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Theory of Almost Algebraic Poincaré Complexes and Local Combinatorial Hirzebruch Formula

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Abstract. We develop a theory of almost algebraic Poincaré complexes to write an analog of the Hirzebruch formula with nonflat coefficients for combinatorial manifolds.

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1. Introduction

The well-know Hirzebruch formula says that for $4k$-dimensional orientable compact closed manifold X the following equation holds

$$\operatorname{sign} X = 2^{2k}\langle L(X), [X]\rangle, \tag{1}$$

where $\operatorname{sign} X = \operatorname{sign}(H^{2k}(X, C), \cup)$ is the signature of nondegenerate quadratic form in the cohomology groups $H^{2k}(X, C)$, defined by $\cup$-product,

$$L(X) = \prod_j \frac{t_j/2}{\operatorname{th}(t_j/2)}$$

is the Hirzebruch characteristic class defined by formal generators t_j by

$$\sigma_k(t_1, \dots, t_n) = c_k(cTX).$$

There are different ways to generalize the Hirzebruch formula mainly for non-simply connected manifolds. Namely, let X be a closed orientable nonconnected manifold and let $\pi = \pi_1(X)$, $f_X: X \to B\pi$ be the canonical mapping defined up to homotopy which induces the isomorphism of fundamental groups $(f_X)_*: \pi_1(X) \to \pi$. Consider a finite-dimensional representation $\rho: \pi \to \mathrm{U}(N)$. Then one can consider the cohomology groups $H^{2k}(X, \rho)$ with the local system of coefficients induced by the representation ρ. Then the $\cup$-product induces a nondegenerate quadratic form in this group. The signature of this form we shall denote by

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$$\operatorname{sign}_\rho X = \operatorname{sign}(H^{2k}(X, \rho), \cup).$$

It is easy to check that

$$\operatorname{sign}_\rho X = 2^{2k} \langle L(X) \operatorname{ch} f_X^* \xi^\rho, [X] \rangle, \tag{2}$$

where $\xi\rho$ is the vector bundle over $B\pi$, induced by the representation ρ.

In spite of both left-hand side and right-hand side parts of formula (2) coinciding with that of (1), this generalization might be useful for further generalizations. Namely, one can at least construct the right-hand side of formula (2) for more general representations of fundamental group π.

1.1. INFINITE-DIMENSIONAL REPRESENTATIONS

Let $C^*[\pi]$ be a C^*-group algebra of the group π. Any unitary representation of the group π can be uniquely extended to a representation $\bar{\rho}$ of the algebra $C^*[\pi]$. Put $A = \operatorname{Im} \bar{\rho}$, $\bar{\rho}\colon C^*[\pi] \to A$.

By ξ^ρ we denote the vector bundle over $B\pi$ with the fiber A, whose transition functions are induced by the action of the group π on the algebra A by the representation ρ. The vector bundle ξ^ρ forms the element of the K-group $\xi^\rho \in K_A(B\pi)$. Now we can write the right-hand part of formula (2):

$$? = 2^{2k} \langle L(X) \operatorname{ch}_A f_X^* \xi^\rho, [X] \rangle \in K_A(\mathrm{pt}) \otimes Q. \tag{3}$$

The left-hand part of the formula can be calculated as a symmetric signature (see [1]) of the manifold X by replacing of rings, induced by the representation ρ, so we obtain the so-called generalized Hirzebruch formula (see [2])

$$\operatorname{sign}_\rho(X) = 2^{2k} \langle L(X) \operatorname{ch}_A f_X^* \xi^\rho, [X] \rangle \in K_A(\mathrm{pt}) \otimes Q.$$

1.2. FINITE-DIMENSIONAL ASYMPTOTIC REPRESENTATIONS

There is a class of vector nonflat bundles which can be defined by so-called almost representations. They are so-called almost flat bundles (see [3]). Namely, the mapping $\sigma\colon \pi \to \mathrm{U}(n)$ is called *ε-almost representation* with respect to a finite subset F, if the following relations hold:

$$\sigma(g^{-1}) = \sigma(g)^{-1} \quad \text{for all } g \in \pi$$

and

$$\|\sigma\|_F = \sup\{\|\sigma(gh) - \sigma(g)\sigma(h)\| : g, h, gh \in F\} \leq \varepsilon.$$

It is more convenient to include the almost representation into a sequence, which is called the asymptotic representation. Namely, let $\sigma = \{\sigma_k\}$ be a sequence of ε_k-almost representations. The sequence σ is called the *asymptotic representation* of the group π (with respect to a finite subset F and to a sequence $\{n_k\}$), if the following relations hold:

$$\lim_{k\to\infty} \varepsilon_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \sup\{\|\sigma_k(g) - \sigma_{k+1}(g)\| : g \in F \subset \pi\} = 0.$$

Then, using the asymptotic representation, the vector bundle ξ_σ over the manifold X and the signature $\mathrm{sign}_\sigma(X)$ can be constructed in a canonical way (see [8]). So we receive again a variant of the Hirzebruch formula

$$\mathrm{sign}_\sigma(X) = 2^{2k}\langle L(X)\mathrm{ch}\,\xi_\sigma, [X]\rangle \in Q.$$

1.3. SMOOTH VERSION OF THE HIRZEBRUCH FORMULA

The left-hand side of the Hirzebruch formula (2) is described in terms of the combinatorial structure of the manifold X. There is a smooth version of this expression as well. Namely, consider the de Rham complex of differential forms on the manifold X with values in the flat vector bundle ξ^ρ:

$$0 \longrightarrow \Omega_0(X, \xi^\rho) \xrightarrow{d} \Omega_1(X, \xi^\rho) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{4k}(X, \xi^\rho) \longrightarrow 0. \tag{4}$$

It is well known that the cohomology groups of the de Rham complex (4) are isomorphic to the cohomology groups $H^*(X, \xi^\rho)$.

Then the $\cup$-product is induced by external product of differential forms, so the Hermitian form which defines the Poincaré duality can be determine by

$$\langle\omega_1, \omega_2\rangle = \int\limits_X \omega_1 \wedge \omega_2. \tag{5}$$

On the other hand, using a Riemannian metric on the manifold X, (ω_1, ω_2), the Poincaré duality (5) can be determined as a bounded operator $*$: $(\omega_1, \omega_2) = \int_X \omega_1 \wedge *\omega_2$, where $*$: $\Omega_k(X) \to \Omega_{n-k}(X)$. Put $\alpha = i^{k(k+1)}*$. Then $\alpha d\alpha - d^*$; $\alpha^2 = 1$. Let

$$\Omega^+(X) = \mathrm{Ker}\,(\alpha - 1); \qquad \Omega^- = \mathrm{Ker}\,(\alpha + 1).$$

It is evident that

$$(d + d^*)(\Omega^+(X)) \subset \Omega^-(X).$$

Consider the elliptic operator

$$D = (d + d^*)\colon \Omega^+(X) \to \Omega^-(X).$$

Then we have $D = \mathrm{sign}(X)$. Using the Atiyah–Singer index formula for the elliptic operator, we have

$$\mathrm{index}(D \otimes \xi) = 2^{2k}\langle L(X)\mathrm{ch}\,\xi, [X]\rangle, \tag{6}$$

for arbitrary vector bundle ξ over the manifold X.

1.4. COMBINATORIAL VERSION OF THE HIRZEBRUCH FORMULA

Similar to smooth version (6), one can put the question about construction the right-hand side of formula (6) using combinatorial terms. If the bundle ξ is induced by a representation of the fundamental group π, then the combinatorial version of the Hirzebruch formula is reduced to formulas (1)–(3). All of them require certain restriction on the bundle ξ: the vector bundle ξ should be flat in the case of classical representation and should be almost flat in the case of asymptotic representation.

The aim of the paper is to construct proper combinatorial objects which can imitate the Poincaré duality in the general case when the vector bundle ξ is arbitrary bundle over the manifold X. The idea of such construction was formulated in the paper by Gromov [7] and goes back to the construction of the Poincaré complexes and the so-called symmetric signature, which were considered in [1] and [2] (see also [6], p. 18).

Here we construct a new algebraic category, which is called *almost algebraic Poincaré complexes*. This category has all the necessary properties for constructing invariants of the signature type for combinatorial manifold with a local coefficient system induced by fibers of a vector bundle ξ over the manifold.

We show that any compact closed combinatorial manifold has a sufficient fine simplicial subdivision, which, in a natural way, induces an almost algebraic Poincaré complex. The signature of the latter can be used as the left-hand side part of the Hirzebruch formula (6). In particular, this formula in some sense gives a new construction for rational Pontryagin classes using only local combinatorial data on the combinatorial manifold X.

In the case when the vector bundle ξ is induced by a classical representation, the correspondent almost algebraic Poincaré complex coincides with the algebraic Poincaré complex from [1], and its signature coincides with the signature of cohomologies with the correspondent local coefficient system.

In the case of asymptotic representation, the almost algebraic Poincaré complex can be constructed from the universal algebraic Poincaré complex over the group algebra of the fundamental group π of the manifold X using the procedure of the change of rings. The signature of the almost algebraic Poincaré complex in this case equals the image of symmetric signature of the manifold X under the change of rings. Of course, we obtain the same Hirzebruch formula as in [3, 4].

Part of the results of this paper appear as a result of fruitful discussions with M. Gromov during the visit of the author to IHES (France) in autumn 1997 and 1998 and is based on the idea by Gromov [7] how to generalize the construction of the author [1] on the case of nonflat bundles. In essence, this work was done jointly and only because of the exceptional correctness of M. Gromov, his name is absent from the list of authors of the present work. In any case, the author considers it to be his duty to express his thanks to M. Gromov.

2. Economic Description of Algebraic Poincaré Complexes

Consider a ring Λ with unit and with unital involution $*$,

$$*^2 = 1, \qquad 1^* = 1, \qquad (\lambda_1\lambda_2)^* = \lambda_2^*\lambda_1^*.$$

Let C be a left Λ-module. Denote by C^* the adjoint module

$$C^* = \mathrm{Hom}_\Lambda(C, \Lambda),$$
$$(\lambda\phi)(x) \stackrel{\text{def}}{=} \phi(x)\lambda^*, \quad \phi \in C^*,\ x \in C,\ \lambda \in \Lambda.$$

If $f\colon C_1 \to C_2$ is a morphism of left modules, then the adjoint homomorphism $f^*\colon C_2^* \to C_1^*$ is defined by the formula

$$f^*(\phi_2)(x_1) \stackrel{\text{def}}{=} \phi_2(f(x_1)), \quad \phi_2 \in C_2^*,\ x_1 \in C_1.$$

Let $q_C\colon C \to C^{**}$ be defined by the formula

$$q_C(x)(\phi) \stackrel{\text{def}}{=} (\phi(x))^*, \quad x \in C,\ \phi \in C^*.$$

The adjoint homomorphism $q_C^*\colon C^{***} \to C^*$, evidently is inverse to the homomorphism q_{C^*}, $q_C^* q_{C^*} = 1_{C^*}$, that is the diagram

$$\begin{tikzcd} C^{***} \arrow[r, "q_C^*"] & C^* \\ C^* \arrow[u, "q_{C^*}"] \arrow[ur, "1_{C^*}"'] & \end{tikzcd}$$

is commutative. In the case when the module C is free or at least projective, then the homomorphism q_C is an isomorphism that allows us to identify the modules C and C^{**}.

