ON SPHERICAL FIBER BUNDLES AND THEIR PL REDUCTIONS

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Twenty years ago, Borel, Bott, Hirzebruch, Milnor, and Thom, among others, studied the structure of the classifying spaces for the orthogonal and unitary groups. From their work, it became clear that the classifying spaces B_{PL}, B_{TOP}, and B_G (B_G is the classifying space for fiber homotopy sphere bundles [14], [20]) contained the answers to many of the problems they raised.

The last ten years have seen a concerted effort to understand these spaces, and the path has been highlighted by several beautiful results: Sullivan's work on G/PL and related spaces leading to the Hauptvermutung for 4-connected manifolds ([18], [21]), Novikov's work on the invariance of the rational Pontrjagin classes ([15]), the work of Kirby-Siebenmann and Lashof-Rothenberg on G/TOP and the triangulation theorem ([7], [9]), and the work of Quillen-Sullivan on the Adams conjecture ([16], [22]).

Recently, in joint work with Brumfiel, we have determined the mod 2 cohomology of B_{PL} and B_{TOP} ([3]). This of course gave the algebraic determination of the unoriented PL-bordism ring and, except in dimension 4, the topological bordism ring.

Here we almost complete the analysis of the structure of $H^*(B_{PL})$ at the prime 2. In particular, at the prime 2, we determine the obstructions to reducing the structure 'group' of a fiber homotopy sphere bundle to TOP or PL. As an application, using the Browder-Novikov theorem, these obstructions determine explicit conditions on a simple-connected Poincare-duality space, which imply that it has the (2-local) homotopy type of a topological or PL-manifold. Also, the result gives the (2-local) structure of the oriented PL-bordism ring $\Omega^{PL}_*(\ , Z_{(2)})$ and, except

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in dimension 4, $\Omega^{TOP}_*(\ , Z_{(2)})$, where $Z_{(2)}$ denotes the integers localized at the prime 2 [24].

The method is to look at the fibrations

$$G/PL \to B_{PL} \to B_G \xrightarrow{\pi} B_{(G/PL)}.$$

The (2-local) structure of B_G is known from [14] and [10]. We next prove

Theorem A. At the prime 2, $B_{(G/PL)}$ is a product

$$E_3 \times \prod_{i=2}^{\infty} K(Z_{(2)}, 4i+1) \times K(Z_2, 4i-1),$$

while $B_{G/TOP}$ is simply a product of Eilenberg-MacLane spaces.

Theorem A has two immediate corollaries.

Corollary B. At the prime 2, the obstructions to the existence of a PL-bundle structure on a fiber homotopy sphere bundle are: (1) a secondary characteristic class in dimension 5, and (2) in odd dimensions, $n \geq 5$, ordinary $Z_{(2)}$- or Z_2-characteristic classes.

Corollary C. $B_{(B_{G/PL})} = E_{3,1} \times \prod_{i=2}^{\infty} K(Z_{(2)}, 4i+2) \times K(Z_2, 4i)$.

Corollary C implies that the fundamental classes in $H^*(B_{G/PL})$ may be taken as primitives.

The next step is to determine the map π^* (actually, this is also the last step, since A implies that standard spectral sequence techniques can now be used to obtain $H^*(B_{PL}, Z_{(2)})$). π factors through the composite

$$B_G \xrightarrow{\rho} B_{G/O} \xrightarrow{\tau} B_{G/PL}.$$

Using the ideas of [14], $H^*(B_{G/O})$ is easily computed and the map ρ^* is unambiguously defined. Thus the evaluation of π^* reduces to the evaluation of τ^*, but, on examining the suspension diagram

$$\begin{array}{ccc} \Sigma G/O & \xrightarrow{\Sigma(\Omega\tau)} & \Sigma G/PL \\ \downarrow \sigma & & \downarrow \sigma \\ B_{G/O} & \xrightarrow{\tau} & B_{G/PL} \end{array},$$

we obtain that τ^* is determined by $(\Omega\tau)^*$. At this stage, we use the proof of the Adams conjecture to obtain a map

$$B_{SO} \xrightarrow{\mathcal{L}} G/O \xrightarrow{\lambda} B_O,$$

so $\lambda \circ \mathcal{L} = \psi^3 - 1$ at the prime 2. Moreover, $\mathcal{L}^*$ and $(\Omega\tau \cdot \mathcal{L})^*$ can be completely determined. Putting this result together with a slight extension of the results of [3], we determine π^*.

