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Signature homology

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Abstract. In this note we give a new construction of Signature homology, and we explain how to associate to any oriented manifold M a characteristic class in $\mathrm{Sig}_*(M)$ which is an integral analog of the L-class. A connection with the Novikov conjecture is explained. Further applications are in the construction of a 2-local characteristic class in the singular cohomology of a topological manifold as well as in the determination of the homotopy type of G/Top.

Contents

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Introduction

This paper is devoted to a new construction of a generalized homology theory denoted by $\mathrm{Sig}_*(-)$ which we call Signature homology (in earlier versions the name "Hirzebruch homology" was also used). This homology functor has been firstly considered by Dennis Sullivan in the proof of the Hauptvermutung (see [Su]) and defined by means of the so called Sullivan-Baas construction. More recently, Matthias Kreck has observed that Signature homology also can be used to state an integral formulation of the Novikov conjecture. In fact, according to Kreck, one can associate to every closed oriented smooth manifold M a fundamental class in $\mathrm{Sig}_*(M)$ and it can be shown easily that the rational reduction of this class coincides with the L-class. This consideration allows then to get an integral formulation of the Novikov conjecture just by requiring the homotopy invariance of the Signature fundamental class for all singular manifolds over $K(\pi, 1)$. Unfortunately this construction is very artificial and cannot be extended to topological manifolds (this is particularly unsatisfactory if one remembers that Novikov has defined rational Pontrjagin classes also for topological manifolds). This paper has the twofold purpose of both providing a more natural construction of Signature homology and of extending the considerations above to the topological category. It is perhaps interesting to notice that, while doing so, we could also prove a generalization of Novikov's theorem about the topological invariance of rational Pontrjagin classes.

This paper is essentially the fruit of a re-elaboration of my PhD thesis written under Matthias Kreck at the Ruprecht-Karls Universität Heidelberg. There are some new considerations, but most results are already contained in the old version (see [Mi]). The first section contains a short introduction to the theory of stratifolds and provides a proof of the transversality theorem. Section 2 is devoted to the definition of H-stratifolds which are the geometric cycles of Signature homology. Section 3 deals with the construction of the Signature homology functor and with the computation of its coefficients. Finally the last section is devoted to the definition of the Signature fundamental class and to its interpretation in terms of other known invariants like Pontrjagin classes.

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1. Stratifolds and transversality

In this section we explain some properties of topological stratifolds. The notion of a stratifold has been firstly introduced by Matthias Kreck in 1998. Through a series of modifications, the term "stratifold" has in the meanwhile come to indicate a different class of spaces, while the original objects are now called p-stratifolds. For simplicity we will however adopt the original terminology. A more detailed treatment of the geometrical properties of stratifolds can be found in Anna Grinberg's work (see [Gr]).

1.1. Stratifolds. Let $(W, \partial W)$ be a pair of spaces with ∂W closed in W, and denote by $\mathring{W}$ the open set $W - \partial W$. If $\delta : \partial W \to (0, +\infty)$ is a continuous function, then we set

$$\big(\partial W \times [0, +\infty)\big)^{<\delta} := \{(x, t) \in \partial W \times [0, +\infty) \mid t < \delta(x)\}.$$

Analogously one defines the sets $\big(\partial W \times [0, +\infty)\big)^{\leqq\delta}$ and $\big(\partial W \times (0, +\infty)\big)^{<\delta}$.

A collar of ∂W is a homeomorphism $c : V \to U$ where V is an open neighborhood of $\partial W \times \{0\}$ of the form $\big(\partial W \times [0, +\infty)\big)^{<\delta}$ and U is an open neighborhood of ∂W in W, so that for any $x \in \partial W$ it is $c(x, 0) = x$. Two collars are called equivalent if they coincide on an open neighborhood of ∂W. An equivalence class of collars will be called a germ of collars.

Definition 1.1. An n-dimensional c-manifold is a pair $(W, \partial W)$ where

– W is a metrizable space;

– $\mathring{W}$ and ∂W are (metrizable topological) manifolds of dimension respectively n and $n - 1$

together with a germ of collars $[c]$. The manifold ∂W is called the boundary of W.

If $c : V \to U$ is a representative of the germ of collars of a c-manifold W, then we denote by π the composition

$$U \xrightarrow{c^{-1}} V \xrightarrow{\pi_1} \partial W.$$

If M is a manifold, then a continuous map

$$f : W \to M$$

is called a c-map if there is a representative of the germ of collars $c : V \to U$ such that for all $x \in U$ it holds

$$f(x) = f\big(\pi(x)\big).$$

Observe that every continuous map can be approximated by a c-map.

Definition 1.2. Let W_1 and W_2 be two c-manifolds. A homeomorphism

$$f : W_1 \to W_2$$

is called an isomorphism if there are representatives of the germs of collars $c_1 : V_1 \to U_1$ and $c_2 : V_2 \to U_2$ such that for all $(x, t) \in V_1$ with $f\big(c_1(x, t)\big) \in U_2$ it holds

$$f\big(c_1(x, t)\big) = c_2\big(f(x), t\big).$$

If W_1 and W_2 are two c-manifolds, then by smoothing the corners one defines the structure of a c-manifold on the product $W_1 \times W_2$.

Let X be an arbitrary topological space. A k-dimensional strat of X is a pair (W, f) where W is a k-dimensional c-manifold, and f is a proper continuous map from W to X such that $f|_{\mathring{W}} : \mathring{W} \to f(\mathring{W})$ is a homeomorphism.

Definition 1.3. An n-dimensional *stratifold* is a pair $(X, \mathscr{P})$ where X is a topological space X, and $\mathscr{P}$ is a sequence of strats $\{W_i, f_i\}_{i \leqq n}$ which satisfy the following conditions:

– $\bigsqcup_i f_i(\mathring{W}_i) = X$;

– $\dim W_i = i$;

– $f_i(\partial W_i) \subset \bigcup_{j \leqq i-1} f_j(W_j)$;

– a subset $U \subset X$ is open if and only if for all i the set $f_i^{-1}(U)$ is open in W_i.

The sequence $\mathscr{P} = \{W_i, f_i\}$ is called the parametrization of X, and the restrictions $f_i|_{\partial W_i}$ are called the attaching maps of X.

For simplicity a stratifold $(X, \mathscr{P})$ will be generally denoted just by X.

If X is a stratifold, then the subspaces

$$\Sigma_k(X) := \bigcup_{i=0}^{k} f_i(W_i) \subset X \quad \text{and} \quad X_k := \Sigma_k(X) - \Sigma_{k-1}(X)$$

are called respectively the k-th skeleton and the k-th stratum of X. A stratifold is said to be purely n-dimensional if X_n is dense in X. The k-th stratum of a stratifold X is by construction a (possibly empty) k-dimensional manifold. The depth of a stratifold X is by definition the difference in dimension of the highest and lowest dimensional non-empty strata.

Definition 1.4. An n-dimensional stratifold X is called oriented if X_{n-1} is empty and the top stratum X_n is oriented.

A standard argument (see [Gr]) shows that the collars of the manifolds W_i define, for any k, a canonical germ of retractions

$$\pi_k : V_k \to X_k$$

where V_k is an open neighborhood of X_k in X.

Example 1.5. Let X and X' be two stratifolds. Then the following spaces inherit a natural structure as stratifolds.

(1) Any open subset $U \subset X$.

(2) The cone over X, if X is compact.

(3) The product $X \times X'$.

Definition 1.6. If X and X' are two n-dimensional stratifolds with stratifications respectively $\mathscr{P} = \{W_i, f_i\}$ and $\mathscr{P}' = \{W_i', f_i'\}$, then an isomorphism from X to X' is a homeomorphism

$$\varphi : X \to X'$$

together with a sequence of isomorphisms of c-manifolds $\varphi_i : W_i(X) \to W_i(X')$ which make the following diagram commutative for every i:

$$\begin{tikzcd} X \arrow[r, "\varphi"] & X' \\ W_i(X) \arrow[u, "f_i"] \arrow[r, "\varphi_i"] & W_i(X'). \arrow[u, "f_i'"'] \end{tikzcd}$$

Now, let X be a topological space and let $\{U_i \mid i \in J\}$ be a family of open subsets of X with the property that every U_i is a stratifold with parametrization $\mathscr{P}_i$. Furthermore assume that there are isomorphisms of stratifolds

$$\psi_{ji} : (U_i \cap U_j, \mathscr{P}_i|_{U_i \cap U_j}) \xrightarrow{\simeq} (U_i \cap U_j, \mathscr{P}_j|_{U_i \cap U_j})$$

which satisfy

$$\psi_{ii} = \mathrm{Id}, \quad \psi_{ij} \circ \psi_{ji} = \mathrm{Id}, \quad \psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \mathrm{Id}.$$

Gluing together the strata of the stratifolds U_i, one proves the following

Lemma 1.7. *There is up to isomorphism a unique parametrization $\mathscr{P}$ of X together with a family of isomorphisms*

$$\phi_i : (U_i, \mathscr{P}|_{U_i}) \xrightarrow{\simeq} (U_i, \mathscr{P}_i)$$

so that for any i, j the diagram

$$\begin{tikzcd} (U_i \cap U_j, \mathscr{P}|_{U_i \cap U_j}) \arrow[d, "\phi_i|_{U_i \cap U_j}"'] \arrow[dr, "\phi_j|_{U_i \cap U_j}"] & \\ (U_i \cap U_j, \mathscr{P}_i|_{U_i \cap U_j}) \arrow[r, "\psi_{ji}"] & (U_i \cap U_j, \mathscr{P}_j|_{U_i \cap U_j}) \end{tikzcd}$$

commutes.

The class of all stratifolds can be turned into a category taking the isomorphisms as morphisms. In particular, for a stratifold X we denote by $\mathrm{Aut}(X)$ the group of all isomorphisms from X to X.

Now we want to explain the notion of a morphism from a stratifold to a manifold.

Definition 1.8. Let X be a stratifold and M a manifold. A morphism is a map $g : X \to M$ with the property that the composition

$$g \circ f_i : W^i \to M$$

is a c-map for all $i \leqq n$.

One can easily show that every continuous map $X \to M$ is homotopic to a morphism.

Definition 1.9. A morphism $f : X \to M$ is called a stratifold bundle if there is a stratifold F so that (X, f, M) is a locally trivial bundle (in the sense of Steenrod, [St]) with fibre F and structure group $\mathrm{Aut}(F)$.

As a consequence of lemma 1.7 we have the following result.

Corollary 1.10. *Let F be a stratifold. If (X, f, M) is a locally trivial bundle with fibre F and group* $\mathrm{Aut}(F)$, *then X admits the structure of a stratifold so that $f : X \to M$ is a stratifold bundle.*

The notion of stratifold bundle allows to distinguish an important class of stratifolds.

Definition 1.11. A stratifold X is called locally trivial if, for each k, there is a representative $\pi_k : V_k \to X_k$ of the germ of retractions which is a stratifold bundle. Furthermore it is called locally conelike if, for any $x \in X_k$, the fibre of π_k over x is the cone over some compact stratifold.

It is straightforward to translate the definition of a c-manifold in the category of stratifolds so that we can also speak of c-stratifolds and c-morphisms.

1.2. Transversality. In this subsection we show how to extend the transversality theorem to the class of locally trivial topological stratifolds.

We begin by recalling the notion of a bicollar. Let (M, N) be a pair of spaces with N closed in M. If $\delta_1 : N \to (-\infty, 0)$ and $\delta_2 : N \to (0, +\infty)$ are two continuous functions, then we set

$${}^{\delta_1<}(N \times \mathbb{R})^{<\delta_2} := \{(x, s) \in N \times \mathbb{R} \,|\, \delta_1(x) < s < \delta_2(x)\}.$$

A bicollar of N is a homeomorphism $\varphi : V \to U$ where V is an open neighborhood of $N \times \{0\}$ of the form ${}^{\delta_1<}(N \times \mathbb{R})^{<\delta_2}$ and U is an open neighborhood of N in M, so that for any $x \in N$ it is $\varphi(x, 0) = x$.

Now, let M be a topological manifold and consider a continuous function

$$\rho : M \to \mathbb{R}.$$

Definition 1.12. A real number $t \in \mathbb{R}$ is called a regular value of ρ, if

– $N := \rho^{-1}(t)$ is an $(n-1)$-dimensional manifold together with a germ of bicollars $[\varphi]$;

– there exists a representative $\varphi : V \to U$ of the germ of bicollars so that it results

$$\rho(\varphi(x,s)) = s + t$$

for all $(x,t) \in V$.

Equivalently, we also say that ρ is transverse at t.

A very sophisticated argument (see [KiSi], [Ma] and [Qu1]) shows that the transversality theorem holds also in the topological category.

Theorem 1.13 (The transversality theorem). *Let A be a closed subset of M and suppose that there is an open neighborhood O of A such that 0 is a regular value of $\rho|_O$. Then there exists a homotopy*

$$H : M \times [0,1] \to \mathbb{R}$$

of $H(-,0) = \rho$ so that

- *$H(x,s) = \rho(x)$ for $x \in A$ and for all $s \in [0,1]$;*
- *0 is a regular value of $H(-,1)$.*

Next, we want to extend the transversality theorem to the class of all c-manifolds. Let $\rho : W \to \mathbb{R}$ be a continuous c-function from an n-dimensional c-manifold to $\mathbb{R}$. A real number $t \in \mathbb{R}$ is a regular value of ρ, if t is a regular value of $\rho|_{\partial W}$ and of $\rho|_{\mathring{W}}$. It follows from the definition that if t is a regular value of ρ, then $Z := \rho^{-1}(t)$ is a c-submanifold of W with boundary $\partial Z := \partial W \cap Z$.

Remark 1.14. Observe that, even though ρ has been assumed to be a c-function, the transversality of $\rho|_{\mathring{W}}$ at a point t does not imply automatically that of $\rho|_{\partial W}$. A counterexample is provided by the map

$$\mathbb{R}^4 \times [0,1) \to Y \times \mathbb{R} \times [0,1) \xrightarrow{\pi_2} \mathbb{R}$$

where Y is the non-manifold constructed in [Bi].

Lemma 1.15. *Let $K \subset W$ be a closed set and let $L \subset \partial W$ be another closed set so that $K \cap \partial W \subset \mathring{L}$ (where $\mathring{L}$ denotes the interior of L). Then there is a representative of the germ of collars $c : V \to U$ so that it holds*

$$x \in K \cap U \quad \Rightarrow \quad \pi(x) \in L.$$

Proof. If $c' : V' \to U'$ is any representative of the germ of collars, then there is by definition a continuous function $\delta' : \partial W \to (0,+\infty)$ such that $V' = (\partial W \times [0,+\infty))^{<\delta'}$.

Since the projection π is a closed map and K is closed, the number

$$m(x) := \min\{t \mid (x,t) \in K\}$$

is defined and for $x \in (\partial W - \mathring{L}) \cap K$ it holds $m(x) > 0$.