Let us give an economic description of algebraic Poincaré complexes as a graded free module equipped with a boundary operator and an operator of Poincaré duality. Consider a chain complex of free Λ-modules C, d:

$$C = \bigoplus_{k=0}^{n} C_k, \qquad d = \bigoplus_{k=1}^{n} d_k, \qquad d_k\colon C_k \to C_{k-1}$$

and a homomorphism of Poincaré duality

$$D\colon C^* \to C, \quad \deg D = n,$$

defining the diagram

$$\begin{tikzcd} C_0 & C_1 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"'] & C_n \arrow[l, "d_n"'] \\ C_n^* \arrow[u, "D_0"] & C_{n-1}^* \arrow[l, "d_n^*"] \arrow[u, "D_1"] & \cdots \arrow[l, "d_{n-1}^*"] & C_0^* \arrow[l, "d_1^*"] \arrow[u, "D_n"] \end{tikzcd}$$

such that the following conditions for the boundary operator and for the Poincaré duality operator hold:

$$\begin{aligned} &d_{k-1}d_k = 0, \\ &d_k D_k + (-1)^{k+1} D_{k-1} d^*_{n-k+1} = 0, \\ &D_k = (-1)^{k(n-k)} D^*_{n-k}. \end{aligned} \tag{7}$$

More of that, assume that the homomorphism of Poincaré duality induces an isomorphism in homologies. Under this condition, the triple (C, d, D) is called an algebraic Poincaré complex. This definition was used by author in [1] and allows the construction of algebraic Poincaré complexes $\sigma(X)$ for arbitrary triangulation of a combinatorial manifold X, setting $\sigma(X) = (C, d, D)$, where $C = C_*(X; \Lambda)$ is the graded chain complex of the manifold X with a local system of coefficients induced by the natural inclusion of the fundamental group $\pi = \pi_1(X)$ into its group ring $\Lambda = C[\pi]$. d is the boundary homomorphism and

$$D = \bigoplus D_k, \quad D_k = \frac{1}{2}\Big(\bigcap[X] + (-1)^{k(n-k)}\Big(\bigcap[X]\Big)^*\Big),$$

where $\bigcap[X]$ is the intersection with the fundamental cycle of the manifold X.

Put

$$F_k = i^{k(k-1)} D_k. \tag{8}$$

The diagram

$$\begin{array}{ccccccc} C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\ \uparrow{\scriptstyle F_0} & & \uparrow{\scriptstyle F_1} & & & & \uparrow{\scriptstyle F_n} \\ C^*_n & \xleftarrow{d^*_n} & C^*_{n-1} & \xleftarrow{d^*_{n-1}} & \cdots & \xleftarrow{d^*_1} & C^*_0 \end{array} \tag{9}$$

satisfies more natural conditions for commutativity of the diagram and adjointness for operators

$$\begin{aligned} &d_k F_k + F_{k-1} d^*_{n-k+1} = 0, \\ &F_k = (-1)^{\frac{n(n-1)}{2}} F^*_{n-k}. \end{aligned} \tag{10}$$

The operator F still induces an isomorphism in homologies. Really, the commutativity condition (7) gives

$$\begin{aligned} 0 &= d_k D_k + (-1)^{k+1} D_{k-1} d^*_{n-k+1} \\ &= d_k i^{k(k-1)} F_k + (-1)^{k+1} i^{(k-1)(k-2)} F_{k-1} d^*_{n-k+1} \\ &= i^{k(k-1)}(d_k F_k + F_{k-1} d^*_{n-k+1}). \end{aligned}$$

From the adjointness condition (7) we have

$$i^{k(k-1)} F_k = (-1)^{k(n-k)} i^{(n-k)(n-k-1)} F^*_{n-k},$$

that is

$$F_k = (-1)^{n(n-1)/2} F^*_{n-k}.$$

Thus put

$$F = \bigoplus_{k=0}^{n} F_k,$$
$$F\colon C^* \to C, \quad \deg F = n.$$

Hence, shortly an algebraic Poincaré complex can be described as a graded free Λ-module C, a bounded operator d of degree -1 and an Poincaré duality operator $F\colon C^* \to C$ of degree n, such that

$$d^2 = 0, \qquad dF + Fd^* = 0,$$
$$F^* = (-1)^{n(n-1)/2} F, \qquad H(F)\colon H(C^*, d^*) \to H(C, d).$$

So we have

$$\begin{aligned} n &= 4k, & F^* &= F,\\ n &= 4k+1, & F^* &= F,\\ n &= 4k+2, & F^* &= -F,\\ n &= 4k+3, & F^* &= -F. \end{aligned}$$

Now consider the so-called cone of the operator F, that is the acyclic complex with respect to the sum of differentials d and F and summary grade of the bycomplex (9):

$$0 \longleftarrow C_0 \xleftarrow{H_1} \begin{matrix} C_1 \\ \oplus \\ C^*_n \end{matrix} \xleftarrow{H_2} \begin{matrix} C_2 \\ \oplus \\ C^*_{n-1} \end{matrix} \xleftarrow{H_3} \cdots$$
$$\cdots \xleftarrow{H_k} \begin{matrix} C_k \\ \oplus \\ C^*_{n-k+1} \end{matrix} \xleftarrow{H_{2l+1}} \begin{matrix} C_{k+1} \\ \oplus \\ C^*_{n-k} \end{matrix} \xleftarrow{H_{k+2}} \cdots \tag{11}$$
$$\cdots \xleftarrow{H_{n-1}} \begin{matrix} C_{n-1} \\ \oplus \\ C^*_2 \end{matrix} \xleftarrow{H_n} \begin{matrix} C_n \\ \oplus \\ C^*_1 \end{matrix} \xleftarrow{H_{n+1}} C^*_0 \longleftarrow 0$$

or

$$0 \longleftarrow A_0 \xleftarrow{H_1} A_1 \xleftarrow{H_2} \cdots \xleftarrow{H_k} A_k \xleftarrow{H_{k+1}} A_{2k+1} \xleftarrow{H_{k+2}} \cdots \xleftarrow{H_n} A_n \xleftarrow{H_{4n+1}} A_{n+1} \longleftarrow 0, \tag{12}$$

where

$$A_k = C_k \oplus C^*_{n-k+1}, \tag{13}$$

$$H_k = \begin{pmatrix} d_k & F_{k-1} \\ 0 & d^*_{n-k+2} \end{pmatrix}.$$

In other words, the complex (13) is a bygraded Λ-module

$$A = \bigoplus_{k=0}^{n+1} A_k = C + C^*,$$

and the boundary operator H is defined by the matrix

$$H = \begin{pmatrix} d & F \\ 0 & d^* \end{pmatrix}.$$

Consider a special case for $n = 4l$. In this case the operator F has even degree and, hence, the operator H has odd degree. Then the diagram (11) looks as follows:

$$0 \longleftarrow C_0 \xleftarrow{H_1} \begin{matrix} C_1 \\ \oplus \\ C_n \end{matrix} \xleftarrow{H_2} \begin{matrix} C_2 \\ \oplus \\ C_{n-1} \end{matrix} \xleftarrow{H_3} \cdots$$

$$\cdots \xleftarrow{H_{2l}} \begin{matrix} C_{2l} \\ \oplus \\ C_{2l+1} \end{matrix} \xleftarrow{H_{2l+1}} \begin{matrix} C_{2l+1} \\ \oplus \\ C_{2l} \end{matrix} \xleftarrow{H_{2l+2}} \cdots \tag{14}$$

$$\cdots \xleftarrow{H_{n-1}} \begin{matrix} C_{n-1} \\ \oplus \\ C_2 \end{matrix} \xleftarrow{H_n} \begin{matrix} C_n \\ \oplus \\ C_1 \end{matrix} \xleftarrow{H_{n+1}} C_0 \longleftarrow 0,$$

that is

$$0 \longleftarrow A_0 \xleftarrow{H_1} A_1 \xleftarrow{H_2} \cdots \xleftarrow{H_{2l}} A_{2l} \xleftarrow{H_{2l+1}} A_{2l+1} \xleftarrow{H_{2l+2}} \cdots \xleftarrow{H_{4l}} A_{4l} \xleftarrow{H_{4l+1}} A_{4l+1} \longleftarrow 0. \tag{15}$$

Consider the decomposition of A into the sum of even and odd components:

$$A = A_{\mathrm{ev}} \oplus A_{\mathrm{odd}}, \qquad A_{\mathrm{ev}} = C_{\mathrm{ev}} \oplus C^*_{\mathrm{odd}}, \qquad A_{\mathrm{odd}} = C_{\mathrm{odd}} \oplus C^*_{\mathrm{ev}}.$$

Then both d and F also decompose into homogeneous components

$$\begin{aligned} d_{\mathrm{ev}}&: A_{\mathrm{ev}} \to A_{\mathrm{odd}}, & d_{\mathrm{odd}}&: A_{\mathrm{odd}} \to A_{\mathrm{ev}}, \\ F_{\mathrm{ev}}&: A_{\mathrm{ev}} \to A_{\mathrm{ev}}, & F_{\mathrm{odd}}&: A_{\mathrm{odd}} \to A_{\mathrm{odd}}. \end{aligned}$$

Evidently

$$A^*_{\mathrm{odd}} = A_{\mathrm{ev}}, \qquad A^*_{\mathrm{ev}} = A_{\mathrm{odd}},$$

and the operator H has the following matrix form:

$$H_{\mathrm{ev}} = \begin{pmatrix} 0 & d_{\mathrm{ev}} \\ d^*_{\mathrm{ev}} & F_{\mathrm{odd}} \end{pmatrix} : A_{\mathrm{ev}} = \begin{matrix} C_{\mathrm{ev}} \\ \oplus \\ C^*_{\mathrm{odd}} \end{matrix} \to \begin{matrix} C^*_{\mathrm{ev}} \\ \oplus \\ C_{\mathrm{odd}} \end{matrix} = A_{\mathrm{odd}} = A^*_{\mathrm{ev}},$$

$$H_{\mathrm{odd}} = \begin{pmatrix} F_{\mathrm{ev}} & d^*_{\mathrm{odd}} \\ d_{\mathrm{odd}} & 0 \end{pmatrix} : A_{\mathrm{odd}} = \begin{matrix} C^*_{\mathrm{ev}} \\ \oplus \\ C_{\mathrm{odd}} \end{matrix} \rightarrow \begin{matrix} C_{\mathrm{ev}} \\ \oplus \\ C^*_{\mathrm{odd}} \end{matrix} = A_{\mathrm{ev}} = A^*_{\mathrm{odd}}\,.$$

Thus we receive the following exact sequence

$$\dots \xleftarrow{H_{\mathrm{ev}}} A_{\mathrm{ev}} \xleftarrow{H_{\mathrm{odd}}} A^*_{\mathrm{ev}} \xleftarrow{H_{\mathrm{ev}}} A_{\mathrm{ev}} \xleftarrow{H_{\mathrm{odd}}} A^*_{\mathrm{ev}} \xleftarrow{H_{\mathrm{ev}}} \dots, \tag{16}$$

which can be split.

Let

$$A_{\mathrm{ev}} = B_0 \oplus B_1, \qquad A^*_{\mathrm{ev}} = B^*_0 \oplus B^*_1 \tag{17}$$

be the decomposition of the free module A_{ev} into the direct sum of two projective summand such that $B_0 = \ker H_{\mathrm{ev}}$.

