oplus o^1$ is trivial.*

PROOF. Let $B(SO_{m+1})$ denote any classifying space for the rotation group. First observe that there exists a fibre bundle over $B(SO_{m+1})$ with fibre S^m and total space $B(SO_m)$. [This is seen as follows. The group SO_{m+1} acts freely on the total space E of its universal bundle, with orbit space $E/SO_{m+1} = B(SO_{m+1})$. Hence the subgroup SO_m also acts freely on E. Define $B(SO_m)$ as the orbit space E/SO_m. It follows easily that $B(SO_m)$ is a bundle over $B(SO_{m+1})$ with fibre $SO_{m+1}/SO_m = S^m$.] Let $\pi : B(SO_m) \to B(SO_{m+1})$ denote the projection map.

Let $f : K \to B(SO_m)$ be a classifying map for the bundle ξ^m. Then $\pi f : K \to B(SO_{m+1})$ is a classifying map for the Whitney sum $\xi^m \oplus o^1$. If πf is null-homotopic then it follows from the covering homotopy theorem that f is homotopic to a map of K into the fibre S^m. Since K has dimension less than m, this implies that f is null-homotopic.[4]

PROOF OF LEMMA 3. If $n \geq 2p + 1$ then any homotopy class of maps from S^p to W contains an imbedding

$$f_0 : S^p \to \text{Interior } W.$$

(See Whitney [**11**, Theorem 2].) Let τ^p denote the tangent bundle of S^p and let ν^{q+1} denote its normal bundle. Then the Whitney sum $\tau^p \oplus \nu^{q+1}$ can be identified with $f_0^*\tau^n$: the tangent bundle of W restricted to S^p.

If $f_0^*\tau^n$ is trivial then it follows that ν^{q+1} is trivial. To see this, let o^k denote the trivial k-dimensional vector space bundle over S^p. Then $o^1 \oplus \tau^p$ is known to be trivial. (Identify o^1 with the normal bundle of S^p in R^{p+1}.) Combining the relations

$$o^1 \oplus \tau^p \approx o^{p+1}, \qquad \tau^p \oplus \nu^{q+1} \approx o^n$$

it follows that $o^{p+1} \oplus \nu^{q+1} \approx o^{n+1}$. Together with Lemma 4 this implies that ν^{q+1} is trivial.

Now take a tubular neighborhood of $f_0(S^p)$ in W. This can be identified with the total space of the normal D^{q+1}-bundle. Hence this neighborhood is diffeomorphic to $S^p \times D^{q+1}$. The resulting imbedding

$$S^p \times D^{q+1} \to W$$

certainly represents the homotopy class of f_0.

Conversely suppose that such an imbedding is given. Since $S^p \times D^{q+1}$ is parallelizable, it follows immediately that $f_0^*\tau^n$ is a trivial bundle. This completes the proof of Lemma 3.

This lemma can also be formulated as follows. Let $B(SO_n)$ denote a classifying space for the rotation group, and let

$$T : W \to B(SO_n)$$

be a classifying map for the tangent bundle of W. Then there exists an imbedding $S^p \times D^{n-p} \to W$ which represents λ if and only if the homomorphism

$$T_* : \pi_p(W) \to \pi_p B(SO_n)$$

annihilates λ.

[2] The sphere S^0 must be oriented as the boundary of D^1.

[3] Compare the definition of "connected sum" of manifolds, as given in Milnor [**5**].

[4] This proof is due to A. Dold.

Combining this assertion with Lemmas 2 and 2′ it is easy to prove the following.

ASSERTION. *Every compact (oriented, differentiable) manifold is χ-equivalent to a manifold W which is connected (if $n \geq 2$), and which satisfies the following condition: the homomorphism*

$$T_* : \pi_p(W) \to \pi_p B(SO_n)$$

is a monomorphism for $1 \leq p \leq n/2 - 1$.

The proof proceeds by induction on p. First surgeries of type $(1, n)$ are used to connect the manifold. Then surgeries of type $(2, n - 1)$ are used to kill its fundamental group, and so on. The kernel of T_* is finitely generated at each stage since W is compact and, for $p > 1$, simply connected.