For this reason, we can define a function δ over the closed set $K \cup (\partial W - \mathring{L})$ setting

$$\delta(x) = \begin{cases} \delta'(x) & \text{if } x \in K, \\ \min\left\{\dfrac{m(x)}{2}, \delta'(x)\right\} & \text{if } x \in \partial W - \mathring{L}. \end{cases}$$

By Tietze's extension theorem, δ can be extended to a continuous function $\partial W \to (0, +\infty)$, which we denote again by δ. Using this new function, we set

$$V := \big(\partial W \times [0, +\infty)\big)^{<\delta},$$
$$c := c'|_V,$$
$$U := c(V).$$

The new collar $c : V \to U$ is by construction equivalent to c'.

Finally observe that, if $\pi(x) \in \partial W - L$ for some $x \in U$, then it must be $\delta\big(\pi(x)\big) < \min\{t \mid (x, t) \in K\}$ and so $x \notin K$. $\square$

The transversality theorem has now the following consequence.

Corollary 1.16. *Let*

$$\rho : W \to \mathbb{R}$$

be a c-function. Furthermore assume that there is a closed c-set $A \subset W$ and an open neighborhood O of A such that 0 is a regular value of $\rho|_O$. Then there exists a homotopy

$$H : W \times [0, 1] \to \mathbb{R}$$

of $H(-, 0) = \rho$ so that:

- *$H(x, s) = \rho(x)$ for all $x \in A$ and all $s \in [0, 1]$;*
- *the map $H(-, s)$ is a c-function for all $s \in [0, 1]$;*
- *0 is a regular value for $H(-, 1)$.*

Proof. Let us choose a representative of the germ of collars $c : V \to U$ such that $\rho(x) = \rho\big(\pi(x)\big)$ for all $x \in U$. Since W and ∂W are normal spaces, there exist two closed sets $K \subset W$ and $L \subset \partial W$ so that

- $A \subset \mathring{K} \subset K \subset O$;
- $K \cap \partial W \subset \mathring{L} \subset L \subset O \cap \partial W$.

By assumption, 0 is a regular value of $\rho|_{O \cap \partial W}$, and so, by theorem 1.13, there is a homotopy

$$h : \partial W \times [0,1] \to \mathbb{R}$$

of $h(-,0) = \rho|_{\partial W}$ such that

- $h(x,s) = \rho(x)$ for all $x \in L$;
- 0 is a regular value of $h(-,1)$.

Now, using lemma 1.15, we can find a representative of the germ of collars so that

$$x \in K \cap U \quad \Rightarrow \quad \pi(x) \in L.$$

Composing with the projection $\pi : U \to \partial W$, we get a homotopy h'

$$U \times [0,1] \xrightarrow{\pi \times \mathrm{id}} \partial W \times [0,1] \xrightarrow{h} \mathbb{R}$$

with the property that

$$h'(x,s) = h\big(\pi(x)\big) = \rho\big(\pi(x)\big) = \rho(x)$$

for each $x \in K \cap U$.

On the closed subspace $B := K \cup c\big(\big(\partial W \times [0,+\infty)\big)^{\leq \delta/2}\big) \subset W$ we define a homotopy h'' setting:

$$h''(x,s) = \begin{cases} \rho(x) & \text{if } x \in K, \\ h'(x,s) & \text{else.} \end{cases}$$

The c-manifold W is a metrizable space and so it is in particular binormal (recall that a space X is called binormal if the product $X \times [0,1]$ is a normal space). According to Borsuk's homotopy extension theorem (see [Sp]), there is an extension

$$\begin{tikzcd}
B \times \{0\} \arrow[r] \arrow[d] & B \times [0,1] \arrow[d] \arrow[ddr, "h''"] & \\
\mathring{W} \times \{0\} \arrow[r] \arrow[drr, "\rho"'] & \mathring{W} \times [0,1] \arrow[dr, dashed, "H'"] & \\
 & & \mathbb{R}
\end{tikzcd}$$

and the function $\rho' := H'(-,1)$ has the following properties:

- ρ' is a c-function;
- $\rho'(x) = \rho(x)$ for all $x \in B$;
- 0 is a regular value of $\rho'|_{\mathring{B}}$.

By construction $\mathring{B} \cap \mathring{W}$ is an open neighborhood of $c\big((\partial W \times (0, +\infty))^{\leqq\delta/3}\big) \cup (A \cap \mathring{W})$ and so, applying the transversality theorem 1.13 to $H'(-,1)$, we get a homotopy

$$H'' : \mathring{W} \times [0,1] \to \mathbb{R}$$

of $\rho'|_{\mathring{W}}$ which fixes $\big(A \cup (\partial W \times [0, +\infty))^{\leqq\delta/3}\big) \cap \mathring{W}$ and such that 0 is a regular value of $H''(-,1)$. The map H'' extends uniquely to a homotopy $H''' : W \times [0,1] \to \mathbb{R}$ with the property that $H'''(-,s)$ is a c-function for all s, and finally we define H as the composition of H' with H'''. $\square$

Now let us pass to the category of stratifolds.

Definition 1.17. Let $\rho : X \to \mathbb{R}$ be a morphism from an n-dimensional stratifold X to $\mathbb{R}$. A number $t \in \mathbb{R}$ is called a regular value of ρ, if

– $Y := \rho^{-1}(t)$ is an $(n-1)$-dimensional stratifold together with a germ of bicollars $[\varphi]$;

– there is a representative of the germ of bicollars $\varphi : V \to U$ with

$$\rho\big(\varphi(x,s)\big) = s + t$$

for all $(x,s) \in V$.

A basic step in the proof of the transversality theorem for stratifolds is given by the following

Lemma 1.18. *Let $\pi : X \to M$ be a stratifold bundle and consider a continuous function $\rho : M \to \mathbb{R}$. If ρ is transverse at zero then the map defined by the composition $\rho \circ \pi$ is also transverse at zero.*

Proof. By definition, the set $N := \rho^{-1}(0)$ is a bicollared submanifold of M, i.e. there is a homeomorphism $\varphi : V \to U$ where V is an open neighborhood of $N \times \{0\}$ in $N \times \mathbb{R}$ and U is an open neighborhood of N in M. Now, the set $E := \pi^{-1}(N)$ is the total space of the bundle

$$\pi|_E : E \to N$$

and therefore, by corollary 1.10, E is a stratifold. Since U is homeomorphic to a space of the form ${}^{\delta_1<}(N \times \mathbb{R})^{<\delta_2}$, we get a bundle isomorphism

$$\begin{array}{ccc} {}^{\delta_1<}(E \times \mathbb{R})^{<\delta_2} & \longrightarrow & \pi^{-1}(U) \\ \Big\downarrow{\scriptstyle \pi|_E \times \mathrm{Id}} & & \Big\downarrow{\scriptstyle \pi|_{\pi^{-1}(U)}} \\ U & = & U \end{array}$$

which defines a bicollar of E in X. $\square$

Using these facts we are now able to prove the following

Theorem 1.19 (Transversality for stratifolds). *Let X be a locally trivial stratifold and let*

$$\rho : X \to \mathbb{R}$$

be a morphism. Moreover assume that there is a closed set A and an open neighborhood O of A so that 0 is a regular value of $\rho|_O$. Then there exists a homotopy

$$H : X \times [0,1] \to \mathbb{R}$$

of $H(-,0) = \rho$ with the following properties:

- $H(x,s) = \rho(x)$ *for all* $x \in A$;
- $H(-,1)$ *is a morphism*;
- 0 *is a regular value of* $H(-,1)$.

Proof. For simplicity we suppose $A = \emptyset$. As explained in the previous subsection, the assumption that X is locally trivial means that, for any i, there is a representative of the germ of projections $\pi_i : V_i \to X_i$ which is a stratifold-bundle. The proposition will be proved by induction on the depth of X, which we denote by $d(X)$.

If the depth of X is zero, then X is a manifold and the proposition is a consequence of the transversality theorem.

Let X be a stratifold of depth $k+1$ and denote by Y the lowest non-empty dimensional stratum of X. Since Y is a manifold, the transversality theorem provides a homotopy of $\rho|_Y$

$$h : Y \times [0,1] \to \mathbb{R}$$

so that $h(-,1)$ is transverse at 0. Now, if $\pi : V \to Y$ is a representative of the germ of projections which is a stratifold-bundle, we define h' as the composition

$$V \times [0,1] \xrightarrow{\pi \times \mathrm{Id}} Y \times [0,1] \xrightarrow{h} \mathbb{R}.$$

The map h' is by construction a homotopy of $\rho|_V$ and it follows from corollary 1.18 that $h'(-,1)$ is transverse at 0. Now, let A, B, and C be three open neighborhoods of Y with

$$\bar{A} \subset B \subset \overline{B} \subset C \subset \overline{C} \subset V.$$

The restriction of h' to $\overline{C}$ can be extended to a homotopy of ρ

$$H' : X \times [0,1] \to \mathbb{R}$$

and, since the map $H'(-,1)|_C$ is a morphism, we can find a homotopy H'' of $H'(-,1)$ which fixes B and so that $\xi := H''(-,1)$ is a morphism. Now, the morphism ξ is by construction transverse at zero on B, and so it follows that in particular $\xi|_{(X-Y)\cap B}$ is transverse at zero.

The open set $X - Y$ is by construction a stratifold of depth k and by inductive assumption there exists a homotopy of $\xi|_{X-Y}$

$$k : (X - Y) \times [0,1] \to \mathbb{R}$$

which fixes $(X - Y) \cap A$ and so that $k(-,1)$ is transverse at 0. The map k can be extended to a homotopy over X setting

$$H''' : X \times [0,1] \to \mathbb{R},$$
$$(x,t) \mapsto \begin{cases} m(x,t) & \text{if } x \in X - Y, \\ \xi(x) & \text{if } x \in A. \end{cases}$$

Finally, we define the homotopy H as the composition of $H' * H'' * H'''$. $\square$

A procedure similar to the one used at the beginning of this subsection can be used to extend the transversality theorem to the class of c-stratifolds.

2. H-stratifolds

In this section we use the notion of a perverse self-dual complex of sheaves due to Markus Banagl (see [Ba1]) to introduce the concept of an H-stratifold.

2.1. Perverse self-dual complexes of sheaves. From now on we will make the following two assumptions:

- All stratifolds are assumed to be locally conelike, oriented, and purely n-dimensional.

- All complexes of sheaves are assumed to be constructible (see [GM] for the definition).

Let X be an n-dimensional stratifold with k-th skeleton Σ_k. For any integer $0 \leqq k \leqq n$, we indicate by U_k the open subset $X - \Sigma_{n-k}$ and by i_k the inclusion

$$U_k \hookrightarrow U_{k+1}.$$

Moreover, let us denote by $D^b(X)$ the derived category of all bounded complexes of sheaves of real vector spaces over X and recall that an orientation of X is the same as an isomorphism

$$\mathfrak{o} : \mathbb{D}^\bullet_{U_2} \xrightarrow{\simeq} \mathbb{R}_{U_2}[n]$$

in the derived category $D^b(U_2)$, where $\mathbb{D}^\bullet_Z$ denotes the Verdier dualizing complex on a space Z.

Definition 2.1. Let $(X, \mathfrak{o})$ be an oriented stratifold. A constructible complex of sheaves $\mathbf{A}^\bullet \in D^b(X)$ is said to be *perverse self-dual* if it satisfies:

(SD1) There is an isomorphism (called normalization):

$$\nu : \mathbf{A}^\bullet|_{U_2} \xrightarrow{\simeq} \mathbb{R}_{U_2}[n].$$

(SD2) $\mathbf{H}^i(\mathbf{A}^\bullet) = 0$, for $i < -n$.

(SD3) $\mathbf{H}^i(\mathbf{A}^\bullet|_{U_{k+1}}) = 0$, for $i > \bar{n}(k) - n$, $k \geqq 2$, where $\bar{n}$ denotes the upper middle perversity.

(SD4) There is an isomorphism $d : \mathscr{D}\mathbf{A}^\bullet[n] \xrightarrow{\simeq} \mathbf{A}^\bullet$ ($\mathscr{D}$ denotes here the Poincaré-Verdier duality functor) such that

$$\mathscr{D}d[n] = (-1)^n \cdot d$$

and the diagram

$$\begin{tikzcd}
\mathbb{R}_{U_2}[n] & \mathbf{A}^\bullet|_{U_2} \arrow[l, "\nu"', "\simeq"] \\
\mathbb{D}^\bullet_{U_2} \arrow[u, "o", "\simeq"'] \arrow[r, "\simeq", "\mathscr{D}\nu[n]"'] & \mathscr{D}\mathbf{A}^\bullet|_{U_2}[n] \arrow[u, "\simeq", "d|_{U_2}"']
\end{tikzcd}$$

commutes.

For any oriented stratifold X, we denote by $\mathrm{SD}(X)$ the full subcategory of $D^b(X)$ whose objects are the perverse self-dual complexes of sheaves over X.

Lemma 2.2. *Any open inclusion $i : U \hookrightarrow X$ induces a functor*

$$i^* : \mathrm{SD}(X) \to \mathrm{SD}(U),$$
$$\mathbf{A}^\bullet \mapsto \mathbf{A}^\bullet|_U.$$

Proof. This is an easy consequence of the fact that there is a natural equivalence of functors

$$i^!(-) \simeq i^*(-). \quad \square$$

Let $\mathbf{E}^\bullet$ and $\mathbf{F}^\bullet$ be two perverse self-dual complexes of sheaves over X, and let us denote by $\mathrm{Hom}_{\mathrm{SD}(X)}(\mathbf{E}^\bullet, \mathbf{F}^\bullet)$ the set of all morphisms from $\mathbf{E}^\bullet$ to $\mathbf{F}^\bullet$ in $\mathrm{SD}(X)$.

Lemma 2.3. *The restriction on the top stratum induces a monomorphism*

$$\mathrm{Hom}_{D^b(X)}(\mathbf{E}^\bullet, \mathbf{F}^\bullet) \to \mathrm{Hom}_{D^b(U_2)}(\mathbf{E}^\bullet|_{U_2}, \mathbf{F}^\bullet|_{U_2}).$$

Proof. The statement follows by an iterated application of [Ba1], Lemma 2.2. $\square$

This lemma has the interesting consequence that the self-duality isomorphism is—if existent—completely determined by the orientation and the normalization.

The problem of determining the structure of the category $\mathrm{SD}(X)$ can be reduced to the determination of the relation between the categories $\mathrm{SD}(U_k)$ and $\mathrm{SD}(U_{k+1})$. For the comfort of the reader we recall here the most important results. We assume for simplicity that n is even, but analogous considerations hold for the case n odd.

Theorem 2.4 (Goresky-MacPherson). *If k is even, then the restriction functor*

$$i_k^* : \mathrm{SD}(U_{k+1}) \to \mathrm{SD}(U_k)$$

is an equivalence of categories whose inverse is given by the functor

$$\tau_{\leqq \bar{m}(k)-n} Ri_{k*}(-) : \mathrm{SD}(U_k) \to \mathrm{SD}(U_{k+1}).$$

The case k odd is more difficult and has been investigated by Banagl in the above cited work.