This decomposition, generally speaking, is not unique but

$$B^*_1 = \mathbf{Ann}(B_0), \qquad B_1 = H_{\mathrm{ev}}/B_0, \qquad B^*_0 = H^*_{\mathrm{ev}}/B^*_1.$$

Then the matrix H_{ev} has the following form:

$$H_{\mathrm{ev}} = \begin{pmatrix} 0 & \alpha_{\mathrm{ev}} \\ 0 & \beta_{\mathrm{ev}} \end{pmatrix}.$$

Since the operator H_{ev} is selfadjoint, $H^*_{\mathrm{ev}} = H_{\mathrm{ev}}$, we have

$$H_{\mathrm{ev}} = \begin{pmatrix} 0 & 0 \\ 0 & \beta_{\mathrm{ev}} \end{pmatrix}, \qquad \beta^* = \beta.$$

Whereas the sequence (16) is exact, we have

$$B^*_1 = \mathrm{Im}\, H_{\mathrm{ev}} = \mathrm{Im}\, \beta_{\mathrm{ev}} = \mathrm{Ker}\, H_{\mathrm{odd}}\,.$$

Hence, the matrix of the operator H_{odd} has the following form:

$$H_{\mathrm{odd}} = \begin{pmatrix} \alpha_{\mathrm{odd}} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \alpha^* = \alpha.$$

Thus, we receive two selfadjoint isomorphisms

$$\beta_{\mathrm{ev}}: B_1 \rightarrow B^*_1, \qquad \alpha_{\mathrm{odd}}: B^*_0 \rightarrow B_0,$$

which are defined functorially, that is independently from the choice of decomposition of the module A_{ev}, into direct sum (17). Consider the direct sum of β_{ev} and α_{odd}:

$$S = \beta_{\mathrm{ev}} \oplus \alpha_{\mathrm{odd}}: (B_1 \oplus B^*_0) = M \rightarrow M^* = (B_1 \oplus B^*_0)^*.$$

Clearly, the module $M = (B_1 \oplus B^*_0)$ is free. Hence, the selfadjoint isomorphism $S: M \rightarrow M^*$ defines an element $[M, S]$ from Hermitian K-theory for the ring Λ:

$$\mathrm{sign}(C, d, D) = [M, S] \in K^h_0(\Lambda). \tag{18}$$

3. Algebraic Poincaré Complexes over C^*-Algebras

In the case when the algebra Λ is a C^*-algebra, formula (18) for calculating the symmetric signature of an algebraic Poincaré complex, can be significally simplified. Let M be a free (finite generated) Λ-module and let $S\colon M^* \to M$ be a selfadjoint isomorphism. Let $(e_1, \dots, e_N)$ be a free basis in the module M.

By using a scalar product in the spaces C_k, we can identify the space C_k^* with the space C_k,

$$\phi_k\colon\ C_k^* \to C_k. \tag{19}$$

Then the formally adjoint operator d_k^* can be identified with the adjoint operator with respect to this scalar product.

Then, by definition (13), the space $A_k = C_k \oplus C_{n-k+1}$ is isomorphic to $A_{n-k+1} = C_{n-k+1} \oplus C_k$ by the isomorphism $T_k\colon A_k \to A_{n-k+1}$, which is defined by the matrix

$$T_k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the operator H_k in the complex (12) satisfies the condition

$$H_k^* = T_k H_{n-k+2} T_{k-1}.$$

Put

$$A = \bigoplus_{k=0}^{n+1} A_k = A_{\mathrm{ev}} \oplus A_{\mathrm{odd}},$$

where

$$A_{\mathrm{ev}} \bigoplus_{k=0}^{2l} A_{2k}, \qquad A_{\mathrm{odd}} = \bigoplus_{k=0}^{2l} A_{2k+1}.$$

Put

$$H = \bigoplus_{k=0}^{n+1} H_k\colon A \to A, \qquad T = \bigoplus_{k=0}^{n+1} T_k\colon A \to A.$$

Then

$$\begin{aligned} H(A_{\mathrm{ev}}) &\subset A_{\mathrm{odd}}, & H(A_{\mathrm{odd}}) &\subset A_{\mathrm{ev}}, \\ T(A_{\mathrm{ev}}) &\subset A_{\mathrm{odd}}, & T(A_{\mathrm{odd}}) &\subset A_{\mathrm{ev}}. \end{aligned}$$

It is clear that

$$H^*(A_{\mathrm{ev}}) \subset A_{\mathrm{odd}}, \qquad H^*(A_{\mathrm{odd}}) \subset A_{\mathrm{ev}}.$$

Since the complex (12) is acyclic, the operator $H + H^*$ is isomorphism. The operator $G = TH + HT$ keeps the space A_{ev} invariant. Put

$$G_{\mathrm{ev}} = G\big|_{A_{\mathrm{ev}}}\colon A_{\mathrm{ev}} \to A_{\mathrm{ev}}. \tag{20}$$

It is clear that the operator G_{ev} is self-adjoint and inverse. More of that, since $A_{\mathrm{ev}} = \bigoplus_{k=0}^{n} C_k =$, we can check that $G_{\mathrm{ev}} = d + d^* + F$.

We can also check that

$$[C, d + d^* + F] = \operatorname{sign}(C, d, D) \in K_0^h(\Lambda). \tag{21}$$

Similar formula in different situation was considered in [5].

For example, for complex numbers (that is, when $\Lambda = C$) for a proper choice of basis in the space $C = \bigoplus_{k=0}^{n} C_k$ the operator d has the following matrix form:

$$d = \begin{pmatrix} 0 & 0 & \tilde{d} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the operator $\tilde{d}$ being invertible.

Then the operator F has the following matrix form:

$$F = \begin{pmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & 0 \\ F_{13} & 0 & 0 \end{pmatrix}$$

and $\tilde{d}F_{13} + F_{31}\tilde{d}^* = 0$. Hence

$$d + d^* + F = \begin{pmatrix} F_{11} & F_{21} & \tilde{d} + F_{31} \\ F_{12} & F_{22} & 0 \\ \tilde{d}^* + F_{13} & 0 & 0 \end{pmatrix}.$$

Therefore

$$\operatorname{sign}(d + d^* + F) = \operatorname{sign}(F_{22}) = \operatorname{sign}(H(F)).$$

4. Almost Acyclic Complexes

Now and below, all linear spaces are assumed to be finite-dimensional over the complex numbers field supplied with a Hermitian structure. Then any linear operator

$$F\colon_1 \to_2 \tag{22}$$

is automatically bounded, that is $\|F\| < \infty$.

We shall consider grade linear spaces $C = \bigoplus_{i=0}^{n} C_i$, usually the gradation will be considered modulo 2, which can be determined by an involutive operator $\eta\colon C \to C$, $\eta_{|C_i} = (-1)^i$. Let us define the so-called (α, A)-*almost chain complex.*

DEFINITION 1. The pair (C, d):

$$(C, d)\colon \quad C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_n} C_n, \tag{23}$$

is called (α, A)-almost chain complex if it satisfies the following conditions:

$$\|d\| \leqslant A, \qquad \|d^2\| \leqslant \alpha.$$

The operator d is called the differential of the almost chain complex (C, d). The pair (α, A) measures the difference of the almost chain complex from the chain complexes. Further, without loss of generality, we assume that $A > 1$ ('large' variable) and $0 < \alpha < 1$ ('small' variable).

Starting from a given pair (α, A), we shall construct new pairs as functions from the first one: $(\alpha', A') = F(\alpha, A)$, where the pair of functions $F = (F_s(\alpha, A), F_l(A))$ is chosen from a class of so-called admissible functions $\mathcal{F}$. By definition, the function $F = (F_s(\alpha, A), F_l(A))$ is called admissible, that is $F = (F_s, F_l) \in \mathcal{F}$, if $F_l(A)$ is a polynomial with positive coefficients of the variable A and its positive rational powers A^r and the function $F_s(\alpha, A)$ is a polynomial with positive coefficients of variables A and α and their positive rational powers A^{r_1}, α^{r_2} with the addition property $F_s(\alpha, A) = 0$, when $\alpha = 0$. Similarly, one can define the admissible functions of the group of 'small' variables α_i and the group of 'large' variables A_j.

Now we define the almost chain homomorphism of almost chain complexes.

DEFINITION 2. The homomorphism

$$f\colon C^{(1)} \to C^{(2)} \tag{24}$$

from (α_1, A_1)-almost chain complex $(C^{(1)}, d^{(1)})$ to (α_2, A_2)-almost chain complex $(C^{(2)}, d^{(2)})$ is called (α_3, A_3)-almost chain homomorphism if f is the graded operator such that

$$\|f\| \leqslant A_3, \qquad \|fd^{(1)} - d^{(2)}f\| \leqslant \alpha_3.$$

On the homogeneous summands this means that the following diagram is almost commute

$$\begin{tikzcd}
C_0^{(1)} \arrow[d, "f_0"] & C_1^{(1)} \arrow[l, "d_1^{(1)}"'] \arrow[d, "f_1"] & \cdots \arrow[l, "d_2^{(1)}"'] & C_n^{(1)} \arrow[l, "d_n^{(1)}"'] \arrow[d, "f_n"] \\
C_0^{(2)} & C_1^{(2)} \arrow[l, "d_1^{(2)}"'] & \cdots \arrow[l, "d_2^{(2)}"'] & C_n^{(2)} \arrow[l, "d_n^{(2)}"']
\end{tikzcd}.$$

Usually, we shall assume that $(\alpha_3, A_3) = F(\alpha_1, \alpha_2; A_1 A_2)$ for an admissible function F.

Consider the *cone* of the mapping (2), that is the new complex

$$\mathrm{Cone}(f) = C^{(1)} \oplus C^{(2)} \tag{25}$$

with differential

$$D\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f) \tag{26}$$

defined as the matrix

$$D = \begin{pmatrix} d^{(1)} & f \\ 0 & -d^{(2)} \end{pmatrix}.$$

It is clear that the pair $(\mathrm{Cone}(f), D)$ is (α_4, A_4)-almost chain complex if the Hermitian structure is defined as the direct sum of Hermitian structures. The variables (α_4, A_4) can be expressed by the formulas

$$A_4 = A_1 + A_2 + A_3, \qquad \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3.$$

This means that there is an admissible function $F \in \mathcal{F}$ such that $(\alpha_4, A_4) = F(\alpha_1, \alpha_2, \alpha_3; A_1, A_2, A_3)$.

DEFINITION 3. We shall say that (α, A)-almost chain complex (C, d) is F-almost acyclic if there is an admissible function $F(\alpha, \delta, A) \in \mathcal{F}$ such that the following condition holds: if $\|dx\| \leqslant \delta\|x\|$, then there is an element y such that

$$\|y\| \leqslant F_l(A)\|x\|, \tag{27}$$

$$\|x - dy\| \leqslant F_s(\alpha, \delta; A)\|x\|. \tag{28}$$

Since $\|dx\| \leqslant A\|x\|$ the second inequality of (16), we can replace for stronger one:

$$\|x - dy\| \leqslant F_s(\alpha, \min\{\delta, A\}; A)\|x\|.$$

Hence, we can choose as an admissible function $F \in \mathcal{F}$ such that all monomials with nontrivial powers of δ have the same power δ^r, $0 < r \leqslant r_0 < 1$ for any sufficiently small number r_0.

Definition 3 can be loosened by considering only x with the condition $\|x\| = 1$. Indeed, if $\|x\| = 0$, that is $x = 0$, the condition of Definition 3 holds automatically. Let x be any element with condition $\|dx\| \leqslant \delta\|x\|$. Put $z = x/\|x\|$. We have $\|dz\| \leqslant \delta = \delta\|z\|$. Therefore, there is an element u such that

$$\|u\| \leqslant A\|z\| = A,$$

$$\|du - z\| \leqslant F_s(\alpha, \delta; A)\|z\| = F_s(\alpha, \delta; A),$$

or

$$\left\| du - \frac{x}{\|x\|} \right\| \leqslant F_s(\alpha, \delta; A).$$

Hence, putting $y = \|x\|u$, we have

$$\|y\| = \|x\| \cdot \|u\| \leqslant A\|x\|,$$
$$\|dy - x\| = \|d(\|x\|u) - x\| \leqslant F_s(\alpha, \delta; A)\|x\|.$$

Finally, we can loosen the conditions of Definition 3 in the following way:

DEFINITION 4. We shall say that (α, A)-almost chain complex (C, d) is F-almost acyclic if there is an admissible function $F(\alpha, \delta; A) \in \mathcal{F}$ and positive real number k such that the following condition holds: if $\|dx\| \leqslant \delta\|x\|$, then there is an element y such that

$$\|y\| \leqslant F_l(A)\|x\|, \tag{29}$$
$$\|x - dy\| \leqslant F_s(\alpha, \delta; A)\|x\|^k. \tag{30}$$

Following Definition 3 (or 4), we can ask what does an F-almost exact sequence of almost homomorphisms mean. For example, the sequence

$$0 \to C_1 \xrightarrow{i} C \xrightarrow{j} C_2 \to 0$$

of almost homomorphisms is called F-almost exact if, as the chain complex, it is F-almost acyclic (α, A)-almost chain complex.

