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On boundary-link cobordism

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0. *Introduction*

An n-dimensional m-component link is an oriented smooth submanifold Σ^n of S^{n+2}, where $\Sigma^n = \Sigma_1^n \cup \ldots \cup \Sigma_m^n$ is the ordered disjoint union of m submanifolds of S^{n+2}, each homeomorphic to S^n. Σ is a *boundary link* if there is an oriented smooth submanifold V^{n+1} of S^{n+2}, $V^{n+1} = V_1^{n+1} \cup \ldots \cup V_m^{n+1}$ the disjoint union of the submanifolds V_i^{n+1}, such that $\partial V_i = \Sigma_i$ $(i = 1, \ldots, m)$. A pair (Σ, V), where Σ is a boundary link and V as above, with each V_i *connected* $(i = 1, \ldots, m)$, is called an n-dimensional *special Seifert pair*. In this paper, we define a notion of cobordism of special Seifert pairs and give an algebraic description of the set (group) of cobordism classes.

Let (Σ, V) and $(\bar{\Sigma}, \bar{V})$ be special Seifert pairs.

Definition 0·1. (Σ, V) and $(\bar{\Sigma}, \bar{V})$ are *w-cobordant* if there are oriented disjoint submanifolds W_i^{n+2} of $S^{n+2} \times I$ $(i = 1, \ldots, m)$ such that

(i) W_i intersects $S^{n+2} \times \{0\}$ and $S^{n+2} \times \{1\}$ transversely at V_i and $\bar{V}_i$, respectively.

(ii) $\partial W_i = V_i \cup M_i \cup \bar{V}_i$, where $M_i \cap V_i = \Sigma_i$, $M_i \cap \bar{V}_i = \bar{\Sigma}_i$ and M_i is homeomorphic to $S^n \times I$.

It is known that, for n even, any two special Seifert pairs are w-cobordant. For a proof, see [**5**] for $m = 1$ and [**3**] or [**1**] for the general case.

Let $B(n, m)$ be the set of w-cobordism classes of $(2n-1)$-dimensional special Seifert pairs with m components. When $m = 1$, $B(n, m)$ coincides with the knot cobordism group which, for $n \geqslant 2$, has been described algebraically by J. Levine in [**7**] as a cobordism of matrices via Seifert forms; this algebraic description was reformulated by Kervaire [**6**] in terms of isometric structures and its structure completely determined by Stoltzfus in [**10**] (see also [**8**]).

Here, we extend the methods of [**7**] and [**6**] to obtain a similar description of $B(n, m)$, for $n \geqslant 3$, $m > 1$. Ambient connected sum of Seifert pairs provides $B(n, m)$ with an abelian group structure, which is shown to be isomorphic to a certain cobordism group $C(\epsilon_n, m)$ of isometric structures to be defined below. This formulation allows us to consider some obstructions to splitting a special Seifert pair up to cobordism, where a pair (Σ, V) is called split if there are disjoint balls B_i^{2n+1} in S^{2n+1} such that $V_i \subset \operatorname{int}(B_i)$ $(i = 1, \ldots, m)$. We show that, for n even, $n > 3$, there are infinitely many linearly independent non-splittable elements in $B(n, m)$.

We remark that Cappell and Shaneson have considered a stronger notion of equivalence of boundary links in [**1**]: Σ and $\bar{\Sigma}$ are equivalent if there are w-cobordant special Seifert pairs (Σ, V) and $(\bar{\Sigma}, \bar{V})$. The group of equivalence classes is computed in terms of their Γ groups [**2**] and they obtain some strong results on the existence of non-splittable cobordism classes of links.

1. *Geometric preliminaries*

Let (Σ, V) be a $(2n-1)$-dimensional special Seifert pair, $n \geqslant 3$.

Definition 1·1. (Σ, V) is a *simple pair* if each V_i is an $(n-1)$-connected manifold $(i = 1, \ldots, m)$.

