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HOMOTOPY INVARIANTS OF NONSIMPLY CONNECTED MANIFOLDS. III. HIGHER SIGNATURES

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Abstract. The homotopy invariance of the higher signatures of nonsimply connected manifolds is proved in this paper. The method of proof is based on the study of absolute invariants of nonsimply connected manifolds similar to algebraic K-theory and on the construction of an analog to intersection theory for Poincaré complexes.

Introduction

The present paper is devoted to a further study of the homotopy invariants of nonsimply connected manifolds which correspond to the obstruction to modifying one manifold until it is homotopically equivalent to another. We will call the following collection of objects a surgery situation: two manifolds M and X, dim $M = \dim X = n$, a vector bundle ξ over the manifold X, a map $f: M \to X$ of degree 1 and an isomorphism ϕ : $\nu(M) \to f^*(\xi)$, where $\nu(M)$ is the normal bundle of the manifold M. As is well known, a surgery situation defines for us a cobordism class in the group $\Omega_n(X, \xi)$. The obstruction to modifying the manifold M until it is homotopically equivalent to the manifold Xis an element of the Wall group $L_n(\pi_1(X))$ denoted by $\theta(M, f, \phi)$.

In the first part we have shown that, roughly speaking, this obstruction is the difference

$$\theta(M, f, \phi) = \sigma(M) - \sigma(X)$$

between two elements $\sigma(M)$ and $\sigma(X)$, each of which now depends only on the manifolds M and X respectively. In addition the element $\sigma(X)$ is a homotopy invariant and is also an invariant of the cobordism of the Eilenberg-Mac Lane space $\Omega_n(K(\pi_1(X), 1))$. More precisely, we have constructed groups $L_n^Q(\pi)$ (which essentially determine the obstruction to modification up to homotopy equivalence, module torsion) and natural homomorphisms $\psi: L_n(\pi) \to L_n^Q(\pi)$ such that $\psi(\theta(M, f, \phi)) = \sigma(M) - \sigma(X)$.

The problem of computing the element $\sigma(M) \in L^Q_n(\pi)$ in terms of suitable invariants remained open. Inasmuch as the element $\sigma(M)$ is an invariant of the cobordism

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 $\Omega_n(K(\pi_1(M), 1))$, it is natural to express this invariant in terms of the characteristic numbers of the manifold M. Novikov [4] and Kasparov [10] have made the conjecture that the invariant $\sigma(M)$ is completely defined by the so-called "higher" signatures of the manifolds

$$\sigma_{x}(M) = \langle L(M) f^{*}(x), [M] \rangle,$$

where L(M) is the total Hirzebruch class for the manifold M, x is an arbitrary element in the rational cohomology of the Eilenberg-Mac Lane space $K(\pi_1(M), 1)$ and $f: M \rightarrow K(\pi_1(M), 1)$ is the natural map which induces an isomorphism of fundamental groups. This is equivalent to the assertion of the homotopy invariance of the "higher" signature.

The present paper is devoted to the proof of this conjecture. Thus our result can be formulated in the following way.

Theorem. In order that the obstruction $\theta(M, f, \phi)$ has finite order in the group $L_n(\pi_1(X))$ it is necessary and sufficient that the "bigher" signatures of the manifolds M and X coincide.

If the rank of the Wall group $L_n(\pi)$ is known, the above theorem enables us to carry out a classification of the smooth structures on a nonsimply connected manifold to within a finite number of smooth structures.

The method of proof is based on the construction of an analog to intersection theory for Poincaré complexes. Let M be a high dimensional manifold, $\pi_1(M) = \pi$, the universal covering of which is highly connected. Let $f: X \to M$ and $g: Y \to M$ be two singular Poincaré complexes. There exists a regular process for constructing the intersection $b: X \cap Y \to M$, and, moreover, if X and Y are smooth manifolds, one obtains as a result a Poincaré complex $X \cap Y$ which is homotopically equivalent to the usual intersection of X and Y. This regular process is based on constructing a series of modifications to the normal bundle of the complex X (which is a smooth manifold with boundary) and has obstruction lying in the Wall group $L_{\dim(X \cap Y)}(\pi_1(X \cap Y))$. In the case when this group is trivial the process of constructing the intersection $X \cap Y$ can be carried out to the end. An example of such a situation would be $\pi_1(X \cap Y) = 1$, dim $X \cap Y = 2k + 1$.

The second essential point in the proof of the homotopy invariance of the "higher" signatures is the construction of the so-called integral absolute invariant $\sigma(M)$ of the manifold M lying in a group denoted by $\Omega_n(\mathbb{Z}[\pi])$ and the proof that the kernel of the map $L_n(\pi) \to \Omega_n(\mathbb{Z}[\pi])$ consists of those elements of finite order.

The plan of the paper is the following. In §1 the essential definitions and theorems are given. §§2-4 are devoted to the algebraic part of the paper, relating to the construction of the invariant $\sigma(M)$. In §§5 and 6 the intersection theory for Poincaré complexes is constructed. Finally the basic conjecture is proved in §7.

We will give the minimum number of references and do not claim originality for the results in \$ 2 and 5.

§1. Summary of essential ideas and results

Let π be a finitely generated group with a finite number of defining relations. Let us denote by $\Lambda = \mathbb{Z}[\pi]$ the group ring of the group π , i.e. the ring of finite integral functions on the group with the following multiplication law:

$$(f * g)x = \sum_{yz=x} f(y)g(z).$$

The group π is naturally embedded in the group of units of the ring Λ by the following formula:

$$g(x) = \begin{cases} 1, & \text{if } x = g, \\ 0, & \text{if } x \neq g., \end{cases}$$

The ring Λ possesses an anticommutative automorphism *: $\Lambda \to \Lambda$ uniquely generated by the map on π given by $*(x) = x^{-1}$.

Let C be an arbitrary right Λ -module. Let C* denote the module of Λ -homomorphisms

$$C^* = \operatorname{Hom}_{\Lambda}(C, \Lambda).$$

The structure of a right Λ -module is given on C^* by

$$(\varphi\lambda)(x) = \lambda^* \varphi(x), \qquad \lambda \subset \Lambda, \qquad x \subset C.$$

There exists a natural homomorphism $C \rightarrow (C^*)^*$ which is an isomorphism if C is a finitely generated projective module. Therefore we will not distinguish between the finitely generated projective modules C and $(C^*)^*$.

If $f: C_1 \to C_2$ is a homomorphism of Λ -modules, then we will denote by $f^*: C_2^* \to C_1^*$ the homomorphism given by

$$f^*(\varphi)(x) = \varphi(f(x)).$$

If C_1 and C_2 are free modules with bases c_1 and c_2 , then any homomorphism $f: C_1 \rightarrow C_2$ is given uniquely by the coefficient matrix A in the expansion of $f(c_1)$ in terms of the basis c_2 . The dual homomorphism f^* is then given in terms of the dual bases c_1^* and c_2^* by the dual matrix $A^* = (a_{ij}^*)$, $a_{ij}^* = a_{ji}^*$ where $A = (a_{ij})$.

The dual to a basis (e_1, \dots, e_n) of the module C is the basis (e'_1, \dots, e'_n) of the module C* for which $e'_i(e_i) = \delta_{ii}$.

Let (C, d) be a complex of free Λ -modules

$$C_0 \leftarrow C_1 \leftarrow \ldots \leftarrow C_n.$$

Then there exists a spectral sequence $E_r^{p,q}$ converging to the homology of the dual complex (C*, d*), the term $E_2^{p,q}$ of which is isomorphic to

$$E_2^{p,q} \approx \operatorname{Ext}_{\Lambda}^q(H_p(C), \Lambda).$$

In particular, if the complex (C, d) is acyclic up to dimension k, then the dual complex (C^*, d^*) is also acyclic up to dimension k. This standard fact can be found in several textbooks (see for example [9]).

We will now formulate a number of facts from the topology of smooth manifolds. Let $f: M \to N$ be an immersion of the manifold M in the manifold N, dim $M < \dim N$. The immersion f is in general position if for any points $x_i \in M$, $f(x_i) = p$, the subspaces $df_{x_i}(TM)$ are linearly independent in TN, where TX is the tangent bundle of the manifold X. Then every immersion f is arbitrarily near to another immersion which is in general position. If the immersion f is in general position, then the dimension of the set of its multiple points is easily estimated in terms of dim M and dim N.

Theorem (Hirsch). The set of regular homotopy classes of immersions $f: M \rightarrow N$ is in one-one correspondence with the set of homotopy classes of embeddings $df: TM \rightarrow f^*TN$. Here dim $M \leq \dim N = 2$.

Let us be given manifolds M and X, a map $f: M \to X$, deg f = 1, a vector bundle ξ over X and an isomorphism $\phi: \nu(M) \to f^*(\xi)$, where $\nu(M)$ is the normal bundle. The triple (M, f, ϕ) determines a cobordism class in the group $\Omega_n(X, \xi)$, $n = \dim M = \dim X$.

Theorem (Wall). With each group π there can be associated groups $L_n(\pi)$ and homomorphisms $\theta: \Omega_n(X, \xi) \to L_n(\pi_1(X))$ such that $\theta(\alpha) = 0$ if and only if the element α has a representative (M, f, ϕ) $\in \alpha$ for which the map f is a homotopy equivalence.

§2. The Wall groups and relations between them

In [3]-[6] Novikov and Wall defined the groups $L_n(\pi)$ which contain the obstruction to modifying one manifold to make it homotopically equivalent to another in a suitable surgery situation.

Let us recall the definition of the groups $L_n(\pi)$.

1. The case n = 2k. Consider a free right Λ -module C and two functions $\lambda(x, y) \in \Lambda$ and $\mu(x) \in \Lambda/\{y - (-1)^k y^*: y \in \Lambda\}$ satisfying the following conditions:

1) The function λ is linear in each variable, i.e.

$$\lambda(x_1 + x_2, y) = \lambda(x_1, y) + \lambda(x_2, y), \quad \lambda(x, y_1 + y_2) = \lambda(x, y_1) + \lambda(x, y_2).$$

- 2) $\lambda(x, ya) = \lambda(x, y)a, a \in \Lambda$.
- 3) $\lambda(x, y) = (-1)^k \lambda(y, x)^*$.
- 4) $\lambda(x, x) = \mu(x) + (-1)^k \mu(x)^*$.
- 5) $\mu(x + y) = \mu(x) + \mu(y) + \lambda(x, y).$
- 6) $\mu(xa) = a^*\mu(x)a$.
- 7) The map $\lambda(*, y): C \to C^*$ is an isomorphism.

The triple (C, λ, μ) is called a quadratic form. If $C = \Lambda \oplus \Lambda$ with basis e, f and the map $\lambda(*, y)$ is represented by the matrix

$$\lambda(*, y) = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix},$$

and further $\mu(e) = \mu(f) = 0$, then (C, λ, μ) is said to be trivial. Two quadratic forms (C_1, λ_1, μ_1) and (C_2, λ_2, μ_2) are considered to be isomorphic if there exists an isomorphism $\phi: C_1 \to C_2$ such that $\lambda_1(x, y) = \lambda_2(\phi(x), \phi(y))$ and $\mu_1(x) = \mu_2(\phi(x))$. The operation of direct sum is introduced on the set of quadratic forms

$$(C_1, \lambda_1, \mu_1) + (C_2, \lambda_2, \mu_2) = (C_1 \oplus C_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).$$

The Grothendieck group generated by the semigroup of isomorphism classes of quadratic forms and factored by the subgroup generated by the trivial quadratic forms is denoted by $L_n(\Lambda)$.

Now let C be a free right Λ -module with basis c and let (C, λ, μ) be a quadratic form satisfying the following additional condition:

8) The isomorphism $\lambda(*, y): C \to C^*$ is a simple isomorphism.

Then (C, c, λ, μ) is said to be a simple quadratic form. Two simple quadratic forms are said to be isomorphic if there exists an isomorphism between the quadratic forms which is a simple isomorphism. The corresponding Grothendieck group generated by the simple quadratic forms is denoted by $L_n^s(\Lambda)$.

2. The case n = 2k + 1. Let $h = (H, \lambda, \mu)$ be a trivial quadratic form and let ϕ : $H \rightarrow H$ be an automorphism which leaves h invariant. Two operations can be introduced on the set of all automorphisms (h, ϕ) :

a) the direct sum

$$(h_1, \varphi_1) \oplus (h_2, \varphi_2) = (h_1 \oplus h_2, \varphi_1 \oplus \varphi_2),$$

b) composition

$$(h, \varphi_1) * (h, \varphi_2) = (h, \varphi_1 \varphi_2).$$

We will discuss the group G generated by the automorphisms (b, ϕ) which satisfies the conditions

(1) $(h_1, \varphi_1) + (h_2, \varphi_2) = (h_1, \varphi_1) \oplus (h_2, \varphi_2),$

(2)
$$(h, \varphi_1) + (h, \varphi_2) = (h, \varphi_1 \varphi_2)$$

and is universal with respect to properties (1) and (2). Let G_0 be the subgroup of the group G which is generated by automorphisms (b, ϕ) of the following type:

1)

$$\varphi = \begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix},$$
2)

$$\varphi = \begin{pmatrix} \Phi^* & 0 \\ 0 & \Phi^{-1} \end{pmatrix},$$
3)

$$\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the factor group G/G_0 is denoted by $L_n(\Lambda)$.

In addition, let the form b be simple and let the automorphisms ϕ and Φ be simple. The corresponding group G/G_0 is denoted $L^s_n(\Lambda)$.

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We define groups $L_n(\Lambda)$ and $L_n^s(\Lambda)$ analogous to the Wall groups $\widetilde{L}_n(\Lambda)$ and $\widetilde{L}_n^s(\Lambda)$ by omitting the quadratic form μ everywhere in the definition, i.e. leaving only the non-degenerate even bilinear form λ . There are natural maps

$$\varphi: L_n(\Lambda) \longrightarrow \widetilde{L}_n(\Lambda), \qquad \varphi^s: L_n^s(\Lambda) \longrightarrow \widetilde{L}_n^s(\Lambda).$$

Theorem 2.1. The homomorphism

$$\varphi \otimes \mathbf{Z}\left[\frac{1}{2}\right] : L_{\mathfrak{a}}(\Lambda) \otimes \mathbf{Z}\left[\frac{1}{2}\right] \rightarrow \widetilde{L}_{\mathfrak{a}}(\Lambda) \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

is a monomorphism for even n.

Proof. Let n = 4k and $(C, \lambda, \mu) = 0$. This means that in a certain basis (e_i, f_i) the matrix of the homomorphism λ has the form

$$\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us show that $\mu(f_i) = 0$. In fact, let

$$x = \sum \alpha_g g, \quad [x] = \mu(f_t) \oplus \Lambda/\{v - v^* : v \in \Lambda\}.$$

Then from $\lambda(f_i, f_i) = 0$ we obtain

$$\alpha_g + \alpha_{g^{-1}} = 0.$$

Separate the group π into three nonintersecting sets

$$\pi = \pi_0 \bigcup \pi_+ \bigcup \pi_-, \quad \pi_0 = \{g \in \pi : g^2 = 1\}, \quad (\pi_+)^{-1} = \pi_-.$$

Then $\alpha_g = 0$ when $g \in \pi_0$. Put $\nu = \sum_{g \in \pi_+} \alpha_g g$. It is not difficult to ascertain that $x = \nu - \nu^*$, i.e. [x] = 0.