Definition 3 is justified by the following theorem:

THEOREM 1. *Let (C, d) be an $F(\alpha, \delta; A)$-almost acyclic complex with*

$$F_s\big(\alpha, \sqrt{\alpha}; A\big) + \big(F_l(A)\big(AF_s(\alpha, \sqrt{\alpha}; A) + \sqrt{\alpha}\,\big)\big)^{1/2} < 1. \tag{31}$$

Then the operator $d + d^*$ is invertible. More of that, the norm of the inverse $(d + d^*)^{-1}$ is estimated with number B such that

$$\begin{aligned} &F_s\left(\alpha, \sqrt{\alpha + A\frac{1}{B}} + \frac{1}{B}; A\right) + \\ &\quad + \left(F_l(A)\left(AF_s\left(\alpha, \sqrt{\alpha + A\frac{1}{B}} + \frac{1}{B}; A\right) + \sqrt{\alpha + A\frac{1}{B}}\right)\right)^{1/2} < 1. \end{aligned} \tag{32}$$

Proof. Consider an element x such that

$$\|(d + d^*)x\| \leqslant \delta\|x\|. \tag{33}$$

We have

$$\|(d^2 + dd^*)x\| \leqslant A\delta\|x\|.$$

Hence

$$\|dd^*x\| \leqslant \|d^2x\| + \|(d^2 + dd^*)x\| \leqslant \alpha\|x\| + A\delta\|x\|$$

and so

$$\|(dd^*x, x)\| \leqslant (\alpha_A\delta)\|x\|^2$$

or

$$\|d^*x\| \leqslant \sqrt{\alpha + A\delta}\,\|x\|.$$

From (33), we obtain

$$\|dx\| \leqslant (\sqrt{\alpha + A\delta} + \delta)\|x\|.$$

From the conditions of acyclicity, for some element y we have that

$$\|x - dy\| \leqslant F_s(\alpha, \sqrt{\alpha + A\delta} + \delta; A)\|x\|, \tag{34}$$
$$\|y\| \leqslant F_l(A)\|x\|. \tag{35}$$

Hence,

$$\|d^*x - d^*dy\| \leqslant AF_s(\alpha, \sqrt{\alpha + A\delta} + \delta; A)\|x\|,$$

or

$$\|d^*dy\| \leqslant F_l(A)\big(AF_s(\alpha, \sqrt{\alpha + A\delta} + \delta; A) + \sqrt{\alpha + A\delta}\,\big)\|x\|$$

and so

$$\|dy\| \leqslant \big(F_l(A)\big(AF_s(\alpha, \sqrt{\alpha + A\delta} + \delta; A) + \sqrt{\alpha + A\delta}\,\big)\big)^{1/2}\|x\|.$$

From (34), we have

$$\|x\| \leqslant \Big(\big(F_l(A)\big(AF_s(\alpha, \sqrt{\alpha + A\delta} + \delta; A) + \sqrt{\alpha + A\delta}\,\big)\big)^{1/2} + \\ + F_s(\alpha, \sqrt{\alpha + A\delta}; A)\Big)\|x\|.$$

Thus, we have

$$\Big(\big(F_l(A)\big(AF_s(\alpha, \sqrt{\alpha + A\delta} + \delta; A) + \sqrt{\alpha + A\delta}\,\big)\big)^{1/2} + \\ + F_s(\alpha, \sqrt{\alpha + A\delta}; A)\Big) \geqslant 1. \qquad \square$$

The inverse statement also holds:

THEOREM 2. *Let (C, d) be a (α, A)-almost chain complex and assume that the operator $d + d^*$ is invertible and $\|(d + d^*)^{-1}\| \leqslant F_l(A)$ for some admissible function $F \in \mathcal{F}$. Then there is an admissible function $F' \in \mathcal{F}$, which does depend only on the choice of F and does not depend on the complex (C, d) and (C, d) is $F'(\alpha, \delta; A)$-almost acyclic complex.*

Proof. Let x be an element such that $\|dx\| \leqslant \delta\|x\|$. Since $d + d^*$ is invertible operator, for some element y we have

$$x = (d + d^*)y, \qquad \|y\| \leqslant F_l(A)\|x\|.$$

Therefore

$$\|dd^*y\| \leqslant \|dx\| + \|d^2y\| \leqslant (\alpha F_l(A) + \delta)\|x\|.$$

Hence

$$\|d^*y\| \leqslant \sqrt{(\alpha F_l(A) + \delta)F_l(A)}\|x\|.$$

Thus, we have

$$\begin{aligned}&\|x - dy\| \leqslant F_l(A)\sqrt{\alpha} + \sqrt{\delta F_l(A)}\|x\|,\\ &\|y\| \leqslant F_l(A)\|x\|.\end{aligned}$$

□

THEOREM 3. *For any admissible function $F \in \mathcal{F}$, there is an admissible function $F' \in \mathcal{F}$ such that for any F-almost exact sequence*

$$0 \to C_1 \xrightarrow{i} C \xrightarrow{j} C_2 \to 0 \tag{36}$$

of (α, A)-almost complexes and (α, A)-homomorphisms such that

$$\begin{aligned}&\|i\| \leqslant A, \qquad \|j\| \leqslant A,\\ &\|id_1 - di\| \leqslant \alpha,\\ &\|jd - d_2j\| \leqslant \alpha,\end{aligned}$$

the following properties hold:

(1) *If (C_1, d_1) and (C_2, d_2) are $F(\alpha, \delta; A)$-almost acyclic chain complexes, then (C, d) is $F'(\alpha, \delta; A)$-almost acyclic chain complex.*
(2) *If (C_2, d_2) and (C, d) are $F(\alpha, \delta; A)$-almost acyclic chain complexes, then (C_1, d_1) is $F'(\alpha, \delta; A)$-almost acyclic chain complex.*
(3) *If (C_1, d_1) and (C, d) are $F(\alpha, \delta; A)$-almost acyclic chain complexes, then (C_2, d_2) is $F'(\alpha, \delta; A)$-almost acyclic chain complex.*

Proof. Let $x \in C$, $\|dx\| \leqslant \delta\|x\|$. Then $\|jdx\| \leqslant A\delta\|x\|$. Hence

$$\|d_2jx\| \leqslant \|jdx\| + \|(d_2j - jd)x\| \leqslant (A\delta + \alpha)\|x\|.$$

Since (C_2, d_2) is an almost acyclic complex, there is an element $y \in C_2$ such that

$$\begin{aligned}&\|y\| \leqslant A\|jx\| \leqslant A^2\|x\|,\\ &\|d_2y - jx\| \leqslant F_s(\alpha, A\delta + \alpha; A)\|x\| = F_s^{(1)}(\alpha, \delta; A)\|x\|.\end{aligned}$$

Since sequence (2) is F-almost exact, there is an element $z \in C$ such that

$$\begin{aligned}&\|z\| \leqslant A\|y\| \leqslant A^3\|x\|,\\ &\|jz - y\| \leqslant F_s(\alpha, 0; A)\|y\| \leqslant F_s(\alpha, 0; A)A^2\|x\|.\end{aligned}$$

Then

$$\begin{aligned}
\|j(dz-x)\| &= \|jdz - jx\| \\
&\leqslant \|d_2jz - jx\| + \|(d_2j - jd)z\| \\
&\leqslant \alpha\|z\| + \|d_2jz - d_2y\| + \|d_2y - jx\| \\
&\leqslant \alpha\|z\| + A\|jz - y\| + F_s^1(\alpha, \delta; A)\|x\| \\
&\leqslant \alpha A^3\|x\| + A^3F_s(\alpha, 0; A)\|x\| + F_s^1(\alpha, \delta; A)\|x\| \\
&\leqslant F_s^{(2)}(\alpha, \delta, A)\|x\|.
\end{aligned}$$

From F-exactness of sequence (2), again there is an element $u \in C_1$ such that

$$\begin{aligned}
&\|u\| \leqslant A\|dz - x\| \leqslant A(\|dz\| + \|x\|) \leqslant A(A^3+1)\|x\|; \\
&\|iu + dz - x\| \leqslant F_s\left(\alpha, F_s^{(2)}(\alpha, \delta; A)\frac{\|x\|}{\|dz-x\|}; A\right)\|dz - x\| \\
&\qquad \leqslant F_s\left(\alpha, F_s^{(2)}(\alpha, \delta; A)\frac{\|x\|}{\|dz-x\|}; A\right)\frac{\|dz-x\|}{\|x\|}\|x\| \\
&\qquad = F_s^{(3)}\left(\alpha, \delta, \frac{\|dz-x\|}{\|x\|}; A\right)\|x\| \\
&\qquad \leqslant F_s^{(3)}(\alpha, \delta, A^3+1; A)\|x\| = F_s^{(4)}(\alpha, \delta; A)\|x\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|diu\| &\leqslant \|diu - d(dz-x)\| + \|d^2z\| + \|dx\| \\
&\leqslant AF_s^{(4)}(\alpha, \delta; A)\|x\| + \alpha A^3\|x\| + \delta\|x\|; \\
\|id_1u\| &\leqslant \|diu\| + \|(id_1 - di)u\| \\
&\leqslant F_s^{(5)}(\alpha, \delta; A)\|x\| = F_s^{(5)}(\alpha, \delta; A)\frac{\|x\|}{\|d_1u\|}\|d_1u\|.
\end{aligned}$$

Again from almost exactness, we have that

$$\begin{aligned}
\|d_1u\| &\leqslant F_s\left(\alpha, F_s^{(5)}(\alpha, \delta; A)\frac{\|x\|}{\|d_1u\|}; A\right)\|d_1u\| \\
&= F_s\left(\alpha, F_s^{(5)}(\alpha, \delta; A)\frac{\|x\|}{\|d_1u\|}; A\right)\frac{\|d_1u\|}{\|x\|}\|x\| \\
&= F_s^6\left(\alpha, \delta; \frac{\|d_1u\|}{\|x\|}, A\right)\|x\| \\
&\leqslant F_s^6(\alpha, \delta; A^2(A^3+1), A)\|x\| \\
&\leqslant F_s^7(\alpha, \delta; A)\|x\|.
\end{aligned}$$

From the acyclicity of (C_1, d_1), we have that there is an element $v \in C_1$ such that

$$\begin{aligned}
&\|v\| \leqslant A\|u\|; \\
&\|d_1v - u\| \leqslant F_s\left(\alpha, F_s^{(7)}t(\alpha, \delta; A)\frac{\|x\|}{\|u\|}; A\right)\|u\|
\end{aligned}$$

or

$$\|id_1v - iu\| \leqslant AF_s\left(\alpha,\, F_s^{(7)}(\alpha, \delta; A)\frac{\|x\|}{\|u\|}; A\right)\|u\|.$$

Hence,

$$\begin{aligned}\|div - iu\| &\leqslant \|div - id_1v\| + \|id_1v - iu\| \\ &\leqslant \left(A\alpha + AF_s\left(\alpha,\, F_s^{(7)}(\alpha, \delta; A)\frac{\|x\|}{\|u\|}; A\right)\right)\|u\|.\end{aligned}$$

Thus

$$\begin{aligned}\|d(z - iv) - x\| &\leqslant \|div - iu\| + \|iu - dz + x\| \\ &\leqslant \left(A\alpha + AF_s\left(\alpha,\, F_s^{(7)}(\alpha, \delta; A)\frac{\|x\|}{\|u\|}; A\right)\right)\|u\| + \\ &\quad + F_s^{(4)}(\alpha, \delta; A)\|x\| \\ &\leqslant \left(A\alpha + AF_s\left(\alpha,\, F_s^{(7)}(\alpha, \delta; A)\frac{\|x\|}{\|u\|}; A\right)\right)\frac{\|u\|}{\|x\|}\|x\| + \\ &\quad + F_s^{(4)}(\alpha, \delta; A)\|x\| \\ &\leqslant F_s^{(8)}\left(\alpha, \delta; \frac{\|u\|}{\|x\|}, A\right)\|x\| \\ &\leqslant F_s^{(9)}(\alpha, \delta; A)\|x\|.\end{aligned}$$

Also, we have that

$$\|z - iv\| \leqslant A^3\|x\| + A^3(A^3 + 1)\|x\| = F_l(A)\|x\|. \qquad \square$$

THEOREM 4. *Let $C = \bigoplus_{ij} C_{ij}$ be a bigraded complex with two almost differentials d_1, d_2 of degrees correspondingly $(1, 0)$ and $(0, 1)$, $d_1d_2 + d_2d_1 = 0$. Let $d = d_1 + d_2$. If $(C, d_1$ is an $F_s(\alpha, \delta; A)$-almost acyclic chain complex, then (C, d) is also an $F_s'(\alpha, \delta; A)$-almost acyclic chain complex for the proper admissible function F'.*

Proof. The restriction of the differential d_1 to a homogeneous summand $C_k = \bigoplus_{j=k} C_{ij}$ also gives an $F_s(\alpha, \delta; A)$-almost acyclic complex. Hence, we can apply Theorem 3. $\square$

COROLLARY 1. *In particular, if (C, d) is an $F_s(\alpha, \delta; A)$-almost acyclic chain complex, then the new complex $(C \otimes C_1, d \otimes 1)$ is also an $F_s(\alpha, \delta; A)$-almost acyclic chain complex.*

Let $Cone(f)$ denote the $F(\alpha, A)$-almost chain complex defined by formulas (15) and (16).