PROPOSITION 1·2. *Any special Seifert pair (Σ, V) is w-cobordant to a simple pair.*

Proof. The arguments of [**5**] show that by performing a finite sequence of surgeries on the V_i's we can make the components Σ_i of the link bound disjoint $(n-1)$-connected submanifolds $\overline{V}_i^{2n}$ of D^{2n+2} $(i = 1, \ldots, m)$. More precisely, there are disjoint connected $(2n+1)$-submanifolds W_i of D^{2n+2} and embeddings $k_i\colon V_i \times I \to W_i$ $(i = 1, \ldots, m)$ such that

(*a*) $W_i \cap S^{2n+1} = V_i$ and $k_i(x, t) = \frac{1}{2}(t+1)\,x$.

(*b*) $\partial W_i = V_i \cup k_i(\partial V_i \times I) \cup \overline{V}_i$ and $\overline{V}_i \cap k_i(\partial V_i \times I) = k_i(\partial V_i \times \{0\})$.

(*c*) $\overline{V}_i$ is $(n-1)$-connected $(i = 1, \ldots, m)$.

(*d*) W_i is obtained from $k_i(V_i \times I)$ by attaching handles of index $\leqslant n$ to $k_i(V_i \times \{0\})$.

As in [**7**], we apply the engulfing theorem of Hirsch and Zeeman, whose hypothesis we now proceed to verify, to embed a $(2n+2)$-ball D_0 in the interior of D^{2n+2} with $D_0 \cap W_i = \overline{V}_i$ $(i = 1, \ldots, m)$. In the notation of [**4**] we let $X = \overline{V}_1 \cup \ldots \cup \overline{V}_m$ and $V = D^{2n+2}$ with cuts along the W_i's. As we have attached only handles of index $\leqslant n$ to $V_1 \cup \ldots \cup V_m$, successive applications of Van Kampen's theorem, Hurewicz theorem and Mayer-Vietoris sequence show that $D^{2n+2} - W$ is n-connected. On the other hand, X is n-collapsible since the $\overline{V}_i$'s are $(n-1)$-connected. This and the n-connectivity of $D^{2n+2} - W$ imply the Dehn cone condition and complete the verification of the hypothesis of the engulfing theorem. By the h-cobordism theorem, there is a diffeomorphism $h\colon D^{2n+2} - \operatorname{int}(D_0) \to S^{2n+1} \times I$. Then, $h(W)$ provides a w-cobordism between (Σ, V) and $(h(\partial \overline{V}), h(\overline{V}))$ which is a simple pair.

We now define an operation in $B(n, m)$ which induces an abelian group structure. Let $x, y \in B(n, m)$ be w-cobordism classes; by Proposition 1·1 we can find simple pairs (Σ, V) and $(\overline{\Sigma}, \overline{V})$ representing x and y, respectively. We choose a collection α of disjoint simple smooth curves α_i $(i = 1, \ldots, m)$ in $S^{2n+1} - \operatorname{int}(V \cup \overline{V})$ connecting Σ_i to $\overline{\Sigma}_i$ so that α_i intersects the links transversely only at its endpoints.

Definition 1·3. A family α of curves as above is called an *allowable collection* for the simple pairs (Σ, V) and $(\overline{\Sigma}, \overline{V})$.

Let $(\Sigma, V) \#_\alpha (\overline{\Sigma}, \overline{V})$ be the simple pair obtained by taking the sum of (Σ, V) and $(\overline{\Sigma}, \overline{V})$ along the allowable collection $\alpha = \{\alpha_i, i = 1, \ldots, m\}$. We set $x + y = [(\Sigma, V) \#_\alpha (\overline{\Sigma}, \overline{V})]$, where the brackets denote w-cobordism class; this operation is well-defined because for a simple pair (Σ, V), $\pi_1(S^{2n+1} - V) = 0$, since each V_i is simply connected.