To describe the main results, consider the Hopf algebra

$$\mathcal{H} = P(p_1, \ldots, p_n \ldots) \otimes E(b_1, \ldots, b_n \ldots) \quad (\text{over } Z_{(2)})$$

with $\psi(p_i) = \Sigma_{r=0}^{i} p_r \otimes p_{i-r}$, $\psi(b_i) = \Sigma_{r=1}^{i}(b_r \otimes p_{i-r} + p_{i-r} \otimes b_r)$, dimension $p_i = 4i$, dimension $b_i = 4i+1$. Introduce a derivation δ by setting $\delta p_i = 8b_i$. In each dimension $4i+1$, there is a primitive in $\mathcal{H}$ of the form b_i + (decomposables). The first few are b_1, $b_2 - b_1p_1$, $b_3 - p_1b_2 - (p_2-p_1^2)b_1 = b_3 - p_1q_2 - p_2q_1$. Indeed, we have inductively

(D) $\quad q_i = b_i - \Sigma q_r p_{i-r}$.

Let s_i be the primitive in $\mathcal{H}$ of dimension $4i$. s_i is given by the Newton formula

$$s_i = p_1 s_{i-1} - p_2 s_{i-2} + \ldots + (-1)^i i(p_i)$$

and $\delta(s_i) = 8iq_i$. Let $\nu(i)$ be the largest power of 2 dividing i. Then we have

Lemma E. $H_*(\mathcal{H}, \delta)$ has non-zero primitive classes $\{q_i\}$ in dimension $4i+1$, and the 2-order of $\{q_i\}$ is $8 \cdot \nu(i)$.

The next result, together with universal facts about modules over the Dyer-Lashof algebra ([10]), completely determines the $Z_{(2)}$-cohomology structure of $H^*(B_G)$.

Lemma F. *There is a (2-local) injection* $j : H_*(\mathcal{K}, \delta) \to H^*(B_G, Z_{(2)})$ *so that, on taking Z_8-coefficients, the composite* $H_*(\mathcal{K}, Z_8) \to H^*(B_G) \to H^*(B_O, Z_8)$ *takes* $\{p_i\}$ *to the mod 8 reduction of the i^{th} Pontrjagin class. Moreover, there is a Hopf algebra $\mathcal{K}$ with derivation δ', so* $H^*(B_G, Z_{(2)}) \cong H^*(\mathcal{K} \otimes \mathcal{K}, \delta \otimes 1 + \varepsilon \otimes \delta')$, *and j is the natural inclusion.*

In Milgram ([23]), a universal surgery class $K_{4*} \in H^{4*}(G/TOP, Z_{(2)})$ was constructed. The class K_{4*} is not primitive,

$$K_{4n} \to 1 \otimes K_{4n} + 8\Sigma K_{4i} \otimes K_{4j} + K_{4n} \otimes 1.$$

It is easy, however, to construct a primitive class

$$k_{4n} = K_{4n} + 4\cdot(\text{decomposable elements}),$$

where the decomposable elements are a polynomial in K_{4j}, $j < n$. Here is our main result ($\pi : B_G \xrightarrow{\rho} B_{G/O} \xrightarrow{\tau} B_{G/TOP}$).

Theorem G. *There is a primitive graded class* $k_{4*+1} \in H^{4*+1}(B_{G/TOP}; Z_{(2)})$ *with the following properties:*

(i) $(\Omega\tau)^*(\sigma k_{4*+1} - k_{4*}) = 0$ in $H^{4*}(G/O; Z_{(2)})$,

(ii) $\pi^*(k_{4i+1}) = 2^{\alpha(i)-1} j(q_i)$, $i \geq 1$.