Now let n = 4k + 2 and $(C, \lambda, \mu) = 0$. Then in some basis (c_i, f_i) the matrix of the homomorphism λ has the form

$$\lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
 ,

and $\mu(f_i) = \sum_{g \in \pi_0} \mu_g \cdot g$. Consider the form $(C \oplus C, \lambda \oplus \lambda, \mu \oplus \mu)$ with basis (e_i, e'_i, f'_i, f'_i) . Choose a new basis

$$a_i = e_i + e'_i, \quad a'_i = e'_i, \quad b_i = f_i, \quad b'_i = f'_i - f_i.$$

Then in the basis (a_i, a_i', b_i, b_i') the form $\lambda \oplus \lambda$ is trivial and

$$\mu\left(a_{i}\right)=\mu\left(b_{i}\right)=0.$$

It remains to show that if the form (λ, μ) on a two-dimensional free module with basis (a, b) has the form

$$\lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
, $\mu(a) = 0$,

then it is trivial. In fact let

$$\mu\left(b\right)=\sum_{g\in\pi_{o}}b_{g}g$$

and let *n* be the number of those elements $g \in \pi_0$ for which $b_g \neq 0$. Let $b_g \neq 0$. Consider the new basis (a, b + ga). Then

$$\mu(b+ga) = \mu(b) + \mu(ga) + \lambda(b, ga) = \mu(b) + g\lambda(b, a) = \mu(b) - g,$$

i.e. the number of elements $g \in \pi_0$ for which $b'_g \neq 0$ is one less. Theorem 2.1 is proved.

§3. Algebraic Poincaré complexes

Let Λ be the group ring of the group π , let $*: \Lambda \to \Lambda$ be the anticommutative automorphism, $*^2 = id$, generated by the map $y^* = y^{-1}$, $y \in \pi$. Consider a chain complex of free right Λ -modules (C, d):

$$C_0 \stackrel{d_1}{\leftarrow} C_1 \stackrel{d_2}{\leftarrow} \dots \stackrel{d_n}{\leftarrow} C_n$$

of length n. If $x \in C_i$ we will write |x| = i. Let homomorphisms

$$D^{\mathbf{k}}: C^* \rightarrow C, \quad \mathbf{k} = 0, 1, \ldots,$$

be given such that $D^k(C^*_{n-i}) \subset C_{i+k}$ and such that the following conditions are fulfilled:

a)

$$D^{k}(d^{*}\varphi) + (-1)^{|\varphi|} dD^{k}(\varphi) + (-1)^{n+k} (D^{k-1}(\varphi) - \overline{D}^{k-1}(\varphi)) = 0,$$

where

$$\overline{D}^{\boldsymbol{k}}(\boldsymbol{\varphi}) = (-1)^{|\boldsymbol{\varphi}|(\boldsymbol{n}+\boldsymbol{k}-|\boldsymbol{\varphi}|)+\boldsymbol{k}} (D^{\boldsymbol{k}})^* (\boldsymbol{\varphi})$$

for $\phi \in C^*_{|\phi|}$.

b) The homomorphism $(D^0)_*: H(C^*) \to H(C)$ is an isomorphism.

Definition 3.1. The system (C, d, D^k) is called an algebraic Poincaré complex with formal dimension n.

Let us consider now a chain complex pair $(C, {}^{0}C, d)$ of free A-modules, i.e. two chain complexes (C, d) of length n + 1 and $({}^{0}C, {}^{0}d)$ of length n, and an embedding $\phi: {}^{0}C \to C$ of the complex ${}^{0}C$ as a direct summand of the complex C. It is not assumed that the boundary homomorphism decomposes in the direct sum. Let homomorphisms $D^{k}: C^* \to C, \ k = 0, 1, \cdots$, be given such that $D^{k}(C^*_{n+1-i}) \subset C_{i+k}$, and let the following conditions be satisfied:

a') The homomorphisms

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$${}^{0}D^{k}(\varphi) = D^{k}d^{*}(\varphi) + (-1)^{|\varphi|} dD^{k}(\varphi) + (-1)^{n+1+k}(D^{k-1}(\varphi) - \overline{D}^{k-1}(\varphi))$$

map the module C^* into the submodule 0C , where

$$\overline{D}^{k}(\varphi) = (-1)^{|\varphi|(k+1+k-|\varphi|)+k} (D^{k})^{\bullet}(\varphi).$$

b') The homomorphism $(D^0)_*: H(C^*) \to H(C/^0C)$ is an isomorphism.

Definition 3.2. The system $(C, {}^{0}C, d, D^{k})$ is said to be an algebraic Poincaré pair with formal dimension n + 1.

Lemma 3.3. Let $(C, {}^{0}C, d, D^{k})$ be an algebraic Poincaré pair with formal dimension n + 1. Then the homomorphisms ${}^{0}D^{k}$ satisfy condition a) for formal dimension n.

Proof. First of all let us compute the homomorphism

$${}^{\scriptscriptstyle 0}\overline{D}^{\boldsymbol{k}}(\boldsymbol{\psi}) = (-1)^{|\boldsymbol{\psi}|(\boldsymbol{\omega}+\boldsymbol{k}-|\boldsymbol{\psi}|)+\boldsymbol{k}} ({}^{\scriptscriptstyle 0}D^{\boldsymbol{k}})^*(\boldsymbol{\psi}).$$

Applying the operation * to a '), we obtain

$$({}^{0}D^{k})^{*}(\psi) = d(D^{k})^{*}(\psi) + (-1)^{2-|\psi|+k} (D^{k})^{*} d^{*}(\psi) + (-1)^{2+1+k} ((D^{k-1})^{*}(\psi) - (\overline{D}^{k-1})^{*}(\psi)) = (-1)^{|\psi|(n+1+k-|\psi|)+k} d\overline{D}^{k}(\psi) + (-1)^{|\psi|(n+k-|\psi|)+k} \overline{D}^{k} d^{*}(\psi) + (-1)^{n+|\psi|(n+k-|\psi|)} (\overline{D}^{k-1}(\psi) - D^{k-1}(\psi)).$$

Consequently

$${}^{0}\overline{D}^{k}(\psi) = \overline{D}^{k}d^{*}(\psi) + (-1)^{|\psi|} d\overline{D}^{k}(\psi) + (-1)^{n+k+1} (D^{k-1}(\psi) - \overline{D}^{k-1}(\psi)).$$
(3.1)

Substitute a') and (3.1) in a):

$$\begin{split} &(D^{k}d^{*}+(-1)^{|\psi|-1}dD^{k}+(-1)^{|\cdot|+1+k}(D^{k-1}-\bar{D}^{k-1}))d^{*}(\psi) \\ &+(-1)^{|\psi|}d(D^{k}d^{*}+(-1)^{|\psi|}dD^{k}+(-1)^{n+1+k}(D^{k-1}-\bar{D}^{k-1}))(\psi) \\ &+(-1)^{\nu+k}(D^{k-1}d^{*}+(-1)^{|\psi|}d\bar{D}^{k-1}+(-1)^{n+k}(D^{k-2}-\bar{D}^{k-2})) \\ &-\bar{D}^{k-1}d^{*}-(-1)^{|\psi|}d\bar{D}^{k-1}-(-1)^{\nu+k}(D^{k-2}-\bar{D}^{k-2}))(\psi) \\ &=\{(-1)^{|\psi|-1}dD^{k}d^{*}+(-1)^{n+1+k}D^{k-1}d^{*}+(-1)^{\nu+k}\bar{D}^{k-1}d^{*} \\ &+(-1)^{|\psi|}d\bar{D}^{k}d^{*}+(-1)^{n+1+k+|\psi|}d\bar{D}^{k-1}+(-1)^{n+k+|\psi|}d\bar{D}^{k-1} \\ &+(-1)^{n+k}D^{k-1}d^{*}+(-1)^{\nu+1+k+|\psi|}d\bar{D}^{k-1}\}(\psi)=0. \end{split}$$

Lemma 3.3 is proved.

Lemma 3.4. Let $(C, {}^{0}C, d, D^{k})$ be an algebraic Poincaré pair and let $\psi: C \rightarrow C/{}^{0}C$ be the natural projection. Then ${}^{0}D^{k}\psi^{*} = 0$.

Proof. According to Definition 3.2 we have $\psi^0 D^k = 0$. It is required to prove that $\psi({}^0D^k)^* = 0$. Inasmuch as the homomorphisms $({}^0D^k)^*$ and ${}^0\overline{D}^k$ differ only in sign on the direct summands, it is sufficient to establish that $\psi^0\overline{D}^k = 0$. For this we transform formula (3.1) by expressing \overline{D}^k in terms of D^k from a'). We have

$$\overline{D}^{k}(\varphi) = D^{k}(\varphi) + (-1)^{n+k} (D^{k+1}d^{*}(\varphi) + (-1)^{|\varphi|} dD^{k+1}(\varphi) - {}^{0}D^{k+1}(\varphi))$$

This implies that

$${}^{0}\overline{D}^{k}(\varphi) = D^{k}d^{*}(\varphi) + (-1)^{n+k}(D^{k+1}d^{*} + (-1)^{|\varphi|+1}dD^{k+1} - {}^{0}D^{k+1})d^{*}(\varphi)$$

+ $(-1)^{|\varphi|}d(D^{k} + (-1)^{n+k}(D^{k+1}d^{*} + (-1)^{|\varphi|}dD^{k+1} - {}^{0}D^{k+1}))(\varphi)$
+ $(-1)^{n+k+1}(D^{k-1} - D^{k-1} - (-1)^{n+k-1}(D^{k}d^{*} + (-1)^{|\varphi|}dD^{k} - {}^{0}D^{k}))(\varphi)$
= $(-1)^{n+k}((-1)^{|\varphi|+1}dD^{k+1} - {}^{0}D^{k+1})d^{*}(\varphi) + (-1)^{|\varphi|+n+k}d(D^{k+1}d^{*} - {}^{0}D^{k+1})(\varphi)$
+ ${}^{0}D^{k}(\dot{\varphi}) = ((-1)^{n+k+10}D^{k+1}d^{*} + (-1)^{|\varphi|+n+k+1}d^{0}D^{k+1} + {}^{0}D^{k})(\varphi).$

Consequently

$$\psi^{0}\overline{D}^{k}(\varphi) = ((-1)^{n+k+1}\psi^{0}D^{k+1}d^{*} + (-1)^{|\varphi|+n+k+1}d\psi^{0}D^{k+1} + \psi^{0}D^{k})(\varphi) = 0.$$

Lemma 3.4 is proved.

Lemma 3.4 shows that the homomorphism ${}^{0}D^{k}$ induces a natural homomorphism

$${}^{0}D^{k}: {}^{0}C^{*} \rightarrow {}^{0}C,$$

satisfying condition a).

Lemma 3.5. The homomorphisms ${}^{0}D^{k}$: ${}^{0}C^{*} \rightarrow {}^{0}C$ satisfy condition b). Thus the system $({}^{0}C, {}^{0}d, {}^{0}D^{k})$ is an algebraic Poincaré complex with formal dimension n.

Proof. It is not difficult to verify that we have the following commutative diagram:

$$\rightarrow H_{i}({}^{0}C) \longrightarrow H_{i}(C) \longrightarrow H_{i}(C/{}^{0}C) \rightarrow$$

$$({}^{0}D^{0})_{*} \uparrow (D^{0})_{*} \psi_{*}^{*} \uparrow \psi_{*}(D^{0})_{*} \uparrow$$

$$\rightarrow H^{n-i}({}^{0}C) \rightarrow H^{n+1-i}(C/{}^{0}C) \rightarrow H^{n+1-i}(C) \rightarrow$$

Inasmuch as $\psi_*(D^0)_*$ is an isomorphism, by a') it follows that $(D^0)_*\psi_*^*$ is also an isomorphism, whence by the Five Lemma it follows that $({}^0D^0)_*$ is an isomorphism.

Definition 3.6. Let $\alpha = (C, {}^{0}C, d, D^{k})$ be an algebraic Poincaré pair with formal dimension n + 1. Then the algebraic Poincaré complex $\beta = ({}^{0}C, {}^{0}d, {}^{0}D^{k})$ with formal dimension n is called the boundary of the algebraic Poincaré pair α and is denoted by $\beta = \partial \alpha$.

Now we are in a position to define the "cobordism" groups $\Omega_n(\Lambda)$. The sum of two Poincaré complexes $\alpha_1 = (C_1, d_1, D_1^k)$ will be the algebraic Poincaré complex

$$\beta = \alpha_1 \cup \alpha_2 = (C_1 \oplus C_2, d_1 \oplus d_2, D_1^k \oplus D_2^k).$$

Definition 3.7. The Grothendieck group generated by the semigroup of algebraic Poincaré complexes with operation the sum \bigcup and factored by the relations $\alpha = 0$ if $\alpha = \partial \beta$ will be called the n-dimensional algebraic Poincaré cobordism group $\Omega_{n}(\Lambda)$.

Lemma 3.8. Let $\alpha = (C, d, D^k)$ be an algebraic Poincaré complex. Then $\beta = (C, d, -D^k) = -\alpha$ in the group $\Omega_n(\Lambda)$.

Proof. We will present an algebraic Poincaré pair $\gamma = (F, {}^{0}F, \delta, H^{k})$ such that $\partial \gamma = \alpha \bigcup \beta$. Put $F_{i} = C_{i} \oplus C_{i-1} \oplus C_{i}$ and ${}^{0}F_{i} = C_{i} \oplus C_{i}$. Let the embedding ϕ : ${}^{0}F_{i} \to F_{i}$ be given by the matrix

$$\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Further, put $\delta = (\delta_j)$, where

$$\delta_{i} = \begin{pmatrix} d_{i} & (-1)^{i} & 0\\ 0 & d_{i-1} & 0\\ 0 & (-1)^{i+1} & d_{i} \end{pmatrix}.$$

It is easy to verify that $\delta_{i-1}\delta_i = 0$. Further, let us define the homomorphisms $H^k = (H_i^k)$, H_i^k : $F_{n+1-i}^* \to F_{i+k}$, by the matrices

$$H_{i}^{k} = \begin{pmatrix} 0 & 0 & 0 \\ (-1)^{n+k+1} D_{i-1}^{k} & (-1)^{(n-i)(k-i)+n+k} (D_{n-k-i+1}^{k-1})^{*} & 0 \\ 0 & (-1)^{n+i+1} D_{i}^{k} & 0 \end{pmatrix}$$

One ascertains by direct substitution that condition a ') is fulfilled; namely,

$${}^{0}H_{i}^{k} = H_{i}^{k}\delta_{..-i+1}^{*} + (-1)^{n-i}\delta_{k+i+1}H_{i+1}^{k} + (-1)^{n+k+1}(H_{i+1}^{k-1} + (-1)^{(n-i)(k-i)+k-1}(H_{k+1-k-i}^{k-1})^{*}),$$
(3.2)

where

$${}^{\scriptscriptstyle 0}H_i^k = \begin{pmatrix} D_i^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -D_i^k \end{pmatrix}.$$

The verification of condition b') is trivial.

Definition 3.9. We will say that two algebraic Poincaré complexes α and β are cobordant if one can find an algebraic Poincaré pair γ such that $\partial \gamma = \alpha \bigcup (-\beta)$.

Lemma 3.10. If the algebraic Poincaré complex α represents zero in the group $\Omega_{\alpha}(\Lambda)$, then one can find an algebraic Poincaré pair β such that $\alpha = \partial \beta$.

Proof. That the element α represents zero in $\Omega_n(\Lambda)$ means that there exist algebraic Poincaré complexes α_1^i , α_2^i and $\beta_j = \partial \gamma_j$ such that

$$\alpha = \sum_{i} \pm (\alpha_{1}^{i} \cup \alpha_{2}^{i} - \alpha_{1}^{i} - \alpha_{2}^{i}) + \sum_{j} \beta_{j}$$

in the free group generated by algebraic Poincaré complexes. Then

$$\alpha + \sum_{i} \alpha_1^i + \sum_{i} \alpha_2^i + \sum_{s} \alpha_1^s \cup \alpha_2^s = \sum_{i} \alpha_1^i \cup \alpha_2^i + \sum_{s} \alpha_1^s + \sum_{s} \alpha_2^s + \sum_{j} \beta_j.$$

Hence it follows that one can find an algebraic Poincaré complex

$$\theta = \bigcup_{i} \alpha_{1}^{i} \bigcup_{i} \bigcup_{i} \alpha_{2}^{i} \bigcup_{s} (\alpha_{1}^{s} \bigcup \alpha_{2}^{s})$$
$$\beta = \bigcup_{j} \beta_{j}, \quad \beta = \partial \gamma,$$

such that

$$\alpha \cup \theta = \theta \cup \beta$$
.