COROLLARY 2. *Let (C, d) be an (α, A)-almost chain complex and $1\colon C \to C$ be an identical homomorphism. Then the cone Cone(1) is an $F_s(\alpha, \delta; A)$-almost acyclic chain complex.*

THEOREM 5. *If Cone(f) and Cone(g) are $F(\alpha, A)$-almost acyclic chain complexes, then Cone(gf) again is an $F'(\alpha, A)$-almost acyclic chain complex for the proper admissible function $F' \in \mathcal{F}$, which does not depend on the choice of complexes C_k and homomorphisms f and g.*

Proof. Let the differentials of the complexes $Cone(f)$, $Cone(g)$ and $Cone(gf)$ be D_1, D_2 and D, correspondingly. We have

$$D_1 = \begin{pmatrix} d_2 & f \\ 0 & -d_1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} d_3 & g \\ 0 & -d_2 \end{pmatrix}, \qquad D = \begin{pmatrix} d_3 & gf \\ 0 & -d_1 \end{pmatrix}.$$

Consider a vector $z = \binom{x_3}{x_1}$ such that $\|Dz\| \leqslant \delta\|z\|$. This means that

$$\|d_3x_3 + gfx_1\| \leqslant \delta\|z\|, \qquad \|d_1x_1\| \leqslant \delta\|z\|.$$

Then

$$\|d_2fx_1\| \leqslant \|fd_1x_1\| + \|(fd_1 - d_2f)x_1\| \leqslant (\delta A + \alpha)\|z\|.$$

Put $z_2 = \binom{x_3}{fx_1}$. Hence

$$\|D_2(z_2)\| = \left\|\begin{pmatrix} d_3x_3 + gfx_1 \\ -d_2fx_1 \end{pmatrix}\right\| \leqslant 2(\delta(1+A) + \alpha)\frac{\|z\|}{\|z_2\|}\|z_2\|.$$

Since the $Cone(g)$ is almost acyclic complex, there is a vector $u = \binom{u_3}{u_2}$ such that

$$\|u\| \leqslant A\|z_2\|,$$
$$\|z_2 - D_2u\| \leqslant F_s\left(\alpha, 2(\delta(1+A)+\alpha)\frac{\|z\|}{\|z_2\|}; A\right)\|z_2\| \leqslant F_s^{(1)}(\alpha, \delta; A)\|z\|$$

or

$$\|x_3 - d_3u_3 - gu_2\| \leqslant F_s^{(1)}(\alpha, \delta; A)\|z\|,$$
$$\|fx_1 + d_2u_2\| \leqslant F_s^{(1)}(\alpha, \delta; A)\|z\|.$$

Consider the vector $z_1 = \binom{u_2}{x_1}$. We have

$$D_1z_1 = \begin{pmatrix} d_2u_2 + fx_1 \\ -d_1x_1 \end{pmatrix}.$$

Hence,

$$\|D_1z_1\| \leqslant F_s^{(1)}(\alpha, \delta; A)\|z\| + \delta\|z\| = F_s^{(2)}(\alpha, \delta; A)\frac{\|z\|}{\|z_1\|}\|z_1\|.$$

Again, from the almost acyclicity of the $Cone(f)$, there is a vector $v = \binom{v_2}{v_1}$ such that

$$\|v\| \leqslant A\|z_1\|,$$
$$\|z_1 - D_1 v\| \leqslant F_s\left(\alpha, F_s^{(2)}(\alpha, \delta; A)\frac{\|z\|}{\|z_1\|}; A\right)\|z_1\| = F_s^{(3)}(\alpha, \delta; A)\|z\|,$$

that is

$$\|u_2 - d_2 v_2 - f v_1\| \leqslant F_s^{(3)}(\alpha, \delta; A)\|z\|,$$
$$\|x_1 + d_1 v_1\| \leqslant F_s^{(3)}(\alpha, \delta; A)\|z\|.$$

Let

$$w = \begin{pmatrix} u_3 + g v_2 \\ v_1 \end{pmatrix}.$$

Then

$$z - Dw = \begin{pmatrix} x_3 - d_3 u_3 - d_3 g v_2 - g f v_1 \\ x_1 + d_1 v_1 \end{pmatrix}.$$

We have

$$\begin{aligned}
&\|x_3 - d_3 u_3 - d_3 g v_2 - g f v_1\| \\
&\quad \leqslant \|x_3 - d_3 u_3 - g u_2\| + \|g u_2 - d_3 g v_2 - g f v_1\| \\
&\quad \leqslant \|x_3 - d_3 u_3 - g u_2\| + A\|u_2 - d_2 v_2 - f v_1\| + \|d_3 g v_2 - g d_2 v_2\| \\
&\quad \leqslant F_s^{(1)}(\alpha, \delta; A)\|z\| + F_s^{(3)}(\alpha, \delta; A)\|z\| + \alpha\|z\| \\
&\quad \leqslant F_s^{(4)}(\alpha, \delta; A)\|z\|;
\end{aligned}$$

and

$$\|x_1 + d_1 v_1\| \leqslant F_s^{(3)}(\alpha, \delta; A)\|z\|.$$

Together, we have

$$\|z - Dw\| \leqslant F_s^{(4)}(\alpha, \delta; A)\|z\| + F_s^{(3)}(\alpha, \delta; A)\|z\| = F_s^{(5)}(\alpha, \delta; A)\|z\|. \qquad \square$$

5. Almost Algebraic Poincaré Coplexes

The definition of almost algebraic Poincaré complexes is given in a way similar to the definition of algebraic Poincaré complexes from Section 2. Namely, let (C, d) be a (α, A)-almost chain complex

$$C = \bigoplus_{i=0}^{n} C_i, \qquad d = \bigoplus d_i, \quad d^2 = 0$$

or

$$C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_n} C_n.$$

It is convenient to use the grading operator

$$\eta: C \to C, \qquad \eta^* = \eta, \qquad \eta^2 = 1, \qquad \eta\big|_{C_k} = (-1)^k.$$

Then we have $d\eta + \eta d = 0$ or, in general, for the homogeneous homomorphism ξ of degree k we have $\xi\eta = (-1)^k \eta\xi$.

DEFINITION 5. The $F(\alpha, \delta; A)$-almost algebraic Poincaré complex of formal dimension n is the pair $M = ((C, d), D)$, where (C, d) is the (α, A)-almost chain complex, that is d is the homogeneous homomorphism of the dimension -1 such that $\|d^2\| \leqslant \alpha$ and $D: C^* \to C$ is the homogeneous homomorphism of the dimension n

$$\begin{tikzcd}
C_0 & C_1 \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"'] & C_n \arrow[l, "d_n"'] \\
C_n^* \arrow[u, "D"'] & C_{n-1}^* \arrow[l, "d_n^*"'] \arrow[u, "D"'] & \cdots \arrow[l, "d_{n-1}^*"'] & C_0^* \arrow[l, "d_1^*"'] \arrow[u, "D"']
\end{tikzcd}$$

such that

$$\|D\| \leqslant A, \qquad \|Dd^* + dD\eta\| \leqslant \alpha, \qquad D^* = D\eta^{n+1}. \tag{37}$$

The second condition of (3) means that if $d^\# = \eta d^*$, then the homomorphism $D: (C^*, d^\#) \to (C, d)$ is an (α, A)-almost chain homomorphism.

More of that, we require that the cone $Cone(D)$ should be the $F(\alpha, \delta; A)$-almost acyclic chain complex.

It is evident that the new complex $(-M) = ((C, d), -D)$ also is an $F(\alpha, \delta; A)$-almost algebraic Poincaré complex (with 'opposite orientation'). This operation is called changing the orientation of the almost algebraic Poincaré complex.

There are two operations in the class of almost algebraic Poincaré complexes – the direct sum and tensor product. The construction of the direct sum is evident. For construction of the tensor product, we should consider two $F(\alpha, \delta; A)$-almost algebraic Poincaré complexes $M_1 = ((C_1, d_1), D_1)$ and $M_2 = ((C_2, d_2), D_2)$, general by of different formal dimensions n_1 and n_2. Then consider the almost chain complex

$$C = C_1 \otimes C_2, \qquad d = d_1 \otimes 1 + \eta_1 \otimes d_2, \qquad \eta = \eta_1 \otimes \eta_2 \tag{38}$$

and the homomorphism

$$D: C^* \to C, \quad D = (D_1 \otimes D_2)\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2), \tag{39}$$

where

$$\zeta(\eta_1, \eta_2) = \tfrac{1}{2}(1 + \eta_1 + \eta_2 - \eta_1\eta_2).$$

THEOREM 6. *There is an admissible function $F' \in \mathcal{F}$ such that the pair $M = ((C, d), D)$, defined by formulas* (3) *and* (21) *is an $F'(\alpha, \delta; A)$-almost algebraic Poincaré complex.*

Proof. First of all we shall check that (C, d) satisfies the properties of the (α, A)-almost chain complex. We have

$$\begin{aligned} d^2 &= (d_1 \otimes 1 + \eta_1 \otimes d_2)(d_1 \otimes 1 + \eta_1 \otimes d_2) \\ &= d_1^2 \otimes 1 + \eta_1^2 \otimes d_2^2 + d_1\eta_1 \otimes d_2 + \eta_1 d_1 \otimes d_2 \\ &= d_1^2 \otimes 1 + 1 \otimes d_2^2. \end{aligned}$$

Therefore $\|d^2\| \leqslant 2\alpha$.