(Here $\alpha(i)$ is the number of ones in the dyadic expansion of i.)

The following corollary shows that the surgery formula for tangential normal maps reduces considerably.

Corollary H. $(\Omega\pi)^*(K_{4n}) = 2^{\alpha(n)-1}\cdot Y_{4n}$, *where* $Y_{4n} \in H^{4n}(G; Z_{(2)})$ *is a class of order 4.*

Corollary J. *There are primitive classes* $\theta_{i,i} \in H^{2^{i+1}-1}(B_G; Z_2)$ *so that a homotopy sphere bundle ξ on a finite complex X admits a reduction to a topological sphere bundle (2-locally) if and only if*

(1) $\theta_{i,i}(\xi)$ *are all* 0, *and*

(2) $2^{\alpha(i)-1} j(q_i)(\xi) = 0$.

Remark K. Recent work of D. Ravenel constructs explicit characteristic classes λ_i which differ from the $\theta_{i,i}$ above only by decomposables. Moreover, his λ_i would be exactly our $\theta_{i,i}$ if they were known to be primitive! Mahowald has shown that, if X is a Poincare-duality space with vanishing Stiefel-Whitney classes, and if ξ is the Spivak normal spherical fibration, then $\theta_{i,i}(\xi)$ is the secondary Wu class associated to the Adams operation $\Phi_{i,i} : H^*(X; Z_2) \to H^*(X; Z_2)$. Here $\theta_{i,i}$ is the secondary operation based on the relation

$$\sum_{0 \leq j \leq i} \chi(Sq^{2^{i+1}-2^j})\chi(Sq^{2^j}) = 0.$$

It is interesting to compare these results with the 'transversality' results of Levitt-Quinn and Brumfiel-Morgan ([17], [4]). Recall that they have used and developed Levitt's ideas on measuring the obstructions to transversality for maps of Poincare-duality spaces in order to construct a fibering

$$B_{STOP} \to B_{SG} \xrightarrow{\mathcal{A}} \mathcal{K},$$

where $\mathcal{K} \simeq B_{(G/TOP)}$. At the prime 2, Brumfiel and Morgan calculate $\mathcal{A}^*$ with Z_2- and Z_4-coefficients, while showing that $8\mathcal{A}^*(K_{4i+1}) \equiv 0$. It was natural to conjecture that $\mathcal{A}$ and j are in some sense the same map, but from G we find

Corollary L. *There is no (2-local) homotopy equivalence*

$$\mu : \mathcal{K} \to B_{G/TOP}$$

for which $\mu \circ \mathcal{A} \quad j$.

Corollary L follows because there is no homotopy equivalence $k : B_{G/TOP} \to B_{G/TOP}$, so $j^*k^*(K_{4i+1})$ has 2-order dividing 8 for all i. However, in a certain sense, the (mod 8) reductions of the obstructions are all that matter, as we see from

Corollary M. Suppose $f : X \to B_{SG}$ lifts to B_{STOP} on the $4n$-skeleton of X. Then f lifts to B_{STOP} (2-locally) on X_{4n+2} if and only if $f^*(L_{4n+1}) = 0$ where $L = 2^{\alpha(n)-1}y$, with y of order 8.

Finally, it is routine but very messy to calculate $H^*(B_{STOP}, Z_{(2)})$ and $H^*(B_{SPL}, Z_{(2)})$, using for example Serre spectral sequence techniques and the complete knowledge of the universal Serre sequences (on the chain level) for the loop path fibering

$$\Omega K(\pi, n) \to E \to K(\pi, n).$$

The proofs of most of the results above follow from [3] after we have proved A. We do this in the next two sections. The third section calculates the composite map $(\Omega\tau\cdot\mathcal{L})^* : H^*(G/PL) \otimes Q \to H^*(BS_O) \otimes Q$, and justifies the coefficient $2^{\alpha(i)-1}$ in Theorem G. Finally, in Section 4, we complete the determination of $(\Omega\tau)^*$. To this end, one splits the homology of G/O in two parts, $H_*(G/O) \approx H_*(BSO) \otimes Y$. The subalgebra Y of $H_*(G/O)$ is abstractly isomorphic to $H_*(\text{Cok } J)$. The primitive class $k_{4i} \in H^{4i}(G/PL)$ evaluates zero on Y, and this, together with the results of Section 3, gives Theorem G.