Adding the algebraic Poincaré complex $(-\theta)$ to the left and right sides, we obtain

$$\alpha \cup \theta = \beta, \quad \theta = \partial \zeta, \quad \beta = \partial \gamma.$$

Lemma 3.10 is now implied by

Lemma 3.11. Let α_1, α_2 and α_3 be algebraic Poincaré complexes. If α_1 is cobordant to α_2 and α_2 is cobordant to α_3 , then α_1 is cobordant to α_3 .

Let us conclude the proof of Lemma 3.10. Inasmuch as $\theta = \partial \xi$, by Lemma 3.8 we can find an algebraic Poincaré pair ξ such that $\partial \xi = \alpha \cup (-\alpha)$. Then $\alpha \cup \theta \cup (-\alpha) = \partial(\zeta \cup \xi)$, i.e. β is cobordant to α ; but β is cobordant to zero, and consequently, by Lemma 3.11, α is cobordant to zero, which was to be proved.

Proof of Lemma 3.11. Let $\alpha_i = ({}_iC, {}_id, {}_iD^k)$, i = 1, 2, 3, and $\beta_i = ({}_iB, {}^0_iB, {}_i\delta, {}_iH^k)$ be algebraic Poincaré pairs such that

$$\begin{pmatrix} {}^{0}_{1}B, {}^{0}_{1}\delta, {}^{0}_{1}H^{k} \end{pmatrix} = ({}_{1}C \oplus {}_{2}C, {}_{1}d \oplus {}_{2}d, {}_{1}D^{k} \oplus {}_{2}D^{k}),$$

$$\begin{pmatrix} {}^{0}_{2}B, {}^{0}_{2}\delta, {}^{0}_{2}H^{k} \end{pmatrix} = ({}_{2}C \oplus {}_{3}C, {}_{2}d \oplus {}_{3}d, - {}_{2}D^{k} \oplus {}_{3}D^{k}),$$

and let $\phi_1: {}_1C \oplus {}_2C \to {}_1B$ and $\phi_2: {}_2C \oplus {}_3C \to {}_2B$ be the corresponding embeddings. Put

$$B = {}_{1}B \oplus {}_{2}B/\varphi({}_{2}C), \qquad {}^{0}B = \varphi({}_{1}C \oplus {}_{3}C),$$

where $\phi: {}_{1}C \oplus {}_{2}C \oplus {}_{3}C \to {}_{1}B \oplus {}_{2}B$ is the diagonal embedding. Then the homomorphisms ${}_{1}\delta \oplus {}_{2}\delta$ and ${}_{1}H^{k} \oplus {}_{2}H^{k}$ induce homomorphisms d and H^{k} on the complex B. The verification of conditions a') and b') is trivial.

Examples of algebraic Poincaré complexes. 1. Let n = 2k and let $C_i = 0$ when $i \neq k$. Then d = 0, $D^k = 0$ when $k \neq 0$, and the homomorphism $D^0: C_k^* \to C_k$ is an isomorphism and satisfies $(-1)^k (D^0)^* = D^0$, i.e. defines a nondegenerate bilinear form (either symmetric or skew-symmetric) on the free A-module C_k . Let us suppose that the algebraic Poincaré complex under consideration is cobordant to zero. Let us examine the conditions which must be satisfied by the form D^0 . In our case as the simplest example of an algebraic Poincaré pair we can take $(F, {}^0F, d, H^k)$:

$$PF_k = F_k = C_k$$
, ${}^{0}F_{k+1} = 0$, $F_i = 0$ when $i \neq k, k+1$.

Consider the diagram



Condition a') becomes

$$\begin{array}{c}
H_{k}^{0}d^{*} + (-1)^{k} dH_{k+1}^{0} = D^{0}, \\
H_{k}^{1}d^{*} + (H_{k+1}^{0} - (H_{k}^{0})^{*}) = 0, \\
H_{k}^{1} = (-1)^{k} (H_{k}^{1})^{*}.
\end{array}$$
(3.3)

Condition b') means that the homomorphism d^* is an epimorphism and that the homomorphism H^0_{k+1} maps Ker d^* isomorphically onto F_{k+1} . In other words,

$$H^0_{k+1} \oplus d^* : {}^{0}F^*_k \to F_{k+1} \oplus F^*_{k+1}$$

is an isomorphism. Let us choose a basis in the module ${}^0F_k^*$ such that the matrix of the homomorphism $H_{k+1}^0 \oplus d^*$ is the identity matrix. Then

$$H^0_{k+1} = (1, 0), \quad d^* = (0, 1),$$

and from (3.3) we obtain

$$D^{0} = \begin{pmatrix} 0 & 1 \\ (-1)^{k} & H_{k}^{1} \end{pmatrix}, \quad H_{k}^{1} = (-1)^{k} (H_{k}^{1})^{*}.$$

2. Now let n = 2k + 1, and let $C_i = 0$ when $i \neq k$, k + 1. Then the algebraic Poincaré complex under consideration can be written as the following diagram:



where the following conditions are fulfilled:

$$D_{k}^{0}d^{*} + (-1)^{k} dD_{k+1}^{0} = 0, \qquad D_{k}^{1}d^{*} + (-1)^{k} (D_{k+1}^{0} - (D_{k}^{0})^{*}) = 0,$$
$$D_{k}^{1}d^{*} + (-1)^{k} (D_{k}^{1})^{*}$$

and the sequence

$$0 \leftarrow C_k \stackrel{\varphi}{\leftarrow} C_{k+1} \bigoplus C_{k+1}^* \stackrel{\psi}{\leftarrow} C_k^* \leftarrow 0, \qquad (3.4)$$

where

$$\varphi = (d, D_k^0), \qquad \qquad \psi = \left(rac{\left(-1
ight)^k D_{k+1}^0}{d^*}
ight),$$

is exact.

Let us define a scalar product on the module $C_{k+1} \oplus C_{k+1}^*$ by assigning it the matrix

$$A = \begin{pmatrix} 0 & 1 \\ (-1)^k & D_k^1 \end{pmatrix},$$

i.e. if $x, y \in C_{k+1}$ and $x', y' \in C_{k+1}^*$ then $\langle x, y \rangle = 0$, $\langle x, x' \rangle = x'(x)$ and $\langle x', y' \rangle = y'(D_k^1(x'))$. The scalar product thus defined is trivial on $\operatorname{Im} \psi$. In fact, let $x, y \in C_k^*$. We have

$$\langle \psi x, \psi y \rangle = \langle (-1)^k D_{k+1}^{\circ}(x), (-1)^k D_{k+1}^{\circ}(y) \rangle + \langle d^*(x), (-1)^k D_{k+1}^{\circ}(y) \rangle + \langle (-1)^k D_{k+1}^{\circ}(x), d^*(y) \rangle + \langle d^*(x), d^*(y) \rangle = (d^*x)((-1)^k D_{k+1}^{\circ}(y)) + ((d^*y)(D_{k+1}^{\circ}(x)))^* + (d^*x)(D_k^1 d^*y) = x((-1)^k dD_{k+1}^{\circ}y) + (y(dD_{k+1}^{\circ}x))^* + x(dD_k^1 d^*y) = x((-1)^k dD_{k+1}^{\circ}y) + x((D_{k+1}^{\circ})^* d^*y) + x(dD_k^1 d^*y) = x((-1)^k dD_{k+1}^{\circ}y + (D_{k+1}^{\circ})^* d^*y + dD_k^1 d^*y)' = x(-D_k^0 d^* + (D_{k+1}^{\circ})^* d^* + dD_k^1 d^*)y = x((-1)^k d(D_k^1)^* d^* + dD_k^1 d^*)y = 0.$$

Choose an embedding

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} : C_k \to C_{k+1} \oplus C_{k+1}^*$$

such that $\phi \circ \chi = id$ and so that the scalar product induced on the module $C_k \oplus C_k^*$ has matrix

$$B = \begin{pmatrix} 0 & 1 \\ (-1)^k & E \end{pmatrix}.$$

The first condition may be satisfied since the sequence (3.4) is exact; and that means that $d\chi_1 + D_k^0\chi_2 = \text{id.}$ The second condition means that $\langle \chi(x), \psi(y) \rangle = y(x)$. This in turn means that

$$y(x) = \langle \chi_1(x), d^*(y) \rangle + \langle \chi_2(x), (-1)^k D^0_{k+1}(y) \rangle + \langle \chi_2(x), d^*(y) \rangle$$

= $d^*(y)(\chi_1(x)) + \chi_2(x)(D^0_{k+1}(y)) + (-1)^k \chi_2(x)(D^1_k d^*(y)) = y(d\chi_1(x))$
+ $y((D^0_{k+1})^* \gamma_2(x)) + (-1)^k y(d(D^1_k)^* \chi_2(x)),$

or

$$\begin{aligned} x &= d\chi_1(x) + (D_{k+1}^0)^* \chi_2(x) + (-1)^k d(D_k^1)^* \chi_2(x) = x - D_k^0 \chi_2(x) \\ &+ (D_{k+1}^0)^* \chi_2(x) + d(D_k^1)^* \chi_2(x) = x + (-1)^{k+1} d(D_k^1)^* \chi_2(x) + (-1)^k d(D_k^1)^* \chi_2(x). \end{aligned}$$

Thus we have constructed an automorphism ϕ of the free module $C_{k+1} \oplus C_{k+1}^*$ given by the matrix

$$\theta = (\psi, \chi),$$

which transforms the trivial bilinear form A into the form B which is of the same type.

§4. Modifications of algebraic Poincaré complexes

Our aim below is to construct the homomorphism

$$\sigma: L_n(\Lambda) \longrightarrow \Omega_n(\Lambda)$$

and to explain its properties. For this we study ways of reducing an algebraic Poincaré complex to the simplest possible form.

Definition 4.1. Let $\alpha = (C, d, D^k)$ and $\alpha' = (C', d', D^k)$ be two algebraic Poincaré complexes and let $f: C \rightarrow C$ be a chain homomorphism such that $D^{k} =$ $(D^k)^*$. Then f is said to be a homotopy equivalence if it induces an isomorphism in homology. The algebraic Poincaré complexes α and α' are then said to be homotopically equivalent.

Lemma 4.2. Homotopically equivalent algebraic Poincaré complexes are cobordant.

Proof. Let $\alpha = (C, {}^{0}C, d, D^{k})$ be an algebraic Poincaré pair and let the algebraic Poincaré complex $\partial \alpha = ({}^{0}C, {}^{0}d, {}^{0}D^{k})$ be homotopically equivalent to $\beta = (B, \delta, H^{k})$, i.e. there exists a homomorphism $f: {}^{0}C \rightarrow B$ satisfying the conditions in Definition 3.12. Let $C = C \oplus {}^{0}C$. Consider the new pair $A = {}^{\perp}C \oplus B$, ${}^{0}A = B$. Define the differential ∂ by

$$\partial(x) = \delta(x), \quad x \in B,$$

 $\partial(x) = gd(x), \quad x \in {}^{\perp}C,$

where the homomorphism $g: {}^{\perp}C \oplus {}^{0}C \to {}^{\perp}C \oplus B$ is defined by g = (id, f). Further, put $F^{\boldsymbol{k}} = \boldsymbol{g} \circ \boldsymbol{D}^{\boldsymbol{k}} \circ \boldsymbol{g}^*.$

We verify that
$$(A, {}^{0}A, \partial, F^{k})$$
 satisfies conditions a') and b') of Definition 3.1. The verification of b') is trivial. To verify a') we need to prove

$$F^{k}\partial^{*} + (-1)^{|\varphi|}\partial F^{k} + (-1)^{n+1+k}(F^{k-1} - \overline{F}^{k-1}) = \begin{pmatrix} H^{k} & 0\\ 0 & 0 \end{pmatrix}$$

when dim $\phi = |\phi|$, or

v

$$gF^{k}g^{\bullet}\partial^{\bullet} + (-1)^{|\varphi|}\partial gF^{k}g^{\bullet} + (-1)^{n+1+k}(gF^{k-1}g^{\bullet} - g\overline{F}^{k-1}g^{\bullet}) = \begin{pmatrix} H^{k} & 0\\ 0 & 0 \end{pmatrix}.$$
 (4.1)

Since $\partial g = g\partial$, we obtain from (4.1)

$$g\begin{pmatrix} {}^{\mathbf{0}}D^{k} & 0\\ 0 & 0 \end{pmatrix}g^{\star} = \begin{pmatrix} H^{k} & 0\\ 0 & 0 \end{pmatrix},$$

as was required. Thus $(A, {}^{0}A, \partial, F^{k})$ is a Poincaré pair. By Lemma 3.8 each algebraic Poincaré complex is cobordant to itself, and that means that $a \bigcup (-a)$ is the boundary of an algebraic Poincaré pair. Consequently if β is homotopically equivalent to α then $\beta \bigcup (-\alpha)$ is the boundary of an algebraic Poincaré pair. Lemma 4.2 is proved.

Let us now define a canonical transformation of an algebraic Poincaré pair $(C, {}^{0}C, d, D^{k})$ corresponding to gluing a handle for manifolds. Let A be a free A-module, let $\beta: A^{*} \rightarrow C^{*}_{n-1}$ be a homomorphism, and let $i \leq [n/2] - 1$, where $d^{*}_{n-i+1}\beta = 0$. We construct a new algebraic Poincaré pair $(\overline{C}, {}^{0}\overline{C}, \overline{d}, \overline{D}^{k})$ by putting

$$\overline{C}_{j} = C_{i} \text{ when } j \neq i+1, n-i-1, n-i,$$

$$\overline{C}_{i} = C_{i} \oplus A \text{ when } j = i+1, n-i-1, n-i,$$
(4.2)

$${}^{0}\overline{C_{j}} = {}^{0}C_{j} \text{ when } j \neq i+1, n-i-1,$$

$${}^{0}\overline{C_{j}} = {}^{0}C_{j} \oplus A \text{ when } j = i+1, n-i-1.$$

$$(4.3)$$

The boundary homomorphism \overline{d} is given by

$$\overline{d}_{i+1} = (d_{i+1}, (-1)^{n+i} {}^{\scriptscriptstyle 0}D_i^{\scriptscriptstyle 0}\beta q),$$

where $q: A \rightarrow A^*$ is some fixed isomorphism,

$$\overline{d}_{i+2} = \begin{pmatrix} d_{i+2} \\ 0 \end{pmatrix}, \qquad \overline{d}_{n-i-1} = (d_{n-i-1}, 0),$$

$$\overline{d}_{n-i} = \begin{pmatrix} d_{n-i} & 0 \\ \beta^* & 1 \end{pmatrix}, \qquad \overline{d}_{i-i+1} = \begin{pmatrix} d_{n-i+1} \\ 0 \end{pmatrix},$$

$$\overline{d}_{j} = d_{j} \text{ for the remaining suffixes } j.$$

Further, put

$$\overline{D}_{i+1}^{0} = \begin{pmatrix} D_{i+1}^{0} & -D_{i+1}^{0} \\ 0 & q^{-1} \end{pmatrix},$$
*
$$\overline{D}_{n-i}^{0} = \begin{pmatrix} D_{n-i}^{0} & 0 \\ (-1)^{(l+1)(n-l)+1} \beta^{*} (D_{i+1}^{0})^{*} & (-1)^{(l+1)(n-l)} q^{-1} \end{pmatrix}$$

All the remaining homomorphisms \overline{D}_i^k are defined with the help of D^k by adding zero homomorphisms for the new summands.

Let us verify that the collection $(\overline{C}, {}^{0}\overline{C}, \overline{d}, \overline{D}^{k})$ thus obtained is an algebraic Poincaré pair.