4. π-manifolds. It is not possible to kill the entire homotopy group $\pi_p(W)$ by this construction, even if the dimension n is large, unless W satisfies some further restriction.

[As an example consider the second homotopy group of the complex projective space $P_{2m}(C)$. The Stiefel-Whitney number $w_2^{2m}[P_{2m}(C)]$ is non-zero; but for any 2-connected manifold V the number $w_2^{2m}[V]$ is zero. Hence $\pi_2(P_{2m})C^{44}$ cannot be killed by the surgery.]

As a first attempt one might try the hypothesis that W is parallelizable. However this condition is too easily destroyed by the surgery.

[For example consider the parallelizable manifold $S^1 \times S^3$. Killing the fundamental group of $S^1 \times S^3$ we obtain a manifold such as S^4 which has positive Euler characteristic; and therefore is not parallelizable.]

As a second attempt consider the following:

DEFINITION. W is a *π-manifold* if the Whitney sum of its tangent bundle and a trivial line bundle is trivial. The notation $(\tau^n \oplus o^1)W$ will be used for this Whitney sum. For example any sphere is a π-manifold.

REMARKS. A trivial vector space bundle of higher dimension could be used in place of the line bundle. (Compare Lemma 4.) If W is imbedded in a high dimensional euclidean space then W is π-manifold if and only if its normal bundle is trivial.[5]

THEOREM 2. *Let W be a π-manifold of dimension $n \geq 2p + 1$. Then any homotopy class in $\pi_p(W)$ is represented by an imbedding*

$$f : S^p \times D^{n-p} \to W$$

such that the new manifold $\chi(W, f)$ is also a π-manifold.

Together with Lemmas 2 and 2′ this clearly implies the following.

COROLLARY. *Any compact π-manifold of dimension n is χ-equivalent to a π-manifold which is $[n/2 - 1]$-connected.*

[5] This assertion is essentially due to J. H. C. Whitehead. It can be proved using Lemma 4 together with the relation $o^1 \oplus \tau^n \oplus \nu^k \approx o^{1+n+k}$.

REMARK 1. In the case of a compact π-manifold V without boundary one can prove the stronger statement that V is χ-equivalent to S^n. (See Milnor **[4]**, Wall **[10]** together with Theorem 1.) This stronger statement is much more difficult to prove.

REMARK 2. It is definitely not true that $\chi(W, f)$ is a π-manifold for any imbedding f. If this were true then any manifold in the trivial cobordism class would have to be a π-manifold. But a counter-example is provided by the disjoint sum $P_2(C) + (-P_2(C))$ where the minus sign stands for reversal of orientation.

PROOF OF THEOREM 2. Since the boundary of W plays no role in this theorem, we may delete it. Then, given any imbedding $f: S^p \times D^{n-p} \to W$, the $(n + 1)$-dimensional manifold $\omega(W, f)$ is defined. (Compare the proof of Theorem 1.) If f is suitably chosen, we will prove that $\omega(W, f)$ is parallelizable. Since $\chi(W, f)$ is one of the two boundaries of $\omega(W, f)$, this will imply that $\chi(W, f)$ is a π-manifold.

Since $n \geq 2p + 1$, any element of $\pi_p(W)$ is represented by some imbedding f_0: $S^p \to W$. Since W is a π-manifold, there exist $n + 1$ linearly independent cross-sections of the bundle $(\tau^n \oplus o^1)W$. Restricting this bundle to $f_0(S^p)$ it splits up into the Whitney sum of $(\tau^p \oplus o^1)f_0(S^p)$ and the normal bundle ν^{n-p}. Note that the bundle $(\tau^p \oplus o^1)S^p$ can be identified with the tangent bundle of R^{p+1}, restricted to the unit sphere. Hence it has $p + 1$ canonical cross-sections, which will be denoted by $e_0, \cdots, e_p$. The corresponding sections of $(\tau^p \oplus o^1)f_0(S^p)$ will be denoted by $f_{0*}e_0, \cdots, f_{0*}e_p$.