Warning. The classes $\tau^*(k_{4i}) \in H^*(G/O; Z_{(2)})$ do not vanish in $H^*(\text{Cok } J; Z_{(2)})$; e.g. $\tau^*(k_{12})$ restricts to a class of order 2 in $H^{12}(\text{Cok } J; Z_{(2)})$.

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§1. The Mahowald orientation

Consider the space $\Omega^n S^{n+1}$. Its cohomology structure has been completely determined in [1], [12]. In particular, we have

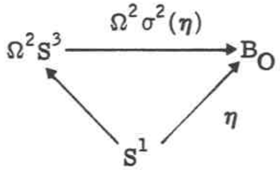
Lemma 1.1. (a) $H^*(\Omega^n S^{n+1}, Z_2) = E(\lambda_1, \ldots, \lambda_I \ldots)$, an exterior algebra on stated generators.

(b) $H_*(\Omega^2 S^3, Z_2) = P(e_1, Q_1 e_1, Q_1 Q_1(e_1), \ldots)$, one generator in each dimension of the form $2^i - 1$.

Consider the non-trivial map

$$S^1 \xrightarrow{\eta} B_0.$$

Since B_O is an infinite loop space, corresponding to η are maps $\sigma^i(\eta) : S^{i+1} \to B^{(i)}_O$ (where $\Omega^i B^{(i)}_O = B_O$). In particular, corresponding to $\sigma^2(\eta) : S^3 \to B^{(2)}_O$, on looping twice we obtain the diagram

(1.2) 

and we have

Theorem 1.3 (M. Mahowald). Let γ be the universal bundle over B_O, and $\gamma = \Omega^2\sigma^2(\eta)^!(\gamma)$. Then at the prime 2, $M(\bar{\gamma}) = K(Z_2, 0)$, the Eilenberg-MacLane spectrum.

Proof. Let $H_*(B_O, Z_2)$ be written as $P(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n \ldots)$, where $\bar{e}_n$ is dual to ω_1^n. Let e be the generator in $H_1(S^1, Z_2)$. Then $\eta_*(e) = \bar{e}_1$. Hence from (1.2), $\Omega^2\sigma^2(\eta)_*(e_1) = \bar{e}_1$. Now by Kochman's result ([8], Theorem 41), $Q_1(\bar{e}_1) = (\bar{e}_3)$, $Q_1(\bar{e}_3) = \bar{e}_7 \ldots \underbrace{Q_1 \ldots Q_1}_{i}(\bar{e}_1) = \bar{e}_{2^{i+1}-1}$.
Thus, since Ω^2 of any map commutes with the action of Q_1, the homology map is determined.

We now pass to Thom spaces. Recall that $\mathfrak{a}(2)^* = P(\xi_1, \xi_3, \xi_7, \ldots, \xi_{2^i-1} \ldots)$, and let q_i be the primitive in $\mathfrak{a}(2)$ dual to ξ_{2^i-1}. Then $Sq^1 = q_0$, $[Sq^2, Sq^1] = q_1, \ldots, q_i = [Sq^{2^i}, q_{i-1}]$.

Lemma 1. 4. $\langle q_i U, \bar{e}^*_{2^{i+1}-1} U_* \rangle = 1$.

Proof. By induction: notice that $q_i U = q_{i-1}(\omega_1^{2^i} U) + Sq^2(\omega_1^{2^i-1} U + TU)$, where $T \subset I(\omega_2, \omega_3, \ldots, \omega_n)$. But $\mathcal{A}(2) I(\omega_2, \ldots, \omega_n \ldots) \subset I(\omega_2, \ldots, \omega_n)$. Hence, mod I, $q_i U = Sq^{2^i} \omega_1^{2^i-1} U = \omega_1^{2^{i+1}-1} U$, and 1. 4 follows.