1. $\overline{d}^2 = 0$. It is sufficient to verify that $\overline{d_i d_{i+1}} = 0$ and $\overline{d_{n-i} d_{n-i+1}} = 0$. In the first case we have

$$\vec{d}_{i}\vec{d}_{i+1} = (d_{i}d_{i+1}, (-1)^{n-i}d_{i}^{0}\beta q) = (0, {}^{0}D_{i-1}^{0}d_{n-i+1}^{*}\beta q) = 0$$

In the second case

$$\overline{d}_{i-i}\overline{d}_{n-i+1} = \begin{pmatrix} d_{i-i}d_{n-i+1} \\ \beta^*d_{n-i+1} \end{pmatrix} = 0.$$

2. The verification of b') presents no difficulty.

3. It is sufficient to verify a') only in those cases when the matrix \overline{D}_i^k has not been constructed trivially from D_i^k .

a) k = 0.

a. 1) We require to prove

$$(-1)^{n-i} \overline{d}_{i+1} \overline{D}_{i+1}^{0} + \overline{D}_{i}^{0} \overline{d}_{n-i+1}^{*} \equiv 0 \mod {}^{0}\overline{C}.$$
(4.5)

We have

$$(-1)^{n-i}\overline{d_{i+1}}\overline{D}_{i+1}^{0} = (-1)^{n-i} (d_{i+1}D_{i+1}^{0}, (-1)^{n-i}D_{i}^{0}\beta - d_{i+1}D_{i+1}^{0}\beta),$$

$$\overline{D}_{i}^{0}\overline{d}_{n-i+1}^{*} = (D_{i}^{0}d_{n-i+1}^{*}, 0).$$

Consequently the expression (4.2) becomes

$${}^{0}\bar{D}_{i}^{0} = ({}^{0}D_{i}^{0}, (-1)^{n-i}D_{i}^{0}d_{n-i+1}^{*}\beta) = ({}^{0}D_{i}^{0}, 0).$$

$$(4.6)$$

a. 2) We require to prove the following

$$(-1)^{n-i-1} \overline{d}_{i+2} \overline{D}_{i+2}^{0} + \overline{D}_{i+1}^{0} \overline{d}_{n-i}^{*} \equiv 0 \mod {}^{0}C.$$
(4.7)

We have

$$(-1)^{n-i-1}\overline{d}_{i+2}\overline{D}_{i+2}^{0} = (-1)^{n-i-1} \begin{pmatrix} d_{i+2}D_{i+2}^{0} & 0\\ 0 & 0 \end{pmatrix},$$
$$\overline{D}_{i+1}^{0}\overline{d}_{n-i}^{*} = \begin{pmatrix} D_{i+1}^{0}d_{n-i}^{*} & 0\\ 0 & q^{-1} \end{pmatrix}.$$

This implies that

$${}^{\mathbf{0}}\overline{D}_{i+1}^{\mathbf{0}} = \begin{pmatrix} {}^{\mathbf{0}}D_{i+1}^{\mathbf{0}} & \mathbf{0} \\ \mathbf{0} & q^{-1} \end{pmatrix}.$$

a. 3) Let us verify that

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$$(-1)^{i+1}\bar{d}_{n-i}\bar{D}_{n-i}^{0} + \bar{D}_{n-i-1}^{0}\bar{d}_{i+2}^{*} \equiv 0 \mod {}^{0}C.$$
(4.9)

We have

$$(-1)^{i+1}\overline{d}_{i-i}\overline{D}_{n-i}^{0} = (-1)^{i+1} \begin{pmatrix} d_{i-i}D_{n-i}^{0} & 0\\ \beta^{*}(D_{n-i}^{0} - (-1)^{(i+1)(n-i)}(D_{i+1}^{0})^{*}) & (-1)^{(i+1)(n-i)}q^{-1} \end{pmatrix}$$

$$\overline{D}_{i-i-1}^{0}\overline{d}_{i+2}^{*} = \begin{pmatrix} D_{n-i-1}^{0}d_{i+2}^{*} & 0\\ 0 & 0 \end{pmatrix}.$$

Consequently

$${}^{0}\overline{D}_{n-i-1}^{0} = \begin{pmatrix} {}^{0}D_{n-i-1}^{0} & 0 \\ \\ \beta^{*}(D_{i:-i}^{0} - (-1)^{(i+1)(n-i)}(D_{i+1}^{0})^{\bullet}) & (-1)^{(i+1)(n+1-i)}q^{-1} \end{pmatrix} \cdot$$

a.4) Let us verify the following

$$(-1)^{i} \bar{d}_{i-i+1} \overline{D}_{n-i+1}^{0} + \overline{D}_{i-i}^{0} \bar{d}_{i+1}^{*} \equiv 0 \mod {}^{0}C.$$
(4.11)

We have:

$$(-1)^{i} \overline{d}_{n-i+1} \overline{D}_{n-i+1}^{0} = (-1)^{i} \begin{pmatrix} d_{i-i+1} D_{n-i+1}^{0} \\ 0 \end{pmatrix},$$

$$\overline{D}_{n-i}^{0} \overline{d}_{i+1}^{*} = \begin{pmatrix} D_{n-i}^{0} d_{i+1}^{*} \\ (-1)^{i(i-i)} \beta^{*} ((^{0} D_{i}^{0})^{*} - (-1)^{n-i} (D_{i+1}^{0})^{*} d_{i+1}^{*}) \end{pmatrix}.$$

This means

$${}^{\scriptscriptstyle 0}\overline{D}_{n-i}^{\scriptscriptstyle 0} = \begin{pmatrix} {}^{\scriptscriptstyle 0}D_{n-i}^{\scriptscriptstyle 0} \\ 0 \end{pmatrix}.$$
(4.12)

b) $k \neq 0$. Since

$$\overline{D}_{i+1}^{0} - (-1)^{(i+1)(n-i)} (\overline{D}_{n-i}^{0})^{*} = \begin{pmatrix} D_{i+1}^{0} - (-1)^{(i+1)(n-i)} (D_{n-i}^{0})^{*} & 0\\ 0 & 0 \end{pmatrix},$$

a') is automatically fulfilled for k = 1. For k > 1 this condition is trivial.

Definition 4.3. The construction of the algebraic Poincaré pair $(\overline{C}, \ ^0\overline{C}, \ \overline{d}, \ \overline{D}^k)$ described above will be called "gluing" on a handle by the map $\beta: A^* \to C^*_{n-1}$.

Definition 4.4. Let (C, d, D^k) be an algebraic Poincaré complex and let β : $A^* \rightarrow C^*_{n-i}$ be a homomorphism such that $d^*_{n-i+1}\beta = 0$. The construction of the new algebraic Poincaré complex $(\overline{C}, \overline{d}, \overline{D^k})$ according to (4.3), (4.4), (4.6), (4.8), (4.10) and (4.12) and the analogous expressions for \overline{D}^k , k > 1, will be called modifying the algebraic Poincaré complex by the map β .

Lemma 4.5. The algebraic Poincaré complex obtained as a result of modifying the algebraic Poincaré complex a is cobordant to a.

Proof. Let y be an algebraic Poincaré pair such that $\partial y = \alpha \bigcup (-\alpha)$. Put

$$\gamma = (C, {}^{o}C, d, D^{k}), {}^{o}C = B \oplus B, {}^{a}a = (B, d, H^{k}).$$

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Let $\beta: A^* \to B^*$ be the map used to carry out the modification. We have to construct a map $\beta': A^* \to C^*$ such that $\beta = \phi\beta'$, where $\phi: B \to C$ is the embedding and $d^*\beta' = 0$. Lemma 3.8 implies that we can take for γ a complex for which there exists a map $\psi: B^* \to C^*$ with $\phi\psi = \text{id}$. Put $\beta' = \psi\beta$. Glue a handle to the pair γ by the map β' . We find that the new algebraic Poincaré pair γ' has boundary $\alpha \cup (-\beta)$. Lemma 4.5 is proved.

Theorem 4.6. Every algebraic Poincaré complex α is cobordant to an algebraic Poincaré complex $\beta = (C, d, D^k)$ for which $C_i = 0$ when $i \neq k, k - 1$ or k + 1 if dim $\alpha = 2k$, and when $i \neq k$ or k + 1 if dim $\alpha = 2k + 1$.

For the proof of this theorem we need

Lemma 4.7. Let $\alpha = (C, d, D^k)$ be an algebraic Poincaré complex such that $H_i(C) = 0$ when $i \leq s < [\dim \alpha/2]$. Then one can find an algebraic Poincaré complex $\overline{\alpha} = (\overline{C}, \overline{d}, \overline{D^k})$ cobordant to it, such that $\overline{C}_i = 0$ when $i \leq s$ and $i \geq \dim \alpha - s$.

Proof. According to Lemma 4.2 it is sufficient to construct a complex $\overline{\alpha}$ which is homotopically equivalent to the complex α . Let A be an acyclic complex of free A-modules, and let $\phi: A \to C$ be an embedding onto a direct summand, i.e. the monomorphism ϕ commutes with the boundary homomorphism d of the complex C. Let $\overline{C} = C/A$, let $\pi: C \to \overline{C}$ be the natural projection and let $\overline{D}^k = \pi D^k \pi^*$. The collection $(\overline{C}, \overline{d}, \overline{D}^k)$ satisfies conditions a) and b) in the definition 3.1 of an algebraic Poincaré complex. Consequently π is a homotopy equivalence between two algebraic Poincaré complexes.

If $H_i(C) = 0$ when $i \le s$, then one can find a direct summand $A \in C_{s+1}$ such that the complex

$$X: C_0 \leftarrow \ldots \leftarrow C_s \leftarrow A$$

is acyclic. By property b) of Definition 3.1 one can find a direct summand $B \subseteq C^*_{n-s-1}$ such that the complex

$$C_n^* \leftarrow \ldots \leftarrow C_{n-s}^* \leftarrow B$$

is acyclic, $n = \dim C$. Consequently the complex

$$Y: B^* \leftarrow C_{n-s} \leftarrow \ldots \leftarrow C_n$$

is also acyclic and the module B^* is a direct summand in C_{n-s-1} . The factor complex $C = C/X \oplus Y$ is then an algebraic Poincaré complex and satisfies the conditions of Lemma 4.7.

Let us now prove Theorem 4.6. We will suppose that $C_i = 0$ when $i \le s$, $i \ge n-s$ and $s \le \lfloor n/2 \rfloor - 1$. We carry out a modification of the complex C with respect to the map $\beta: A^* \to C^*_{n-s+1}$, where $A = C_{n-s+1}$ and β is the identity map. It is not difficult to verify that as a result of this modification we obtain a new algebraic Poincaré complex $(\overline{C}, \overline{d}, \overline{D}^k)$ for which $H_{s+1}(\overline{C}) = 0$. For the rest we apply Lemma 4.7 and use induction on s. Theorem 4.6 is proved. **Corollary 4.8.** The groups $\Omega_n(\Lambda)$ and $\Omega_{n+4}(\Lambda)$ are isomorphic when $n \ge 4$.

We will now construct the homomorphisms

$$\psi:\widetilde{L}_n(\Lambda)\to\Omega_n(\Lambda).$$

If n = 2k, then to each even bilinear form (H, λ) we associate the algebraic Poincaré complex (C, d, D^k) :

$$C_k = H$$
, $C_i = 0$, $i \neq k$, $d = 0$, $D^0 = \lambda$, $D^s = 0$, $s \ge 1$.

Lemma 4.9. If (H, λ) is the trivial form, then $\psi(H, \lambda) = 0$.

Proof. Let

$$H = A \oplus A^*, \quad \lambda = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}.$$

Put $C_k = H$, ${}^0C_k = H$, $C_{k+1} = A$, ${}^0C_{k+1} = 0$, $C_i = {}^0C_j = 0$ for all remaining *i* and *j*, and let

$$d_{k+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D_k^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad D_{k+1}^0 = (1, 0), \quad D^k = 0, \quad k \ge 1.$$

The collection $(C, {}^{0}C, d, D^{k})$ thus obtained is an algebraic Poincaré pair with formal dimension 2k + 1 and with boundary $\psi(H, \lambda)$. Lemma 4.9 is proved.

Let n = 2k + 1, and let ϕ be an automorphism which leaves invariant the trivial form

$$\lambda = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

on the module $H = A \oplus A^*$. Express the automorphism ϕ in the form

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{pmatrix}.$$

Put $C_k = A$, $C_{k+1} = A^*$, $d_{k+1}^* = \phi_2$, $D_{k+1}^0 = (D_k^0)^* = \phi_1$ and $D^k = 0$, $k \ge 1$. Then (C, d, D^k) is an algebraic Poincaré complex. We put $\psi(\phi) = (C, d, D^k)$.

Lemma 4.10. If ϕ represents the trivial element in the group $\widetilde{L}_n(\Lambda)$, then $\psi(\phi) = 0$.

Proof. By definition the trivial element of the group $\widetilde{L}_n(\Lambda)$ is represented by automorphisms of the form

$$I_1 = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{*-1} \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

and their combinations.

Let (C, d, D^k) be an algebraic Poincaré complex such that $C_i = 0$ for $i \neq k, k + 1$, and $D^k = 0$ for $k \ge 1$. We construct a new complex $(\overline{C}, \overline{d}, \overline{D}^k)$ by putting $\overline{C}_k = C_k$, $\overline{C}_{k+1} = C_{k+1}^*$, $\overline{d}_{k+1} = D_k^0$, $\overline{D}_k = d_{k+1}$ and $\overline{D}^k = 0$. We will show that these two Poincaré complexes are cobordant. For this it is sufficient to prove that it is possible to carry out a modification by a map $\beta: A \to C_{k+1}^*$. If this is so, then, putting $A = C_{k+1}^*$ and $\beta = 1$, we obtain a new complex $H_k = C_k \oplus C_{k+1}$, $H_{k+1} = C_{k+1} \oplus C_{k+1}^*$,

$$\delta_{k+1}=egin{pmatrix} d_{k+1}&D_k^{
m o}\ 1&0 \end{pmatrix}.$$

Factoring by the acyclic subcomplex $(\delta_{k+1}(C_{k+1}), C_{k+1})$, we obtain the complex $(\overline{C}, \overline{d}, \overline{D}^k)$.

Let us now glue on a handle by the map $\beta: A \to C^*_{k+1}$. Let $(C, {}^0C, d, D^k)$ be an algebraic Poincaré pair, dim C = 2k + 2, such that $D^k = 0$, $k \ge 1$, and $\beta: A^* \to C^*_{k+1}$. Put

$$\overline{C}_{k} = C_{k} \oplus A, \quad {}^{0}\overline{C}_{k} = {}^{0}C_{k} \oplus A, \qquad \overline{C}_{k+1} = C_{k+1} \oplus A^{*} \oplus A^{*}$$
$${}^{0}\overline{C}_{k+1} = {}^{0}C_{k+1} \oplus A^{*}, \qquad \overline{d}_{k+1} = \begin{pmatrix} d_{k+1} & {}^{0}D_{k}^{0}\beta & 0\\ \beta^{*} & 0 & 1 \end{pmatrix},$$
$$\overline{D}_{k+1}^{0} = \begin{pmatrix} D_{k+1}^{0} & 0 & -D_{k+1}^{0}\beta\\ 0 & 0 & (-1)^{k}\\ -\beta^{*}D_{k+1}^{0} & 1 & \beta^{*}D_{k+1}^{0}\beta \end{pmatrix}.$$

Then

$${}^{0}\overline{D}^{0}_{k} = \begin{pmatrix} {}^{0}D^{0}_{k} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad {}^{0}\overline{D}^{0}_{k+1} = \begin{pmatrix} {}^{0}D^{0}_{k+1} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus the modification in dimension k is well defined.