Taking the pair formed by the trivial m-link in S^{2n+1} and m disjoint bounding discs as the zero element in $B(n, m)$ $(n \geqslant 3)$, and arguing as in (ii) of Theorem 3·3 below, we see that for $x \in B(n, m)$ and (Σ, V) a simple pair representing x, $(-\Sigma, -V)$ represents its inverse, where $(-\Sigma, -V)$ is the simple pair obtained from (Σ, V) by reversing the orientations of the components of Σ, V and S^{2n+1}. This completes the description of the abelian group structure of $B(n, m)$.

2. *Isometric structures*

Let m be a positive integer and $\epsilon = \pm 1$.

Definition 2·1. An (ϵ, m)-symmetric isometric structure over $\mathbb{Z}$ is a $(2m+1)$-tuple $(M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)$, where M_i is a finitely generated free $\mathbb{Z}$-module, $\langle\ ,\ \rangle_i$ is an ϵ-symmetric bilinear form on M_i $(i = 1, \ldots, m)$ and $t\colon M \to M$ is an endomorphism of $M = \bigoplus_{i=1}^m M_i$, satisfying:

(i) $\langle\ ,\ \rangle_i$ is unimodular $(i = 1, \ldots, m)$.

(ii) $\langle t(x), y\rangle = \langle x, (1-t)\,y\rangle$, $\forall\, x, y \in M$, where $\langle\ ,\ \rangle = \bigoplus_{i=1}^m \langle\ ,\ \rangle_i$.

Remark. (ii) implies $\langle t(x_i), x_j\rangle = -\epsilon\langle t(x_j), x_i\rangle$, if $x_i \in M_i$, $x_j \in M_j$ and $i \neq j$, since $\langle x_i, x_j\rangle = 0$.

Definition 2·2. An (ϵ, m)-symmetric isometric structure $\sigma = (M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)$ is *metabolic*, if there are submodules N_i of M_i $(i = 1, \ldots, m)$ satisfying:

(i) $N_i = N_i^\perp$, where $N_i^\perp = \{x_i \in M_i \mid \langle x_i, n_i\rangle_i = 0, \forall\, n_i \in N_i\}$.

(ii) $N = \bigoplus_{i=1}^m N_i$ is invariant under t.

$(N_1, \ldots, N_m)$ is called a *metabolizer* for σ.

Let $\sigma = (M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)$ and $\tau = (N_1, \ldots, N_m, (\ ,\ )_1, \ldots, (\ ,\ )_m, s)$ be isometric structures.

(*a*) *Addition* is defined by

$$\sigma + \tau = (M_1 \oplus N_1, \ldots, M_m \oplus N_m, \langle\ ,\ \rangle_1 \oplus (\ ,\ )_1, \ldots, \langle\ ,\ \rangle_m \oplus (\ ,\ )_m, t \oplus s).$$

(*b*) σ is *isomorphic* to τ, if there are isomorphisms $f_i\colon M_i \to N_i$ $(i = 1, \ldots, m)$ such that $\langle x_i, y_i\rangle_i = (f_i(x_i), f_i(y_i))_i$, $\forall\, x_i, y_i \in M_i$ and $f \circ t = s \circ f$, where $f = \bigoplus_{i=1}^m f_i$.

Definition 2·3. The isometric structures σ and τ are *cobordant* if there are metabolic structures η_1 and η_2 such that $\sigma + \eta_1$ is isomorphic to $\tau + \eta_2$.

Cobordism determines an equivalence relation in the set of (ϵ, m)-symmetric isometric structures. The set of equivalence classes, $C(\epsilon, m)$, forms an abelian group under the previously defined addition with zero element represented by metabolic structures and the inverse of $[(M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)]$ given by $[(M_1, \ldots, M_m, -\langle\ ,\ \rangle_1, \ldots, -\langle\ ,\ \rangle_m, t)]$, where the brackets denote equivalence class.†

To an isometric structure σ there is associated a *Seifert form* which is the bilinear form $\theta\colon M \times M \to \mathbb{Z}$, $M = \bigoplus_{i=1}^m M_i$, defined by $\theta(x, y) = \langle t(x), y\rangle$.