Now we shall check that D satisfies condition (3). Previously, we checked that

$$\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2)(d_1^* \otimes 1) = (d_1^* \otimes \eta_2)\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2).$$

Really, we have

$$\begin{aligned} &\tfrac{1}{2}(1 \otimes 1 + (-1)^{n_1}\eta_1 \otimes 1 + 1 \otimes \eta_2 + (-1)^{n_1}\eta_1 \otimes \eta_2)(d_1^* \otimes 1) \\ &\quad = (d_1^* \otimes 1)\tfrac{1}{2}(1 \otimes 1 - (-1)^{n_1}\eta_1 \otimes 1 + 1 \otimes \eta_2 - (-1)^{n_1}\eta_1 \otimes \eta_2) \\ &\quad = (d_1^* \otimes \eta_2)\tfrac{1}{2}(1 \otimes 1 + (-1)^{n_1}\eta_1 \otimes 1 + 1 \otimes \eta_2 + (-1)^{n_1}\eta_1 \otimes \eta_2). \end{aligned}$$

Similarly, we have

$$\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2)(1 \otimes d_2^*) = ((-1)^{n_1}\eta_1 \otimes d_2^*)\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2)$$

and

$$\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2)(d_1 \otimes 1) = (d_1 \otimes \eta_2)\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2),$$

$$\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2)(1 \otimes d_2) = ((-1)^{n_1}\eta_1 \otimes d_2)\zeta((-1)^{n_1}\eta_1 \otimes 1, 1 \otimes \eta_2).$$

Now we have

$$\begin{aligned} &Dd^* + dD\eta \\ &\quad = (D_1 \otimes D_2)\zeta(d_1^* \otimes 1 + \eta_1 \otimes d_2^*) + \\ &\qquad + (d_1 \otimes 1 + \eta_1 \otimes d_2)(D_1 \otimes D_2)\zeta\eta \\ &\quad = (D_1 \otimes D_2)(d_1^* \otimes \eta_2 + (-1)^{n_1}\eta_1^2 \otimes d_2^*)\zeta + \\ &\qquad + (d_1 \otimes 1 + \eta_1 \otimes d_2)(D_1 \otimes D_2)\eta\zeta \\ &\quad = ((D_1 \otimes D_2)(d_1^* \otimes \eta_2 + (-1)^{n_1} \otimes d_2^*) + \\ &\qquad + (d_1 \otimes 1 + \eta_1 \otimes d_2)(D_1 \otimes D_2)\eta)\zeta \\ &\quad = ((D_1 d_1^* \otimes D_2\eta_2 + (-1)^{n_1} D_1 \otimes D_2 d_2^*) + \\ &\qquad + (d_1 D_1 \eta_1 \otimes D_2\eta_2 + \eta_1 D_1 \eta_1 \otimes d_2 D_2 \eta_2))\zeta \\ &\quad = ((D_1 d_1^* \otimes D_2\eta_2 + (-1)^{n_1} D_1 \otimes D_2 d_2^*) + \\ &\qquad + (d_1 D_1 \eta_1 \otimes D_2\eta_2 + (-1)^{n_1} D_1 \otimes d_2 D_2 \eta_2))\zeta \\ &\quad = ((D_1 d_1^* + d_1 D_1 \eta_1) \otimes D_2\eta_2)\zeta + \\ &\qquad + (-1)^{n_1}(D_1 \otimes (D_2 d_2^* + d_2 D_2 \eta_2))\zeta. \end{aligned}$$

Hence

$$\begin{aligned}
&\|Dd^* + dD\eta\| \\
&\quad \leqslant \|D_1 d_1^* + d_1 D_1 \eta_1\| \cdot \|D_2\| \cdot \|\zeta\| + \|D_2 d_2^* + d_2 D_2 \eta_2\| \cdot \|D_1\| \cdot \|\zeta\| \\
&\quad \leqslant 2\alpha A = F_s(\alpha; A).
\end{aligned}$$

Finally, we should prove that the $Cone(D)$ is an $F'(\alpha, \delta; A)$-almost acyclic chain complex. The homomorphism D is the $F(\alpha; A)$-almost chain complex $D\colon (C^*, d^\#) \to (C, d)$, where $d^\# = -d^*\eta$.

We shall split the homomorphism D into a composition of two homomorphisms:

$$C^* = C_1^* \otimes C_2^* \xrightarrow{B_1} C_1^* \otimes C_2 \xrightarrow{B_2} C_1 \otimes C_2 = C,$$

where

$$B_2 = D_1 \otimes 1, \qquad B_1 = (1 \otimes D_2)\zeta, \qquad D = B_1 \circ B_2. \tag{40}$$

Define the almost differential

$$\tilde{d}\colon C_1^* \otimes C_2 \to C_1^* \otimes C_2$$

as

$$\tilde{d} = \eta_1 d_1^* \otimes 1 + (-1)^{n_1} \eta_1 \otimes d_2.$$

Then

$$\|\tilde{d}^2\| = \|(\eta_1 d_1^* \otimes 1 + (-1)^{n_1} \eta_1 \otimes d_2)^2)\| = \|\eta_1 d_1^* \eta_1 d_1^* \otimes 1 + 1 \otimes d_2^2\| \leqslant 2\alpha,$$

$$\begin{aligned}
&\|dB_2 - B_2 \tilde{d}\| \\
&\quad = \|(d_1 \otimes 1 + \eta_1 \otimes d_2)(D_1 \otimes 1) - (D_1 \otimes 1)(\eta_1 d_1^* \otimes 1 + (-1)^{n_1} \eta_1 \otimes d_2)\| \\
&\quad = \|(d_1 D_1 \otimes 1 + \eta_1 D_1 \otimes d_2) - (D_1 \eta_1 d_1^* \otimes 1 + (-1)^{n_1} D_1 \eta_1 \otimes d_2)\| \\
&\quad \leqslant \|d_1 D_1 \otimes 1 - D_1 \eta_1 d_1^* \otimes 1\| + \|\eta_1 D_1 \otimes d_2 - (-1)^{n_1} D_1 \eta_1 \otimes d_2\| \\
&\quad = \|d_1 D_1 - D_1 \eta_1 d_1^*\| \leqslant \alpha.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\|B_1 d^\# - \tilde{d} B_1\| \\
&\quad = \|(1 \otimes D_2)\zeta\eta(d_1^* \otimes 1 + \eta_1 d_2^*) - \\
&\qquad - (\eta_1 d_1^* \otimes 1 + (-1)^{n_1} \eta_1 \otimes d_2)(1 \otimes D_2)\zeta\| \\
&\quad = \|(1 \otimes D_2)\eta(d_1^* \otimes \eta_2 + (-1)^{n_1} \otimes d_2^*)\zeta - \\
&\qquad - (\eta_1 d_1^* \otimes D_2 + (-1)^{n_1} \eta_1 \otimes d_2 D_2)\zeta\| \\
&\quad = \|(\eta_1 d_1^* \otimes D_2 + (-1)^{n_1} \eta_1 \otimes D_2 \eta_2 d_2^*)\zeta - \\
&\qquad - (\eta_1 d_1^* \otimes D_2 + (-1)^{n_1} \eta_1 \otimes d_2 D_2)\zeta\| \\
&\quad = \|(-1)^{n_1} \eta_1 \otimes D_2 \eta_2 d_2^* - (-1)^{n_1} \eta_1 \otimes d_2 D_2\| \\
&\quad = \|D_2 \eta_2 d_2^* - d_2 D_2\| \leqslant \alpha.
\end{aligned}$$

Thus we can apply Theorem 5. For this, we should prove that both homomorphisms (40) induced almost acyclic chain complexes $Cone(B_1)$ and $Cone(B_2)$. Consider the first complex $Cone(B_1)$:

$$C_1^* \otimes C_2^* \xrightarrow{B_1} C_1^* \otimes C_2.$$

The differential of the $Cone(B_1)$ is defined by the matrix

$$\begin{aligned}
G_1 &= \begin{pmatrix} \tilde{d} & B_1 \\ 0 & \eta d^* \end{pmatrix} \\
&= \begin{pmatrix} \eta_1 d_1^* \otimes 1 + (-1)^{n_1}\eta_1 \otimes d_2 & (1 \otimes D_2)\zeta \\ 0 & \eta(d_1^* \otimes 1 + \eta_1 \otimes d_2^*) \end{pmatrix} \\
&= \begin{pmatrix} \eta_1 d_1^* \otimes 1 & 0 \\ 0 & \eta_1 d_1^* \otimes \eta_2 \end{pmatrix} + \begin{pmatrix} (-1)^{n_1}\eta_1 \otimes d_2 & (1 \otimes D_2)\zeta \\ 0 & 1 \otimes \eta_2 d_2^* \end{pmatrix} \\
&= \begin{pmatrix} \eta_1 d_1^* \otimes 1 & 0 \\ 0 & \eta_1 d_1^* \otimes \eta_2 \end{pmatrix} + \\
&\quad + \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} (-1)^{n_1}\eta_1 \otimes d_2 & (1 \otimes D_2) \\ 0 & (-1)^{n_1}\eta_1 \otimes \eta_2 d_2^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \\
&= L + K.
\end{aligned}$$

Here

$$\begin{aligned}
K &= \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} (-1)^{n_1}\eta_1 \otimes d_2 & (1 \otimes D_2) \\ 0 & (-1)^{n_1}\eta_1 \otimes \eta_2 d_2^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} K_1 \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
K_1 &= \begin{pmatrix} (-1)^{n_1}\eta_1 \otimes d_2 & (1 \otimes D_2) \\ 0 & (-1)^{n_1}\eta_1 \otimes \eta_2 d_2^* \end{pmatrix} \\
&= \Lambda \begin{pmatrix} 1 \otimes d_2 & (1 \otimes D_2) \\ 0 & 1 \otimes \eta_2 d_2^* \end{pmatrix} \Lambda,
\end{aligned}$$

$$\Lambda = \begin{pmatrix} \zeta((-1)^{n_1}\eta_1, \eta_2) & 0 \\ 0 & \zeta((-1)^{n_1}\eta_1, (-1)^{n_2}\eta_2) \end{pmatrix}.$$

Therefore, K defines the differential of an $F_s(\alpha, \delta; A)$-almost acyclic chain complex since the $Cone(D_2)$ is an $F_s(\alpha, \delta; A)$-almost acyclic chain complex. Thus, by Theorem 4, the cone $Cone(B_1)$ is an $F_s'(\alpha, \delta; A)$-almost acyclic chain complex.

Consider the second complex $Cone(B_2)$: $C_1^* \otimes C_2 \xrightarrow{B_2} C_1 \otimes C_2 = C$. The differential of the $Cone(B_2)$ is defined by the matrix

$$\begin{aligned} G_2 &= \begin{pmatrix} d & B_2 \\ 0 & -\tilde{d} \end{pmatrix} \\ &= \begin{pmatrix} d_1 \otimes 1 + \eta_1 \otimes d_2 & D_1 \otimes 1 \\ 0 & -\eta_1 d_1^* \otimes 1 - (-1)^{n_1} \eta_1 \otimes d_2 \end{pmatrix} \\ &= \begin{pmatrix} d_1 \otimes 1 & D_1 \otimes 1 \\ 0 & -\eta_1 d_1^* \otimes 1 \end{pmatrix} + \begin{pmatrix} \eta_1 \otimes d_2 & 0 \\ 0 & -(-1)^{n_1} \eta_1 \otimes d_2 \end{pmatrix} \\ &= K + L, \end{aligned}$$

where

$$K = \begin{pmatrix} d_1 \otimes 1 & D_1 \otimes 1 \\ 0 & -\eta_1 d_1^* \otimes 1 \end{pmatrix},$$

$$L = \begin{pmatrix} \eta_1 \otimes d_2 & 0 \\ 0 & -(-1)^{n_1} \eta_1 \otimes d_2 \end{pmatrix},$$

$$KL + LK = 0.$$

Hence, by Theorem 4, the cone $Cone(B_2)$ is an $F_s'(\alpha, \delta; A)$-almost acyclic chain complex. □

DEFINITION 6. Let

$$M_1 = ((C_1, d_1), D_1) \quad \text{and} \quad M_2 = ((C_2, d_2), D_2)$$

be two $F(\alpha, \delta; A)$-almost algebraic Poincaré complexes, generally of different formal dimensions n_1 and n_2. Then the pair $M = ((C, d), D)$, defined by formulas (3) and (21) is called the tensor product and we note $M = M_1 \otimes M_2$.

In similar way, we define an almost algebraic Poincaré with boundary.