Consequently, if A is an automorphism, then

$$\psi(A) = \psi(I_3 A). \tag{4.13}$$

Let us show that

$$\psi(A) = \psi(I_1 A). \tag{4.14}$$

Let

$$A = \begin{pmatrix} X & U \\ Y & V \end{pmatrix}, \quad I_1 = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}.$$

Then

$$I_{\mathbf{1}}A = \begin{pmatrix} X + \varphi Y & * \\ Y & * \end{pmatrix}.$$

Interpreting the homomorphism ϕ as a chain homotopy between two homomorphisms X and $X + \phi Y$, one can construct an algebraic Poincaré pair, just as in Lemma 3.8, the boundary of which will be $\psi(A) - \psi(I_1A)$. Namely, we put

$$\begin{split} \overline{C}_{k} &= C_{k} \oplus C_{k}, \quad \overline{C}_{k+1} = C_{k+1} \oplus C_{k} \oplus C_{k+1}, \qquad \overline{C}_{k+2} = C_{k+1}, \\ {}^{0}\overline{C}_{k} &= C_{k} \oplus C_{k}, \qquad {}^{0}\overline{C}_{k+1} = C_{k+1} \oplus C_{k+1}, \qquad {}^{0}\overline{C}_{k+2} = 0, \\ \delta_{k+1} &= \begin{pmatrix} d_{k+1} & (-1)^{k+1} & 0 \\ 0 & (-1)^{k} & d_{k+1} \end{pmatrix}, \qquad \delta_{k+2} = \begin{pmatrix} (-1)^{k} \\ d_{k+1} \\ (-1)^{k+1} \end{pmatrix}, \\ H_{k}^{0} &= \begin{pmatrix} 0 \\ (-1)^{k}D_{k}^{0} \end{pmatrix}, \quad H_{k+2}^{0} = (D_{k+1}^{0}, 0), \\ H_{k+1}^{0} &= \begin{pmatrix} \varphi & 0 & 0 \\ 0 & (-1)^{k+1}D_{k+1}^{0} & 0 \end{pmatrix}, \quad H_{k}^{1} = \begin{pmatrix} 0 \\ (-1)^{k+1}D_{k}^{0} \end{pmatrix}, \\ H_{k+1}^{1} &= (0, (-1)^{k+1}D_{k+1}^{0}, 0). \end{split}$$

Then

$${}^{0}H_{k} = \begin{pmatrix} D_{k}^{0} + (-1)^{k+1}d_{k+1}\phi & 0 & 0\\ 0 & 0 & -D_{k}^{0} \end{pmatrix}, \quad {}^{0}H_{k+1}^{0} = \begin{pmatrix} D_{k+1}^{0} + \varphi d_{k+1}^{*} & 0\\ 0 & 0\\ 0 & -D_{k+1}^{0} \end{pmatrix},$$

as required.

Applying (4.13) and (4.14) successively, we complete the proof of Lemma 4.10.

Lemma 4.11. The homomorphism

$$\psi: \widetilde{L}_{4k+2}(\Lambda) \to \Omega_{4k+2}(\Lambda)$$

is a monomorphism.

Proof. Every bilinear form $\lambda = -\lambda^*$ is an even form. In fact, let $\lambda = (\lambda_{ij}), \lambda_{ii} = \sum_{g \in \pi} \alpha_g g$. The condition $\lambda_{ii} = -\lambda^*_{ii}$ means that $\alpha_g = 0$ when $g \in \pi_0$, and $\alpha_g = -\alpha_{g-1}$. Put

$$\mu = (\mu_{ij}), \quad \mu_{ij} = \lambda_{ij} \text{ when } i < j,$$

$$\mu_{ii} = \sum_{g \in \pi_+} lpha_g g, \quad \mu_{ij} = 0 ext{ when } i > j.$$

It is clear that $\lambda = \mu - \mu^*$. Thus if $\lambda \in L_{4k+2}(\Lambda)$ and $\phi_2(\lambda) = 0$, then one can find matrices H_1 and H_2 , $H_i^* = -H_i$ such that

$$X^{*} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & H_{1} \end{pmatrix} X = \begin{pmatrix} 0 & 1 \\ -1 & H_{2} \end{pmatrix}.$$

Since the forms H_1 and H_2 are even, for a suitable choice of the matrix X one can obtain

$$X^* \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

as was required to be proved.

Theorem 4.12. The homomorphism

$$(\psi \circ \varphi) \otimes \mathbf{Z}\left[\frac{1}{2}\right] : L_{a}(\Lambda) \otimes \mathbf{Z}\left[\frac{1}{2}\right] \to \Omega_{a}(\Lambda) \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

is a monomorphism.

Proof. When n = 4k + 2 the assertion of the theorem follows from Theorem 2.1 and Lemma 4.11.

In [8] Shaneson constructed a homomorphism

$$\beta: L_n(\Lambda) \to L^s_{a+1}(\Lambda[z, z^{-1}]),$$

which whom combined with another homomorphism (also constructed by him)

$$\alpha: L^{s}_{n+1}(\Lambda[z, z^{-1}]) \longrightarrow L_{n}(\Lambda)$$

becomes the identity.

Let $\overline{\beta}: \Omega_n(\Lambda) \to \Omega_{n+1}(\Lambda[z, z^{-1}])$ be the natural homomorphism generated by the tensor product of an algebraic Poincaré complex α by a fixed chain complex for the circle. It is not difficult to see that the following diagram is commutative:

$$\begin{array}{ccc} L_{a}(\Lambda) \xrightarrow{\beta} L_{n+1}^{s}(\Lambda[z,z^{-1}]) \\ \stackrel{\psi \circ \phi \downarrow}{\to} & \stackrel{\overline{\beta}}{\to} & \stackrel{\psi \circ \phi}{\to} \\ \Omega_{n}(\Lambda) \xrightarrow{\beta} & \Omega_{n+1}(\Lambda[z,z^{-1}]) \end{array}$$

It was proved in [7] that the homomorphism $\gamma \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism. Consequentquently $\beta \otimes \mathbb{Z}[\frac{1}{2}]$ is a monomorphism, and therefore the fact that $(\psi \circ \phi) \otimes \mathbb{Z}[\frac{1}{2}]$ is a monomorphism for n + 1 follows from the fact that it is a meromorphism for dimension *n*. Theorem 4.12 is proved.

§5. Geometrical Poincaré complexes

Let us consider a finite CW-complex X, $\pi = \pi_1(X)$. Recall that Λ denotes the group ring of the group π , $\Lambda = \mathbb{Z}[\pi]$. Let X be the universal covering space of X. We may suppose that the group π acts on X freely and simplicially (cell-like) for

some simplicial subdivision of X which covers a certain simplicial subdivision of X. Then the chain groups $C_r(\widetilde{X})$ are naturally provided with: a) the structure of a right Λ -module, and b) a basis which makes the Λ -modules $C_r(X)$ into free Λ -modules. As a basis c_r for the Λ -module $C_r(\widetilde{X})$ one simplex (cell) should be chosen in each inverse image of the simplexes (cells) of X. The basis c_r is defined uniquely up to a choice of the ordering and orientation of the cells and multiplication by elements of the group π . The complex $C_*(\widetilde{X}) = \sum_r C_r(\widetilde{X})$ is a differential complex of Λ -modules.

Let B be a right Λ -module. Put

$$H^*(X; B) = H(\operatorname{Hom}_{\Lambda}(C_*(X), B)).$$

In order to introduce the concept of homology with coefficients in the Λ -module B it is necessary to introduce the structure of a left Λ -module in B. Put $\lambda(b) = b\lambda^*$, $\lambda \in \Lambda$, $b \in B$. The module B then acquires the structure of a left Λ -module and is then denoted by B^t . Put

$$H_{\star}(X; B) = H(C_{\star}(X) \otimes_{\Lambda} B^{\iota}).$$

Now let us define \cap , the intersection between the homology and the cohomology of a complex. Let $\Delta: X \to X \times X$ be the diagonal map. The map Δ is not simplicial but has a simplicial approximation. Let Δ_0 be a simplicial approximation to Δ . If $T: X \times X \to X \times X$ is a permutation of coordinates then Δ_0 and $T\Delta_0$ are homotopic, i.e. there exists a simplicial map

$$\Delta_1: X \times I \to X \times X$$

such that $\Delta_1 | X \times 0 = \Delta_0$ and $\Delta_1 | X \times 1 = T\Delta_0$. Further, if $S: I \to I$ is given by S(t) = 1 - t, then Δ_1 and $T\Delta_1 S$ coincide on $X \times 0 \bigcup X \times 1$ and are homotopic, i.e. there exists a simplicial map

$$\Delta_2: X \times I \times I \longrightarrow X \times X$$

such that $\Delta_2 | X \times I \times 0 = \Delta_1$ and $\Delta_2 | X \times I \times 1 = T\Delta_1 S$. In general one can construct a sequence of simplicial maps

$$\Delta_n: X \times I^n \to X \times X$$

satisfying the following:

$$\Delta_n | X \times I^{n-1} \times 0 = \Delta_{n-1}, \qquad \Delta_n | X \times I^{n-1} \times 1 = T \Delta_{n-1} S_{n-1}, \Delta_n | X \times \partial I^{n-1} \times t = \text{const.} \quad t \in I.$$

where $S_{n-1}: X \times I^{n-1} \to X \to I^{n-1}$ is given by

$$S_{n-1}(x, t_1, ..., t_{n-1}) = (x, 1-t_1, ..., 1-t_{n-1}).$$

It is easy to verify that Δ_n and $T\Delta_n S_n$ coincide on $X \times \partial I^n$.

Thus Δ_0 induces a map of chain complexes

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$$(\Delta_0)_{\bullet}: C_{\bullet}(\widetilde{X}) \longrightarrow C_{\bullet}(\widetilde{X} \times \widetilde{X}) \approx C_{\bullet}(\widetilde{X}) \otimes_{\mathbb{Z}} C_{\bullet}(\widetilde{X}).$$

The homomorphism $(\Delta_0)_*$ is a homomorphism of Λ -modules if we introduce in $C_*(\widetilde{X}) \oplus_Z C_*(\widetilde{X})$ the Λ -module structure induced by the diagonal embedding $(\Delta_0)_*$: $\pi \to \pi \times \pi$.

Each map Δ_n induces a homomorphism of the complexes of homogeneous degree (+n), and the complexes are chain homotopic:

$$(\Delta_n)_*: C_*(\widetilde{X}) \to C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(\widetilde{X}),$$

$$(5.1)$$

$$-(\Delta_n)_* d_k + d_{k+n} (\Delta_n)_* = (-1)^{k+n-1} ((\Delta_{n-1})_* - (T\Delta_{n-1}S_{n-1})_*).$$

Let us fix on an element $\xi \in C_*(\widetilde{X})$. Then $(\Delta_0)_*(\xi)$ defines a homomorphism

$$\alpha((\Delta_0)_*(\xi)): \operatorname{Hom}_{\Lambda}(C_*(\widetilde{X}); \Lambda) \to C_*(\widetilde{X})$$

as follows. Let

$$(\Delta_0)_*(\xi) = \sum a_i \otimes b_i, \ \varphi \in \operatorname{Hom}_{\Lambda}(C_*(\tilde{X}), \Lambda);$$

then

$$\varphi \to \sum b_i \otimes \varphi(a_i) \in C_*(\tilde{X}) \otimes_\Lambda \Lambda.$$

Clearly, if we introduce the structure of a right Λ -module on the group $\operatorname{Hom}_{\Lambda}(C_*(\widetilde{X}), \Lambda)$ by the formula

$$(\varphi\lambda)(a) = \lambda^{\bullet}\varphi(a),$$

then $\alpha((\Delta_0)_*(\xi))$ is a Λ -module homomorphism. Thus, putting $\beta(\xi) = \alpha((\Delta_0)_*(\xi))$, we have defined

 $\beta: C_*(\widetilde{X}) \longrightarrow \operatorname{Hom}_{\Lambda}(C^*(\widetilde{X}), C_*(\widetilde{X}) \otimes_{\Lambda} \Lambda).$

Here $C^*(\widetilde{X}) = \operatorname{Hom}_{\Lambda}(C_*(\widetilde{X}), \Lambda).$

Lemma 5.1. Let \mathbb{Z}^t be the group of integers endowed with the structure of a left (and simultaneously right) Λ -module generated by the augmentation $\pi \to 1$. Then the homomorphism β decomposes into $\beta = \beta_1 \circ \beta_2$, where $\beta_2 \colon C_*(\widetilde{X}) \otimes_{\Lambda} \mathbb{Z}^t$ is an epimorphism and

$$\beta_{\mathbf{1}}: C_{\bullet}(\widetilde{X}) \otimes_{\Lambda} \mathbf{Z}^{t} \to \operatorname{Hom}_{\Lambda}(C^{\bullet}(\widetilde{X}), C_{\bullet}(\widetilde{X}) \otimes_{\Lambda} \Lambda).$$

Proof. It is sufficient to verify that $\alpha((\Delta_0)_*(\xi)) = \alpha((\Delta_0)_*(\xi g)), g \in \pi$. Let $\phi \in C^*(\widetilde{X})$ and $(\Delta_0)(\xi) = \sum a_i \otimes b_i$. Then $(\Delta_0)_*(\xi g) = \sum a_i g \otimes b_i g$. We have

$$(\Delta_0)_*(\xi)(\varphi) = \sum b_i \otimes \varphi(a_i),$$

$$(\Delta_0)_*(\xi g)(\varphi) = \sum b_i g \otimes \varphi(a_i g) = \sum b_i g \otimes (\varphi(a_i) g)$$

$$= \sum b_i g \otimes g^{-1} \varphi(a_i) = \sum b_i \otimes \varphi(a_i).$$

Lemma 5.1 is proved.

Note that the analogous assertion is true for the homomorphisms $\alpha((\Delta_n)_*(\xi))$.

Lemma 5.2. The homomorphisms

$$[\alpha((\Delta_0)_*(\xi)]_k \text{ and } (-1)^{k(n-k)} [\alpha((T\Delta_0)_*(\xi))]_k,$$

where dim $\xi = n$, are dual.

Proof. Let us fix a certain free basis (e_{α}) in the Λ -module $C_*(\widetilde{X})$. Then the elements $(\Delta_0)_*(\xi)$ and $(T\Delta_0)_*(\xi)$ will have, respectively, the forms

$$(\Delta_{0})_{*}(\xi) = \sum_{i,\alpha,\beta} (\lambda_{i\alpha} \otimes \mu_{i\beta}) e_{\alpha} \otimes e_{\beta},$$
$$(T\Delta_{0})_{*}(\xi) = \sum_{i,\alpha,\beta} (-1)^{\dim e_{\alpha}\dim e_{\beta}} (\mu_{i\beta} \otimes \lambda_{i\alpha}) e_{\beta} \otimes e_{\alpha}.$$

Let ϕ_{γ} be the basis in the group $C^{k}(\widetilde{X})$ which is dual to e_{a} , i.e.

Then

$$\begin{aligned} \alpha((\Delta_{\mathfrak{g}})_{*}(\xi))(\varphi_{\Upsilon}) &= \sum_{\beta} e_{\beta} \dot{A}_{\beta\Upsilon}, \\ \alpha((T\Delta_{\mathfrak{g}})_{*}(\xi))(\varphi_{\Upsilon}) &= \sum_{\beta} e_{\beta} B_{\beta\Upsilon}, \end{aligned}$$

where

$$A_{\beta\gamma} = \sum_{i} \mu_{i\beta} \lambda_{i\gamma}^{*}, \quad B_{\beta\gamma} = (-1)^{k(i-k)} \sum_{i} \lambda_{i\beta} \mu_{i\gamma}^{*}.$$

Clearly the matrices $A = ||A_{\beta\gamma}||$ and $(-1)^{k(n-k)}B = (-1)^{k(n-k)}||B_{\beta\gamma}||$ are dual, i.e. $B = (-1)^{n(n-k)}(A^t)^*$. Lemma 5.2 is proved.

Theorem 5.3. Let $\xi \in C_n(\widetilde{X}) \otimes_{\Lambda} \mathbb{Z}^t$ be a cycle and let $\widetilde{\xi} \in C_*(\widetilde{X})$ be an element covering ξ . Then the homomorphisms

$$\bigcap \xi = \alpha((\Delta_0)_*(\overline{\xi})), \qquad D_i = \alpha((\Delta_i)_*(\xi))$$

are uniquely defined by the element ξ and satisfy the following conditions:

a) $d_k \circ (\bigcap \xi) - (-1)^{k+1} (\bigcap \xi) d_{i-k}^*;$

b)
$$d_{k+i}D_i + (-1)^{k+1}D_i d_{n-k}^* = (-1)^{n+i-1}D_{i-1} - (-1)^{k(n+i)+n}D_{i-1}^*$$
;
if $i = 1$ we put $\bigcap \xi = D_0$ in b).