LEMMA 5. *The $n + 1$ independent cross-sections $c_0, \cdots, c_n$ of $(\tau^n \oplus o^1)W$ can be chosen in such a way that, for each point $w = f_0(u)$ of $f_0(S^p)$, the first $p + 1$ of these vectors are given by*

$$c_i(w) = f_{0*}e_i(u), \qquad i = 0, \cdots, p.$$

Assuming this lemma for the moment, the proof of Theorem 2 proceeds as follows. We may assume that the remaining $n - p$ vectors, $c_{p+1}(W), \cdots, c_n(W)$ are normal to the submanifold $f_0(S^p) \subset W$. Hence these vector fields determine a specific product structure for the normal bundle ν^{n-p}. Making use of this product structure, a tubular neighborhood of $f_0(S^p)$ can be identified with $S^p \times D^{n-p}$. This gives the required imbedding f: $S^p \times D^{n-p} \to W$.

ASSERTION. If f is chosen in this way, then $\omega(W, f)$ is parallelizable.

To prove this assertion, identify $(\tau^n \oplus o^1)W$ with the restriction to W of the tangent bundle of $\omega(W, f)$. Thus $c_0, \cdots, c_n$ can be considered as independent cross-sections of $\tau^{n+1}\omega(W, f)$ which are defined only on W. We must extend these cross-sections throughout $\omega(W, f)$.

Consider first the problem of extending over the disk $D^{p+1} \times 0 \subset L \subset \omega(W, f)$. The boundary of this disk is $S^p \times 0 = f_0(S^p) \subset W$, and the sections $c_0, \cdots, c_p$ have been chosen so that, on $S^p \times 0$, they reduce to the standard cross-sections

$e_0, \cdots, e_p$ of $\tau^{p+1}(D^{p+1} \times 0)$. Thus the first $p + 1$ cross-sections can certainly be extended over $D^{p+1} \times 0$.

But the remaining cross-sections $c_{p+1}, \cdots, c_n$, on $S^p \times 0$, are just the standard basis for the normal bundle of $D^{p+1} \times 0$ in L. Hence these can be extended over $D^{p+1} \times 0$. This extension can be carried out so that all $n + 1$ vectors remain independent.

Thus the tangent bundle of $\omega(W, f)$, restricted to $W \cup (D^{p+1} \times 0)$, is a trivial bundle. To complete the argument it is only necessary to observe that $W \cup (D^{p+1} \times 0)$ is a deformation retract of $\omega(W, f)$. This can be proved in two steps as follows.

STEP 1. The manifold $\omega(W, f)$ can be deformed into the subset $W \cup$ (Closure L) by a deformation which leaves $W \cup (D^{n+1} \times 0)$ pointwise fixed. (The proof is not difficult.)

STEP 2. The space $W \cup$ (Closure L) has $W \cup (D^{n+1} \times 0)$ as deformation retract. In fact a retraction can be defined as follows, for $(x, y) \in$ Closure L:

$$(x, y) \to (x, 0) \quad \text{if} \quad ||x|| \leq 1,$$

$$(x, y) \to (x, y(||x||^2 - 1)^{1/2}/||y||) \quad \text{otherwise}.$$

This retraction is clearly homotopic to the identity.

This proves that $W \cup (D^{n+1} \times 0)$ is a deformation retract of $\omega(W, f)$, and therefore proves that $\omega(W, f)$ is parallelizable.

To prove Theorem 2 observe that $\chi(W, f)$ is imbedded as a boundary of $\omega(W, f)$. Hence $(\tau^n \oplus o^1)\chi(W, f)$ can be identified with the tangent bundle of $\omega(W, f)$ restricted to $\chi(W, f)$. Since this is trivial it follows that $\chi(W, f)$ is a π-manifold. This completes the proof of Theorem 2, except for Lemma 5.