DEFINITION 7. An $F(\alpha, \delta; A)$-almost algebraic Poincaré complex with boundary of the formal dimension $n + 1$ is a triple $((C, d), (C_0, d_0), D)$, where the pairs (C, d) and (C_0, d_0) are (α, A)-almost chain graded complexes, the inclusion on the direct summand $i: C_0 \to C$ is an (α, A)-almost chain homomorphism that is $\|id_0 - di\| \leqslant \alpha$. Let $j: C \to C/C_0$ be the natural projection and let $k: C/C_0 \to C$ be the bounded operator which is left inverse to the projection j on the orthogonal supplement $(i(C_0))^\perp$. Finally, let $l: C \to C_0$ be the orthogonal projection. Put $\bar{d} = jdk$ which is (α, A)-almost differential on C/C_0. Let $D: C^* \to C$ be the homogeneous homomorphism of the formal dimension $n + 1$ such that

$$\|j(Dd^* + dD\eta)\| \leqslant \alpha \qquad D^* = D\eta^{n+2}. \tag{41}$$

Assume that the homomorphisms

$$jD: C^* \to C/C_0, \qquad Dj^*: (C/C_0)^* \to C \tag{42}$$

induce $F(\alpha, \delta; A)$-almost acyclic chain complexes on the cones $Cone(jD)$ and $Cone(Dj^*)$.

Definition 7 gives the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & C_0 \arrow[r, "i"] & C \arrow[r, "j"] & C/C_0 \arrow[r] & 0 \\ 0 \arrow[r] & (C/C_0)^* \arrow[r, "j^*"] \arrow[ur, "Dj^*"] & C^* \arrow[r, "i^*"] \arrow[u, "D"] \arrow[ur, "jD"'] & C_0^* \arrow[r] & 0. \end{tikzcd} \tag{43}$$

Remark. The second condition in (3) is implied from the first one for sufficiently small α, but we prefer to avoid additional difficulties in our considerations.

LEMMA 1. *Let $M = ((C, d), (C_0, d_0), D)$ be an $F(\alpha, \delta; A)$-almost algebraic Poincare complex with boundary of the formal dimension $n + 1$. Put*

$$D_0 = l(Dd^* + dD\eta)l^*, \qquad D_0: C_0^* \to C_0.$$

Then the pair $((C_0, d_0), D_0)$ is an $F'(\alpha, \delta; A)$-almost algebraic Poincaré complex (without boundary) of the formal dimension n for some admissible function $F' \in \mathcal{F}$ which does not depend on the choice of M.

Proof. It easy to check that $il + kj = 1$. From (3) we have $\|(Dd^* + dD\eta)j^*\| \leqslant \alpha$. Hence

$$\begin{aligned} &\|Dd^* + dD\eta - iD_0i^*\| \\ &\quad \leqslant \|(il + kj)(Dd^* + dD\eta - iD_0i^*)(l^*i^* + j^*k^*)\| \\ &\quad \leqslant \|il(Dd^* + dD\eta)l^*i^* - iliD_0i^*l^*i^*\| + 2\alpha \\ &\quad = \|il(Dd^* + dD\eta)l^*i^* - ilil(Dd^* + dD\eta)l^*i^*l^*i^*\| + 2\alpha = 2\alpha. \end{aligned}$$

In the diagram (43) we can substitute the right space C/C_0 for the homotopy equivalent one $Cone(i) = C \oplus C_0$:

$$\begin{tikzcd} & & & C \oplus C_0 \arrow[d] & \\ 0 \arrow[r] & C_0 \arrow[r, "i"] & C \arrow[r, "j"] \arrow[ur, "\gamma"] & C/C_0 \arrow[r] & 0 \\ 0 \arrow[r] & (C/C_0)^* \arrow[r, "j^*"] \arrow[ur, "Dj^*"] & C^* \arrow[r, "i^*"] \arrow[u, "D"] \arrow[ur, "jD"'] & C_0^* \arrow[r] & 0. \end{tikzcd} \tag{44}$$

Then the diagram

$$\begin{tikzcd} C \arrow[r, "\gamma"] & Cone(i) = C \oplus C_0 \\ (C/C_0)^* \arrow[r, "j^*"] \arrow[u, "Dj^*"] & C^* \arrow[u, "\beta"] \end{tikzcd} \tag{45}$$

with

$$\gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} D \\ -D_0i^* \end{pmatrix}$$

is almost commutative, hence it can be completed to the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & (C, d) & \xrightarrow{\gamma} & (C \oplus C_0, d_c) & \xrightarrow{\gamma'} & (C_0, d_0) & \to & 0 \\ & & \big\uparrow{\scriptstyle Dj^*} & & \big\uparrow{\scriptstyle \beta} & & \big\uparrow{\scriptstyle D_0} & & \\ 0 & \to & ((C/C_0)^*, -\tilde{d}^*\eta) & \xrightarrow{j^*} & (C^*, -d^*\eta) & \xrightarrow{i^*} & (C_0^*, -d_0^*\eta) & \to & 0, \end{array}$$

where

$$\gamma' = (0, -1), \qquad d_c = \begin{pmatrix} d & 0 \\ 0 & d_0 \end{pmatrix}.$$

Hence, to prove that the $Cone(D_0)$ is almost acyclic, it is sufficient to check that the middle homomorphism β induces an almost acyclic cone.

Consider the following almost commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & (C_0 \oplus C_0, \tilde{d}_0) & \xrightarrow{(i\oplus 1)} & (C \oplus C_0, d_c) & \xrightarrow{(j,0)} & (C/C_0, \tilde{d}) & \to & 0 \\ & & \big\uparrow{\scriptstyle 0} & & \big\uparrow{\scriptstyle \beta} & & \big\uparrow{\scriptstyle jD} & & \\ 0 & \to & 0 & \xrightarrow{0} & (C^*, -d^*\eta) & \xrightarrow{1} & (C^*, -d^*\eta) & \to & 0, \end{array} \tag{46}$$

where

$$\tilde{d}_0 = \begin{pmatrix} d_0 & -1 \\ 0 & d_0 \end{pmatrix}.$$

The first and third terms in diagram (46) form almost acyclic cones. Hence, the middle terms also are almost acyclic. □

Denote $\partial M = (C_0, d_0)$. We shall say that almost algebraic Poincaré complex ∂M is bordant to zero. In the common case, consider two almost algebraic Poincaré complexes M_1 and M_2 of formal dimension n. We shall say that these two almost algebraic Poincaré complexes are bordant if there is an almost algebraic Poincaré complex with boundary W of formal dimension $n+1$ such that $\partial W = M_1 \oplus (-M_2)$.

LEMMA 2. *Let M_1, M_2 be two almost algebraic Poincaré complexes with boundary of formal dimension – $n + 1$. Let $f\colon \partial M_1 \to \partial M_2$ be an almost isomorphism such that f^{-1} also is almost isomorphism. Then the so-called 'connected sum' $M = (M_1 \oplus M_2)/\{x \sim f(x)\}$ also is an almost algebraic Poincaré complex* (*without boundary*) *of formal dimension $n + 1$.*

This new almost algebraic Poincaré complex M is denoted by $M = M_1 \cup_f M_2$.

More generally, there is an analog of the connected sum for almost algebraic Poincaré complexes with boundary.

LEMMA 3. *Let M_1, M_2 be two almost algebraic Poincaré complexes with boundary of formal dimension $n + 1$. Assume that the following splitting holds:*

$$\partial M_1 = M_{11} \cup M_{12}, \qquad \partial M_2 = M_{21} \cup M_{22},$$

and there is an almost isomorphism $f: M_{12} \to M_{22}$. *Then the 'connected sum'* $M = (M_1 \oplus M_2)/\{x \sim f(x)\}$ *is an almost algebraic Poincaré complex with boundary of formal dimension* $n+1$. *More of that, the boundary* ∂M *also is the connected sum*

$$\partial M = M_{11} \cup_{f_0} M_{21}, \quad \text{where } f_0 = f|_{\partial M_{12} = \partial M_{11}}.$$

The bordism relation clearly is an equivalence relation for a proper choice of admissible functions. More of that, the bordism relation is compatible with operations of the direct sum and tensor product. Namely, we have the following lemma:

LEMMA 4. *Let* M_1 *and* M_2 *be two almost algebraic Poincaré complexes. If* $\partial M_2 = 0$, *then* $M_1 \otimes M_2$ *is an almost algebraic Poincaré complex with boundary* $\partial M_1 \otimes M_2$.

In the common case, $M_1 \otimes M_2$ *is an almost algebraic Poincaré complex with boundary*

$$\partial(M_1 \otimes M_2) = (\partial M_1 \otimes M_2) \cup_f (M_1 \otimes \partial M_2),$$

where

$$f\colon \partial M_1 \otimes \partial M_2 \to \partial M_1 \otimes \partial M_2$$

is identity.

For the construction of an invariant similar to the signature, we shall introduce a class of elementary almost algebraic Poincaré complexes.

DEFINITION 8. The almost algebraic Poincaré complex

$$M = ((C, d), D), \quad C = \bigoplus_{k=0}^{n} C_k,$$

of formal dimension n is called an elementary almost algebraic Poincaré complex if $C_k = 0$, except one or two middle dimensions. In the case $n = 2k$ the elementary algebraic Poincaré complex coincides with an algebraic Poincaré complex considered in [1]. In the case of $n = 4k$ for sufficient small α, the homomorphism $D: C^*_{2k} \to C_{2k}$ is an invertible selfadjoint operator. Hence, put $\operatorname{sign}(M) = \operatorname{sign}(D)$. In the case $n = 4k+2$, the homomorphism $D: C^*_{2k+1} \to C_{2k+1}$ is an invertible skew-adjoint operator. Therefore put $\operatorname{sign}(M) = \operatorname{sign}(iD)$.

LEMMA 5. *Any almost algebraic Poincaré complex* M *is bordant to an elementary almost algebraic Poincaré complex* M_e *for a proper choice of the admissible function* $F \in \mathcal{F}$ *which does not depend on the choice of,* M *for a given formal dimension* n.

This lemma allows us to define the signature for an arbitrary almost algebraic Poincaré complex for a proper relation between α and A.

DEFINITION 9. Let M be an almost algebraic Poincaré complex of formal dimension n and let M_e be an elementary almost algebraic Poincaré complex which is bordant to M. If $n = 2k$, then put sign M = sign M_e. In the case $n \neq 2k$ we put sign $M = 0$.

The following statement justifies the definition:

LEMMA 6. *Definition* (9) *is correct in that it does not depend on the choice of elementary almost algebraic Poincaré complex M_e which is bordant to M.*

Thus we obtain the signature function defined on the classes of bordant almost algebraic Poincaré complexes, which satisfies the following natural conditions:

LEMMA 7. *The following relations hold:*

$$\operatorname{sign}(M_1 \oplus M_2) = \operatorname{sign}(M_1) + \operatorname{sign}(M_2),$$
$$\operatorname{sign}(M_1 \otimes M_2) = \operatorname{sign}(M_1) \cdot \operatorname{sign}(M_2).$$

6. Construction of an Algebraic Poincaré Complex Associated with a Combinatorial Manifold Equipped with a Vector Bundle

Let X be a compact oriented combinatorial manifold of dimension n and let $\bar{V}$ be a vector bundle over – X. The vector bundle $\bar{V}$ is determined by a continuous projector-valued function $P(x)$, $x \in X$. This means that there is a trivial vector bundle $\bar{N}$, $\bar{N} = X \times C^N$ and a continuous projector-valued mapping

$$P \colon X \times C^N \to X \times C^N. \tag{47}$$

Let us fix a simplicial structure on the manifold X. Denote this by

$$nei(X) = \max_{\sigma \subset X} \#\{\tau \colon \sigma \cap \tau \neq \emptyset\},$$

and

$$s(X) = \min_{\sigma \subset X}(\operatorname{diam}(\sigma)).$$

LEMMA 8. *There is a number $L > nei(X)$ such that for any $\varepsilon > 0$ there is a simplicial subdivision X_∂ on the manifold X such that $s(X_\delta) \leqslant \varepsilon$, $nei(X_\delta) \leqslant L$.*