Let $\mathbf{A}^\bullet$ be a perverse self-dual complex of sheaves over the open set U_k, with k odd and set $s := \bar{n}(k) - n$. The lifting obstruction of $\mathbf{A}^\bullet$ is by definition the complex of sheaves

$$\mathscr{O}(\mathbf{A}^\bullet) := \mathbf{H}^s(Ri_{k*}\mathbf{A}^\bullet)[-s] \in D^b(U_{k+1}).$$

Definition 2.5. A Lagrangian structure on $\mathbf{A}^\bullet$ is a morphism

$$\phi : \mathscr{L} \to \mathscr{O}(\mathbf{A}^\bullet)$$

which induces injections $\mathbf{H}^*(\phi) : \mathbf{H}^*(\mathscr{L}) \to \mathbf{H}^*\big(\mathscr{O}(\mathbf{A}^\bullet)\big)$ and such that some distinguished triangle on ϕ is a null-bordism for the perverse self-dual lifting obstruction (see [Ba1], Def. 2.3).

For the application there is an alternative approach to Lagrangian structures (see [Ba1], Remark 2.4) which is particularly useful. Let i and j denote respectively the inclusions $U_k \hookrightarrow U_{k+1}$ and $\Sigma := U_{k+1} - U_k \hookrightarrow U_{k+1}$ and set $\mathbf{H} := \mathbf{H}^s(j^*Ri_*\mathbf{A}^\bullet) \simeq \mathscr{O}(\mathbf{A}^\bullet)|_\Sigma$.

By [Ba1], Lemma 2.3, the self-duality isomorphism d induces an isomorphism $\delta : \mathscr{D}\mathscr{O}(\mathbf{A}^\bullet)[n+1] \xrightarrow{\simeq} \mathscr{O}(\mathbf{A}^\bullet)$ and therefore a non-singular pairing

$$\mathbf{H} \otimes \mathbf{H} \to \mathbb{R}_\Sigma.$$

A subsheaf $\mathbf{E} \subset \mathbf{H}$ is called Lagrangian if, for every $x \in \Sigma$, the stalk $\mathbf{E}_x$ is a Lagrangian subspace of $\mathbf{H}_x$. The connection between Lagrangian structures and Lagrangian subsheaves is explained by the following lemma due to Banagl.

Lemma 2.6. *The map*

$$\left\{ \begin{matrix} \text{Lagrangian structures} \\ \text{of } \mathbf{A}^\bullet \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Lagrangian subsheaves} \\ \text{of } \mathbf{H} \end{matrix} \right\},$$
$$(\mathscr{L}, \phi) \mapsto \big(\mathbf{H}^s(\phi)(\mathscr{L})\big)|_\Sigma \subset \mathscr{O}(\mathbf{A})|_\Sigma$$

is a bijection.

A morphism of Lagrangian structures is by definition a commutative diagram in $D^b(U_{k+1})$

$$\begin{array}{ccc} \mathscr{L}_A & \xrightarrow{\phi_A} & \mathscr{O}(\mathbf{A}^\bullet) \\ \big\downarrow & & \big\downarrow \scriptstyle{\mathscr{O}(f)} \\ \mathscr{L}_B & \xrightarrow{\phi_B} & \mathscr{O}(\mathbf{B}^\bullet) \end{array}$$

for some $f : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$. It follows from the functoriality of the lifting obstruction that the composition of morphisms of Lagrangian structures is well defined and thus that the Lagrangian structures form a category denoted by $\mathrm{Lag}(U_{k+1} - U_k)$.

The categories $\mathrm{SD}(U_k)$ and $\mathrm{Lag}(U_{k+1} - U_k)$ can be used to construct a new category which is called the twisted product category and which is denoted by $\mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k)$. By definition this is the category whose objects are the pairs

$$\big(\mathbf{A}^\bullet, \phi : \mathscr{L} \to \mathscr{O}(\mathbf{A}^\bullet)\big) \in \mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k),$$

and whose morphisms are the pairs (f, g) with first component a morphism $f \in \mathrm{Hom}_{D^b(U_k)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ and second component a commutative square

$$\begin{array}{ccc} \mathscr{L}_A & \xrightarrow{\phi_A} & \mathscr{O}(\mathbf{A}^\bullet) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \scriptstyle{\mathscr{O}(f)} \\ \mathscr{L}_B & \xrightarrow{\phi_B} & \mathscr{O}(\mathbf{B}^\bullet). \end{array}$$

If $\mathbf{A}^\bullet$ is a perverse self-dual complex of sheaves on U_{k+1}, then there is a constructive way to extract from $\mathbf{A}^\bullet$ a lagrangian structure on $i_k^*\mathbf{A}^\bullet$. This procedure allows to define a functor

$$\Lambda : \mathrm{SD}(U_{k+1}) \to \mathrm{Lag}(U_{k+1} - U_k)$$

and Banagl's main result can be thus formulated as follows.

Theorem 2.7 (Banagl). *The functor*

$$(i_k^*, \Lambda) : \mathrm{SD}(U_{k+1}) \to \mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k)$$

is an equivalence of categories whose inverse is denoted by $\boxplus$.

Putting together Goresky-MacPhersons's and Banagl's results one obtains the following fundamental result (see [Ba1], Theorem 2.10).

Theorem 2.8. *Let X be an n-dimensional stratifold. If n is even, then there is an equivalence of categories*

$$\mathrm{SD}(X) \simeq \mathrm{Const}(U_2) \rtimes \mathrm{Lag}(U_4 - U_3) \rtimes \cdots \rtimes \mathrm{Lag}(U_{n-2} - U_{n-3}) \rtimes \mathrm{Lag}(U_n - U_{n-1}).$$

If n is odd, then there is an equivalence of categories

$$\mathrm{SD}(X) \simeq \mathrm{Const}(U_2) \rtimes \mathrm{Lag}(U_3 - U_2) \rtimes \cdots \rtimes \mathrm{Lag}(U_{n-1} - U_{n-2}) \rtimes \mathrm{Lag}(U_{n+1} - U_n).$$

2.2. *H*-stratifolds. Let X be an n-dimensional oriented stratifold. By definition an H-structure over X is a pair $\mathscr{S} = (\mathbf{A}^\bullet, v)$ where $\mathbf{A}^\bullet$ is a perverse self-dual complex of sheaves over X and v is a normalization of $\mathbf{A}^\bullet$. If $\mathscr{S}_1 = (\mathbf{A}_1^\bullet, v_1)$ and $\mathscr{S}_2 = (\mathbf{A}_2^\bullet, v_2)$ are two H-structures over X, then an isomorphism of H-structures

$$\varphi : \mathscr{S}_1 \to \mathscr{S}_2$$

is an isomorphism of complexes of sheaves $\varphi : \mathbf{A}_1^\bullet \to \mathbf{A}_2^\bullet$, for which the diagram

$$\begin{tikzcd} \mathbf{A}_1^\bullet|_{U_2} \arrow[rr, "\varphi|_{U_2}"] \arrow[dr, "v_1"'] & & \mathbf{A}_2^\bullet|_{U_2} \arrow[dl, "v_2"] \\ & \mathbb{R}_{U_2}[n] & \end{tikzcd}$$

commutes.

Definition 2.9. An *H-stratifold* is a pair $(X, \mathscr{S})$, where X is an oriented topological stratifold and $\mathscr{S}$ is an H-structure over X.

If $(X, \mathscr{S})$ is an H-stratifold, then we denote by $-(X, \mathscr{S})$ the H-stratifold $(-X, \mathscr{S})$ obtained reversing the orientation of X and considering $\mathscr{S}$ as an H-structure over $-X$.

Let $\varphi : X \to Y$ be an orientation-preserving isomorphism of stratifolds and let $\mathscr{S} = (\mathbf{A}^\bullet, v)$ be an H-structure on Y.

Lemma/Definition 2.10. *The pair $\varphi^*\mathscr{S} = (\varphi^*\mathbf{A}^\bullet, \varphi^* v)$ is an H-structure on X which is called the pull-back of $\mathscr{S}$.*

Proof. We have to check that $\varphi^*\mathbf{A}^\bullet$ is a perverse self-dual complex of sheaves over X. Axiom (SD1) is satisfied with the normalization $\varphi^* v$. Axioms (SD2) and (SD3) are satisfied since it holds respectively

$$\mathbf{H}^i(\varphi^*\mathbf{A}^\bullet) \simeq \varphi^*\mathbf{H}^i(\mathbf{A}^\bullet)$$

and

$$\mathbf{H}^i\big((\varphi^*\mathbf{A}^\bullet)|_{U_{k+1}(X)}\big) \simeq \varphi^*\mathbf{H}^i(\mathbf{A}^\bullet|_{U_{k+1}(Y)}).$$

Finally, axiom (SD4) is satisfied using the self-duality isomorphism

$$\mathscr{D}(\varphi^*\mathbf{A}^\bullet)[n] \simeq \varphi^!(\mathscr{D}\mathbf{A}^\bullet[n]) \xrightarrow{\simeq} \varphi^!\mathbf{A}^\bullet \simeq \varphi^*\mathbf{A}^\bullet. \quad \square$$

Let H-stratifolds $(X_1, \mathscr{S}_1)$ and $(X_2, \mathscr{S}_2)$ be two H-stratifolds. An H-isomorphism

$$\varphi : (X_1, \mathscr{S}_1) \xrightarrow{\simeq} (X_2, \mathscr{S}_2)$$

is a pair (φ_1, φ_2), where $\varphi_1 : X_1 \xrightarrow{\simeq} X_2$ is an orientation-preserving isomorphism of stratifolds and $\varphi_2 : \mathscr{S}_1 \xrightarrow{\simeq} \varphi_1^* \mathscr{S}_2$ is an isomorphism of H-structures.

Observe that an isomorphism of H-stratifolds is automatically compatible with the self-duality isomorphisms. In fact applying lemma 2.3 one can prove the following

Lemma 2.11. *For any isomorphism of H-stratifold*

$$\varphi = (\varphi_1, \varphi_2) : \big(X_1, (\mathbf{A}_1^\bullet, v_1)\big) \xrightarrow{\simeq} \big(X_2, (\mathbf{A}_2^\bullet, v_2)\big),$$

the diagram

$$\begin{array}{ccc} \mathscr{D}\mathbf{A}_1^\bullet[n] & \xleftarrow[\simeq]{\mathscr{D}\varphi_2[n]} & \mathscr{D}\varphi_1^*\mathbf{A}_2^\bullet[n] \simeq \varphi_1^*\mathscr{D}\mathbf{A}_2^\bullet[n] \\ {\scriptstyle d_1}\Big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\Big\downarrow{\scriptstyle \varphi_1^* d_2} \\ \mathbf{A}_1^\bullet & \xrightarrow[\varphi_2]{\simeq} & \varphi_1^*\mathbf{A}_2^\bullet \end{array}$$

commutes.

Remark 2.12. The class $\mathfrak{C}$ of all H-stratifolds can be turned into a category with the H-isomorphisms as morphisms.

Example 2.13. Let M be an oriented n-dimensional topological manifold. The trivial sheaf $\mathbb{R}_M[n]$ is a perverse self-dual complex of sheaves with the self-duality isomorphism

$$\mathscr{D}(\mathbb{R}_M[n])[n] \simeq \mathscr{D}\mathbb{R}_M \simeq \mathbb{D}_M^\bullet \xrightarrow{\simeq} \mathbb{R}_M[n].$$

Definition 2.14. If $\mathscr{S} = (\mathbf{A}^\bullet, v)$ is an H-structure over a stratifold X and U is any open subset of X, then we denote by $\mathscr{S}|_U$ the H-structure $(\mathbf{A}^\bullet|_U, v|_U)$.

2.3. Product structures. The purpose of this subsection is to define the product of two H-stratifolds. In the second part we specialize to the case when one factor is the real line and we consider the problem of determining all H-structures on a bicollared substratifold of an H-stratifold.

Consider two oriented stratifolds X_1 and X_2 of dimension m and n respectively, and let π_1, π_2 denote the projections of $X_1 \times X_2$ to the first and second factors. For $i = 1, 2$, consider furthermore the map p_i defined by the restriction of π_i to the top stratum $U_2(X_1 \times X_2) = U_2(X_1) \times U_2(X_2)$.

A central role in the definition of the product-structure is played by the tensor product of complexes of sheaves. It is perhaps convenient to recall here that, if $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are complexes of sheaves of real vector spaces, then, by [GM], Section 1.9, there is an isomorphism

$$\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet \simeq \mathbf{A}^\bullet \otimes \mathbf{B}^\bullet.$$

Using [Bo], Corollary V,10.26, an orientation of $X_1 \times X_2$ is induced by the orientations of X_1 and X_2 through the isomorphism

$$\mathbb{D}^\bullet_{X_1\times X_2} \simeq \pi_1^*\mathbb{D}^\bullet_{X_1} \overset{L}{\otimes} \pi_2^*\mathbb{D}^\bullet_{X_2}.$$

In fact, if $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are the orientations of X_1 and X_2 respectively, then $p_1^*\mathfrak{o}_1 \overset{L}{\otimes} p_2^*\mathfrak{o}_2$ induces an isomorphism

$$\begin{aligned}(\mathbb{D}^\bullet_{X_1\times X_2})|_{U_2(X_1\times X_2)} &\simeq p_1^*(\mathbb{D}^\bullet_{X_1}|_{U_2(X_1)}) \overset{L}{\otimes} p_2^*(\mathbb{D}^\bullet_{U_2(X_2)})\\ &\xrightarrow{\simeq} p_1^*(\mathbb{R}_{U_2(X_1)}[m]) \overset{L}{\otimes} p_2^*(\mathbb{R}_{U_2(X_2)}[n])\\ &\simeq \mathbb{R}_{U_2(X_1\times X_2)}[m] \overset{L}{\otimes} \mathbb{R}_{U_2(X_1\times X_2)}[n]\\ &\simeq \mathbb{R}_{U_2(X_1\times X_2)}[m+n].\end{aligned}$$

Now, let $\mathscr{S}_1 = (\mathbf{A}_1^\bullet, \nu_1)$ and $\mathscr{S}_2 = (\mathbf{A}_2^\bullet, \nu_2)$ be two H-structures over X_1 and X_2 respectively.

Lemma/Definition 2.15. *The pair*

$$\mathscr{S}_1 \times \mathscr{S}_2 := (\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet, p_1^*\nu_1 \overset{L}{\otimes} p_2^*\nu_2)$$

is an H-structure over $X_1 \times X_2$. The H-stratifold $(X_1 \times X_2, \mathscr{S}_1 \times S_2)$ is called the product of $(X_1, \mathscr{S}_1)$ with $(X_2, \mathscr{S}_2)$.