PROOF OF LEMMA 5. Start with any $n + 1$ independent cross-sections $b_0, \cdots, b_n$ of $(\tau^n \oplus o^1)W$. For each $u \in S^p$ the required vectors $f_0 . e_i$ at $w = f_0(u)$ can be expressed as linear combinations

$$f_0 . e_i(u) = \sum z_{ij}(u) b_j(w).$$

The correspondence $u \to (z_{ij}(u))$ defines a map z from S^p to the Stiefel manifold consisting of all $(p + 1) \times (n + 1)$ matrices of rank $p + 1$. But this Stiefel manifold is $(n - p - 1)$-connected;[6] hence z is null-homotopic. Using the covering homotopy theorem it follows that z can be lifted to a null-homotopic map $\bar{z}$ which carries S^p into the Stiefel manifold of all non-singular $(n + 1) \times (n + 1)$ matrices. Let $\bar{z}_{ij}(u)$ denote the ijth component of $\bar{z}(u)$. Then $\bar{z}_{ij} = z_{ij}$ for $0 \leq i \leq$ p.

Now define new sections $c_0, \cdots, c_n$ as follows. Let $c_i(w) = b_i(w)$ outside of some neighborhood of $f_0(S^p)$, and let

$$c_i(w) = \sum \bar{z}_{ij}(u) b_j(w) \qquad \text{for} \quad w = f_0(u).$$

[6] See Steenrod [8, §25.6] together with §7.7 and §12.9.

The null-homotopy of $\bar{z}$ can be used to extend throughout the neighborhood. This completes the proof of Lemma 5 and of Theorem 2.

5. k-parallelizable manifolds. The next step will be to extend the above procedure so that it applies to something more general than a π-manifold.

DEFINITION. A bundle ξ over a space B is *k-trivial* if for every complex k of dimension $\leq k$, and every map $f: K \to B$, the induced bundle $f^*\xi$ is trivial. In case B itself is a complex, this is equivalent to the requirement that the restriction of ξ to the k-skeleton of B should be trivial.

DEFINITION. A manifold is *k-parallelizable* if its tangent bundle is k-trivial. Thus every (orientable) manifold is 1-parallelizable. For large values of k, this condition implies that the manifold is actually parallelizable.

THEOREM 3. *Let W be a compact manifold which is k-parallelizable, $1 \leq k < n$. Then W is χ-equivalent to a k-parallelizable manifold W' which is c-connected where $c =$* Min $(k, [n/2 - 1])$.

PROOF. Suppose by induction that W is $(p - 1)$-connected with $1 \leq p \leq$ Min $(k, [n/2 - 1])$. Consider some triangulation of Interior W, together with its dual cell subdivision. Let W^r denote the r-skeleton of one of these cell subdivisions and W_r the r-skeleton of the other. Then the skeleton W^k is a deformation retract of (Interior $W) - W_{n-k-1}$. Therefore the open manifold (Interior $W) - W_{n-k-1}$ is parallelizable, and hence is a π-manifold. Since $p \leq k$, any element of $\pi_p(W)$ is represented by a map of S^p into the skeleton.

$$W^k \subset \text{(Interior } W) - W_{n-k-1}.$$

Since $n \geq 2p + 1$ it follows from Theorem 2 that this homotopy class is represented by an imbedding

$$f: S^p \times D^{n-p} \to \text{(Interior } W) - W_{n-k-1}$$

such that the manifold $\chi(($Interior $W) - W_{n-k-1}, f)$ is again a π-manifold.

But f can also be considered as an imbedding of $S^p \times D^{n-p}$ in W. We assert that the manifold $\chi(W, f)$ is k-parallelizable. The above argument shows that

$$\text{(Interior } \chi(W, f)) - W_{n-k-1} = \chi(\text{(Interior } W) - W_{n-k-1}, f)$$

is a π-manifold. But any map of a k-dimensional complex K into $\chi(W, f)$ can be deformed into a map

$$g: K \to \text{(Interior } \chi(W, f)) - W_{n-k-1},$$

since the set W_{n-k-1} is a countable union of imbedded simplexes, each having codimension $\geq k + 1$. Hence $g^*(\tau^n \oplus o^1) = (g^*\tau^n) \oplus o^1$ is a trivial bundle. Using Lemma 4 and the assumption that $k < n$, this implies that $g^*\tau^n$ is trivial. Therefore $\chi(W, f)$ is k-parallelizable.

Since W is compact and $(p - 1)$-connected it follows that $\pi_p(W)$ is finitely generated. Hence a finite number of iterations of this construction will kill the pth homotopy group. This completes the proof of Theorem 3.