Next, notice that, if we define a map $\phi : \mathcal{A}(2) \to H^*(M(\gamma))$ by $\alpha \to \alpha U$, then using the multiplication in B_0 to give $M(\gamma) \wedge M(\gamma) \to M(\gamma)$, the diagram

(1. 5)
$$\begin{array}{ccc} \mathcal{A}(2) & \xrightarrow{\Delta} & \mathcal{A}(2) \otimes \mathcal{A}(2) \\ \downarrow \phi & & \downarrow \phi \otimes \phi \\ H^*(M(\gamma)) & \longrightarrow & H^*(M(\gamma)) \wedge M(\gamma) \\ \downarrow & & \downarrow \\ H^*(M(\bar{\gamma})) & \longrightarrow & H^*(M(\bar{\gamma}) \wedge M(\bar{\gamma})) \end{array}$$

commutes. This implies that the dual map

$$\phi_* : H_*(M(\bar{\gamma})) \to \mathcal{A}(2)^*$$

is a map of algebras. Lemma 1. 4 now shows that $\phi_*(\underbrace{Q_1 \ldots Q_1}_{i}(e_1)U_*) = \xi_{2^{i+1}-1} + d$, where $d \in I(\xi_1 \ldots \xi_{2^i-1})$. From this, it follows that ϕ_* and hence ϕ are isomorphisms, and this gives 1. 3.

Generalizing (1. 2) slightly, we have the diagram

(1. 6) $$S^1 \to \Omega^2 S^3 \to \Omega^3 S^4 \to \Omega^4 S^5 \to \ldots \to \Omega^n S^{n-1} \to B_0.$$

Let $\bar{\gamma}_r = \Omega^r \Sigma^r (\eta)^! (\gamma)$. Then we have the Thom spaces $M(\bar{\gamma}_r)$ and the maps

(1. 7) $$M(\bar{\gamma}) \to M(\bar{\gamma}_3) \to M(\bar{\gamma}_4) \to \ldots,$$

and, by [2], each $M(\bar{\gamma}_i)$ is a wedge of Eilenberg-MacLane spaces. Moreover, the map

$$M(\bar{\gamma}_i) \wedge M(\bar{\gamma}_i) \to M(\bar{\gamma}_i)$$

induced by Whitney bundle sum is the result of the H-map loop sum: $\Omega^i S^{i+1} \times \Omega^i S^{i+1} \to \Omega^i S^{i+1}$, which has more and more structure as $i \to \infty$.

Definition 1. 8. A differentiable manifold M^n has a Mahowald orientation if the classifying map for its normal bundle

$$\nu : M^n \to B_0$$

factors through $\Omega^r \Sigma^r(\eta)$ for some r. We say it has a primitive Mahowald orientation if it factors through $\Omega^2 \Sigma^2(\eta)$.

Theorem 1. 9. Let M^n have a Mahowald orientation. Then if ω is any positive-dimensional characteristic class of $\nu(M^n)$, we have $\omega^2 = 0$. In particular, $V^2(M^n) = 1$, where V is the total Wu class of M. Moreover, M^n is a Z_2-manifold. (Since $x^2 = 0$ for all $x \in H^*(\Omega^r S^{r+1}, Z_2)$, and e_1^* is an integral class, so $V_1(M)$ is always the restriction of an integral class.)

Theorem 1. 10. Let $x \in H_n(X; Z_2)$. Then x admits a realization by a manifold and map (M^n, f) with a primitive Mahowald orientation. Moreover, up to Mahowald-oriented bordism, (M^n, f) is unique.

Proof. Let $\mathfrak{M}_*(X)$ be the bordism of X with respect to primitively Mahowald-oriented manifolds. Then $\mathfrak{M}_*(X) \cong \pi_*(X \wedge M(\bar{\gamma})) \cong H_*(X; Z_2)$ ([5]), and the result follows.