Proposition 2·4. *If $N_i \subset M_i$ $(i = 1, \ldots, m)$ are submodules with rank $M_i = 2$ rank N_i and $\theta(x, y) = 0$, $\forall\, x, y \in N = \bigoplus_{i=1}^m N_i$ then σ is metabolic.*

Proof. $\langle t(x), y\rangle = \langle x, (1-t)\,y\rangle$ can be rewritten as $\theta(x, y) + \epsilon\,.\,\theta(y, x) = \langle x, y\rangle$. Hence, if $x, y \in N_i$, $\langle x, y\rangle_i = \langle x, y\rangle = 0$ and therefore $N_i \subset N_i^\perp$. On the other hand, from the exact sequence

$$0 \to N_i^\perp \to M_i \xrightarrow{\text{Ad}} \text{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z}) \to 0,$$

where $\text{Ad}(m_i) \in \text{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z})$ is given by $\text{Ad}(m_i)(n_i) = \langle n_i, m_i\rangle$, $n_i \in N_i$, it follows

† $C(\epsilon, 1)$ coincides with the group $C^\epsilon(Z)$ of **[6]** and **[10]**.

that rank N_i = rank $N_i^\perp$, since 2 . rank N_i = rank M_i. As we can assume that $N_i \subset M_i$ is a pure submodule, i.e. M_i/N_i is torsion free, we conclude that $N_i = N_i^\perp$. It remains to show that N is t-invariant; if $x, y \in N$, $\theta(x,y) = \langle t(x), y\rangle = 0$ and hence $t(x) \in N^\perp = (\oplus N_i)^\perp = \oplus N_i^\perp = \oplus N_i = N$.

PROPOSITION 2·5. *Let* $\sigma = (M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)$ *and* $\tau = (N_1, \ldots, N_m, (\ ,\ )_1, \ldots, (\ ,\ )_m, s)$ *be isometric structures. If* τ *and* $\sigma+\tau$ *are metabolic, so is* σ.

Proof. The argument is a simple generalization of that given in proposition 1·6 of [**10**]. Let $H = (H_1, \ldots, H_m)$ and $F = (F_1, \ldots, F_m)$ be metabolizers for τ and $\sigma+\tau$, respectively. We first show that we can assume $H_i \subset F_i$ $(i = 1, \ldots, m)$.

Let $\mathscr{L}$ be the set of m-tuples $(L_1, \ldots, L_m)$ of submodules L_i of $M_i \oplus N_i$, with $H_i \subseteq L_i$, $L_i \subseteq L_i^\perp$ $(i = 1, \ldots, m)$ and $\bigoplus_{i=1}^m L_i$ $t \oplus s$-invariant, ordered by componentwise inclusion. Let $L = (L_1, \ldots, L_m)$ be a maximal element of $\mathscr{L}$; then

$$L + (F \cap L^\perp) = (L_1 + (F_1 \cap L_1^\perp), \ldots, L_m + (F_m \cap L_m^\perp)) \in \mathscr{L}$$

and satisfies $L \leqslant L + (F \cap L^\perp)$, so that $L_i = L_i + F_i \cap L_i^\perp$ $(i = 1, \ldots, m)$ by maximality. It follows that $(F_i + L_i)^\perp = F_i^\perp \cap L_i^\perp = F_i \cap L_i^\perp \subset L_i$ and in particular, $L_i^\perp \subset (F_i + L_i) \cap L_i^\perp = L_i + F_i \cap L_i^\perp = L_i$, i.e. L is a metabolizer for $\sigma+\tau$ containing H.