6. Killing the middle homotopy group. Consider compact manifolds W of dimension $2m$. Using Theorem 2 or 3 it may be possible to kill the homotopy groups $\pi_p(W)$ in dimensions $p < m$. This section will study the possibility of killing $\pi_m(W)$.

REMARK. If one succeeds in obtaining a manifold W' which is m-connected, then the homotopy type of W' is almost determined by that of Boundary W. The Poincaré duality theorem implies that[7] $H_i(W', \text{Boundary } W) = 0$ for $m \leq i \leq 2m - 1$; and hence that $H_i(W')$ is isomorphic to H_i (Boundary W) for $m \leq i < 2m - 1$. As an example, if Boundary W is vacuous, then it follows easily that W' has the homotopy type of a $2m$-sphere.

In order to kill $\pi_m(W)$, the first problem is to represent a given homotopy class by an imbedded sphere.

LEMMA 6. *If W is a simply connected manifold of dimension $2m > 4$, then every element of $\pi_m(W)$ is represented by an imbedding $f_0 : S^m \to W$.*

More generally, if V is a compact connected manifold of dimension m, the proof will show that any homotopy class of maps $V \to W$ is represented by an imbedding. For the special case $W = R^{2m}$ this result is due to Whitney [**12**]. The present proof makes use of Whitney's method. I do not know whether the assertion is true in the case $m = 2$.

First any map can be approximated by an immersion $f : V \to W$ which has no singularities other than a finite number of double points:

$$f(a_1) = f(a_2), \qquad f(a_3) = f(a_4), \cdots,$$

where $a_1, \cdots, a_{2k}$ are distinct points of V. (See Whitney, [**11**, Theorem 2].) Since V is connected there exists an arc joining a_1 to a_2 in V which misses the points $a_3, \cdots, a_{2k}$. This arc projects into a simple closed curve C in W which is differentiable except for one angle.

Identify C with a simple closed curve in R^2, having one angle. Since W is simply connected of dimension ≥ 5, the imbedding of C in W can be extended to an imbedding g of R^2 in W. Now a tubular neighborhood N of $g(R^2)$ in W will be diffeomorphic to R^{2m}.

Consider the immersion $f^{-1}(N) \to N$. This has a double point, $f(a_1) = f(a_2)$, where a_1 and a_2 belong to the same component of $f^{-1}(N)$. Apply the construction described by Whitney [**12**, Theorems 3, 4] to this immersion. This has the effect of first adding one double point and then removing two double points. The map f is altered only in a compact subset of $f^{-1}(N)$. Therefore the modified immersion $f^{-1}(N) \to N$ gives rise to a new immersion $f' : V \to W$ which has only $k - 1$ double points.

[7] Integer coefficients are to be understood.

Since the set $N \subset W$ is homeomorphic to R^{2m} it follows easily that f' is homotopic to f. Iterating this construction k times we obtain an imbedding f_0 of V in W, which completes the proof.

The next problem is to decide whether the normal bundle ν^m of $f_0(S^m)$ is trivial. This turns out to be a difficult question for m odd. Hence we will concentrate on the case m even.

LEMMA 7. *Let W be m-parallelizable of dimension $2m$ with m even. Let $f_0 : S^m \to W$ be an imbedding which represents the homology class[7] $\beta \in H_m(W)$. Then the normal bundle ν^m of $f_0(S^m)$ is trivial if and only if the intersection number $\langle \beta, \beta \rangle$ is zero.*

[Here the notation $\langle \alpha, \alpha' \rangle$ stands for the intersection number of two homology classes $\alpha \in H_i(W)$, $\alpha' \in H_{2m-i}(W)$. See Lefschetz [**2**].]