Remark 1. 11. A good way of regarding the Mahowald orientation is as an explicit inverse of the Thom class map $M(\bar{\gamma}) \xrightarrow{U} K(Z_2, 0)$. But the miracle is that it actually is a Thom space, and hence satisfies transversality.

§2. The proof of Theorem A

The strategy is to calculate in the Eilenberg-Moore spectral sequence passing from $\mathrm{Tor}_{H_*(G/PL, Z_2)}(Z_2, Z_2)$ to $H_*(B_{G/PL}, Z_2)$. Of course, our attack must be made using manifolds. This suggests the use of the Eilenberg-Moore spectral sequence passing from

(2.1) $\mathrm{Tor}_{\Omega_*(G/PL,\ Z_2)}(\Omega_*(pt,\ Z_2),\ \Omega_*(pt,\ Z_2))$

to $\Omega_*(B_{G/PL};\ Z_2)$. There is a Hurewicz map of spectral sequences, which at the E^2-level is the change of rings map

(2.2) $\tilde{h}: \mathrm{Tor}_{\Omega_*(G/PL,\ Z_2)}(\Omega_*(pt,\ Z_2),\ \Omega_*(pt,\ Z_2)) \to \mathrm{Tor}_{H_*(G/PL,\ Z_2)}(Z_2,\ Z_2)$

induced by the Hurewicz map $h: \Omega_*(G/PL,\ Z_2) \to H_*(G/PL,\ Z_2)$ and the augmentation $h: \Omega_*(pt) \to H_*(pt)$. On the other hand, the Mahowald orientation induces a ring map $H_*(G/PL,\ Z_2) \to \Omega_*(G/PL,\ Z_2)$, which in turn induces a map of Eilenberg-Moore spectral sequences, which at E^2 is

$\tau: \mathrm{Tor}_{H_*(G/PL,\ Z_2)}(Z_2; Z_2) \to \mathrm{Tor}_{\Omega_*(G/PL,\ Z_2)}(\Omega_*(pt,\ Z_2),\ \Omega_*(pt,\ Z_2))$,

and $h \circ \tau = 1$. Moreover, this is true at all levels, and, in fact, the differentials in the bordism spectral sequence are completely determined by the differentials in the homology sequence from τ.

Generally speaking, any differential going to filtration 1 in these Eilenberg-Moore spectral sequences is determined by a matric Massey product being defined and non-zero in $H_*(G/PL,\ Z_2)$ or $\Omega_*(G/PL,\ Z_2)$. Indeed, if we have $x \in \langle A_1, \ldots, A_n\rangle$ and x not contained in any smaller product, then $d_{n-1}(|A_1|\ldots|A_n|) = \{x\}$ represents a non-zero differential.

Next, we observe that the Eilenberg-Moore spectral sequences are sequences of differential Hopf algebras, by results of A. Clark, and it is easy to see that, if any differentials are non-zero, the first such is d_{2^i-1} for some i, and there must be an element $y = |x_1|\ldots|x_{2^i}|$ with $d_{2^i-1}(y) \neq 0$. Moreover, except in certain cases coming from the peculiarities of the space $E_2 \subset G/PL$, these x_i are all equal to a single element $x \in H_*(G/PL,\ Z_2)$ (the remaining cases give terms $|x|y|x|y|\ldots|x|y|$).

Lemma 2.3. Suppose $x \in H_*(G/PL,\ Z_2)$, represented by a Mahowald-oriented manifold M^i, and map $f: M^i \to G/PL$ with resulting surgery problem a homotopy equivalence. Then if $\underbrace{\langle x, x, \ldots, x\rangle}_{2^i}$ is defined, we have $\langle\langle x, \ldots, x\rangle, k_{(i+1)2^i-2}\rangle = 0$, where k is the Kervaire class.