Proof. Observe first of all that, for $p = (x_1, x_2) \in X_1 \times X_2$, it holds

$$\begin{aligned}(1)\qquad \big(\mathbf{H}^i(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet)\big)_p &\simeq \mathbf{H}^i\big((\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet)_p\big)\\ &\simeq \mathbf{H}^i\big((\pi_1^*\mathbf{A}_1^\bullet)_p \overset{L}{\otimes} (\pi_2^*\mathbf{A}_2^\bullet)_p\big)\\ &\simeq \mathbf{H}^i\big((\mathbf{A}_1^\bullet)_{x_1} \overset{L}{\otimes} (\mathbf{A}_2^\bullet)_{x_2}\big)\\ &\simeq \bigoplus_{a+b=j} \mathbf{H}^a\big((\mathbf{A}_1^\bullet)_{x_1}\big) \otimes \mathbf{H}^b\big((\mathbf{A}_2^\bullet)_{x_2}\big)\end{aligned}$$

where the last step is a consequence of the algebraic Künneth formula. According to [Bo], V,10.25, if $\mathbf{L}_1^\bullet \in D^b(X_1)$ and $\mathbf{L}_2^\bullet \in D^b(X_2)$ are two constructible complexes of sheaves, then there is an isomorphism

$$\pi_1^*\mathscr{D}_X\mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{B}^\bullet \simeq R\,\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}^\bullet, \pi_2^!\mathbf{B}^\bullet).$$

Now, let us show that $\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet \in D^b(X_1 \times X_2)$ is a perverse self-dual complex of sheaves. Axiom (SD1) is of course satisfied with the normalization

$$p_1^*\nu_1 \overset{L}{\otimes} p_2^*\nu_2.$$

Axioms (SD2) and (SD3) can be checked looking at the stalks and using formula (1). Since both $\mathbf{A}_1^\bullet$ and $\mathbf{A}_2^\bullet$ satisfy (SD2), it follows, for $i < -m-n$,

$$\mathbf{H}^i(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet) = 0,$$

and so $\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet$ satisfies (SD2). In order to show (SD3), let us consider an integer $k \geqq 2$ and a point $p = (x_1, x_2) \in U_{k+1}(X_1 \times X_2)$. The structure of stratifold on the product space $X_1 \times X_2$ has the property that for any integer $k \geqq 2$ there exists a partition of k of the form $k = \alpha + \beta$, so that $x_1 \in U_\alpha(X_1)$ and $x_2 \in U_\beta(X_2)$. Now, for any $i > \bar{n}(k) - m - n$ and any partition $a + b = i$, it results

$$a + b > \bar{n}(k) - m - n \geqq \bar{n}(\alpha) + \bar{n}(\beta) - m - n$$

and so it must also hold

$$a > \bar{n}(\alpha) - m \quad \text{or} \quad b > \bar{n}(\beta) - n.$$

On the other hand, $\mathbf{A}_1^\bullet$ and $\mathbf{A}_2^\bullet$ satisfy (SD3), and consequently it must be

$$\big(\mathbf{H}^a(\mathbf{A}_1^\bullet)\big)_{x_1} = 0 \quad \text{or} \quad \big(\mathbf{H}^b(\mathbf{A}_2^\bullet)\big)_{x_2} = 0.$$

Finally, using formula (1), we obtain:

$$\big(\mathbf{H}^i(\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet)\big)_p \simeq \bigoplus_{a+b=i} \big(\mathbf{H}^a(\mathbf{A}_1^\bullet)\big)_{x_1} \otimes \big(\mathbf{H}^b(\mathbf{A}_2^\bullet)\big)_{x_2} = 0.$$

The last to point to prove is the existence of a self-duality isomorphism

$$d : \mathscr{D}(\pi_1^* \mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet)[m+n] \to \pi_1^* \mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet.$$

Using (among other facts) the identity provided by [Bo], Theorem V,10.25, we define d as the composition of isomorphisms

$$
\begin{aligned}
(2) \qquad \mathscr{D}(\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet) &= R\mathbf{Hom}^\bullet(\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet, \mathbb{D}_{X_1 \times X_2}^\bullet) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet, \pi_1^* \mathbb{D}_{X_1}^\bullet \overset{L}{\otimes} \pi_2^* \mathbb{D}_{X_2}^\bullet) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^* \mathbf{A}_2^\bullet, \pi_1^* \mathbb{D}_{X_1}^\bullet \overset{L}{\otimes} \pi_2^* \mathbb{D}_{X_2}^\bullet)\big) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet\big(\pi_2^* \mathbf{A}_2^\bullet, \pi_1^*(\mathscr{D}_{X_1} \mathbb{R}_{X_1}) \overset{L}{\otimes} \pi_2^* \mathbb{D}_{X_2}^\bullet)\big)\big) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet\big(\pi_2^* \mathbf{A}_2^\bullet, R\mathbf{Hom}^\bullet(\pi_1^* \mathbb{R}_{X_1}, \pi_2^! \mathbb{D}_{X_2}^\bullet)\big)\big) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet\big(\pi_2^* \mathbf{A}_2^\bullet, R\mathbf{Hom}^\bullet(\mathbb{R}_{X_1 \times X_2}, \pi_2^! \mathbb{D}_{X_2})\big)\big) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^* \mathbf{A}_2^\bullet, \pi_2^! \mathbb{D}_{X_2}^\bullet)\big) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, \pi_2^! R\mathbf{Hom}^\bullet(\mathbf{A}_2^\bullet, \mathbb{D}_{X_2}^\bullet)\big) \\
&\simeq R\mathbf{Hom}^\bullet\big(\pi_1^* \mathbf{A}_1^\bullet, \pi_2^!(\mathscr{D}_{X_2} \mathbf{A}_2^\bullet[n])\big)[-n] \\
&\overset{\simeq}{\to} R\mathbf{Hom}^\bullet(\pi_1^* \mathbf{A}_1^\bullet, \pi_2^! \mathbf{A}_2^\bullet)[-n] \\
&\simeq \pi_1^*\big(\mathscr{D}_{X_1}(\mathbf{A}_1^\bullet)[m]\big) \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet[-m-n] \\
&\overset{\simeq}{\to} \pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet[-m-n].
\end{aligned}
$$

The compatibility of d with the orientation of $X_1 \times X_2$ and with the normalization of $\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet$ follows from the naturality of the construction. $\square$

Observe that the product has the following two properties:

– The switch map induces an isomorphism

$$(X_1, \mathscr{S}_1) \times (X_2, \mathscr{S}_2) \xrightarrow{\simeq} (-1)^{\dim X_1 \cdot \dim X_2} (X_2, \mathscr{S}_2) \times (X_1, \mathscr{S}_1).$$

– There is a canonical isomorphism

$$\big((X_1 \times \mathscr{S}_1) \times (X_2, \mathscr{S}_2)\big) \times (X_3, \mathscr{S}_3) \xrightarrow{\simeq} (X_1, \mathscr{S}_1) \times \big((X_2, \mathscr{S}_2) \times (X_3, \mathscr{S}_3)\big).$$

Remark 2.16. If the second factor is an oriented n-dimensional manifold M (with the trivial H-structure $\mathbb{R}_M[n]$), then this construction becomes much easier and can be explained as follows: according to [KaSc], Prop. 3.3.2, there is a natural equivalence of functors

$$\pi_1^!(-) \simeq \pi_1^*(-) \otimes \pi_2^* \mathbb{D}_M^\bullet.$$

In particular the orientation of M induces a natural equivalence

$$\pi_1^!(-) \simeq \pi_1^*(-) \otimes \pi_2^* \mathbb{D}_M^\bullet \xrightarrow{\simeq} \pi_1^*(-) \otimes \mathbb{R}_{X \times M}[n] \simeq \pi_1^*[n](-)$$

and so we get an isomorphism

$$\pi_1^! \mathbf{A}_1^\bullet \simeq \pi_1^* \mathbf{A}_1^\bullet[n] \simeq \pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbb{R}_M[n].$$

Next we want to restrict our attention to the case $M = \mathbb{R}$. This case is particularly interesting, since, it plays an important role in the definition of the boundary operator for the Mayer-Vietoris sequence of signature homology. Let us assume $\mathbb{R}$ to be endowed with a fixed orientation, and denote by j and j_2 the inclusions $X = X \times \{0\} \hookrightarrow X \times \mathbb{R}$ and $U_2 \hookrightarrow X \times \mathbb{R}$ respectively.

Lemma 2.17. *If $\mathscr{S} = (\mathbf{A}^\bullet, v)$ is any H-structure over $X \times \mathbb{R}$, then there is a unique (up to isomorphism) H-structure $j^! \mathscr{S}$ over X such that*

$$\mathscr{S} \simeq \pi^! j^! \mathscr{S}.$$

Proof. According to the remark above, the orientation of $\mathbb{R}$ induces an isomorphism $\pi^!(-) \simeq \pi^*[1](-)$. Moreover, according to [Ba1], Lemma 5.2, there is a natural identification $\pi^* j^*(-) \simeq \mathrm{Id}$ and consequently it results

$$j^![1](-) \simeq j^! \pi^* j^*[1](-) \simeq j^! \pi^! j^* \simeq j^*(-).$$

The H-structure $j^! \mathscr{S}$ is defined setting:

$$j^! \mathscr{S} := (j^! \mathbf{A}^\bullet, j_2^! v).$$

The orientation of X is here given by the isomorphism $j_2^!(\mathfrak{o})$. As usual, we only have to show that $j^! \mathbf{A}^\bullet$ is a perverse self-dual complex of sheaves.

- (SD1) is clear since a normalization is given by

$$j_2^!(v) : j_2^!(\mathbf{A}^\bullet|_{U_2}) \xrightarrow{\simeq} j_2^!(\mathbb{R}_{U_2})[n] \simeq \mathbb{R}_{U_2 \cap X}[n-1].$$

- (SD2) and (SD3) are consequences of the fact that $j^!\mathbf{A}^\bullet \simeq j^*\mathbf{A}^\bullet[-1]$.

- The self-duality isomorphism is given by the composition

$$\mathscr{D}(j^!\mathbf{A}^\bullet)[n-1] \simeq j^*\mathscr{D}(\mathbf{A}^\bullet)[n-1] \simeq j^!\mathscr{D}(\mathbf{A}^\bullet)[n] \xrightarrow{\simeq} j^!\mathbf{A}^\bullet.$$

The only thing left to show is the isomorphism

$$\pi^! j^! \mathscr{S} \simeq \mathscr{S}$$

but this is just a consequence of the functorial identification

$$\pi^! j^!(-) \simeq \pi^* j^*(-) \simeq \mathrm{Id}. \quad \square$$

2.4. Existence of a small subclass. This subsection is devoted to the construction of a small subcategory $\mathscr{C}_0$ of $\mathscr{C}$ which has the property that every H-stratifold is isomorphic to an H-stratifold in $\mathscr{C}_0$. This construction is based on the following

Lemma 2.18. *For a fixed n-dimensional stratifold X, there is a small subclass $\mathrm{SD}_0(X) \subset \mathrm{SD}(X)$ such that any perverse self-dual complex of sheaves over X is isomorphic to a complex of sheaves in $\mathrm{SD}_0(X)$.*

Proof. We proceed by induction on the codimension of the strata of X. By definition, every perverse self-dual complex of sheaves on U_2 is isomorphic to the trivial complex $\mathbb{R}_{U_2}[n](\mathfrak{o}, \mathbb{R}_{U_2}[n], v)$ and so we can set $\mathrm{SD}_0(U_2) = \mathrm{Const}(U_2)$. Now, suppose to have already defined $\mathrm{SD}_0(U_k)$ and let us show how to construct $\mathrm{SD}(U_{k+1})$.

Let k be even. According to Goresky-MacPherson (see [GM]), the restriction functor i_k^* induces an equivalence of categories $\mathrm{SD}(U_{k+1}) \simeq \mathrm{SD}(U_k)$ and the subclass $\mathrm{SD}_0(U_{k+1})$ can be defined as the preimage under i_k^* of $\mathrm{SD}_0(U_k)$.

Now let k be odd. According to Banagl (see [Ba1]), there is an equivalence of categories

$$\mathrm{SD}(U_{k+1}) \simeq \mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k).$$

By inductive assumption, there is a small class $\mathrm{SD}_0(U_k) \subset \mathrm{SD}(U_k)$ so that every complex $\mathbf{A}^\bullet \in \mathrm{SD}(U_k)$ is isomorphic to a complex in $\mathrm{SD}_0(U_k)$. Let $\mathrm{Lag}_0(U_{k+1} - U_k)$ be the subclass of $\mathrm{Lag}(U_{k+1} - U_k)$ defined setting:

$$\mathrm{Lag}_0(U_{k+1} - U_k) := \{\mathscr{L} \subset \mathscr{O}(\mathbf{A}^\bullet) \mid \mathbf{A}^\bullet \in \mathrm{SD}_0(U_k), \mathscr{L} \in \mathrm{Lag}(U_{k+1} - U_k)\}.$$

In other words an element of $\mathrm{Lag}_0(U_{k+1} - U_k)$ is a Lagrangian subsheaf of $\mathscr{O}(\mathbf{A}^\bullet)$, for some $\mathbf{A}^\bullet \in \mathrm{SD}_0(U_k)$. The class $\mathrm{Lag}_0(U_{k+1} - U_k)$ is by construction a set and, moreover, every Lagrangian structure is isomorphic to an element of $\mathrm{Lag}_0(U_{k+1} - U_k)$. In fact, if $(\mathscr{L}, \phi)$

is a Lagrangian structure over $\mathbf{A}^\bullet \in \mathrm{SD}(U_k)$, then there exist a complex $\mathbf{B}^\bullet \in \mathrm{SD}_0(U_k)$ and an isomorphism $\alpha : \mathbf{A}^\bullet \xrightarrow{\simeq} \mathbf{B}^\bullet$. The Lagrangian structure $(\mathscr{L}, \phi)$ is now isomorphic to $\big(\mathscr{L}, \mathcal{O}(\alpha) \circ \phi\big)$ and consequently, since the map $\mathcal{O}(\alpha) \circ \phi : \mathscr{L} \to \mathcal{O}(\mathbf{B}^\bullet)$ induces an injection in cohomology, $\mathscr{L}$ can be identified up to isomorphism with its image under $\mathcal{O}(\alpha) \circ \phi$ (here we are using the fact that the cohomology of $\mathscr{L}$ is concentrated in degree s, and that for this reason $\mathscr{L}$ is canonically isomorphic to $\mathbf{H}^s(\mathscr{L})[-s]$). Finally, $\mathrm{SD}_0(U_{k+1})$ is defined as the set of all complexes of sheaves of the form $\mathbf{A}^\bullet \boxplus \mathscr{L}$ for $\mathbf{A}^\bullet \in \mathrm{SD}_0(U_k)$ and $\mathscr{L} \in \mathrm{Lag}_0(U_{k+1} - U_k)$. $\square$

A consequence of this result is the

Proposition 2.19. *There is a small subcategory $\mathscr{C}_0 \subset \mathscr{C}$ of the category of H-stratifolds so that every H-stratifold is isomorphic to an element of $\mathscr{C}_0$.*

Proof. Let $\mathscr{C}_0$ be the class defined by

$$\mathscr{C}_0 := \{(X, \mathscr{S}) \mid X \in \mathscr{K}_0, \mathscr{S} = (\mathbf{A}^\bullet, v) \text{ with } \mathbf{A}^\bullet \in \mathrm{SD}_0(X)\}$$

where $\mathscr{K}_0$ is a small subclass of the class of all stratifolds, such that every stratifold is isomorphic to an object in $\mathscr{K}_0$ (in the case of compact stratifolds this can be proved by showing that a compact stratifold can be embedded in some Euclidean space).