PROOF. Let $[\nu^m] \in \pi_{m-1}(SO_m)$ denote the homotopy class which corresponds to the bundle ν^m. (See Steenrod [**8**, §18].) Just as in the proof of Lemma 3 it is seen that $\nu^m \oplus o^{m+1}$ is a trivial bundle and hence, by Lemma 4, that $\nu^m \oplus o^1$ is trivial. Therefore $[\nu^m]$ is annihilated by the homomorphism

$$i_* : \pi_{m-1}(SO_m) \to \pi_{m-1}(SO_{m+1}).$$

Now consider the exact sequence

$$\pi_m(S^m) \xrightarrow{\partial} \pi_{m-1}(SO_m) \xrightarrow{i_*} \pi_{m-1}(SO_{m+1}).$$

The homomorphism ∂ carries a generator of $\pi_m(S^m)$ into the class $[\tau^m]$ which corresponds to the tangent bundle of S_m. Hence $[\nu^m]$ must be equal to some multiple $k[\tau^m]$ where k is an integer.

The Euler class $X(\xi^m)$ of an m-dimensional vector space bundle can be defined as the first obstruction to the existence of a non-zero cross-section. Note that the correspondence

$$[\xi^m] \to X(\xi^m)$$

defines a homomorphism from $\pi_{m-1}(SO_m)$ to the infinite cyclic group $H^m(S^m)$. [Proof: Let $X \in H^m(B(SO_m))$ denote the universal Euler class and let $g : S^m \to B(SO_m)$ denote a classifying map for ξ^m. Thus $X(\xi^m) = g^*X$. This is clearly an additive function of the homotopy class $\{g\} \in \pi_m(B(SO_m)) \approx \pi_{m-1}(SO_m)$.]

For the tangent bundle τ^m of S^m the class $X(\tau^m)$ is known to be twice a generator of $H^m(S^m)$. (The hypothesis that m is even comes in here.) Since $[\nu^m] = k[\tau^m]$ it follows that $X(\nu^m)$ is equal to $2k$ times a generator.

This argument proves that the group Image ∂ = kernel i_* generated by $[\tau^m]$ is infinite cyclic. An element $[\xi^m]$ in this group is zero if and only if the Euler class $X(\xi^m)$ is zero.

But for the normal bundle ν^m, the class $X(\nu^m)$ can be interpreted as the intersection number $\langle \beta, \beta \rangle$ multiplied by a generator of $H^m(S^m)$. For given a normal vector field with only finitely many zeros, we can deform $f_0(S^m)$ along these vectors to obtain a new imbedding which intersects $f_0(S^m)$ at only finitely many places. The multiplicity of each such intersection is equal to the index of the corresponding zero of the normal vector field.

Thus ν^m is trivial if and only if $\langle\beta, \beta\rangle$ equals zero, which completes the proof.

REMARK. This argument also proves that $\langle\beta, \beta\rangle$ is equal to $2k$ and hence is always an even number. This fact will be important later.

Lemma 7 raises the following question: Does there exist a non-zero element β of $H_m(W)$ such that the intersection number $\langle\beta, \beta\rangle$ is zero?

Assume now that W is compact and $(m-1)$-connected of dimension $2m$, and furthermore assume that the boundary of W has no homology in dimensions m, $m-1$. Then the Poincaré duality theorem implies that $H_m(W)$ is a free abelian group. [The group $H_m(W)$ is isomorphic to $H_m(W, \text{Boundary } W)$ and hence, by duality, to $H^m(W)$. Since $H_{m-1}(W) = 0$, the universal coefficient theorem implies that $H^m(W)$ is free abelian.]

Since m is even the correspondence $\alpha \to \langle\alpha, \alpha\rangle$ for $\alpha \in H_m(W)$ defines a quadratic form with integer coefficients. The determinant of this quadratic form is ± 1. (Compare Lefschetz [2, p. 178] or Milnor [3, Lemma 1].) Our question now becomes: does the quadratic form of W have a non-trivial zero?

LEMMA 8. *A quadratic form φ with integer coefficients and with determinant ± 1 has a non-trivial zero if and only if it is indefinite.*

Both Lemma 8 and Lemma 9 (which follows) are immediate consequences of [**3**, Theorems 1, 2]. The more direct proofs given below are due to H. Sah. The basic reference for these proofs is B. Jones [**1**].

PROOF OF LEMMA 8. If φ is positive definite or negative definite then it clearly has only the trivial zero. Assume that φ is indefinite of rank r. That is, if ν denotes the number of negative terms when φ is diagonalized over the rational numbers, assume that $0 < \nu < r$.