(The proof is in three steps: (1) $\langle x, \ldots, x\rangle$ can be constructed in $\Omega_*(G/PL$, Mahowald orientation). Moreover, (2) each piece of the Massey product can be assumed to have corresponding surgery problem a homology equivalence. Then the surgery problem over $\langle x, \ldots, x\rangle$ is a homology equivalence, hence has Kervaire invariant 0. But (3),

$$K\langle x, \ldots, x\rangle = \langle f^*(k_*)\cdot V^2,\ [\langle x, \ldots, x\rangle]\rangle$$
$$= \langle f^*(k_{(i+1)2^i-2}),\ [\langle x, \ldots, x\rangle]\rangle = 0,$$

since $V^2 \equiv 1$ in a Mahowald-orientable manifold.)

There are only three types of homology classes which fail to admit Mahowald-orientable representatives satisfying the hypotheses of 2.3 those dual to K_{4i}, k_{4i+2}, and $Sq^1(k_{4i+2})$. For these, we require a more delicate argument. Recall that, if $\langle x_{4i+2}, k_{4i+2}\rangle = 1$, then also $\langle Q_2(x_{4i+2}), k_{8i+6}\rangle = 1$ ([10]), where Q_2 is the Araki-Kudo operation ([1]). Also, the classes y dual to K_{4i} satisfy $Q_0(y) = Q_1(y) = Q_2(y) = 0$, while those dual to k_{4i} and $Sq^1(k_{4i})$ satisfy $Q_0(y) = Q_1(y) = 0$.

We then have

Lemma 2.4. Let y be one of the three types of classes above, and suppose

$$\langle\underbrace{\langle y, \ldots, y\rangle}_{2^i},\ k_{4j+2}\rangle = 1.$$

Then k_{8j+6} evaluates 1 on a strictly shorter Massey product.

Proof. Using the higher Mahowald orientations $M(\bar{\gamma}_i)$, we have that the theory $H_*(G/PL,\ M(\bar{\gamma}_i))$ admits (homology) $\cup_j$-products for $j \leq i - 1$. Thus we can apply the result of [13] and

$$(2.5)\quad Q_2\langle x, \ldots, x\rangle \subset \left\langle (Q_2(x), Q_1(x), Q_0(x)) \begin{pmatrix} Q_0(x) & 0 & 0 \\ Q_1(x) & Q_0(x) & 0 \\ Q_2(x) & Q_1(x) & Q_0(x) \end{pmatrix} \cdots \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \end{pmatrix} \right\rangle.$$

But in all these cases, $Q_1(x) = Q_0(x) = 0$, hence this Massey product is easily seen to contain 0. Thus $Q_2\langle x, \ldots, x\rangle \subset$ (indet), which runs over Massey products of length $2^j - 1$.

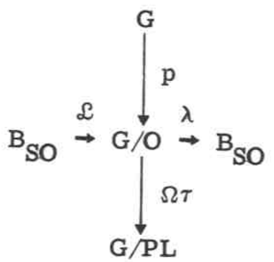
Remark 2. 6. Actually, the hypothesis of 2. 4 can be weakened to $Q_0(y) = Q_1(y) = 0$.

Now the proof of A is fairly routine. It is easily checked that the first time a differential can occur, it must hit a $(k_{4j+1})_*$. Suppose then that $d_{2^i-1}(|x|\ldots|x|) = (k_{4j+2})_*$. Then $Q_2((k_{4j+2})_*)$ is $d_{2^{i-s_1}-1}(z)$, $Q_2(Q_2(k_{4j+2})_*) = d_{2^{i-s_1-s_2}-1}(z')\ldots$ for $s_1, s_2 \ldots$ greater than 0. Finally, we will obtain $d_1(z^r) = (k_{4l+2})_*$ for some l, but this is impossible due to the fact that k_{4l+2} is primitive in $H^*(G/PL, Z_2)$ ([19]). This contradiction shows that $(k_{4j+2})_*$ is a surviving cycle for each j. Hence $E^2 = E^\infty$, and A follows.

§3. The Adams map $\mathcal{L} : B_{SO} \to G/O$

Consider the diagram (localized at 2)

(3. 1)



where $\lambda \circ \mathcal{L}$ $\psi^3 - 1$.