Let G_i be the projection of L_i on the component M_i of $M_i \oplus N_i$. We will verify the hypothesis of Proposition 2·4 for $(G_1, \ldots, G_m)$ to conclude that it is a metabolizer for σ. By the exact sequences

$$0 \to H_i \to L_i \to G_i \to 0 \quad (i = 1, \ldots, m)$$

it follows that rank M_i = 2 . rank G_i, since rank $(M_i \oplus N_i)$ = 2 . rank L_i and rank N_i = 2 . rank H_i. To conclude the proof, it suffices to show that $G_i \subset G_i^\perp$, for this implies that $\theta(x,y) = \langle t(x), y\rangle = 0$, $\forall\ x, y \in G = \bigoplus_{i=1}^m G_i$, since G is t-invariant. Let $g_i \in G_i$ be the projection of $(g_i, n_i) \in L_i$; as $H_i \subset L_i = L_i^\perp$, $n_i \in H_i^\perp = H_i$ and therefore $(g_i, 0) \in L_i = L_i^\perp$, *a fortiori*, $g_i \in G_i^\perp$.

COROLLARY 2·6. *An isometric structure* σ *represents the zero element of* $C(\epsilon, m)$ *if and only if it is metabolic.*

3. *The main theorem*

Let (Σ, V) be a special Seifert pair, where Σ is a $(2n-1)$-dimensional m-component boundary link in S^{2n+1}. Let M_i be the torsion-free part of $H_n(V_i)$ and $\langle\ ,\ \rangle_i$: $M_i \times M_i \to \mathbb{Z}$ the intersection pairing of V_i, which is a $(-1)^n$-symmetric unimodular bilinear form and $M = \bigoplus_{i=1}^m M_i$. As in [**9**], there is a pairing θ: $M \times M \to Z$, defined by $\theta(\alpha, \beta) = L(\alpha, \beta^+)$, where L denotes linking number and β^+ is obtained by translating β in the positive normal direction, satisfying

$$\theta(\alpha,\beta) + (-1)^n\, \theta(\beta,\alpha) = \langle \alpha, \beta\rangle, \tag{3·1}$$

where $\langle\ ,\ \rangle = \bigoplus_{i=1}^m \langle\ ,\ \rangle_i$. Since $\langle\ ,\ \rangle$ is unimodular, θ defines a unique endomorphism t: $M \to M$ such that $\langle t(\alpha), \beta\rangle = \theta(\alpha,\beta)$, for any $\alpha, \beta \in M$. Accordingly, (3·1) can be rewritten as $\langle t(\alpha), \beta\rangle = \langle \alpha, (1-t)\beta\rangle$. In other words, to the special Seifert pair (Σ, V) we have assigned the isometric structure

$$\sigma(\Sigma, V) = (M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t).$$

This allows us to define a map

$$\phi_n^m : B(n, m) \to C(\epsilon_n, m)$$

by $\phi_n^m(x) = [\sigma(\Sigma, V)]$, where (Σ, V) is a simple pair representing the w-cobordism class x and $\epsilon_n = (-1)^n$.

PROPOSITION 3·2. *ϕ_n^m is a well-defined homomorphism.*

Our main result can now be stated as

THEOREM 3·3. *ϕ_n^m is an isomorphism, for $n \geq 3$.*

Proof of 3·2. If (Σ, V) and $(\overline{\Sigma}, \overline{V})$ are simple pairs and α an allowable collection of curves (see Definition 1·3), it follows from the Mayer–Vietoris sequence that $\sigma((\Sigma, V) \#_\alpha (\overline{\Sigma}, \overline{V})) = \sigma(\Sigma, V) + \sigma(\overline{\Sigma}, \overline{V})$, i.e. σ is additive. Hence it suffices to show that if (Σ, V) is a w-null-cobordant simple pair, $\sigma(\Sigma, V)$ is metabolic. A w-null-cobordism of (Σ, V) gives a $(2n+1)$-submanifold $W = W_1 \cup \ldots \cup W_m$ of D^{2n+2} (each W_i connected) intersecting $\partial D^{2n+2} = S^{2n+1}$ transversely at V, such that $\partial W_i = V_i \cup D_i$, where D_i is a $2n$-disc properly embedded in D^{2n+2} with $V_i \cap D_i = \Sigma_i$ $(i = 1, \ldots, m)$.