Proof. Let $\xi \in C_n(\widetilde{X}) \otimes_{\Lambda} \mathbb{Z}^t$, and let $\widetilde{\xi}$ be an element in the module $C_n(\widetilde{X})$ which goes into the element ξ under the map induced by the augmentation $\epsilon \colon \Lambda \to \mathbb{Z}$. Since $d_n(\xi) = 0$, we have $\epsilon d_n(\widetilde{\xi}) = 0$. From (5.1) we obtain A. S. MIŠČENKO

$$d_{n+i}(\Delta_{i})_{*}(\tilde{\xi}) - (\Delta_{i})_{*}(d_{n}\tilde{\xi}) = (-1)^{l+n-1} \{ (\Delta_{i-1})_{*}(\tilde{\xi}) - (T\Delta_{i-1}S_{i-1})_{*}(\tilde{\xi}) \}, d_{n}(\Delta_{0})_{*}(\tilde{\xi}) = (\Delta_{0})_{*}(d_{n}\tilde{\xi}).$$
(5.2)

By Lemma 5.1, since $\epsilon d_n(\vec{\xi}) = 0$,

$$\alpha\left(\left(\Delta_{i}\right)_{*}\left(d_{\mathbf{r}}\widetilde{\boldsymbol{\xi}}\right)\right)=0,\quad i\geqslant0.$$

By the definition of $\bigcap \xi$ and D_i , if $\phi \in C^k(\widetilde{X})$, then

$$D_i(\varphi) = \sum_{\alpha} b^i_{\alpha} (\varphi(a^i_{\alpha}))^*, \qquad \cap \xi(\varphi) = \sum_{\alpha} b_{\alpha} (\varphi(a_{\alpha}))^*,$$

where $(\Delta_i)_*(\xi) = \sum_a a_a^i \otimes b_a^i$. Then, using Lemma 5.2 on the right side of formula (5.2), we have

$$(-1)^{i+n-1} (D_{i-1}(\varphi) - (-1)^{(i-1)+k(n+i-1-k)} D_{i-1}^{*}(\varphi))$$

= $\alpha (d_{n+i}(\Delta_i)_{*}(\tilde{\xi}))(\varphi) = \sum_{\alpha, \beta} b_{\alpha}^{i} (\varphi (d x_{\alpha}^{i}))^{*} + \sum_{\alpha} (-1)^{\dim a_{\alpha}^{i}+1} d b_{\alpha}^{i} \varphi (a_{\alpha}^{i})$
= $D_{i} (d \varphi) + (-1)^{k+1} d D_{i} (\varphi),$

which proves property b) of the theorem. Property a) is proved similarly. Theorem 5.3 is proved.

Definition 5.4. A finite complex X, $\pi_1(X) = \pi$, is said to be an oriented Poincaré complex if there exists a cycle $\xi \in C_n(\widetilde{X}) \oplus_{\Lambda} \mathbb{Z}^t$ such that

$$\bigcap \, \xi : C^*(\widetilde{X}) \to C_*(\widetilde{X})$$

induces an isomorphism of homology groups

$$(\bigcap \xi)_*: H^i(X; \Lambda) \longrightarrow H_{n-i}(X; \Lambda).$$

If, in addition, the torsion of the homomorphisms $\mathbf{0} \xi$ is zero, then X is said to be a simple oriented Poincaré complex.

This presupposes the following choice of bases in the modules $C_*(\widetilde{X})$ and $C^*(\widetilde{X})$: in the module $C_*(\widetilde{X})$ we choose as basis one cell complex \widetilde{X} in each orbit of the group π , and in the module $C^*(\widetilde{X})$ we choose the dual basis. It is not difficult to verify that the arbitrariness in the choice of basis for $C_*(X)$ does not alter the torsion of $\mathbf{n} \boldsymbol{\xi}$ in the group $K_1(\Lambda)$.

Let (X, Y) be a CW-complex pair, let $i: Y \in X$ be the embedding, and let $\pi_1(X) = \pi_1(Y) = \pi$. Then there is an analogous theorem. Let $\xi \in C_n(\widetilde{X}) \otimes_{\Lambda} \mathbb{Z}^t$ be a cycle relative to the module $C_{n-1}(\widetilde{Y}) \otimes_{\Lambda} \mathbb{Z}^t$ and let $\widetilde{\xi} \in C_n(\widetilde{X})$ be an element covering ξ , i.e. $\epsilon(\widetilde{\xi}) = \xi$.

Theorem 5.5. An element ξ uniquely determines the homomorphisms

$$\bigcap \xi : C^{k}(\widetilde{X}) \to C_{n-k}(\widehat{X}),$$
$$D_{i} : C^{k}(\widetilde{X}) \to C_{n-k+i}(\widetilde{X}),$$

satisfying conditions a) and b) of Theorem 5.3 relative to the submodule $C_*(\widetilde{Y})$, i.e.

a)
$$((d_k \cap \xi) + (-1)^{k+1} (\mathbf{n} \xi) d_k)(\phi) \in C_{n-k}(\widetilde{Y}),$$

b) $(d_{k+i}D_i + (-1)^{k+1}D_i d_k + (-1)^{n+i}D_{i-1} + (-1)^{(n+i)(n+i-k)+n+k+1}D_{i-1}^*)(\phi) \in C_{n-k+i-1}(\widetilde{Y}).$

The proof of Theorem 5.5 is analogous to the proof of Theorem 5.3, and we will omit it.

Definition 5.6. A pair (X, Y) is said to be an oriented Poincaré pair if there exists a cycle $\xi \in C_n(\widetilde{X}, \widetilde{Y}) \oplus_{\Lambda} \mathbb{Z}^t$ such that the homomorphism $\bigcap \xi$ induces an isomorphism in homology

$$(\bigcap \xi)_*$$
: $H^{\mathcal{R}}(X; \Lambda) \to H_{n-k}(X, Y; \Lambda).$

Lemma 5.7. If (X, Y) is a Poincaré pair with a fundamental cycle ξ of dimension n, then the complex Y is an oriented Poincaré complex with fundamental cycle $d\xi$ of dimension (n - 1).

Proof. Property a) of Theorem 5.3 can be made more precise in the following way: if $\xi \in C_*(X) \otimes_{\Lambda} \mathbb{Z}^t$ is a chain, then

$$d_k \circ (\bigcap \xi) + (-1)^{k+1} (\bigcap \xi) d_k = \bigcap (d\xi).$$
(5.3)

Let us prove that the homomorphisms

$$p \circ \bigcap \xi : C^*(\bar{X}) \to C_*(\bar{X}, \bar{Y}),$$
$$(\bigcap \xi) \circ p^* : C^*(\bar{X}, \bar{Y}) \to C_*(\bar{X}),$$
$$\bigcap (d\xi) : C^*(\bar{Y}) \to C_*(\bar{Y})$$

induce maps of the exact cohomology sequence of the pair (X, Y) into an exact homology sequence. Here

$$i_*: C_*(\widetilde{Y}) \to C_*(\widetilde{X}), \qquad p: C_*(\widetilde{X}) \to C_*(\widetilde{X}, \widetilde{Y})$$

are the natural maps of the chain groups. Note that the diagram

is commutative. Let $x \in C^*(\widetilde{Y})$, dx = 0 and $y = i^*(x)$. Then

$$((\cap \xi) p^*) d[x] = \cap \xi(dy) = \cap (d\xi)(y) \pm d(\cap \xi(y))$$
$$= i_* \cap (d\xi)(x) \pm d(\cap \xi(y)).$$

Finally, let $x \in C^*(\widetilde{X})$. Then

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 $\bigcap (d\xi) i^*(x) = (\bigcap \xi) (dx) \pm d(\bigcap \xi) (x) = \pm d(\bigcap \xi(x)).$

This proves that each square in the diagram is commutative. On the other hand, part b) of Theorem 5.5 implies that $(\mathbf{n} \xi)p^*$ and $(\mathbf{n} \xi)^*p^*$ are chain homotopic, i.e. coincide at the homology level. Consequently we have obtained a map between exact sequences in which every two out of three maps are isomorphisms. Applying the Five Lemma, we obtain the assertion of Lemma 5.7.

Definition 5.8. If for an oriented Poincaré pair (X, Y) the isomorphisms $\bigcap \xi$ and $\bigcap d\xi$ are simple, then the pair (X, Y) is said to be a simple oriented Poincaré pair.

We note the well-known fact that every smooth (PL)-manifold (with boundary) is a simple Poincaré complex (simple Poincaré pair).

Lemma 5.9 ([1], Theorem 2.2). Every finite Poincaré complex X with formal dimension n is homotopically equivalent to a finite n-dimensional complex.

Lemma 5.10. Every finite Poincaré complex X of formal dimension $n \ge 3$ is bomotopically equivalent to a closed domain W with smooth boundary in Euclidean space \mathbb{R}^{N+n} , $N \ge n+1$. Furthermore, $\pi_1(\partial W) \approx \pi_1(W)$.

Let us study the cohomology of a manifold W using Lemma 5.10. Let \widetilde{W} be the universal covering space of W. Then

$$H^{i}(W; \Lambda) \approx H_{n-i}(W; \Lambda) \approx H^{N+i}(W, \partial W; \Lambda)$$

Lemma 5.11. Multiplication by an element ξ is defined for every $\xi \in H^n(W, \mathbb{Z})$ and is a cohomology homomorphism

$$\bigcup \xi: H^{\iota}(W; B) \longrightarrow H^{\iota+n}(W; B).$$

Proof. Let us consider the chain homomorphism

 $(\Delta_{\mathbf{0}})_*: C_*(\widetilde{W}) \to C_*(\widetilde{W}) \otimes_{\mathbf{Z}} C_*(\widetilde{W}).$

Let $(\Delta_0)_*(x) = \sum_a a_a \otimes b_a$ and let $\xi: C_*(\widetilde{W}) \to \mathbb{Z}$ be a cocycle. Put

$$\zeta(x) = (\bigcup \xi)(\eta)(x) = \sum_{\alpha} \xi(a_{\alpha}) \eta(b_{\alpha}).$$

Then ζ is a A-homomorphism from the module $C_*(\widetilde{W})$ into the module B. In fact,

$$\zeta(\mathbf{x}\mathbf{g}) = \sum_{\alpha} \xi(a_{\alpha}\mathbf{g}) \eta(b_{\alpha}\mathbf{g}) = \sum_{\alpha} \xi(a_{\alpha}) \eta(b_{\alpha}) \mathbf{g} = \zeta(\mathbf{x}) \mathbf{g}$$

It is not difficult to verify that

$$\bigcup \xi: C^*(X; B) \to C^*(X; B)$$

is a chain homomorphism and in homology it does not depend on the cocyle ξ representing it.

Lemma 5.12. Let W be a manifold with boundary which is also a domain in \mathbb{R}^{N+n} and a Poincaré complex of formal dimension n. There exists a cocycle $\eta \in H^{N}(W, \partial W; \mathbb{Z})$ such that

 $\bigcup \eta: H^{i}(W; \Lambda) \to H^{N+i}(W, \partial W; \Lambda)$

is an isomorphism.

Proof. Let $\xi \in H_n(W; \mathbb{Z})$ be a fundamental cycle of the Poincaré complex W and let $\zeta \in H_{N+n}(W, \partial W_j; \mathbb{Z})$ be a fundamental cycle of the manifold with boundary W. Put $\eta = (\mathbf{n} \zeta)^{-1}(\zeta)$. Then

$$(\bigcup \eta)(x) = (\bigcap \zeta)^{-1} ((\bigcap \zeta)(\bigcup \eta)(x))$$
$$= (\bigcap \zeta)^{-1} (\bigcap (\bigcap \zeta(\eta))(x)) = (\bigcap \zeta)^{-1} (\bigcap \xi)(x).$$

Consequently $(\mathbf{U} \eta)$ induces an isomorphism in homology. Lemma 5.12 is proved.

The converse assertion is also true.

Lemma 5.13. Let W be a manifold with boundary which is also a domain in \mathbb{R}^{N+n} . If there exists a cocycle $\eta \in H^{N}(W, \partial W; \mathbb{Z})$ such that

$$\bigcup \eta: H^*(W; \Lambda) \to H^*(W, \partial W; \Lambda)$$

is an isomorphism, then the complex W is an oriented Poincaré complex of formal dimension n.

The proof is analogous to the proof of Lemma 5.12.

Lemma 5.14. Let W be a domain in \mathbb{R}^{N+n} with smooth boundary ∂W , N > n. The following conditions are equivalent:

a) The Serre fibration associated with the embedding $\partial W \subset W$ is a spherical fibration with fiber S^{N+1} and with the fundamental group acting trivially on the homology of the fiber.

b) The manifold W is an oriented Poincaré complex with fundamental cocycle $\eta \in H^N(W, \partial W; \mathbb{Z})$.

Proof. (a) \Rightarrow (b). Let us consider the spectral sequence of the pair of fibrations

$$(D^N, S^{N-1}) \to (W, \partial W) \to W$$

for cohomology with coefficients in the module Λ . Then

 $E_2^{N, q}(W, \partial W; \Lambda) = H^q(W; \Lambda), \qquad E_2^{p, q}(W, \partial W; \Lambda) = 0 \text{ when } p \neq N.$

Consequently

$$H^{q}(W, \ \partial W; \ \Lambda) \approx E_{\infty}^{N,q}(W, \ \partial W) \approx E_{2}^{N,q}(W, \ \partial W) = H^{q}(W; \ \Lambda);$$

moreover, this isomorphism is realized by multiplication by an element $\zeta \in H^{N}(D^{N}, S^{N-1}; \mathbb{Z})$. Put $\eta = \zeta \bigcup 1$, where $1 \in H^{0}(W; \mathbb{Z})$, and apply Lemma 5.13.

(b) \Rightarrow (a). Since the cohomology $H^{i}(W; \Lambda)$ is isomorphic to the compact cohomology, the assertion reduces to a computation of the cohomology of the fiber using the spectral sequence for the fibration. The latter assertion is implied by the fact that the restriction $H^{N}(W, \partial W; \mathbb{Z}) \rightarrow H^{N}(D^{N}, S^{N-1}; \mathbb{Z})$ is an epimorphism.

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Let us now apply the results of §4 to a geometric Poincaré complex.

Theorem 5.15. Let X be an oriented geometric Poincaré complex. Then there is an element $a(X) \in \Omega_n(\Lambda)$, uniquely associated with X, generated by the homomorphisms

$$\bigcap \xi : C^{k}(\widetilde{X}) \to C_{n-k}(\widetilde{X}), \qquad D_{i} : C^{k}(\widetilde{X}) \to C_{n-k+i}(\widetilde{X}).$$

Proof. It is only required to verify that the element a(X) does not depend on the choice of the cocycle ξ , the homotopies Δ_i and the simplicial decomposition of the complex X. In the first and second cases it is easy to construct a simplicial decomposition of $X \times I$ related to the cocycle η on the complex $X \times I$ and the homotopies $\widetilde{\Delta}_i$ on the complexes $X \times I \times I^i$ such that we obtain an algebraic Poincaré pair with boundary $a(X) \bigcup (-a(X'))$. In the third case there exists a simplicial map $f: X \to X'$ from one simplicial structure to the other, and we apply Lemma 4.2.

Theorem 5.16. Let $\phi: L_n(\Lambda) \to \Omega_n(\Lambda)$ be the natural map constructed in §4. Let M^n be a smooth manifold, $\pi_1(M^n) = \pi$, $\Lambda = \mathbb{Z}[\pi]$, let ξ be a fiber bundle over the manifold M^n and let $(X, f, \phi) \in \Omega_n(M^n, \xi)$ be a triple, where $f: X \to M^n$ is a map of the manifold X of degree 1. Let $\phi: \nu(X) \to f^*(\xi)$ be an isomorphism and let $\theta(X, f, \phi) \in L_n(\Lambda)$ be the obstruction to modifying (X, f, ϕ) to a homotopy equivalence. Then

$$\varphi\left(\theta(X, f, \varphi)\right) = a\left(M\right) - a\left(X\right).$$

The proof is trivial (see, for example, [11]).

§6. Intersection theory for Poincaré complexes

In this section we discuss the question of defining the intersection of Poincaré complexes lying in a manifold. We must require the intersection to satisfy a number of conditions. Namely, let X_1 and X_2 be Poincaré complexes, M a smooth manifold, let $f_i: X_i \to M$ be continuous maps and let $(f_i)_*: \pi_1(X_i) \to \pi_1(M)$ be isomorphisms. We want to define a canonical construction of a Poincaré complex $(Y, g) = (X_1, f_1) \cap (X_2, f_2)$, dim $Y = \dim X_1 + \dim X_2 - \dim M$, such that the condition $(X_1, f_1) = \partial(W, f)$ implies $(Y, g) = \partial((W, f) \cap (X_2, f_2))$.