CASE 1. $r \geqq 5$. According to [**1**, Corollary 27d], φ has a non-trivial zero over the rational numbers (that is in $H_m(W; Q)$). Clearing denominators it follows that φ has a non-trivial zero over the integers.

CASE 2. $r = 3$ or 4. According to [**1**, Theorem 14], φ has a non-trivial zero over the p-adic integers for p odd. It clearly has a non-trivial zero over the real numbers. The Hasse symbol $c_2(\varphi)$ can be computed as on pages 38–39 of [**1**], and turns out to be $+1$ for $\nu = 1, 2$ and -1 for $\nu = 3$. The determinant of φ is clearly equal to $(-1)^\nu$. Now using [**1**, Theorem 14] it follows that φ has a non-trivial zero over the 2-adic numbers. Together with the Hasse-Minkowski theorem, [**1**, Theorem 27], this implies that φ has a rational zero.

CASE 3. $r = 2$. Then the determinant of φ is -1, and the conclusion follows by an elementary argument. (See [**1**, Theorem 14a].) This completes the proof of Lemma 8

The *index* of a quadratic form is defined as $r - 2\nu$. (For a diagonalized form this is the number of positive terms minus the number of negative ones.) The index of the quadratic form $\alpha \to \langle\alpha, \alpha\rangle$ of W turns out to be invariant under the χ-construction. (Compare the Corollary to Theorem 1.) Thus in order to have any hope of killing $H_m(W)$ we must assume that this index is zero.

Now suppose that W is m-parallelizable. Then the quadratic form of W takes on only even values. This follows from the remark after Lemma 7, together with Lemma 6 and the Hurewicz theorem. (For an alternative proof, compare [**3**, Lemma 3].)

The conditions which we have obtained for the quadratic form of W turn out to be sufficient to characterize this form. Let U denote the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA 9. *Consider a quadratic form over the integers with determinant ± 1 and index 0, which takes on only even values. The matrix of this form, with respect to a suitable basis, is*[8] diag $(U, \cdots, U)$.

PROOF. It will be convenient to use the notation $\alpha \to \langle\alpha, \alpha\rangle$, $\alpha \in H$, for this quadratic form. According to Lemma 8 there exists a non-zero element β_1 of H such that $\langle\beta_1, \beta_1\rangle = 0$. We may assume that β_1 is indivisible. Since the determinant is ± 1, it follows that there exists an element α of H with $\langle\beta_1, \alpha\rangle = 1$. Define

$$\beta_2 = \alpha - \tfrac{1}{2}\langle\alpha, \alpha\rangle\beta_1.$$

It follows that $\langle\beta_1, \beta_2\rangle = 1$ and $\langle\beta_2, \beta_2\rangle = 0$. Thus the matrix $\|\beta_i, \beta_j\|$ for $i, j = 1, 2$ is equal to U.

Let H' denote the set of α in H with $\langle\beta_1, \alpha\rangle = \langle\beta_2, \alpha\rangle = 0$. Since the determinant of U is ± 1 it follows that H splits into the direct sum of H' and the free abelian group generated by β_1 and β_2. By induction on the rank we may choose a basis $\beta_3, \cdots, \beta_r$ for H' which has the required form. This completes the proof of Lemma 9.

THEOREM 4. *Let W be m-parallelizable and $(m-1)$-connected of dimension $2m$ where m is even $\neq 2$. Suppose that the quadratic form of W has index zero, and that the boundary of W has no homology in dimensions m, $m-1$. Then W is χ-equivalent to an m-connected manifold.*

PROOF. According to Lemma 9 there exists a basis $\beta_1, \cdots, \beta_r$ for $H_m(W)$ so that the intersection matrix takes the form diag $(U, \cdots, U)$. By Lemma 6 and the Hurewicz theorem, there exists an imbedding

$$f_0 : S^m \to \text{Interior } W$$

which represents the homology class β_1. By Lemma 7 the normal bundle of

[8] Here diag$(U, \cdots, U)$ denotes the $r \times r$ matrix with $r/2$ copies of U along the diagonal, and zeros elsewhere.