Let $j_* : H_n(V) \to H_n(W)$ be the map induced by inclusion and $H = \ker j_*$. If α and β are n-cycles in V representing elements of H, we can find $(n+1)$-chains γ and η in W such that $\partial(\gamma) = \alpha$ and $\partial(\eta) = \beta$. Then $L(\alpha, \beta^+) = \gamma \cdot \eta^+$, where $\cdot$ denotes intersection number and η^+ is the translation of η along the normal field to W in D^{2n+2} which extends the normal field to V in S^{2n+1}. Since γ and η^+ are disjoint, $\theta(\alpha, \beta) = L(\alpha, \beta^+) = 0$. Letting $j_i : H_n(V_i) \to H_n(W_i)$ $(i = 1, \ldots, m)$ be the homomorphisms induced by $V_i \subset W_i$, we have $H = \ker j_* = \bigoplus_{i=1}^m \ker j_i$. As in lemma 2 of [7], $\operatorname{rank} H_n(V_i) = 2 . \operatorname{rank}(\ker j_i)$; therefore, setting $N_i = \ker j_i$ $(i = 1, \ldots, m)$, the above discussion shows that $\theta(\alpha, \beta) = 0$ for any $\alpha, \beta \in N = \bigoplus_{i=1}^m N_i$ and $\operatorname{rank} N_i = \frac{1}{2} \operatorname{rank} H_n(V_i)$. By Proposition 2·4, $\sigma(\Sigma, V)$ is metabolic.

Proof of Theorem 3·3

(i) *ϕ_n^m is onto.* Let $(M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)$ be an (ϵ_n, m)-symmetric isometric structure and $\theta : M \times M \to \mathbb{Z}$ the bilinear form on $M = \bigoplus_{i=1}^m M_i$ given by $\theta(x, y) = \langle t(x), y \rangle$. It suffices to construct a simple pair (Σ, V) such that θ corresponds to its Seifert form and $(M_i, \langle\ ,\ \rangle_i)$ to the intersection pairing of V_i $(i = 1, \ldots, m)$.

Let θ_i be the restriction of θ to M_i; from (3·1) it follows that

$$\theta_i(x, y) + (-1)^n \theta_i(y, x) = \langle x, y \rangle_i \quad (\forall\ x, y \in M_i).$$

In theorem II.3 of [5] it is shown that there is a pair $(\overline{\Sigma}_i, \overline{V}_i)$, where $\overline{\Sigma}_i$ is a $(2n-1)$-knot and $\overline{V}_i$ an $(n-1)$-connected Seifert surface for $\overline{\Sigma}_i$, such that $(M_i, \langle\ ,\ \rangle_i)$ corresponds to the intersection pairing of $\overline{V}_i$ and θ_i to its Seifert form. Let $(\overline{\Sigma}, \overline{V})$ be the split simple pair whose ith component is $(\overline{\Sigma}_i, \overline{V}_i)$. Then the Seifert form of $(\overline{\Sigma}, \overline{V})$ is $\overline{\theta} = \bigoplus_{i=1}^m \theta_i$. To conclude the proof, we adjust the linkage of the handles of $\overline{V}_i$ and $\overline{V}_j$ $(i \neq j)$, so as to get a pair (Σ, V) having θ as its Seifert form. This can be accomplished by proceeding as in the proof of the theorem II.3 of [5], since $\theta(x, y) - \overline{\theta}(x, y) = 0$ if $x, y \in M_i$ for some i and $\theta(x_i, x_j) - \overline{\theta}(x_i, x_j) = (-1)^{n+1}(\theta(x_j, x_i) - \overline{\theta}(x_j, x_i))$, if $x_i \in M_i$, $x_j \in M_j$ $(i \neq j)$.