We will restrict ourselves to the case when (X_2, f_2) is a smooth manifold.

Let X be a domain in \mathbb{R}^{N+n} with smooth boundary which is also a Poincaré complex of dimension *n*. Let Y be a smooth manifold of dimension k and M a smooth manifold of dimension *m*. Let $f: X \to M$ and $g: Y \to M$ be continuous (smooth) maps inducing isomorphisms of the fundamental groups. Put

$$W_0 = M \times X \times Y, \quad W_1 = X \times Y,$$

$$W_2 = Y \times X, \quad g_1 = ((f \times id) \Delta) \times id,$$

$$g_2 = T (((g \times id) \Delta) \times id),$$

where $T: M \times Y \times X \to M \times X \times Y$ is a permutation of the coordinates.

Lemma 6.1. The manifolds W_0, W_1, W_2 with boundary are oriented Poincaré complexes. Moreover, if $\eta_i \in H^N(W_i, \partial W_i; \mathbb{Z})$ are fundamental cocycles, then

$$\eta_i = g_i^*(\eta_0), \quad i = 1, 2,$$

where Z is the trivial module over the group $\pi_1(W_i)$.

Lemmas 6.1 and 5.14 imply

Corollary 6.2. The diagrams

$$\begin{array}{ccc} \partial W_i \to \partial W_0 \\ \downarrow & \downarrow \\ W_i \to W_0 \end{array}$$

induce commutative diagrams of Serre fibrations with homotopically equivalent fibers which are homotopy spheres S^{N-1} .

Lemma 6.3. Let $\alpha_i \in \pi_N(W_i, \partial W_i)$ be the elements corresponding to the fundamental cycle of the fiber of the spherical fibrations $\partial W_i \subset W_i$. The elements α_i can be realized as embedded discs

$$\Psi_i: (D^N, S^{N-1}) \to (W_i, \partial W_i).$$

The embedding g_i can be changed by regular homotopies to embeddings in general position, where

$$\varphi_0 = g_i \circ \varphi_i, \quad i = 1, 2.$$

Proof. First of all, by changing the embedding g_i we can ensure that $g_i \phi_i = \phi_0$. After this we move the embedding g_i into general position leaving g_i fixed on $\text{Im } \phi_i$.

Lemma 6.3 tells us that $W_3 = W_1 \cap W_2$ is a manifold with boundary, and, moreover, the embedding $b_i: W_3 \subset W_i$ induces an epimorphism

$$(h_i)_*: \pi_N(W_3, \partial W_3) \rightarrow \pi_N(W_i, \partial W_i), \quad i = 1, 2.$$

Lemma 6.4. Let $\pi_2(W_0) = 0$. Then the embeddings g_i can be chosen such that W_3 is connected and the composition

$$\pi_1(W_3) \xrightarrow{(h_i)_*} \pi_1(W_i) \xrightarrow{(g_i)_*} \pi_1(W_0) \xrightarrow{\rho} \pi_1(M)$$

is a monomorphism.

Proof. Let us show that W_3 can be supposed connected. Let the points x_0 and x_1 lie in different connected components of W_3 . Join x_0 to x_1 by paths y_1 and y_2 lying in W_1 and W_2 respectively. Then the closed path $y_1y_2^{-1}$ defines an element $\alpha \in \pi_1(W_0)$. Since

$$(g_1)_* \oplus (g_2)_* : \pi_1(W_1) \oplus \pi_1(W_2) \to \pi_1(W_0)$$

is an epimorphism, the paths γ_1 and γ_2 can be altered so that $\alpha = 0$. This means that there exists an embedding of the disc $D^2 \subset W_0$ with $\partial D^2 = \gamma_1 \cup \gamma_2$. Deform the embedding g_1 so as to remain fixed on the boundary of a neighborhood of γ_1 and in

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such a way that γ_1 is taken across the disc D^2 to γ_2 . Then the manifold W_3 has undergone a modification with respect to the embedded zero-dimensional sphere (x_0, x_1) .

Suppose now that

$$(h_1)_* \oplus (h_2)_* : \pi_1(W_3) \rightarrow \pi_1(W_1) \oplus \pi_1(W_2)$$

is a monomorphism. Let us show that, for example, $\psi = \rho(g_1)_*(b_1)_*$ is a monomorphism. In fact,

$$\pi_1(W_1) = \pi_1(X) \oplus \pi_1(Y),$$

$$\pi_1(W_0) = \pi_1(M) \oplus \pi_1(X) \oplus \pi_1(Y),$$

and $(g_i)_*(\alpha \oplus \beta) = (f_*(\alpha), \alpha, \beta)$. Thus if $(b_1)_*(x) = (\alpha, \beta)$, then $\psi(x) = f_*(\alpha)$. On the other hand, since $g_1b_1 = g_2b_2$, it follows that $(b_2)_*(x) = (\alpha', \beta')$ implies

$$(f_*(\alpha), \alpha, \beta) = (g_*(\alpha'), \beta', \alpha'),$$

i.e. $(b_2)_*(x) = (\beta, \alpha)$ and $f_*(\alpha) = g_*(\beta)$.

In this way the fact that $(b_i)_* \oplus (b_2)_*$ is a monomorphism implies that $(b_1)_*$ and $(b_2)_*$ are also monomorphisms. If $x \neq 0$ then $(b_1)_*(x) = (\alpha, \beta) \neq 0$. If $\alpha = 0$, then $\beta \neq 0$, i.e. $g_*(\beta) \neq 0$, which means that $f_*(\alpha) \neq 0$. Thus $\psi(x) \neq 0$. If $\alpha \neq 0$, then, since f_* is a monomorphism, $f_*(\alpha) = \psi(x) \neq 0$.

Consequently it is sufficient to show that $(b_1)_* \oplus (b_2)_*$ is a monomorphism. Let $\alpha \in \pi_1(W_3)$ and $(b_1)_*(\alpha) = (b_2)_*(\alpha) = 0$. Let us realize the element α as an embedded curve $\gamma_3: S^1 \to W_3$ and extend the mapping γ_3 to embeddings of discs $\gamma_i: D^2 \to W_i$. Since $\pi_2(W_0) = 0$, it follows that $\gamma = \gamma_1 \cup \gamma_2: S^2 \to W_0$ extends to an embedding of the disc $\gamma_0: D^3 \to W_0$. Let ξ_3 be the normal bundle to $\gamma_3(S^1)$ in the manifold W_3 and let ξ_i be the normal bundle to $\gamma_i(D^2)$ in the manifolds W_i . It is clear that the bundles ξ_i are trivial. Put $\xi_i | S^1 = \xi_3 + \nu_i$. Then $\xi_3 + \nu_1 + \nu_2 + 2$ is the normal bundle to the embedding of S^1 in W_0 . Consequently the bundle $\xi_3 + \nu_1 + \nu_2 + 2$ is trivial, whence it follows immediately that the bundles ξ_3 and ν_i are trivial. Let us show that a neighborhood U of the disc $\gamma_0(D^3)$ can be expressed as a direct product $\gamma_0(D^3) \times R_1 \times R_2 \times R_3$ is a neighborhood of $\gamma_2(D^2)$ in W_2 and

$$\gamma_3(S^1) \times 0 \times R_2 \times 0 = \gamma_1(D^2) \times R_1 \times R_2 \times 0 \cap \gamma_2(D^2) \times 0 \times R_2 \times R_3$$

is a neighborhood of $\gamma_3(S^1)$ in W_3 . In fact, let η be the normal (trivial) bundle of the disc $\gamma_0(D^3)$. The bundle $\eta | \gamma_3(S^1)$ decomposes as the sum of trivial bundles $\xi_3 + \nu_1 + \nu_2$ and $\eta | \gamma_2(D^2)$ decomposes into the sum $\xi_2 + \nu_1$; finally, $\eta | \gamma_1(D^2)$ decomposes into the sum $\xi_1 + \nu_2$. Extend, in a trivial way, the sub-bundles ν_1 and ν_2 from, the discs $\gamma_2(D^2)$ and $\gamma_1(D^2)$ respectively to the whole of the sphere $\gamma(S^2)$. Then ξ_3 has a natural (trivial) extension over the sphere $\gamma(S^2)$. Furthermore, we can extend the sub-bundles ν_1 and ν_2 to a sub-bundle over $\gamma_0(D^3)$, the complement of $\nu_1 \oplus \nu_2$ being an extension of ξ_3 over $\gamma_0(D^3)$. It is now not difficult to carry out a regular isotopy of W_1 , deforming the disc $\gamma_1(D^2)$ across $\gamma_0(D^3)$ to $\gamma_2(D^2)$ leaving fixed the boundary $\gamma_0(S^1)$. As a result of this isotopy the intersection $W_3 = W_1 \cap W_2$

undergoes a Morse modification by which the one-dimensional cycle $\alpha \in \pi_1(W_3)$ becomes an element homotopic to zero. Lemma 6.4 is proved.

Let dim X = p, dim Y = q and dim M = n, let p + q - n be odd, and let N > p + q - n > 0. Let $\pi_i(M) = 0$ when $2 \le i \le \frac{1}{2}(p + q - n) + 2$.

Theorem 6.5. The manifold $W_3 = W_1 \cap W_2$ can be chosen in such a way that the homomorphisms of "multiplication"

$$\bigcup \eta_3: H^i(W_3; Q) \to H^{N+i}(W_3, \partial W_3; Q)$$

by the cocycle $\eta_3 = b_1^*(\eta_1) = b_2^*(\eta_2) \in H^N(W_3, \partial W_3, \mathbb{Z})$ is an isomorphism.

Proof. First of all consider a single elementary deformation of the embedding g_1 : $W_1 \rightarrow W_0$. Let us consider a half disc (D_+^{k+2}, D^{k+1}) whose boundary (D^{k+1}, S^k) is divided by the equator into two parts

$$(D^{k+1}, S^k) = (D^{k+1}_+, D^k_+) \cup (D^{k+1}_-, D^k_-),$$

 $(D^{k+1}_+, D^k_+) \cap (D^{k+1}_-, D^k_-) = (D^k, S^{k-1}).$

Let an embedding

$$\varphi: (D_+^{k+2}, D^{k+1}) \subset (W_0, \partial W_0)$$

be given such that

$$\varphi(D_+^{k+1}, D_+^k) \subset (W_1, \partial W_1), \qquad \varphi(D_-^{k+1}, D_-^k) \subset (W_2, \partial W_2).$$

Then there exists an isotopy of the embedding g_1 corresponding to a deformation of the pair (D_+^{k+1}, D_+^k) across the disc (D_+^{k+1}, D^{k+1}) into the pair (D_-^{k+1}, D_-^k) . As a result of the deformation the manifold W_3 undergoes a Morse modification corresponding to the embedding of the pair

$$\varphi: (D^k, S^{k-1}) \subset (W_3, \partial W_3).$$

Lemma 6.6. The embedding g_1 can be chosen in such a way that $\pi_i(W_3, \partial W_3) = 0$ when $2 \le i \le N - 1$.

Proof. Apply induction. Let $\pi_i(W_3, \partial W_3) = 0$ when $2 \le i \le k$. Then $H_i(W_3, \partial W_3; \Lambda) = 0$ when $1 \le i \le k$. Consequently $H^i(W_3, \partial W_3; \Lambda) = 0$ when $1 \le i \le k$, and hence $H_1(W_3; \Lambda) = 0$ when $N + p + q - n - k \le i \le N + p + q - n = r$.

Lemma 6.7. Let W be a manifold with boundary, dim W = N, and $H_i(\widetilde{W}, \partial \widetilde{W}) = 0$ when i < n. Then W is homotopically equivalent to a complex of dimension N - n.

Using Lemma 6.7, we see that W_3 is homotopically equivalent to a complex Y of dimension d = N + p + q - n - k + 1. Moreover, we may suppose that Y is a subcomplex of the manifold W_3 . Let us consider an element $\alpha \in \pi_{k+1}(W_3, \partial W_3)$ realized as an immersion of a sphere in general position

$$\alpha: (D^{k+1}, S^k) \to (W_3, \partial W_3).$$

Then the subcomplex Z of multiple points of the immersion α is of dimension $\leq d' = 2(k+1) - N - p - q + n$. Then $d + d' = k + 3 \leq r$, where $r = \dim W_3$. Consequently we may suppose that $\alpha(Z) \cap Y = \emptyset$. Since Y is homotopically equivalent to W_3 , then $W_3 \setminus Y$ is diffeomorphic to $\partial W_3 \times [0, 1)$ [2]. Thus we may ensure that $\alpha(Z)$ lies arbitrarily near the boundary of the manifold W_3 . Throwing away small neighborhoods of the boundaries of W_0 , W_1 and W_2 , we obtain a realization of the element α as an embedded disc

$$\alpha : (D^{k+1}, S^k) \to (W_3, \partial W_3).$$

In order to apply an elementary deformation it is necessary to extend α to an embedding of a pair

$$\overline{\alpha}: (D^{k+3}, S^{k+2}) \subset (W_0, \partial W_0),$$

which is possible since

$$\pi_{k+2}(W_i, \ \partial W_i) = \pi_{k+2}(W_c, \ \partial W_0) = 0.$$

Lemma 6.6 is proved.

Proof of Lemma 6.7. Let C(W) be the chain complex of free A-modules for the manifold W, and let $C(W, \partial W)$ be the relative chain complex. The condition $H_i(\widetilde{W}, \partial W) = 0$ means that the complex $C(W, \partial W)$ is acyclic up to dimension n - 1. Consequently the boundary homomorphism

$$d_n: C_n(W, \partial W) \to C_{n-1}(W, \partial W)$$

maps $C_n(W, \partial W)$ onto a direct summand F in $C_{n-1}(W, \partial W)$; moreover, F is stably free. By Poincaré duality the same thing holds for the coboundary homomorphism

$$d_{N-n+1}^* : C_{N-1}^*(W) \to C_{N-n+1}^*(W).$$

In other words, $C_{N-n}^*(W)$ may be expressed as a direct sum $F_1^* \oplus F_2^*$ of stably free modules $d_{N-n+1}^*(F_1^*) = 0$, and $d_{N-n+1}^*|F_2^*$ is a monomorphism onto a direct summand in $C_{N-n+1}^*(W)$. Thus the chain complex C(W) can be expressed as two complexes

$$C_0(W) \leftarrow \dots \leftarrow C_{N-n+1}^{d_{N-n+1}} C_{N-n-1}(W) \leftarrow F_1 \leftarrow 0,$$

$$0 \leftarrow F_2 \leftarrow C_{N-n+1}(W) \leftarrow \dots \leftarrow C_N(W) \leftarrow 0,$$

where the first complex is homotopically equivalent to C(W) and the second is acyclic. Let F_3 be a free module such that $F_3 \oplus F_2$ is free. Then the complex

$$C_{0}(W) \leftarrow \dots \leftarrow C_{N-n-2}(W) \xleftarrow{\overline{a}_{N-n-1}} C_{N-n-1}(W) \oplus F_{3} \oplus F_{2} \leftarrow \overline{a}_{N-n} \leftarrow F_{1} \oplus F_{3} \oplus F_{2} \leftarrow 0, \qquad \overline{d}_{N-n-1} = (d_{N-n-1}, 0, 0),$$

$$\bar{d}_{N-n} = \begin{pmatrix} d_{N-n} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}$$

is homotopically equivalent to C(W).