$f_0(S^m)$ is trivial. Hence f_0 can be extended to an imbedding

$$f : S^m \times D^m \to W.$$

Let $W' = \chi(W, f)$, $W_0 = W - \text{Interior } f(S^m \times D^m)$.

First consider the exact sequence

$$0 \to H_m(W_0) \to H_m(W) \xrightarrow{j_*} H_m(W \bmod W_0) \xrightarrow{\partial} \cdots .$$

By excision it is seen that $H_m(W \bmod W_0)$ is infinite cyclic. A generator of this group has intersection number ± 1 with the cycle $f_0(S^m)$. Thus for $\alpha \in H_m(W)$ the image $j_*(\alpha)$ is equal to the intersection number $\langle \alpha, \beta_1 \rangle$ multiplied by a generator of $H_m(W \bmod W_0)$. Hence $H_m(W_0)$ is isomorphic to the subgroup of $H_m(W)$ generated by the elements $\beta_1, \beta_3, \beta_4, \cdots, \beta_r$; with β_2 omitted. It is easily verified that W_0 is $(m - 1)$-connected.

Next consider the exact sequence

$$\cdots \to H_{m+1}(W' \bmod W_0) \xrightarrow{\partial} H_m(W_0) \to H_m(W') \to 0.$$

In this case the group $H_{m+1}(W' \bmod W_0)$ is infinite cyclic. The homomorphism ∂ carries a generator into the homology class represented by the imbedding

$$u \to f(u, v_0)$$

of S^m in W_0. But this is clearly the homology class β_1. Therefore $H_m(W')$ is a free abelian group with a basis corresponding to the elements $\beta_3, \beta_4, \cdots, \beta_r$. The manifold W' is $(m - 1)$-connected.

The effect of this construction on W is to replace the sphere $f_0(S^m)$ by a sphere of dimension $m - 1$. Any map of an m-dimensional complex into W' can be deformed so as to miss this $(m - 1)$-sphere, and hence can be deformed into the subspace W_0. This implies:

(1) that W' is also m-parallelizable and

(2) that the intersection number of any two elements of $H_m(W')$ is equal to the intersection number of two corresponding elements of $H_m(W)$. Thus the quadratic form of W' has matrix diag $(U, \cdots, U)$, with rank $r - 2$.

Now iterate this construction $r/2$ times. The resulting manifold W'' will still be $(m - 1)$-connected and the group $H_m(W'')$ will be zero. Together with the Hurewicz theorem, this completes the proof of Theorem 4.

Remark. If the hypothesis that W has index zero is replaced by the weaker hypothesis that the quadratic form of W is indefinite, then it is at least possible to reduce the rank of $H_m(W)$. The above argument shows that the intersection matrix of $H_m(W)$ has the form diag (U, X), for a suitable choice of basis. Applying surgery one obtains a manifold W' with intersection matrix X.

In conclusion we consider the corresponding problem for m odd. Then the quadratic form of W is replaced by a skew symmetric bilinear form. The following result is well known. (See Veblen [9, p. 183].) Let U' denote the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Assertion. Every skew symmetric bilinear form with determinant ± 1 has matrix diag $(U', \cdots, U')$, for a suitable choice of basis.

This takes the place of Lemma 9, and is easier to prove. Unfortunately, however, Lemma 7 is not so easy to replace, except in two special cases.

If m is equal to 3 or 7 then the group $\pi_{m-1}(SO_m)$ is zero. Hence any m-sphere in a $2m$-dimensional manifold has trivial normal bundle. This makes it possible to prove the following analog of Theorem 4.

Theorem 4'. *If W is compact and $(m - 1)$-connected of dimension $2m$, with $m = 3$ or 7, and if*

$$H_{m-1}(\text{Boundary } W) = H_m(\text{Boundary } W) = 0,$$

then W is χ-equivalent to an m-connected manifold.

The proof goes just as before, with the assertion concerning skew symmetric forms in place of Lemma 9.

For m odd but unequal to 1, 3, 7 the situation is more difficult, and will be considered in a later paper.

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