(ii) *ϕ_n^m is injective.* We argue as in lemma 5 of [7]. Let (Σ, V) be a simple pair with $\phi_n^m([\Sigma, V]) = 0$; by Corollary 2·6, $\sigma(\Sigma, V)$ is metabolic and therefore $H_n(V_i)$ has a basis

$\alpha_1^i, \ldots, \alpha_{r_i}^i, \beta_1^i, \ldots, \beta_{r_i}^i$, $i = 1, \ldots, m$, such that $\theta(\alpha_s^i, \alpha_t^j) = 0$ for $1 \leqslant i, j \leqslant m$, $1 \leqslant s \leqslant r_i$, $1 \leqslant t \leqslant r_j$. Since $\langle \alpha_s^i, \alpha_t^i \rangle_i = \theta(\alpha_s^i, \alpha_t^i) + (-1)^n . \theta(\alpha_t^i, \alpha_s^i) = 0$, α_s^i can be represented by disjoint embedded spheres $S_{i,s}$ in V_i ($1 \leqslant i \leqslant m$, $1 \leqslant s \leqslant r_i$); in D^{2n+2} the spheres $S_{i,s}$ bound disjoint embedded discs $D_{i,s}$, for the intersection numbers $D_{i,s} \cdot D_{j,t} = \theta(\alpha_s^i, \alpha_t^j) = 0$ so that we can apply Whitney's procedure to remove possible intersections. Let ν_i be a normal field to V_i in S^{2n+1}; as $\theta(\alpha_s^i, \alpha_s^i) = 0$, ν_i can be extended to $D_{i,s}$ in D^{2n+2} to yield a field $\nu_{i,s}$, and choosing trivializations for the orthogonal complement to $\nu_{i,s}$ along $D_{i,s}$, we obtain n-handles $h_{i,s}$ ($1 \leqslant i \leqslant m$, $1 \leqslant s \leqslant r_i$). Then by performing r_i surgeries on V_i along $h_{i,s}$ ($1 \leqslant s \leqslant r_i$) we obtain disjoint submanifolds Δ_i of D^{2n+2} bounded by Σ_i ($i = 1, \ldots, m$), which are actually $2n$-discs since each Δ_i is contractible and $n \geqslant 3$. The trace of the surgeries provides the required w-null-cobordism of (Σ, V).

4. *Split cobordism*

In this section we consider some obstructions to splitting a special Seifert pair up to cobordism. Recall that a pair (Σ, V) is called split if there are mutually disjoint balls $B_i^{2n+1} \subset S^{2n+1}$ such that $V_i \subset \operatorname{int}(B_i)$ ($i = 1, \ldots, m$).

Let $\overline{S}_i \colon C(\epsilon_n, m) \to C(\epsilon_n, 1)$ ($i = 1, 2$) be the homomorphisms defined by

$$\overline{S}_1(\sigma) = [(M, \langle\ ,\ \rangle, t)]$$

and
$$\overline{S}_2(\sigma) = [(M_1, \langle\ ,\ \rangle_1, t_1)] + \ldots + [(M_m, \langle\ ,\ \rangle_m, t_m)],$$

were $(M_1, \ldots, M_m, \langle\ ,\ \rangle_1, \ldots, \langle\ ,\ \rangle_m, t)$ is an isometric structure representing σ, $M = \bigoplus_{i=1}^m M_i$, $\langle\ ,\ \rangle = \bigoplus_{i=1}^m \langle\ ,\ \rangle_i$ and $t_i = p_i \circ t \circ j_i$, $j_i \colon M_i \to M$ being the inclusion and $p_i \colon M \to M_i$ the projection of M on to the ith factor ($i = 1, \ldots, m$). For $n \geqslant 3$, the homomorphisms $\overline{S}_i$ can be viewed geometrically as homomorphisms $S_i \colon B(n, m) \to B(n, 1)$ under the isomorphisms obtained in Theorem 3·3, i.e.

$$S_i = (\phi_n^1)^{-1} \cdot \overline{S}_i \cdot \phi_n^m \quad (i = 1, 2). \tag{4·1}$$

A simple argument shows that

PROPOSITION 4·2. *If (Σ, V) is a split cobordant special Seifert pair, $S_1([\Sigma, V]) = S_2([\Sigma, V])$.*

The above proposition allows us to establish the existence of non-splittable Seifert pairs by means of knot cobordism invariants. For simplicity, we consider the case of 2-component Seifert pairs, the generalization to an arbitrary number of components being straightforward.