Let $(X, \mathscr{S})$ be any H-stratifold and take an isomorphism $\varphi : Y \xrightarrow{\simeq} X$ with $Y \in \mathscr{K}_0$. The pull-back construction provides an H-structure $\varphi^*\mathscr{S}$ over Y with the property that $(Y, \varphi^*\mathscr{S})$ is isomorphic to $(X, \mathscr{S})$ as an H-stratifold. Now, according to Lemma 2.18, there is an isomorphism $\alpha : \mathbf{B}^\bullet \xrightarrow{\simeq} \varphi^*\mathbf{A}^\bullet$ with $\mathbf{B}^\bullet \in \mathrm{SD}_0(Y)$ and in particular it results

$$\varphi^*\mathscr{S} \simeq \mathscr{T} := \big(\mathbf{B}^\bullet, (\varphi|_{U_2})^*(v) \circ (\alpha|_{U_2})\big).$$

Collecting these facts together, we get an isomorphism

$$(X, \mathscr{S}) \simeq (Y, \varphi^*\mathscr{S}) \simeq (Y, \mathscr{T}) \in \mathscr{C}_0$$

and so the only thing left to show is that the class $\mathscr{C}_0$ is small, but this is clear since the orientations of a stratifold are a set, and the same holds for the class of all normalizations of a fixed complex. $\square$

2.5. Collared *H*-stratifolds. In this subsection we introduce the notion of an H-stratifold with boundary or, more precisely, of a collared H-stratifold.

Let $(X, \partial X)$ be a pair of spaces with ∂X closed in X, and suppose that $(\mathring{X}, \mathscr{S})$ and $(\partial X, \partial\mathscr{S})$ are two H-stratifolds of dimension n and $n-1$ respectively. Moreover, denote by i and j the inclusions in X of $\mathring{X}$ and ∂X, and by π the projection $\partial X \times (0, +\infty) \to \partial X$.

Definition 2.20. A collar of ∂X is a pair (c, φ) where

– $c : V \to U$ is a collar of ∂X as a topological stratifold;

– φ is an isomorphism of H-stratifolds

$$\varphi : \left(V - \partial X \times \{0\}, (\pi^! \partial \mathscr{S})|_{V - \partial X \times \{0\}}\right) \xrightarrow{\simeq} (U - \partial X, \mathscr{S}|_{U - \partial X})$$

whose first component φ_1 is equal to $c|_{V - \partial X \times \{0\}}$. Here $\pi^! \partial \mathscr{S}$ is the product structure on $\partial X \times (0, +\infty)$.

Two collars $(c, \varphi) : V \to U$ and $(c', \varphi') : V' \to U'$ are said to be equivalent if there is an open subset $V'' \subset V \cap V'$, such that $(c, \varphi)|_{V'} = (c, \varphi')|_{V''}$. An equivalence class of collars is called a germ of collars. If ∂X is compact, then it is possible to assume the collar to be of constant length, that is to say of the form

$$(c, \varphi) : \partial X \times [0, +\varepsilon) \to U.$$

Definition 2.21. A collared H-stratifold is a pair of spaces $(X, \partial X)$, where $(\mathring{X}, \mathscr{S})$ is an n-dimensional H-stratifold, ∂X is a closed subspace and an $(n-1)$-dimensional H-stratifold with H-structure $\partial \mathscr{S}$, together with a germ of collars $[(c, \varphi)]$. The H-stratifold $(\partial X, \partial \mathscr{S})$ is called the boundary of $(X, \partial X)$.

Example 2.22. Let $(X, \mathscr{S})$ be an H-stratifold with $\mathscr{S} = (\mathbf{A}^\bullet, v)$.

(1) The product $X \times [0, 1]$ is a collared H-stratifold with boundary

$$(X, \mathscr{S}) + (-X, \mathscr{S}),$$

where $-X$ denotes the stratifold obtained reversing the orientation of X and the H-structure on $X \times (0, 1)$ is given by the product structure $\pi^! \mathscr{S}$ described in subsection 3.3.3.

(2) Let i denote the inclusion of $X \times (0, 1)$ in CX. *If* there exists an H-structure $\mathscr{T}$ on CX so that $i^* \mathscr{T}$ is isomorphic to $\pi^! \mathscr{S}$, then $(CX, \mathscr{T})$ is a collared H-stratifold whose boundary is isomorphic to $(X, \mathscr{S})$. We will see in the next subsection under which conditions such an H-structure $\mathscr{T}$ exists.

The H-structure on the boundary of a collared H-stratifold X can be deduced directly from the H-structure on the interior of X, as we are going to show. In order to simplify the notation, we assume the collar to be of constant length, but the same argument applies in the general case. Using the collar we can restrict our attention to a space of the form $X \times [0, +\varepsilon)$ and we denote by i, j, π the maps indicated in the following diagram:

$$\begin{tikzcd}
X \times (0, +\varepsilon) \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & X \times [0, +\varepsilon) \\
& X. \arrow[ur, hook, "j = j_0"'] &
\end{tikzcd}$$

Furthermore let p denote the projection $X \times [0, +\varepsilon) \to X$. As an easy consequence of the Vietoris-Begle theorem one has the following identity (see [Bo], V,10.22).

Lemma 2.23. *For any complex $\mathbf{A}^\bullet \in D^b(X)$ it holds*

$$Ri_* \pi^* \mathbf{A}^\bullet \simeq p^* \mathbf{A}^\bullet.$$

In particular it follows that there is a natural equivalence

$$j^* Ri_* \pi^* \mathbf{A}^\bullet \simeq \mathbf{A}^\bullet.$$

Now, it is showed in [Ba1], Lemma 4.1, that there is an equivalence of functors

$$j^* Ri_* \pi^*(-) \simeq j^! Ri_! \pi^*(-)[1]$$

and thus, for any complex of sheaves $\mathbf{A}^\bullet \in D^b(X)$, we get an isomorphism

$$\mathbf{A}^\bullet \xrightarrow{\simeq} j^! Ri_! \pi^! \mathbf{A}^\bullet.$$

Corollary 2.24. *Let X be a collared H-stratifold with boundary $(\partial X, \partial \mathscr{S})$ and denote by i and j the inclusions of $\mathring{X}$ and of ∂X in X. If $\mathscr{S} = (\mathbf{A}^\bullet, v)$ is the H-structure on $\mathring{X}$, then there is an isomorphism*

$$(\partial X, \partial \mathscr{S}) \simeq (\partial X, \delta \mathscr{S})$$

where $\delta \mathscr{S} = \big(j^! Ri_! \mathbf{A}^\bullet, j_2^! Ri_{2!}(v)\big)$ is the H-structure on the boundary defined in [Ba1], *Section* 4.2.

Observe that the product of two H-stratifolds defined in subsection 2.3 can be extended to the case when one of the factors is a collared H-stratifold.

The final part of this subsection is devoted to the gluing of H-stratifolds along the boundary. Let X be an n-dimensional oriented stratifold, Y_1 and Y_2 be two open subsets such that $X = Y_1 \cup Y_2$ and assume Y_1 and Y_2 endowed with the induced orientations.

Lemma 2.25. *Let $\mathscr{S}_1 = (\mathbf{B}_1^\bullet, v_1)$ and $\mathscr{S}_2 = (\mathbf{B}_2^\bullet, v_2)$ be two H-structures over Y_1 and Y_2 and furthermore suppose that there is an isomorphism of H-stratifolds*

$$\varphi : (Y_1 \cap Y_2, \mathscr{S}_1|_{Y_1 \cap Y_2}) \to (Y_1 \cap Y_2, \mathscr{S}_2|_{Y_1 \cap Y_2}).$$

Under these assumptions there is up to isomorphism a unique H-structure $\mathscr{S} = (\mathbf{A}^\bullet, v)$ over X together with isomorphisms $\psi_j : \mathscr{S}|_{Y_j} \to \mathscr{S}_j$ which make the diagram

$$\begin{tikzcd} \mathscr{S}_1|_{Y_1\cap Y_2} \arrow[rr, "\varphi"] & & \mathscr{S}_2|_{Y_1\cap Y_2} \\ & \mathscr{S}|_{Y_1\cap Y_2} \arrow[ul, "\psi_1|"] \arrow[ur, "\psi_2|"'] & \end{tikzcd}$$

commute.

Proof. An H-structure consists essentially of a perverse self-dual complex of sheaves and we will show how to define such a complex of sheaves by constructing inductively a sequence of complexes $\mathbf{A}_k^\bullet \in \mathrm{SD}(U_k)$ together with isomorphisms

$$(\psi_j)_{U_k} : \mathbf{A}_k^\bullet|_{U_k \cap Y_j} \xrightarrow{\simeq} \mathbf{B}_j^\bullet|_{U_k(Y_j)}.$$

Let us write ψ_j instead of $(\psi_j)_{U_k}$ in order to simplify the notation.

For $k = 2$, let us set

$$\mathbf{A}_2^\bullet := \mathbb{R}_{U_2}[n] \in \mathrm{SD}(U_2).$$

The isomorphisms ψ_1 and ψ_2 can be easily defined setting $\psi_j := \nu_j^{-1}$ and the commutativity of the diagram

$$\begin{tikzcd} \mathbf{B}_1^\bullet|_{U_2(Y_1 \cap Y_2)} \arrow[rr, "\varphi|"] & & \mathbf{B}_2^\bullet|_{U_2(Y_1 \cap Y_2)} \\ & \mathbf{A}^\bullet|_{Y_1 \cap Y_2} \arrow[ul, "\psi_1|"] \arrow[ur, "\psi_2|"'] & \end{tikzcd}$$

is now just a consequence of the definition of isomorphism of H-structures.

Now, assume to have already defined $\mathbf{A}_k^\bullet \in \mathrm{SD}(U_k)$, ψ_1, and ψ_2, and consider the inclusions

$$i : U_k \hookrightarrow U_{k+1},$$
$$i' : U_k(Y_1 \cap Y_2) \hookrightarrow U_{k+1}(Y_1 \cap Y_2),$$
$$j : \Sigma = U_{k+1} - U_k \hookrightarrow U_{k+1}.$$

In order to define $\mathbf{A}_{k+1}^\bullet$ we have to distinguish two cases.

- For k even, we set

$$\mathbf{A}_{k+1}^\bullet := \tau_{\leqq \bar{m}(k)-n}\, Ri_*\, \mathbf{A}_k^\bullet \in \mathrm{SD}(U_{k+1}).$$

The functor $\tau_{\leqq \bar{m}(k)-n}\, Ri_*(-)$ is by [GM] the inverse of $i^* : \mathrm{SD}(U_{k+1}) \to \mathrm{SD}(U_k)$, and consequently $\mathbf{A}_{k+1}^\bullet$ is a perverse self-dual complex of sheaves over U_{k+1}. The isomorphisms $(\psi_j)_{U_{k+1}}$ are defined through the composition

$$\mathbf{A}_{k+1}^\bullet|_{Y_j} = (\tau_{\leqq \bar{m}(k)-n}\, Ri_*\, \mathbf{B}^\bullet)|_{U_{k+1}(Y_j)} \xrightarrow[\simeq]{\tau_{\leqq \bar{m}(k)-n}\, Ri_*(\psi_j)} \mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)}.$$

- Consider now the case k odd. By Banagl's main result (see theorem 2.8), there are two natural isomorphisms

$$\mathbf{B}_1^\bullet|_{U_{k+1}(Y_1)} \simeq \mathbf{B}_1^\bullet|_{U_k(Y_1)} \boxplus \Lambda(\mathbf{B}_1^\bullet) \quad \text{and} \quad \mathbf{B}_2^\bullet|_{U_{k+1}(Y_2)} \simeq \mathbf{B}_2^\bullet|_{U_k(Y_2)} \boxplus \Lambda(\mathbf{B}_2^\bullet)$$

where $\Lambda(\mathbf{B}_j^\bullet) = (\mathscr{L}_j, \phi_j)$ is the "canonical" Lagrangian structure over $\mathbf{B}_j^\bullet|_{U_k}$. Identifying $\mathbf{B}_j^\bullet|_{U_k(Y_j)}$ with $\mathbf{A}^\bullet|_{U_k(Y_j)}$ through the isomorphism ψ_j, one has

$$\mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)} \simeq (\mathbf{A}^\bullet|_{U_k(Y_j)}) \boxplus \Lambda(\mathbf{B}_j^\bullet).$$

On the other hand, restricting φ to $U_{k+1}(Y_1 \cap Y_2)$ one obtains an isomorphism

$$\mathbf{B}_1^\bullet|_{U_k(Y_1 \cap Y_2)} \boxplus \Lambda(\mathbf{B}_1^\bullet)|_{U_{k+1}(Y_1 \cap Y_2)} \xrightarrow[\simeq]{i'^*\varphi \boxplus \Lambda(\varphi)} \mathbf{B}_2^\bullet|_{U_k(Y_1 \cap Y_2)} \boxplus \Lambda(\mathbf{B}_2^\bullet)|_{U_{k+1}(Y_1 \cap Y_2)}.$$

By the inductive assumption, the diagram

$$\begin{tikzcd} \mathbf{B}_1^\bullet|_{U_k(Y_1\cap Y_2)} \arrow[rr, "i'^*\varphi"] & & \mathbf{B}_2^\bullet|_{U_k(Y_1\cap Y_2)} \\ & \mathbf{A}_k^\bullet|_{U_k(Y_1\cap Y_2)} \arrow[ul, "\psi_1|"] \arrow[ur, "\psi_2|"'] & \end{tikzcd}$$

commutes, and this allows to identify $\varphi|_{U_{k+1}(Y_1\cap Y_2)}$ with the isomorphism

$$\begin{tikzcd} \mathscr{L}_1|_{U_{k+1}(Y_1\cap Y_2)} \arrow[r, "\phi_1"] \arrow[d, "\alpha"', "\simeq"] & \mathcal{O}(\mathbf{A}_k^\bullet|_{Y_1\cap Y_2}) \arrow[d, "1"] \\ \mathscr{L}_2|_{U_{k+1}(Y_1\cap Y_2)} \arrow[r, "\phi_2"'] & \mathcal{O}(\mathbf{A}_k^\bullet|_{Y_1\cap Y_2}) \end{tikzcd}$$

where α is given by $\Lambda(\alpha)$.