Let e_1, \dots, e_l be a basis for the module $F_3 \oplus F_2$, let f_1, \dots, f_k be a basis for $C_{N-n}(W) = F_1 \oplus F_2$ and let b_1, \dots, b_s be a basis for F_3 . It is not difficult to verify that the restriction of \overline{d}_{N-n} : $F_1 \oplus F_2 \to C_{N-n-1}(W)$ coincides with d_{N-n} . Consider the complex X given by $[W]_{N-n-1} \vee \bigvee_{i=1}^l S_i^{N-n-1}$, where $[W]_{N-n-1}$ is the (N-n-1)-dimensional skeleton of the manifold W. We glue discs D^{N-n} onto the complex X by maps

$$\varphi_i: S_i^{N-n-1} \to X,$$

 $1 \leq i \leq k + s$, putting

$$\varphi_i = \psi_i \bigvee \chi_i$$
 when $i \leqslant k$,

where ψ_i glues the *i*th cell of W onto the skeleton $[W]_{N-n-1}$ and χ_i is induced by the composition of the projection and the embedding

$$F_1 \oplus F_2 \rightarrow F_2 \oplus F_2 \oplus F_3$$
, $\varphi_i = * \lor \text{id when } k+1 \leqslant i \leqslant k+s$.

We obtain a complex $Y = X \bigcup_{\phi_i} D_i^{N+n}$. If $F: X \to W$ is a composition of contractions of spheres to points and embeddings, then F extends to a map $F': Y \to W$ which is a homotopy equivalence.

Lemma 6.8. The embedding g_1 can be chosen in such a way that

 $(h_i)_*: H_s(W_3, \partial W_3; \Lambda) \to H_s(W_i, \partial W_i; \Lambda)$

is a monomorphism when $s \leq N + \frac{1}{2}(p + q - n - 1)$.

Proof. Lemma 6.7 implies that W_3 is homotopically equivalent to a subcomplex Y of dimension d = p + q - n. Thus every element of $\pi_i(W_3, \partial W_3)$, where $i \le N + \frac{1}{2}(p + q - n - 1)$, can be realized as an embedded disc. The proof of Lemma 6.8 is carried out by induction in which the groups $H_{s+1}(b_i)$ are successively diminished. Let us note that it is sufficient to show that the direct sum $(b_1)_* \oplus (b_2)_*$ is a monomorphism, since $b_1 = b_2$ in the representation of the manifolds as $W_1 = W_2 = X \times Y$.

Thus let $H_{s+1}(b_1)$ be the first nontrivial group. Then an element α in it may be realized as an embedding

$$\alpha: (D_{+}^{s+1}, D_{-}^{s+1}, D^{s}) \longrightarrow (W_{1}, W_{2}, W_{3}).$$

Since $\pi_i(M) = 0$ when $i \leq \frac{1}{2}(p + q - n - 1) + 2$, the map $g_2: W_2 \to W$ induces an

epimorphism of the homotopy group

$$(g_2)_*: \pi_{s+1}(W_2, \partial W_2) \rightarrow \pi_{s+1}(W_0, \partial W_0).$$

By changing, if necessary, the map $\alpha | D_{-}^{s+1}$ we may ensure that

$$\alpha: (D^{s+1}, S^s) \to (W_0, \partial W_0)$$

is homotopic to zero. This means that there exists an extension of α to a map of the disc

$$\overline{\alpha}: (D^{s+2}, D^{s+1}) \longrightarrow (W, \partial W),$$

and we arrive at an elementary deformation. Lemma 6.8 is proved.

Let us turn now to the proof of Theorem 6.5. Lemma 6.8 implies that

$$(h_i)^*$$
: $H^s(W_i, \partial W_i; \mathbb{Z}) \rightarrow H^s(W_3, \partial W_3; \mathbb{Z})$

is an isomorphism when $s \le N + \frac{1}{2}(p + q - n - 1)$. This means that

$$\bigcup \eta_3: H^s(W_3; \mathbb{Z}) \to H^{N+s}(W_3, \partial W_3; \mathbb{Z})$$

is an epimorphism when $s \leq \frac{1}{2}(p+q-n-1)$. In fact, if $x \in H^{N+s}(W_3, \partial W_3; \mathbb{Z})$ then $x = (b_1)^*(z_1)$, and $z_1 = z \cup \eta_1$. Consequently $x = (b_1)^*(z) \cup (b_1)^*(\eta_1) = (b_1)^*(z) \cup \eta$.

Lemma 6.9. The embedding g_1 can be chosen in such a way that

$$(h_i)_*: H_s(W_3; \Lambda) \to H_s(W_i; \Lambda)$$

is a monomorphism when $s \leq \frac{1}{2}(p + q - n - 1)$.

The proof is analogous to the proof of Lemma 6.8, and applies elementary deformations, the construction of which makes use of the following:

Lemma 6.10. Let ξ be the trivial fiber bundle over the disc D^{k+2} , let ξ_1 be a sub-bundle over D^{k+1}_+ , let ξ_2 be a sub-bundle over the disc D^{k+1}_- and let

$$\xi_3 = \xi_1 \cap \xi_2, \ \xi_1 \mid S^k = \xi_3 \oplus \nu_1,$$

 $\xi_2 | S^k = \xi_3 \oplus v_2, \quad \xi | S^k = \xi_3 \oplus v_1 \oplus v_2, \text{ dim } \xi_3, \text{ dim } v_1, \text{ dim } v_2 \ge 2k+3.$ Then there exist trivial sub-bundles η_1, η_2, η_3 over D^{k+2} such that

$$\begin{split} \eta_1 \oplus \eta_2 \oplus \eta_3 &= \xi, \quad (\eta_1 \oplus \eta_3) \, | \, D^{k+1}_+ = \xi_1, \\ (\eta_2 \oplus \eta_3) \, | \, D^{k+1}_+ &= \xi_2, \quad \eta_1 \, | \, S^k = v_1, \quad \eta_2 \, | \, S^k = v_2, \quad \eta_3 \, | \, S^k = \xi_3. \end{split}$$

Proof. Let $\overline{\nu}_1$ be the complement of ξ_2 over D_-^{k+1} and $\overline{\nu}_2$ the complement of ξ_1 over D_+^{k+1} . Extend $\overline{\nu}_1$ over D_+^{k+1} as a sub-bundle of ξ_1 in such a way that we obtain as a result a trivial bundle η_1 . This is possible since dim $\xi_1 > 2(k+1) + 1$. We construct η_2 similarly. Then their complement η_3 will also be a trivial bundle over $S^{k+1} = \partial D^{k+2} = D_+^{k+1} \cup D_-^{k+1}$. Further we extend all three bundles η_1, η_2, η_3 over the whole disc D^{k+2} .

Note that elementary deformations of dimension $s \leq \frac{1}{2}(p + q - n - 1)$ do not change the homology of the manifold $(W_3, \partial W_3)$ up to dimensions $N + \frac{1}{2}(p + q - n - 1)$ inclusive. Thus Lemma 6.9 implies that

$$(h_i)^*$$
: $H^s(W_i; \mathbb{Z}) \rightarrow H^s(W_3; \mathbb{Z})$

is an epimorphism when $s \leq \frac{1}{2}(p + q - n - 1)$, and by Lemma 6.8

$$(h_i)^*: H^s(W_i, \partial W_i; \mathbb{Z}) \to H^s(W_3, \partial W_3; \mathbb{Z})$$

is a monomorphism. So, if $x \in H^{s}(W_{3}; \mathbb{Z}), x \neq 0$, then $x = (b_{1})^{*}(x_{1})$ and $x_{1} \neq 0$. Therefore $x_{1} \cup \eta_{1} \neq 0$ and consequently $x \cup \eta = (b_{1})^{*}(x_{1} \cup \eta_{1}) \neq 0$.

Thus we have proved Theorem 6.5 for all $i \leq \frac{1}{2}(p + q - n - 1)$. For the remaining dimensions Theorem 6.5 is implied by the fact that the Poincaré duality homomorphism is self-dual.

Remark 6.11. Theorem 6.5 is false in a more general formulation. The obstruction to the existence of the embedding g_1 for which W_3 would be a Poincaré complex of formal dimension p + q - n reduces to the obstruction to the existence of an embedding g such that

$$(h_{i})_{*}: H_{\frac{1}{2}(p+q-n-1)} (W_{3}; \Lambda) \to H_{\frac{1}{2}(p+q-n-1)} (W_{i}; \Lambda),$$

$$(h_{i})_{*}: H_{N+\frac{1}{2}(p+q-n-1)} (W_{3}, \partial W_{3}; \Lambda) \to H_{N+\frac{1}{2}(p+q-n-1)} (W_{i}, \partial W_{i}; \Lambda)$$

are monomorphisms, and this obstruction lies in the Wall group $L_{p+q-n}(\pi)$, $\pi = \pi_1(M)$. In our case (p + q - n odd) this obstruction is zero.

§7. Homotopy invariance of the higher signatures

Definition 7.1. Let $x \in H^*(K(\pi, 1); Q)$, let M be an orientable smooth closed manifold, $\pi_1(M) = \pi$, let $f_M: M \to K(\pi, 1)$ be the natural map inducing an isomorphism of fundamental groups and let L(M) be the complete Hirzebruch class of the manifold M. Put

$$\sigma_x(M) = \langle f_M^*(x) L(M), [M] \rangle$$

The numbers $\sigma_x(M)$ are called the "higher" signatures of the manifold M.

The aim of the present section is to prove the following theorem.

Theorem 7.2. Let M and M' be smooth orientable closed manifolds, let h: $M' \rightarrow M$ be a homotopy equivalence and let $\pi_1(M) = \pi$ and $f_{M'} = f_M \circ h$. Then for any $x \in H^*(K(\pi, 1); Q)$ we have

$$\sigma_x(M) = \sigma_x(M').$$

Proof. The higher signatures $\sigma_x(M)$ are cobordism invariants $(M, f_M) \in \Omega_{SO}(K(\pi, 1) \otimes Q)$.

Lemma 7.3. Let the signature homomorphism

 $\sigma:\Omega_{SO}(K(\pi, 1))\otimes_{\Omega_{SO}}L_{*}(1)\otimes Q \to \Omega(\Lambda)\otimes Q,$

be a monomorphism, where Λ is the group ring of the group π . Then Theorem 7.2 is true.

Proof. Let $h: M' \to M$ be a homotopy equivalence, and let $f_M = f_M \circ h$. Then

$$\sigma\left((M, f_M)\right) = \sigma\left((M', f_{M'})\right).$$

Consequently $[M, f_M] = [M', f_M']$ in the group $\Omega_{SO}(K(\pi, 1)) \oplus \Omega_{SO} L_*(1) \otimes Q$, i.e. $\sigma_x(M) = \sigma_x(M')$.

Lemma 7.3 shows that it is sufficient to prove that the signature homomorphism

$$\sigma: \Omega_{SO}(K(\pi, 1)) \otimes_{\Omega_{SO}} L_{*}(1) \otimes Q \to \Omega(\Lambda) \otimes Q.$$

is a monomorphism.

Lemma 7.4. If there exists an element $0 \neq (M, f) \in \Omega_{SO}(K(\pi, 1))$ such that $\sigma(M, f) = 0$, then one can find another element $0 \neq (M', f') \in \Omega_{SO}(K(\pi, 1))$ such that $\sigma(M', f') = 0$ and $f'([M']) \neq 0$ in the group $H_*(K(\pi, 1); Q)$.

Proof. Choose a basis of elements (N_{α}, f_{α}) in the Ω_{SO} -module $\Omega_{SO}(K(\pi, 1)) \otimes Q$ such that $(f_{\alpha})_{*}([N_{\alpha}])$ forms a basis in $H_{*}(K(\pi, 1); Q)$. Then [M, f] decomposes as a sum

$$[M, f] = \sum_{\alpha} \lambda_{\alpha} [N_{\alpha}, f_{\alpha}],$$

 $\lambda_a \in \Omega_{SO}$. We order the suffixes α in the order of increasing dimension of the manifolds N_a . We choose the largest suffix α_0 for which $\sigma(\lambda_{a_0}) \neq 0 \in L_*(1)$. Then

$$[M', f'] = \sum_{\alpha \leqslant \alpha_0} \lambda_{\alpha} [N_{\alpha}, f_{\alpha}]$$

satisfies the condition $\sigma(M', f') = 0$. In fact,

$$\sigma(M', f') = \sum_{\alpha \leqslant \alpha_a} \sigma(\lambda_\alpha) \sigma(N_\alpha, f_\alpha)$$

= $\sigma(M, f) - \sum_{\alpha > \alpha_a} \sigma(\lambda_\alpha) \sigma(N_\alpha, f_\alpha) = \sigma(M, f),$

since by definition of α_0 the second term is zero $(\sigma(\lambda_a) = 0$ when $\alpha > \alpha_0)$. Further, we put $\mu_a = [CP^2]^k \sigma(\lambda_a)$ if dim $\lambda_a = 4k$, and $\mu_a = 0$ if dim $\lambda_a \neq 0$ (4). Put

$$[M'', f''] = \sum_{\alpha \leqslant \alpha_0} \mu_{\alpha} [N_{\alpha}, f_{\alpha}].$$

Then

$$\sigma(M'', f'') = \sum_{\alpha \leqslant \alpha_o} \sigma(\mu_\alpha) \sigma(N_\alpha, f_\alpha).$$

Since $\sigma(\mu_{\alpha}) = \sigma(\lambda_{\alpha})$, we have $\sigma(M'', f'') = 0$. Let $\dim \lambda_{\alpha_0} = 4k_0$. Put $\nu_{\alpha} = [CP^2]^{k-k_0}\sigma(\lambda_0)$ and

$$(M''', f''') = \sum_{\alpha \leqslant \alpha_0} \nu_{\alpha} [N_{\alpha}, f_{\alpha}].$$

Obviously $\sigma(M'', f'') = 0$. On the other hand,

$$f_*^{\prime\prime\prime}([M^{\prime\prime\prime}]) = \sum_{\dim v_{\alpha}=0} v_{\alpha}(f_{\alpha})_*([N_{\alpha}]) \neq 0.$$

Lemma 7.4 is proved.

Lemma 7.5. For any integer n one can find a manifold (Z, f_Z) such that the map $f_Z: Z \to K(\pi, 1)$ is a bomotopy equivalence up to dimension n.

Proof. We take as the manifold Z a parallelizable manifold of dimension > 2n. By surgery we transform this into another manifold Z' for which $\pi_i(Z') = 0$ when $2 \le i \le n$.

Let us turn now to the proof of Theorem 7.2. From Lemmas 7.4 and 7.5 we obtain a map $f: M \to Z$, $\pi_i(Z) = 0$ for $2 \le i \le 3 \dim M$, and $f_*(M) \ne 0$ in the group $H_*(Z, Q)$, $\sigma(M, f) = 0$. Consider a fiber bundle ξ over the sphere S^4 such that $\lambda = L(\lambda) \ne 0$ and $J(\xi) = 0$, and consider a map of degree 1 from the manifold P^4 , $f: p^4 \to S^4$ with an isomorphism $\phi: \nu(P^4) \to f^*(\xi)$. The triple (P^4, f, ϕ) induces another triple

$$\alpha = (M \times P^4, f \times id, \phi \times id).$$

The obstruction to surgery $\theta(\alpha) \in L_{m+4}(\pi)$, $m = \dim M$, is of finite order. In fact,

$$\psi: L_{m+4}(\pi) \to \Omega_{m+4}(\Lambda) \otimes Q$$

has a finite kernel (Theorem 4.12). Also

$$\psi(\theta(\alpha)) = \sigma(M \times P^4) - \sigma(M \times S^4) = \sigma(M)\sigma(P^4) - \sigma(M)\sigma(S^4) = 0.$$

Thus by taking the sum of a finite number of manifolds P^4 we may suppose that $\theta(\alpha) = 0$. Consequently there exists a Poincaré pair $(W, \partial W)$, $\pi_1(\partial W) = \pi$, and a map $f_W: W \to K(\pi, 1)$ such that $W = M \times P^4$.

Choose a singular manifold $g: Y \to Z$ of dimension complementary to M such that $\pi_1(Y) = \pi$ and such that the intersection number of Y and M is not zero. This is possible because $f_*([M]) \neq 0$ in the group $H_*(Z; Q)$. Then the conditions of Theorem 6.5 are satisfied. Thus the intersection $(U, \partial U) = (W, \partial W) \cap Y$ is a Poincaré pair with respect to the augmentation $\pi(U) \to 1$, and $\partial U = P^4$. We have obtained a contradicton, since $\sigma(P^4) = \lambda \neq 0$. This contradiction proves Theorem 7.2.

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