For arbitrary simplex $\sigma = (a_0, a_1, \ldots, a_k)$, denote by $x(\sigma)$ its central point σ,

$$x(\sigma) = \sum_{j=0}^{k} \frac{1}{n+1} a_j.$$

Assume that the simplicial structure is sufficiently fine such that, for any simplex σ and for any of its face τ, we have the inequality

$$\|P(x(\sigma)) - P(x(\tau))\| \leqslant \varepsilon.$$

Denote by $(C(X, \bar{N}), \bar{d})$ the chain complex with coefficients in the trivial vector bundle $\bar{N}$:

$$C(X, \bar{N}) = C(X) \otimes C^N, \quad \bar{d} = d \otimes C^N.$$

Then the projector-valued function (47) induces the projector $\bar{P}$ in the space $C(X, \bar{N})$:

$$\bar{P}(\sigma \otimes v) = \sigma \otimes P(x(\sigma))v, \quad v \in C^N.$$

Let us define the almost chain complex $(C(X, \bar{V}, \bar{P}), d_{\bar{V},\bar{P}})$ by:

$$C(X, \bar{V}, \bar{P}) = \operatorname{Im} \bar{P}, \quad d_{\bar{V},\bar{P}} = \bar{P}\bar{d}\bar{P}.$$

Let $D: C^*(X) \to C(X)$ be the Poincaré duality homomorphism on the manifold X. By extending it to the chain complex with coefficients in the trivial vector bundle

$$\bar{D} = D \otimes C^N: C^*(X, \bar{N}) \to C(X, \bar{N}),$$

we obtain the algebraic Poincaré complex $((C(X, \bar{N}), \bar{d}), \bar{D})$. Then the almost algebraic Poincaré complex is defined as:

$$D_{\bar{V},\bar{P}}: C^*(X, \bar{V}, \bar{P}) \to C(X, \bar{V}, \bar{P}),$$
$$D_{\bar{V},\bar{P}} = \bar{P}\bar{D}\bar{P}.$$

Still, the pair $((C(X, \bar{V}, \bar{P}), d_{\bar{V},\bar{P}}), D_{\bar{V},\bar{P}})$ determines the almost homomorphism of almost chain complexes.

In order to have that the pair $((C(X, \bar{V}, \bar{P}), d_{\bar{V},\bar{P}}), D_{\bar{V},\bar{P}})$ determines the almost algebraic Poincaré complex, one should check that the cone defined by the homomorphism $D_{\bar{V},\bar{P}}$ makes up an almost acyclic complex. The following statement answers this question.

THEOREM 7. *Consider an oriented combinatorial manifold X and a vector bundle $\bar{V}$ defined by a projector-valued function, P. Then there are such numbers L and $\varepsilon > 0$ that, if the simplicial structure on the manifold X satisfies the conditions $nei(X) \leqslant L$, $\operatorname{diam}(X) \leqslant \varepsilon$, then the pair*

$$M(X, \bar{V}, \bar{P}) = ((C(X, \bar{V}, \bar{P}), d_{\bar{V},\bar{P}}), D_{\bar{V},\bar{P}})$$

is the $F(\varepsilon, L)$-almost algebraic Poincaré complex for the proper admissible function $F \in \mathcal{F}$ which does not depend on the choice of simplicial subdivision on the manifold M.

If X_δ is a fine subdivision with the condition nei$(X_\delta) \leqslant L$, then the almost algebraic Poincaré complex $M(X_\delta, \bar{V}, \bar{P})$ is bordant to the almost algebraic Poincaré complex $M(X, \bar{V}, \bar{P})$.

If P' is another projector-valued function, which defines the same vector bundle $\bar{V}$, then there is a number $\varepsilon_1 < \varepsilon$ such that for a simplicial subdivision X_δ with conditions nei$(X_\delta) < L$, $\operatorname{diam}(X_\delta) \leqslant \varepsilon_1$, the almost algebraic Poincaré complex $M(X_\delta, \bar{V}, \bar{P}')$ is bordant to the almost algebraic Poincaré complex $M(X, \bar{V}, \bar{P})$.

Finally, the following formula similar to the Hirzebruch formula holds:

$$\operatorname{sign}(M(X, \bar{V}, \bar{P})) = 2^{2k}\big\langle L(X)\operatorname{ch}(\bar{V}), [M]\big\rangle. \tag{48}$$

Proof. In reality, we need to prove the statement of the Theorem for the more general case of manifolds with boundary.

Firstly, notice that the manifold X can be decomposed into a finite family of handles. Each handle is homeomorphic to an n-dimensional cube $I^n = I \times I \times \cdots \times I$. Therefore, it is sufficient to prove that the theorem is true for unit interval I and to apply Lemma 4. Thus, the complex $M(I^n, \bar{V}, \bar{P})$ is the almost algebraic Poincaré complex with boundary. Hence, due to Lemma 2, the sphere S^n with the vector bundle $\bar{V}$ induces an almost algebraic Poincaré complex. Further, we apply Lemma 3 for gluing the manifold with boundary by the handles.

Thus, the function sign $M(X, \bar{V}, \bar{P})$ in reality is a function on the bordism group

$$\operatorname{sign}\colon \Omega_*\Big(\bigcup BU(n)\Big) \to Z. \tag{49}$$

To prove the analog of the Hirzebruch formula, it is sufficient to check some algebraic properties of the function (49): namely, additivity with respect to a nonconnected sum of manifolds, additivity with respect to a direct sum of vector bundles and multiplicativity with respect to a Cartesian product of manifolds and tensor product of vector bundles.

All these properties follow from Lemma 7.

Thus, we should check formula (48) only in the case when $X = T^2$ is the two-dimensional torus and $\bar{V}$ is the Hopf bundle over the two-dimensional torus. In this case, there is a simple description of the Hopf bundle.

LEMMA 9. *There is an asymptotic representation ρ of the fundamental group $\pi_1(T^2) = Z \times Z$ such that the correspondent vector bundle $\bar{V}_\rho$ over the torus T^2 has a nontrivial first characteristic Chern class.*

Therefore, it is sufficient to check formula (48) for the vector bundle $\bar{V}_\rho$ induced by the asymptotic representation ρ. This case is the direct consequence of the general Hirzebruch formula for asymptotic representations (see [2, 8–10]), and the fact that the signature $\operatorname{sign}(M(X, \bar{V}_\rho, \bar{P}))$, defined above, coincides with the image of the symmetric signature of the nonconnected manifold for the asymptotic representation ρ.

Really, let $\rho = \{\rho_k\}$ be an asymptotic representation of the fundamental group $\pi = \pi_1(X, x_0)$ of the manifold X with fixed point $x_0 \in X$. Let us fix a simplicial subdivision on the manifold X and a polygonal curve γ_x for each vertex $a \in X$ which connects the initial vertex x_0 with the vertex x. Consider the chart atlas $\mathcal{U} = \{U_a\}$, compounded with the star of vertices $U_a = \operatorname{star}(a)$. Then, on the intersection of two charts $U_{a_0a_1} = U_{a_0} \cap U_{a_1}$, let us consider an 'almost transition function'

$$\varphi_{a_0a_1} = \rho_k(g_{a_0a_1}), \quad g_{a_0a_1} = \gamma_{a_0} \cdot (a_0, a_1) \cdot \gamma_{a_1}^{-1} \in \pi.$$

This means that the bundle $\bar{V}_{\rho_k}$ is defined by the transition functions $\psi^k_{a_0a_1}(x)$, $x \in U_{a_0a_1}$, such that

$$\|\varphi_{a_0a_1} - \psi^k_{a_0a_1}(x)\| = \varepsilon_k \to 0.$$

There is a construction that is equivalent to (48) for the almost algebraic Poincaré complex. Namely put

$$C(X, \bar{V}_{\rho_k}) = \bigoplus_{\sigma} \bar{V}_{\rho_k}(x(\sigma)),$$

where $x(\sigma)$ is the central point of the simplex σ. Let $a(\sigma)$ be a vertex of the simplex σ. Then the bounded operator d is defined by the formula

$$d\colon \bar{V}_{\rho_k}(\sigma) \to \bigoplus_{\sigma_j} \bar{V}_{\rho_k}(\sigma_j),$$

$$d = \bigoplus_j (-1)^j \varphi_{a(\sigma)a(\sigma_j)},$$

where σ_j is the jth face of the simplex σ.

Similarly, one can construct the Poincaré duality operator D.

One can also check that this new definition gives the almost algebraic Poincaré complex

$$(C(X, \bar{V}_{\rho_k}), d, D), \tag{50}$$

which is bordant to (48).

On the other hand, the complex (50) is the tensor product of the universal algebraic Poincaré complex of the manifold X, supplied with a simplicial structure and a asymptotic representation ρ_k. □

References

1. Mishchenko, A. S.: Homotopic invariants of non simply connected manifolds. 1. Rational invariants, *Izvest. Acad. Sci. USSR Ser. Math.* **34**(3) (1970), 501–514, in Russian; English translation: *Math. USSR Izvest.* **4**(3) (1970), 506–519.
2. Mishchenko, A. S. and Solovjov, Ju. P.: Representations of Banach algebras and the Hirzebruch type formulas, *Math. Sb. N.S.* **111**(2) (1980), 209–226, in Russian; English translation: *Math. USSR Sb.* **39**(2) (1981).

3. Connes, A., Gromov, M. and Moscovoci, H.: Conjecture de Novicov et fibrés presque plats, *C. R. Acad. Sci. Paris Sér. I* **310** (1990), 273–277.
4. Hilsum, M. and Skandalis, G.: Invariance par homotopie de la signature a coefficients dans un fibre presque plat, *J. Reine Angew. Math.* **423** (1992), 72–99.
5. Roe, J.: *Index Theory, Coarse Geometry, and Topology of Manifolds*, CBMS, Regional Conf. Ser. in Math. **90**, Conf. Board. Math. Sci., Washington, DC, 1996, p. 62.
6. Ferry, S. C., Ranicki, A. and Rosenberg, J.: A history and survey of the Novikov conjecture, In: *Funct. Anal. on the Eve of the 21st Century*, Vol. II, Progr. in Math. 132, Birkhauser, Basel; *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 1 (Oberwolfach, 1993), London Math. Soc. Lecture Notes Ser. 226, Cambridge Univ. Press, 1995, pp. 7–66.
7. Gromov, M.: Positive curvature, macroscopic dimension, spectral gaps and higher signatures, In: *Funct. Anal. on the Eve of the 21st Century*, Vol. II, Progr. in Math. 132, Birkhauser, Basel, 1995.
8. Manuilov, V. M. and Mishchenko, A. S.: Relations between asymptotic and Fredholm representations, Preprint IHES (1997).
9. Mishchenko, A. S. and Noor, M.: Asymptotic representations of discrete groups, In: *Lie Groups and Lie Algebras. Their Representations, Generalizations and Applications*, Math. Appl. 433. Kluwer Acad., Dordrecht, 1998, pp. 299–312; *Math. Rev.*: 1630585.
10. Manuilov, V. M. and Mishchenko, A. S.: Asymptotic and Fredholm representations of discrete groups, *Math. Sb.* **189**(10) (1998), 53–74, in Russian; English transl. *Sb. Math.* **189**(10) (1998), 1485–1504.
11. Mishchenko, A. S.: *Local Combinatorial Hirzebruch Formula*, International Congress of Mathematicians, Abstracts of Short Communications and Poster Sessions, Berlin, 18–27 August 1998, p. 94.
12. Mishchenko, A. S.: Local combinatorial Hirzebruch formula, In: *International Conference Dedicated to the 90th Anniversary of L. S. Pontryagin, Abstracts, Algebra, Geometry and Topology*, 31 August–6 September (1998), pp. 119–121.
13. Mishchenko, A. S.: Theory of almost algebraic Poincaré complexes and local combinatorial Hirzebruch formula, Preprint IHES/M/98/84 (1998).
14. Mishchenko, A. S.: Local combinatorial Hirzebruch formula, *Trudy MIAN* **224** (1999), 249–263, in Russian; English transl.: *Proc. Steklov Inst. Math.* **224** (1999), 1–13.