If we set $\mathbf{H} = \mathbf{H}^s j^*\mathcal{O}(\mathbf{A}_k^\bullet)$ and $\mathbf{E}_j = \mathbf{H}^s j^*(\mathscr{L}_j)|_{U_{k+1}(Y_j)}$, then we get a diagram

$$\begin{tikzcd} \mathbf{E}_1|_{Y_1\cap Y_2} \arrow[r, "\gamma_1"] \arrow[d, "\beta"', "\simeq"] & \mathbf{H}|_{Y_1\cap Y_2} \arrow[d, "1"] \\ \mathbf{E}_2|_{Y_1\cap Y_2} \arrow[r, "\gamma_2"'] & \mathbf{H}|_{Y_1\cap Y_2}. \end{tikzcd}$$

Now, we can glue the sheaves $\mathbf{E}_1$ and $\mathbf{E}_2$ through β and we obtain thus a sheaf $\mathbf{E}$; the maps γ_1 and γ_2 extend to an injection $\gamma : \mathbf{E} \hookrightarrow \mathbf{H}$. The image $\gamma(\mathbf{E})$ is a Lagrangian subsheaf and, by lemma 2.6, this determines a Lagrangian structure $(\mathscr{L}, \phi)$ over $\mathbf{A}_k^\bullet$ such that, for $j = 1, 2$, there is an isomorphism $(\mathscr{L}, \phi)|_{U_{k+1}(Y_j)} \simeq \Lambda(\mathbf{B}_j^\bullet)$. The complex $\mathbf{A}_{k+1}^\bullet$ is finally defined by setting

$$\mathbf{A}_{k+1}^\bullet := \mathbf{A}_k^\bullet \boxplus (\mathscr{L}, \phi).$$

The isomorphisms

$$(\psi_j)_{U_{k+1}} : \mathbf{A}_{k+1}^\bullet|_{Y_j} \xrightarrow{\simeq} \mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)}$$

are defined through the compositions

$$\mathbf{A}_{k+1}^\bullet|_{Y_j} \simeq \mathbf{A}_k^\bullet|_{U_k(Y_j)} \boxplus (\mathscr{L}, \phi)|_{U_{k+1}(Y_j)} \simeq \mathbf{B}_j^\bullet|_{U_k(Y_j)} \boxplus \Lambda(\mathbf{B}_j) \simeq \mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)}. \quad \square$$

The preceding lemma allows to prove the following result.

Proposition 2.26. *Let $(X, \mathscr{S})$ and $(X', \mathscr{S}')$ be two H-stratifolds, and suppose that there is an orientation-reversing isomorphism*

$$\varphi : (\partial X, \partial\mathscr{S}) \xrightarrow{\simeq} (\partial X', \partial\mathscr{S}').$$

Then there is up to isomorphism a unique H-structure over $X \underset{\partial X \equiv \partial X'}{\cup} X'$ which restricts to $\mathscr{S}$ over X and to $\mathscr{S}'$ over X'.

Proof. The stratifold $X \cup X'$ is naturally decomposed in the union of the two open sets

$$Y_1 := \mathring{X} \cup \mathring{X}',$$
$$Y_2 := \partial X \times (-\varepsilon, +\varepsilon).$$

Both Y_1 and Y_2 are naturally endowed with an H-structure and these two structures are isomorphic if restricted on the intersection. Applying the previous lemma, we can thus obtain an H-structure over $X \cup X'$ which extends $\mathscr{S}$ and $\mathscr{S}'$. $\square$

3. Signature homology

This section is devoted to the construction of the Signature homology functor $\mathrm{Sig}_*(-)$ and to the investigation of some of its properties. In particular we show that $\mathrm{Sig}_*(-)$ is a multiplicative homology theory and we compute its coefficients.

3.1. The functor $\mathrm{Sig}_*(-)$. In order to simplify the notation let us indicate an H-stratifold by its underlying stratifold. An n-dimensional singular H-stratifold over a topological space X is a pair (S, f) where S is an n-dimensional closed H-stratifold and

$$f : S \to X$$

is a continuous map. We denote by $\mathscr{C}^n(X)$ the class of all n-dimensional singular H-stratifolds over X. Two singular H-stratifolds $(S, f), (S', f') \in \mathscr{C}^n(X)$ are called isomorphic if there is an isomorphism of H-stratifolds

$$\varphi : S \xrightarrow{\simeq} S'$$

such that, if φ_1 denotes the first component of φ, then the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi_1"] \arrow[d, "f"'] & S' \arrow[dl, "f'"] \\ X. \end{tikzcd}$$

If (S, f) is a singular H-stratifold, then we denote by $-(S, f)$ the singular H-stratifold $(-S, f)$, where $-S$ is the H-stratifold obtained reversing the orientation of S.

Definition 3.1. Two singular H-stratifolds $(S, f), (S', f') \in \mathscr{C}^n(X)$ are called bordant, if there exists a pair (T, g) where T is a collared compact H-stratifold and g is a map $T \to X$ so that

$$(\partial T, g|_{\partial T}) \simeq (S, f) + (-S', f').$$

Using the standard argument one sees that the bordism of H-stratifolds is an equivalence relation (observe that for transitivity one needs to glue H-stratifolds along the boundary as explained in the last part of subsection 2.5) and let us denote by $[S, f]$ the bordism class of (S, f). Furthermore we denote by $\mathrm{Sig}_n(X)$ the quotient set of $\mathscr{C}^n(X)$ under bordism (due to proposition 2.19 there are no set-theoretical problems here).

Definition 3.2. The Abelian group $\mathrm{Sig}_n(X)$ with the sum defined by the disjoint union of bordism classes is called the n-th *Signature homology* group of X.

If $g : X \to Y$ is any continuous map, then we can associate to g a group homomorphism

$$g_* : \mathrm{Sig}_n(X) \to \mathrm{Sig}_n(Y),$$
$$[S, f] \mapsto [S, g \circ f].$$

In particular, this assignment allows to define a multiplicative functor

$$\mathrm{Sig}_*(-) : \mathrm{Top} \to \mathrm{Ab}^{\mathbb{Z}},$$
$$X \mapsto \mathrm{Sig}_*(X) := \bigoplus_n \mathrm{Sig}_n(X)$$

where $\mathrm{Ab}^{\mathbb{Z}}$ denotes the category of graded Abelian groups. Furthermore, since every oriented topological manifold can be naturally realized as an H-stratifold we also obtain a multiplicative functor

$$\Omega_*^{TOP}(-) \to \mathrm{Sig}_*(-).$$

Now, we want to show that $\mathrm{Sig}_*(-)$ is a multiplicative homology theory (here we are following a general strategy developed by Matthias Kreck in [Kr]).

The proof of homotopy invariance of $\mathrm{Sig}_*(-)$ is identical to the usual one and will therefore be omitted. So we only have to consider the Mayer-Vietoris sequence.

Proposition 3.3. *Let U and V be open subsets of a space X. Then there is an exact sequence of Abelian groups*

$$\cdots \to \mathrm{Sig}_n(U \cap V) \to \mathrm{Sig}_n(U) \oplus \mathrm{Sig}_n(V) \to \mathrm{Sig}_n(U \cup V) \to \mathrm{Sig}_{n-1}(U \cap V) \to \cdots.$$

Proof. All morphisms except the boundary operator are clear by functoriality. To define the boundary operator, consider a singular H-stratifold $(S, f) \in \mathscr{C}^n(X)$. The subspaces $A_S := f^{-1}(X - V)$ and $B_S := f^{-1}(X - U)$ are closed and disjoint in S, and for this reason there is a morphism

$$\rho : S \to \mathbb{R}$$

with $\rho(A_S) = +1$ and $\rho(B_S) = -1$.

Applying the transversality theorem 1.19, one gets a homotopy of ρ relative to $A_S \cup B_S$

$$h : S \times [0, 1] \to \mathbb{R}$$

with the property that 0 is a regular value of $\sigma = h(-, 1)$. The subset $Z := \sigma^{-1}(0) \subset S$ is an $(n-1)$-dimensional stratifold and there is a bicollar

$$i : Z \times (-\varepsilon, +\varepsilon) \hookrightarrow S$$

such that $\sigma\big(i(x,t)\big) = t$. We indicate by $j : Z \to Z \times (-\varepsilon, +\varepsilon)$ the inclusion $x \mapsto (x, 0)$ and by $\pi : Z \times (-\varepsilon, +\varepsilon) \to Z$ the projection on the first factor.

Since i is an open embedding, the H-structure of S can be pulled back to an H-structure $i^*\mathscr{S}$ over $Z \times (-\varepsilon, +\varepsilon)$ and, by lemma 2.17, there exists a unique H-structure $\mathscr{T} = j^! i^* \mathscr{S}$ over Z so that it holds

$$\pi^! \mathscr{T} \simeq i^* \mathscr{S}.$$

Finally, if Z denotes the H-stratifold $(Z, \mathscr{T})$, we define

$$\begin{aligned} d : \mathrm{Sig}_n(U \cup V) &\to \mathrm{Sig}_{n-1}(U \cap V), \\ [S, f] &\mapsto [Z, f|_Z]. \end{aligned}$$

The proof that d is well defined as well as that of the exactness of the Mayer-Vietoris sequence are analogous to the ordinary ones and therefore left to the reader. $\square$

Putting together the considerations above we obtain the

Theorem 3.4. *The functor* $\mathrm{Sig}_*(-)$ *is a multiplicative homology theory and there is a natural transformation of multiplicative homology theories*

$$\Omega_*^{TOP}(-) \to \mathrm{Sig}_*(-).$$

3.2. The coefficients of $\mathrm{Sig}_*(-)$. In this subsection we show that the signature of an H-stratifold defined by Banagl allows to construct a ring isomorphism

$$\mathrm{Sig}_*(\mathrm{pt}) \simeq \mathbb{Z}[t]$$

where the degree of the variable t is equal to 4.

Let S be a compact $(4k)$-dimensional H-stratifold and denote by $\mathscr{S} = (\mathbf{A}^\bullet, v)$ the H-structure of S. The self-duality isomorphism $d : \mathscr{D}\mathbf{A}^\bullet[4k] \xrightarrow{\simeq} \mathbf{A}^\bullet$ induces by Verdier duality an isomorphism in hypercohomology

$$\mathscr{H}^{-2k}(S, \mathbf{A}^\bullet) \simeq \mathscr{H}^{-2k}(S, \mathscr{D}\mathbf{A}^\bullet[4k]) \simeq \mathscr{H}^{2k}(S, \mathscr{D}\mathbf{A}^\bullet) \simeq \mathrm{Hom}\big(\mathscr{H}^{-2k}(S, \mathbf{A}^\bullet), \mathbb{R}\big)$$

or, equivalently, a non-degenerate symmetric bilinear form

$$\mathscr{H}^{-2k}(S, \mathbf{A}^\bullet) \otimes \mathscr{H}^{-2k}(X, \mathbf{A}^\bullet) \to \mathbb{R}.$$

Following Banagl, we call the index of this pairing the *signature* of S. If the dimension of S is not divisible by 4, one sets $\mathrm{sig}(S) = 0$.

The following three properties of the signature can be easily deduced from the definitions:

(1) $\operatorname{sig}(S + S') = \operatorname{sig}(S) + \operatorname{sig}(S')$;

(2) $\operatorname{sig}(-S) = -\operatorname{sig}(S)$;

(3) if S and S' are isomorphic H-stratifolds, it results

$$\operatorname{sig}(S) = \operatorname{sig}(S').$$

Moreover, the signature is multiplicative with respect to the product of H-stratifolds.

Proposition 3.5. *If S_1 and S_2 are two H-stratifolds then it holds*

$$\operatorname{sig}(S_1 \times S_2) = \operatorname{sig}(S_1) \cdot \operatorname{sig}(S_2).$$

Proof. Let $\mathscr{S}_1 = (\mathbf{A}_1^\bullet, v_1)$ and $\mathscr{S}_2 = (\mathbf{A}_2^\bullet, v_2)$ denote the H-structure on S_1 and S_2 respectively. It follows from [Bo], Theorem V,10.19, that there is an isomorphism of complexes of real vector spaces

$$\Gamma(S_1; \mathbf{A}_1^\bullet) \overset{L}{\otimes} \Gamma(S_2; \mathbf{A}_2^\bullet) \simeq \Gamma(S_1 \times S_2; \pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet),$$

and so, by the algebraic Künneth formula, there is an isomorphism

$$\mathscr{H}^k(S_1 \times S_2, \pi_1^* \mathbf{A}_1^\bullet \otimes \pi_2^* \mathbf{A}_2^\bullet) \simeq \bigoplus_{i+j=k} \mathscr{H}^i(S_1, \mathbf{A}_1^\bullet) \otimes \mathscr{H}^j(S_2; \mathbf{A}_2^\bullet).$$

Finally, one can apply the usual argument used to show the multiplicativity of the signature of a manifold. □

Another fundamental property of the signature is given by the next proposition.

Proposition 3.6. *If S is a $(4k+1)$-dimensional H-stratifold with boundary, then the signature of ∂S is zero.*

Proof. Let $\mathscr{S}$ and $\partial\mathscr{S}$ denote the H-structure of $\mathring{S}$ and ∂S respectively. By corollary 2.24, there is an isomorphism of H-stratifolds

$$(\partial S, \partial\mathscr{S}) \simeq (\partial S, \delta\mathscr{S})$$

where $\delta\mathscr{S}$ is the H-structure defined in [Ba1], Section 4.2. In particular, since $\operatorname{sig}(\partial S, \delta\mathscr{S})$ is zero by [Ba1], Corollary 4.1, it results

$$\operatorname{sig}(\partial S, \partial\mathscr{S}) = \operatorname{sig}(\partial S, \delta\mathscr{S}) = 0. \quad \square$$

The proposition above implies that the signature can be used to define a homomorphism of graded rings

$$\gamma : \operatorname{Sig}_*(\{\mathrm{pt}\}) \to \mathbb{Z}[t],$$
$$[S] \mapsto \operatorname{sig}(S) \cdot t^{\dim S/4}.$$

Remark 3.7. If M is a $4k$-dimensional compact oriented manifold, then it results

$$\mathscr{H}^{-2k}(X, \mathbb{R}_M[4k]) = H^{2k}(X, \mathbb{R})$$

and in particular the signature of M as an oriented manifold equals the signature of $(M, \mathbb{R}_M[4k], \mathrm{Id})$ as an H-stratifold.

Proposition 3.8. *The ring homomorphism γ is an isomorphism.*

Proof. The map γ is evidently surjective and therefore we only have to prove its injectivity. Since the case $n = 0$ is trivial, we can assume the dimension of S to be strictly positive. The general strategy will be to show that, if S is an n-dimensional H-stratifold with $\gamma(S) = 0$, then there exists an H-structure on the cone over S, so that $S \simeq \partial(CS)$. Let v be the vertex point of CS, and denote by π, i and j the maps indicated in the diagram:

$$\begin{tikzcd} S \times (0,1) \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathring{C}S & \{v\} \arrow[l, hook', "j"'] \\ S. & & \end{tikzcd}$$

Using the notation introduced in subsection 2.1, one has

$$U_{n+1} = S \times (0,1) \quad \text{and} \quad U_{n+2} = \mathring{C}S.$$

We have already seen in subsection 3.5 that the problem is to extend the product structure

$$(\pi^! \mathbf{A}^\bullet, \pi^! v)$$

over $S \times (0,1)$ to an H-structure over $\mathring{C}S$. If n is odd, this can always be done applying theorem 2.4 and so it is enough to consider the case $n = 2m$.

Now, since we have supposed $\mathrm{sig}(S) = 0$, it follows that there exists a Lagrangian subsheaf

$$L \subset \mathscr{H}^{-m}(S, \mathbf{A}^\bullet)$$

and we have to show how such a Lagrangian subspace gives rise to a Lagrangian structure on $\pi^! \mathbf{A}^\bullet$. Note that a Lagrangian subspace always exists if $4 \nmid n$.