In [7], J. Levine defines a signature invariant for knot cobordism as follows: let $(M, \langle\ ,\ \rangle, t)$ be an $(\epsilon, 1)$-symmetric isometric structure and $\theta, \theta' \colon M \times M \to \mathbb{Z}$ the bilinear forms given by $\theta(x, y) = \langle t(x), y \rangle$, $\theta'(x, y) = \theta(y, x)$ (θ is the Seifert form); the signature is a continuous function $\Gamma_\theta \colon S_\theta \to \mathbb{Z}$, where S_θ is the unit circle S^1 with the zeros of $\det(\xi . \theta + \theta')$ removed, given by $\Gamma_\theta(\xi) =$ signature of B_ξ,

$$B_\xi = \begin{cases} \dfrac{\xi\theta + \theta'}{1 + \xi}, & \xi \neq -1 \\[2ex] i(\theta' - \theta), & \xi = -1, \end{cases}$$

which vanishes for metabolic structures and is additive, i.e. $\Gamma_{\theta_1+\theta_2} = \Gamma_{\theta_1}+\Gamma_{\theta_2}$ on $S_{\theta_1} \cap S_{\theta_2}$.

Let $\eta_k = (M_1, M_2, \langle\ ,\ \rangle_1, \langle\ ,\ \rangle_2, t)$, k a positive integer, be the $(1,2)$-symmetric isometric structure with $M_1 = \mathbb{Z} \oplus \mathbb{Z}$, $M_2 = \mathbb{Z} \oplus \mathbb{Z}$, such that its Seifert form is given in the canonical basis by the matrix

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ \hline 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

If θ_1 and θ_2 are the Seifert forms of $\overline{S}_1(\eta_k)$ and $\overline{S}_2(\eta_k)$ respectively, their matrices in the canonical basis are

$$\theta_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can check that $\Gamma_{\theta_1}(i) = 2$ and $\Gamma_{\theta_2}(i) = 0$. According to Proposition 4·2, $(\phi_n^2)^{-1}(\eta_k)$ is a non-splittable w-cobordism class, n even, $n > 3$. Actually, it is possible to get a stronger result; if $sB(n,2)$ is the subgroup of $B(n,2)$ formed by the splittable cobordism classes and $B^*(n,2) = B(n,2)/sB(n,2)$, arguing as in §26 of [7] we can show that

THEOREM 4·3. *The family* $\{(\phi_n^2)^{-1}(\eta_k),\ k > 0\}$ *is linearly independent in* $B^*(n,2)$, n *even*, $n > 3$.

Remarks.

(1) Each component of $(\phi_n^2)^{-1}(\eta_k)$ in Theorem 4·3 represents the trivial element of $B(n,1)$.

(2) Arguing as above, we can obtain from each knot cobordism invariant an obstruction to split cobordism as follows.

Let G be an abelian group and $\rho: C(\epsilon_n, 1) \to G$ a homomorphism (i.e. a $(2n-1)$-dimensional knot cobordism invariant); define $\chi_\rho: B(n,m) \to G$ by $\chi_\rho = \rho \cdot \phi_n^1 \cdot (s_1 - s_2)$. Then from (4·2), if (Σ, V) is split cobordant, $\chi_\rho([\Sigma, V]) = 0$. This can be applied, for example, to the Alexander invariant (characteristic polynomial) and the other invariants obtained in [**7**, **8**, **10**].

(3) (*Added in proof*). The author has just learned that J. Duval has studied boundary links from a Seifert surface viewpoint and obtained an alternative description of the F_m-link cobordism groups of Cappell and Shaneson, as announced in [**11**].

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