As we have seen in lemma 2.6, a Lagrangian structure over $\pi^! \mathbf{A}^\bullet$ is the same as a Lagrangian subsheaf

$$\mathbf{H} := \mathbf{H}^{\bar{n}(n+1)-(n+1)}(j^* Ri_* \pi^! \mathbf{A}^\bullet).$$

In our case, however, $\mathbf{H}$ is a sheaf over $\{v\}$ and so it can be identified with the vector space $\mathbf{H}_v$. Since we have assumed $n = 2m$, it results

$$\bar{n}(n+1) - (n+1) = \left[\frac{2m+1-1}{2}\right] - 2m - 1 = -m - 1.$$

Substituting this expression and the canonical identification $\pi^! \simeq \pi^*[1]$, we can also write

$$\mathbf{H} \simeq \mathbf{H}^{-m-1}(j^* Ri_* \pi^* \mathbf{A}^\bullet[1]) \simeq \mathbf{H}^{-m}(j^* Ri_* \pi^* \mathbf{A}^\bullet).$$

By slightly adapting the proof of lemma 2.23 one can easily show that, if $\mathbf{A}^\bullet \xrightarrow{\simeq} \mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$, then the resolution

$$\pi^*\mathbf{A}^\bullet \xrightarrow{\simeq} \pi^*\mathbf{I}^\bullet$$

can be used to compute Ri_*. Using this fact, the vector space $\mathbf{H}$ can be identified with the stalk at v of the sheaf $\mathbf{H}^{-m}(i_*\pi^*\mathbf{I}^\bullet)$.

On the other hand, the stalk at v of $\mathbf{H}^{-m}(i_*\pi^*\mathbf{I}^\bullet)$ is isomorphic to the $(-m)$-th cohomology of the complex of vector spaces $(i_*\pi^*\mathbf{I}^\bullet)_v$. The latter is by definition equal to

$$\lim_{U \ni v} \Gamma(U, i_*\pi^*\mathbf{I}^\bullet) = \lim_{U \ni v} \Gamma\big(i^{-1}(U), \pi^*\mathbf{I}^\bullet\big) \simeq \lim_{\varepsilon \to 0} \Gamma\big(S \times (0,\varepsilon), \pi^*\mathbf{I}^\bullet\big),$$

where the last isomorphism follows from the compactness of S.

Since $\pi^*\mathbf{I}^\bullet$ is constant on the fibres, one has

$$\lim_{\varepsilon \to 0} \Gamma\big(S \times (0,\varepsilon), \pi^*\mathbf{I}^\bullet\big) \simeq \lim_{\varepsilon \to 0} \Gamma(S, \mathbf{I}^\bullet) \simeq \Gamma(S, \mathbf{I}^\bullet).$$

In particular, this computation shows that we can identify $\mathbf{H}$ with the $(-m)$-th cohomology space of the complex $\Gamma(S, \mathbf{I}^\bullet)$ or, in other words, that there is an isomorphism

$$\mathbf{H} \simeq \mathscr{H}^{-m}(S, \mathbf{A}^\bullet).$$

It follows from the definition of the bilinear form on $\mathbf{H}$ (see [Ba1], Lemma 2.4 and [GM], Section 5.2), that the diagram

$$\begin{tikzcd}
\mathbf{H} \otimes \mathbf{H} \arrow[r] \arrow[d] & \mathscr{H}^{-m}(S, \mathbf{A}^\bullet) \otimes \mathscr{H}^{-m}(S, \mathbf{A}^\bullet) \arrow[dl] \\
\mathbb{R} &
\end{tikzcd}$$

commutes. This means that the Lagrangian subspace $L \subset \mathscr{H}^{-m}(S, \mathbf{A}^\bullet)$ induces a Lagrangian subspace $\mathbf{L} \subset \mathbf{H}$ and thus a Lagrangian structure on $\pi^!\mathbf{A}^\bullet$. Finally, by theorem 2.7, there is an H-structure over $\mathring{C}S$ extending the product structure over $S \times (0,1)$ and therefore it results

$$S = \partial(CS). \quad \square$$

4. The Signature fundamental class of a manifold

In the following pages we show how to use Signature homology to construct a characteristic class for closed oriented manifolds, and we explain the connection between the Novikov conjecture and $\mathrm{Sig}_*(-)$.

Let M be an n-dimensional topological oriented closed manifold.

Definition 4.1. The Signature fundamental class of M is by definition the element

$$[M] := [M, \mathrm{Id}] \in \mathrm{Sig}_n(M)$$

where M has the trivial H-structure.

It follows from the definition that the Signature fundamental class is invariant under orientation-preserving homeomorphisms. In order to understand the information carried on by the characteristic class, we will restrict our attention to the case of a smooth manifold.

4.1. Signature homology with rational coefficients. Let us denote by $\Omega_*(-)$ the smooth oriented bordism and by Ω_* its ring of coefficients. Moreover consider the natural transformation

$$\Omega_*(-) \xrightarrow{u} \mathrm{Sig}_*(-)$$

given by regarding an oriented smooth manifold as an H-stratifold with the trivial H-structure and denote by u_* the ring homomorphism induced on the coefficients. The ring $\mathbb{Z}[t]$ is an Ω_*-module with the multiplication induced by the genus

$$\Omega_* \xrightarrow{\tau} \mathbb{Z}[t],$$
$$[M^n] \mapsto \mathrm{sig}(M) \cdot t^{n/4}.$$

It is well known that for any space X there is an isomorphism

$$\Omega_*(X) \otimes \mathbb{Q} \simeq H_*(X; \Omega_* \otimes \mathbb{Q}).$$

In particular the functor $\Omega_*(-) \otimes_\tau \mathbb{Q}[t]$ is isomorphic to singular homology with coefficients in $\mathbb{Q}[t]$ and so it is a homology theory. Now, the product of a singular manifold with an H-stratifold induces a family of natural homomorphisms

$$\Omega_*(X) \otimes_{u_*} \mathrm{Sig}_*(\mathrm{pt}) \to \mathrm{Sig}_*(X),$$
$$[M, f] \otimes_{u_*} [S] \mapsto [M \times S, f \circ \pi_1].$$

After tensoring with $\mathbb{Q}$ and precomposing with

$$1 \otimes (\gamma \otimes \mathbb{Q})^{-1} : \Omega_*(X) \otimes_{u_*} \mathbb{Q}[t] \xrightarrow{\simeq} \Omega_*(X) \otimes_{u_*} \mathrm{Sig}_*(\mathrm{pt}) \otimes \mathbb{Q}$$

we get by the comparison theorem that the transformation

$$\Omega_*(X) \otimes_\tau \mathbb{Q}[t] \to \mathrm{Sig}_*(X) \otimes \mathbb{Q}$$

is an isomorphism.

On the other side Hirzebruch's L-class (see [MS]) allows to define a natural transformation

$$\lambda : \Omega_*(X) \to H_*(X; \mathbb{Q}[t]),$$
$$[M, f] \mapsto f_*\big(\Delta L(M)\big)$$

where Δ is the Poincaré duality isomorphism

$$H^k(M) \to H_{n-k}(M)$$

and

$$L(M) = \sum_{k \geqq 0} L_k(p_1, \dots, p_k) \cdot t^k \in H^0(M; \mathbb{Q}[t])$$

is the total L-class of M.

Passing to rational coefficients one gets an isomorphism

$$\Omega_*(-) \otimes_\tau \mathbb{Q}[t] \xrightarrow{\simeq} H_*(-; \mathbb{Q}[t]),$$

and collecting all these facts together one has the following

Proposition 4.2. *There exists an equivalence*

$$\varphi : \mathrm{Sig}_*(X) \otimes \mathbb{Q} \xrightarrow{\simeq} H_*(X; \mathbb{Q}[t])$$

such that the diagram

$$\begin{tikzcd} \Omega_*(X) \arrow[r, "u \otimes \mathbb{Q}"] \arrow[d, "\lambda"'] & \mathrm{Sig}_*(X) \otimes \mathbb{Q} \arrow[dl, "\varphi"] \\ H_*(X; \mathbb{Q}[t]) & \end{tikzcd}$$

commutes. Here $u \otimes \mathbb{Q}$ denotes the composition of the transformation $u : \Omega_*(-) \to \mathrm{Sig}_*(-)$ defined above with the inclusion $\mathrm{Sig}_*(-) \to \mathrm{Sig}_*(-) \otimes \mathbb{Q}$.

In particular for a smooth oriented n-dimensional manifold M one has

Corollary 4.3. *If M is an n-dimensional smooth oriented manifold M, then it holds*

$$\varphi_*([M, \mathrm{Id}]) = \Delta L(M) \in H_m(M; \mathbb{Q}[t]).$$

Since the rational Pontrjagin classes of a manifold M determine and at the same time are determined by the total L-class of M, one can also interpret this result as follows.

Meta-theorem. *The rational Signature fundamental class of a manifold contains the same information as the rational Pontrjagin classes.*

For example, according to a theorem of Dold and Milnor, the rational Pontrjagin classes of a manifold are not homotopy invariant and so we see that the Signature fundamental class cannot be a homotopy invariant.

4.2. The L-class of an H-stratifold. The construction of the isomorphism φ above can be made more explicit by introducing the homology L-class of an H-stratifold. This procedure is based on Thom's work on the combinatorial invariance of Pontrjagin classes (see [MS]) and has also been analyzed by Banagl in [Ba2].

Let Y be a compact H-stratifold of dimension n and let

$$f :^{n} \to S^{r}$$

be a morphism, with $n - r = 4i$. By a modification of the transversality theorem there is at least one point y of S^r which is a regular value of f. The inverse image $f^{-1}(y)$ is a compact oriented $4i$-dimensional H-stratifold whose signature is independent of y and will be therefore indicated by $\sigma(f)$. The integer $\sigma(f)$ depends only on the homotopy class of f. Furthermore, if $4i < (n-1)/2$, then the correspondence $f \mapsto \sigma(f)$ defines a homomorphism

$$\pi^{r}(Y) \to \mathbb{Z}$$

where $\pi^r(Y)$ denotes the r-th cohomotopy group of Y. According to Serre (see [MS]), the homomorphism

$$\pi^{n-4i}(Y) \to H^{n-4i}(Y)$$

is a rational isomorphism and therefore σ induces a homomorphism

$$H^{n-4i}(Y) \to \mathbb{Q}$$

or equivalently a class $l_i(Y) \in H_{n-4i}(Y;\mathbb{Q})$. Putting these homology classes together one can define the element

$$l(Y) \in H_n(X;\mathbb{Q}[t])$$

which is called the homology L-class of Y (observe that, due to the failure of Poincaré duality, there only exists a *homology* L-class). The class $l(Y)$ allows to re-define the isomorphism φ of the previous section as follows:

$$\begin{gathered} \mathrm{Sig}_*(X) \otimes \mathbb{Q} \to H_*(X;\mathbb{Q}[t]), \\ [Y, f] \mapsto f_* l(Y). \end{gathered}$$

Finally it is important to observe that by a result of Banagl the L-class of an H-stratifold does not depend on the chosen H-structure (see [Ba2]).

4.3. An integral formulation of the Novikov conjecture. In this subsection we want to show that the Novikov conjecture for a group π is equivalent to the homotopy invariance of the rational Signature fundamental class for singular manifolds over $K(\pi,1)$.

Let π be any discrete group, and let us fix any rational cohomology class $x \in H^*\big(K(\pi,1);\mathbb{Q}\big)$. By definition the higher signature sig_x of a singular manifold (M,α) over $K(\pi,1)$ is the rational number

$$\mathrm{sig}_x(M,\alpha) = \langle L(M) \cup \alpha^* x, [M]\rangle = \big\langle x, \alpha_*\big(\Delta L(M)\big)\big\rangle.$$

Moreover recall that the number sig_x is said to be homotopy invariant if for every singular manifold (M,α) and for every orientation-preserving homotopy equivalence $f : N \to M$ it holds

$$\mathrm{sig}_x(M,\alpha) = \mathrm{sig}_x(N, \alpha \circ f).$$

The Novikov conjecture. All higher signatures are homotopy invariant.

Before we come to the announced connection between the Novikov conjecture and the signature fundamental class of a manifold we need to explain what we mean by homotopy invariance of the latter.

Definition 4.4. The Signature fundamental class is homotopy invariant for a group π if for every pair (M,α), and for every orientation-preserving homotopy equivalence $f : N \to M$, it results

$$[M,\alpha] = [N, \alpha \circ f] \in \mathrm{Sig}_n\big(K(\pi,1)\big).$$

This terminology allows to formulate the following

Proposition 4.5. *The Novikov conjecture for a group π is equivalent to the homotopy invariance of the rational Signature fundamental class for π.*

Proof. If (M,α) is a singular manifold over $K(\pi,1)$, then one has

$$\mathrm{sig}_x(M,\alpha) = \big\langle x, \alpha_*\big(\Delta L(M)\big)\big\rangle = \langle x, \varphi_n([M,\alpha])\rangle$$

and thus it is clear that it results

$$\mathrm{sig}_x(M,\alpha) = \mathrm{sig}_x(N, \alpha \circ f)$$

for any $x \in H^*\big(K(\pi,1);\mathbb{Q}\big)$, if and only if

$$[M,\alpha] = [N, \alpha \circ f] \in \mathrm{Sig}_n\big(K(\pi,1)\big) \otimes \mathbb{Q}. \quad \square$$

The proposition above suggests that an integral version of the Novikov conjecture can be obtained requiring the homotopy invariance of the Signature fundamental class.

Integral Novikov problem (M. Kreck). Determine all discrete groups π for which the Signature fundamental class is homotopy invariant.

Unfortunately nothing is known about this generalization of the Novikov conjecture. However, if one replaces the homotopy invariance with the topological invariance then the statement is always true (this follows from the topological invariance of the signature fundamental class).

4.4. Signature homology at odd primes. Applying the Landweber exact functor theorem, we show in this subsection that the $\mathbb{Z}[1/2]$-localization of Signature homology is isomorphic to connective KO-theory.

By the Landweber exact functor theorem (see [La], Example 3.4), the tensor product

$$\Omega_*(-) \otimes_\tau \mathbb{Z}[1/2][t, t^{-1}]$$

is a homology theory and therefore u induces an isomorphism

$$\Omega_*(-) \otimes_\tau \mathbb{Z}[1/2][t, t^{-1}] \xrightarrow{\simeq} \mathrm{Sig}_*(-) \otimes \mathbb{Z}[1/2][t^{-1}].$$

On the other hand, the map of spectra

$$MSpin \to MSO$$

is a $\mathbb{Z}[1/2]$-equivalence and so one can define a map $v : MSO \to KO[1/2]$ through the composition

$$MSO[1/2] \to MSpin[1/2] \to KO[1/2]$$

where the last map is induced by the Atiyah-Bott-Shapiro $MSpin$-orientation of KO-theory. The map v defines a natural transformation

$$\Omega_*(-) \to KO_*(-)[1/2].$$

According to a theorem of Sullivan (see [MM]), the ring homomorphism v_* induced by the transformation v for $X = \{\mathrm{pt}\}$ coincides with the ring homomorphism τ. In particular, it results

$$\Omega_*(-) \otimes_\tau \mathbb{Z}[1/2][t, t^{-1}] \simeq \Omega_*(-) \otimes_{v_*} \big(KO_*(\mathrm{pt})[1/2]\big).$$

and so, applying again the Landweber exact functor theorem, we get an isomorphism

$$\Omega_*(-) \otimes_{v_*} KO_*(\mathrm{pt})[1/2] \xrightarrow{\simeq} KO_*(-)[1/2].$$

The diagram

$$\begin{tikzcd}
\Omega_*(-) \otimes_\tau \mathbb{Z}[1/2][t, t^{-1}] \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{Sig}_*(-) \otimes \mathbb{Z}[1/2][t^{-1}] \arrow[dl, dashed, "\simeq"'] \\
KO_*(-)[1/2] &
\end{tikzcd}$$

provides an isomorphism

$$\mathrm{Sig}_*(-) \otimes \mathbb{Z}[1/2][t^{-1}] \simeq KO_*(X)[1/2]$$

and passing to the connected coverings one concludes the proof of the following

Proposition 4.6. *There is an isomorphism*

$$\mathrm{Sig}_*(-) \otimes \mathbb{Z}[1/2] \xrightarrow{\simeq} ko_*(-)[1/2].$$

4.5. Signature homology at the prime 2. In this subsection we show that the 2-localization of the signature homology is isomorphic to singular homology with coefficients in the ring $\mathbb{Z}_{(2)}[t]$.

What we need is the following result due to Wall (see [CF]).

Theorem 4.7. *There is a natural equivalence of functors*

$$\Omega_*(X) \otimes \mathbb{Z}_{(2)} \simeq H_*(X; \Omega_* \otimes \mathbb{Z}_{(2)}).$$

From the previous fact it follows

Corollary 4.8. *There is a natural equivalence of functors*

$$\varphi^{-1} : H_*(-; \mathbb{Z}_{(2)}[t]) \xrightarrow{\simeq} \mathrm{Sig}_*(-) \otimes \mathbb{Z}_{(2)}.$$

Proof. First of all observe that the isomorphism of theorem 4.7 implies the existence of a natural equivalence

$$\Omega_*(-) \otimes_\tau \mathbb{Z}_{(2)}[t] \simeq H_*(-; \Omega_*) \otimes_\tau \mathbb{Z}_{(2)}[t]$$

where the tensor products are taken as Ω_*-modules.

Now the desired isomorphism is given by the composition of the following transformations:

$$\begin{tikzcd}
H_*(-) \otimes \mathbb{Z}_{(2)}[t] \arrow[r, "\simeq"] \arrow[rrdd, dashed] & H_*(-) \otimes \Omega_* \otimes_\tau \mathbb{Z}_{(2)}[t] \arrow[r] & H_*(-; \Omega_*) \otimes_\tau \mathbb{Z}_{(2)}[t] \arrow[d, "\simeq"'] \\
 & & \Omega_*(-) \otimes_\tau \mathbb{Z}_{(2)}[t] \arrow[d] \\
 & & \mathrm{Sig}_*(-) \otimes \mathbb{Z}_{(2)}.
\end{tikzcd}$$

Finally it is easy to check that the induced natural transformation is an isomorphism on the coefficients. $\square$

Observe that in the 2-local setting it is not possible to find a direct proof of the isomorphism above which uses the L-class of an H-stratifold. In fact, if one tries to mimic the construction of section 4.2, one sees that rational coefficients are really necessary in order to use Serre's theorem.

The importance of the result above is explained by the following

Corollary 4.9. *For any closed oriented topological manifold M there exists a cohomology class $L'(M) \in H^{4*}(M; \mathbb{Z}_{(2)})$ with the property that the evaluation of $L'(M)$ on the fundamental class of M is the signature of M. Moreover $L'(M)$ is a topological invariant.*

Proof. The class $L'(M)$ can be defined as the image of the signature fundamental class $[M] \in \mathrm{Sig}_n(M) \otimes \mathbb{Z}_{(2)}$ under the isomorphism

$$\mathrm{Sig}_n(M) \otimes \mathbb{Z}_{(2)} \xrightarrow{\varphi} H_n(M; \mathbb{Z}_{(2)}[t]) \xrightarrow{\Delta^{-1}} H^0(M; \mathbb{Z}_{(2)}[t]) = \bigoplus_{k=0}^{\infty} H^{4k}(M; \mathbb{Z}_{(2)}). \quad \square$$

4.6. The relation between $\mathrm{Sig}_*(-)$ and the classifying space for surgery. In this last subsection we use the signature homology to obtain an integral formulation of Kirby-Siebenmann's theorem about the homotopy structure of G/Top. The reader is referred to [MM] and [KiSi] for more details.

Let Top_n denote the topological group of homeomorphisms $f : \mathbb{R}^n \to R^n$, $f(0) = 0$ and let G_n denote the topological monoid of homotopy equivalences of S^{n-1}. The natural map $Top_n \to G_n$ induces a sequence of maps $BTop_n \to BG_n$ on the classifying spaces and after stabilization one gets a map

$$BTop \to BG.$$

Let G/Top be the fibre of the map above and recall that G/Top is the classifying space for surgery problems on simply connected topological manifolds.

Now, let

$$\begin{array}{ccc} \nu(M) & \xrightarrow{\hat{f}} & \xi \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{f} & X \end{array}$$

be the normal map associated to a map $\gamma : X \to G/Top$. By the process of surgery we can always assume f to be a homotopy equivalence for n odd and to be $(n/2) - 1$ connected if n is even. The obstruction to complete the surgery for n even is given by an Arf invariant with values in $\mathbb{Z}/2$ if $n = 4k + 2$ and by $\mathrm{sig}(M) - \mathrm{sig}(X) \in 8\mathbb{Z}$ if $n = 4k$. In particular this procedure allows to define a map

$$s : [X^n, G/Top] \to L_n(0)$$

where $L_*(0) = L_1(0), L_2(0), \ldots$ is the four-periodic sequence

i	1	2	3	4	5	6	7	8	9	...
$L_i(0)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	...

By taking $X = S^n$ we see that the surgery obstruction defines a map

$$s : \pi_n(G/Top) \to L_n(0).$$

One of the main results in the theory of topological manifolds is the following theorem of Kirby and Siebenmann (see [KiSi]).

Theorem 4.10 (Kirby and Siebenmann). *The surgery map*

$$s : \pi_n(G/Top) \to L_n(0)$$

is an isomorphism for all $n \geqq 1$.

In particular, it follows from the theorem above that the homotopy groups of G/Top are given by the formula

$$\pi_n(G/Top) \simeq \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \pmod 4,\ n \geqq 1, \\ \mathbb{Z}/2 & \text{for } n \equiv 2 \pmod 4, \\ 0 & \text{else.} \end{cases}$$

Now we want to look deeper into the surgery obstructions and their relation with the signature homology.

Lemma 4.11. *The surgery obstruction*

$$[X^{4k}, G/Top] \xrightarrow{s} L_{4k}(0)$$

factorizes through $\mathrm{Sig}_{4k}(G/Top)$ *as showed in the following diagram*:

$$\begin{tikzcd} {[X^{4k}, G/Top]} \arrow[r, "s"] \arrow[d] & \mathbb{Z} \\ \mathrm{Sig}_{4k}(G/Top). \arrow[ur, dashed, "s_I"'] & \end{tikzcd}$$

Proof. Let $[\gamma] : X \to G/Top$ be a homotopy class. By the homotopy invariance of signature homology, $[\gamma]$ defines a unique bordism class $[X, \gamma] \in \mathrm{Sig}_{4k}(G/Top)$ and so we only have to show that, if (X, γ) is zero-bordant in $\mathrm{Sig}_{4k}(G/Top)$, then it holds

$$s(X, \gamma) = 0.$$

Let us assume that (X, γ) is the boundary of a pair (W, F) where W is a closed $4k+1$-dimensional H-stratifold and F is a map from W into G/Top which is an extension of γ. The maps F and γ define stable topological bundles λ over W and η over X respectively so that it holds $\lambda|_{\partial W} = \eta$. Both of these stable bundles are trivial as a spherical fibration and so taking representatives we have a fiber homotopy equivalence

$$\begin{tikzcd} \lambda \arrow[r, "\simeq", "c"'] \arrow[d, "\pi"'] & W \times \mathbb{R}^l \arrow[d] \\ W \arrow[r, "1"] & W. \end{tikzcd}$$

Now, by the transversality theorem we can assume c to be transverse to $W \times 0$ and we can set $T := c^{-1}(W \times 0)$. In particular, if $M \to X$ denotes a representative of the surgery problem associated to γ, then we see that M is bordant to ∂T and so that it is zero-bordant. Finally, this fact together with the bordism-invariance of the signature imply

$$s([\gamma]) = \mathrm{sig}(M) - \mathrm{sig}(X) = 0. \quad \square$$

The case $n = 4k + 2$ can be treated in a similar way: in fact one can prove that the Kervaire invariant is a bordism invariant and thus one gets the following factorization

$$\begin{tikzcd}
{[X^{4k+2}, G/Top]} \arrow[r, "s"] \arrow[d] & \mathbb{Z}/2 \\
\mathfrak{N}^{Top}_{4k+2}(G/Top) \arrow[ur, dashed, "s_K"'] &
\end{tikzcd}$$

where $\mathfrak{N}^{Top}_*(-)$ denotes the non-oriented topological bordism.

Next we need the following result from Anderson and Kainen (see [Yo]).

Theorem 4.12. *Let $h_*(-)$ be a multiplicative homology theory with the property that multiplication induces an isomorphism*

$$h_*(\mathrm{pt}) \xrightarrow{\simeq} \mathrm{Hom}\big(h_{-*}(\mathrm{pt}), \mathbb{Z}\big).$$

Then for any X there is a short exact sequence

$$0 \to \mathrm{Ext}\big(h_{*-1}(X), \mathbb{Z}\big) \to h^*(X) \to \mathrm{Hom}\big(h_*(X), \mathbb{Z}\big) \to 0.$$

Unfortunately the signature homology functor does not satisfy the conditions of the theorem above. However it is quite easy to overcome this problem by considering the periodic homology theory associated to $\mathrm{Sig}_*(-)$. This new functor, which we denote by $\overline{\mathrm{Sig}}_*(-)$, is obtained by formally inverting the class of $[CP^2]$ or more precisely by considering the ring homomorphism $\mathrm{Sig}_*(\mathrm{pt}) \to \mathbb{Z}[t, t^{-1}]$ induced by $[CP^2] \mapsto t$ and then by setting

$$\overline{\mathrm{Sig}}_*(X) := \mathrm{Sig}_*(X) \otimes_{\mathrm{Sig}_*(\mathrm{pt})} \mathbb{Z}[t, t^{-1}].$$

Now, the homology theory $\overline{\mathrm{Sig}}_*(-)$ satisfies the condition of theorem 4.12 and so it follows that every homomorphism from $\overline{\mathrm{Sig}}_*(X)$ to $\mathbb{Z}$ lifts to an (in general not unique) element in $\overline{\mathrm{Sig}}^*(X)$.

Moreover, if

$$\gamma : N \to G/Top$$

is a surgery problem over N and if M is a closed oriented manifold, then the surgery obstruction of the problem

$$M \times N \xrightarrow{\pi_2} N \xrightarrow{\gamma} G/Top$$

satisfies

$$s_I(M \times N, \gamma \circ \pi_2) = s_I(N, \gamma) \cdot \mathrm{sig}(M).$$

In particular this formula implies that the homomorphism s_I factorizes through $\overline{\mathrm{Sig}}_{4n}(G/Top)$ and so we get a map

$$\begin{tikzcd} s_I : \mathrm{Sig}_{4n}(G/Top) \arrow[r] \arrow[d] & \mathbb{Z} \\ \overline{\mathrm{Sig}}_{4n}(G/Top). \arrow[ur, dashed] & \end{tikzcd}$$

According to Kainen's theorem the homomorphism

$$s_I : \overline{\mathrm{Sig}}_{4n}(G/Top) \to \mathbb{Z}$$

can be lifted to an element of

$$\overline{K}_{4n} \in \overline{\mathrm{Sig}}^{4n}(G/Top) = \overline{\mathrm{Sig}}^{0}(G/Top) = [G/Top, \Omega^\infty \overline{\mathrm{Sig}}]$$

where $\Omega^\infty \overline{\mathrm{Sig}}$ is the Ω^∞-space of the spectrum of $\overline{\mathrm{Sig}}_*(-)$. On the other hand, since $\mathrm{Sig}_*(-)$ corresponds to the (-1)-connected cover of $\overline{\mathrm{Sig}}_*(-)$, it follows that there is an equivalence

$$\Omega^\infty \overline{\mathrm{Sig}} \simeq \Omega^\infty \mathrm{Sig}.$$

Notice that an explicit construction of $\Omega^\infty \mathrm{Sig}$ as a semi-simplicial set can be obtained using the so called Quinn's construction (see [Qu2]). The space $\Omega^\infty \mathrm{Sig}$ has the form $\Sigma \times \mathbb{Z}$ and since G/Top is connected we get a homotopy class

$$G/Top \xrightarrow{\overline{K}} \Sigma.$$

A similar construction for the Kervaire invariant provides a family of homotopy classes

$$K_{4n-2} : G/Top \to K(\mathbb{Z}/2, 4n-2), \quad \text{for } n \geqq 1.$$

It is perhaps useful to notice that this construction requires the existence of an isomorphism

$$\mathfrak{N}_*^{Top}(X) \simeq H_*(X; \mathfrak{N}_*^{Top}),$$

which follows, for instance, from a more general result of Pazhitnov and Rudyak (see [PR]).

Taking representatives for the homotopy classes $\overline{K}$ and K_{4n-2} we can define a continuous map

$$\phi : G/Top \to \Sigma \times \prod_{n \geqq 1}^{\infty} K(\mathbb{Z}/2, 4n-2).$$

The main result of this last part can now be stated.

Theorem 4.13. *The map ϕ is an homotopy equivalence.*

Proof. Since we know that the homotopy groups of G/Top are given by the formula

$$\pi_n(G/Top) \simeq \begin{cases} \mathbb{Z} & \text{for } n \equiv 0 \ (\mathrm{mod}\, 4), \\ \mathbb{Z}/2 & \text{for } n \equiv 2 \ (\mathrm{mod}\, 4), \\ 0 & \text{else,} \end{cases}$$

it is clear that in order to prove the theorem it is enough to show that ϕ induces isomorphisms on homotopy groups in every dimension. In particular we know that the generators $\imath_{2n} \in \pi_{2n}(G/Top)$ are specified by the condition

$$s_I(S^{4n}, \imath_{4n}) = 1, \quad s_K(S^{4n-2}, \imath_{4n-2}) = 1$$

so that $\overline{K}$ and K_{4n+2} evaluate to 1 on the homotopy generators in every dimension and the theorem follows. $\square$

As a corollary one gets in particular the following two results due to Sullivan and Kirby-Siebenmann.

Corollary 4.14. *There are the following homotopy equivalences*:

(1) *at two*

$$G/Top_{(2)} \simeq \prod_{n \geqq 1} K(\mathbb{Z}_{(2)}, 4n) \times K(\mathbb{Z}/2, 4n-2);$$

(2) *at odd primes*

$$G/Top[1/2] \simeq BO[1/2].$$

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