Lemma 3. 2. With coefficients $Z_{(2)}$ of Z_2, $\mathcal{L}^*$ is surjective. Indeed, $\mathcal{L}^* : H^*(G/O)/\text{Torsion} \to H^*(B_{SO})/\text{Torsion}$ is an isomorphism.

We outline two proofs.

Proof. (1) Using the Bockstein spectral sequence, we find that $\{p_*(e_2 \underline{*} e_2)^{2^i}\}^*$ are integral cohomology generators. Now $p_*(e_2 \underline{*} e_2)^* = (p_*(e_1 \underline{*} e_1)^*)^2$ mod 2, and, more generally,

$$\left[\left(p_*(e_1 \underline{*} e_1)^{2^i}\right)^*\right]^2 = \left[p_*(e_2 \underline{*} e_2)^{2^i}\right]^*.$$

In dimension 2, we know that $\mathcal{L}_*(\eta) = p_*(\{\eta^2\})$ in homotopy. Hence, in cohomology, $\mathcal{L}^*((p_*(e_1 \underline{*} e_1))^*) = \omega_2$. Next, consider $Sq^1(\omega_2) = \omega_3 = \mathcal{L}^*(p_*(e_2 \underline{*} e_1)^*)$ and

$$Sq^2(Sq^1\omega_2) = \omega_2\omega_3 + \omega_5 ;$$

but $(e_1 \underline{*} e_1)^* \cup Sq^1(e_1 \underline{*} e_1)^* = (e_3 \underline{*} e_2)^* + Sq^1[(e_1 \underline{*} e_1)^2]$, while $Sq^2(Sq^1\omega_2) = (e_3 \underline{*} e_2)^* + (e_4 \underline{*} e_1)^*$. Thus using the relation

$$Sq^1\omega_4 = \omega_2\omega_3 + Sq^2Sq^1\omega_2,$$

we have $\mathcal{L}^*[p_*((e_1 \underline{*} e_1)^2)]^* = \omega_4 + \alpha$, where α is a decomposable in ker (Sq^1).

Similarly, using $Sq^4(Sq^1\omega_4) = \omega_4\omega_5 + \omega_9 + \ldots$, we obtain $\mathcal{L}^*[p_*((e_1 \underline{*} e_1)^4)]^* = \omega_8 + \alpha'$, $\ldots$, $\mathcal{L}^*\left[p_*\left((e_1 \underline{*} e_1)^{2^i}\right)\right]^* = \omega_{2^{i+1}} + \alpha''$. On the other hand, as a ring over $\mathcal{A}(2)$, the ω_{2^i} generate $H^*(B_{SO})$. 3. 2 now follows easily.

(2) The splitting theorem of Sullivan, $G/O = B_{SO} \times \text{Cok } J$, is proved by using the difference of the two canonical B_O-orientations on G/O to construct a map $\gamma : G/O \to B_{SO}^{\otimes}$, and $(\gamma \cdot \mathcal{L})^*$ is seen to be an isomorphism.

Lemma 3. 3. $\mathcal{L}^*$ is a (rational) map of Hopf algebras.

Proof. λ is a map of H-spaces as is $\psi^3 - 1 = \lambda \cdot \mathcal{L}$. Thus the deviation of $\mathcal{L}^*$ from an H-map is contained in the torsion homology of G/O, and 3. 3 follows.

Theorem 3. 4. For the primitive class

$k_{4i} \in H^*(G/PL, Z_{(2)})$

($k_{4i} \equiv K_{4i}$ modulo decomposables), we have

$$(\lambda \circ \mathcal{L})^*(k_{4i}) = 2^{\alpha(i)-1} \cdot s_i$$

in $H^*(B_{SO}, Z_2)/_{Torsion}$, where s_i is the primitive in $H^{4i}(B_{SO}, Z_{(2)})/_{Torsion}$.

Proof. In homotopy, the generator γ_i in $\pi_{4i}(B_{SO})$ maps to the free generator β_i in $\pi_{4i}(G/O)$, and this generator maps to $q_i K_{4i}$ where K_{4i} is the generator in $\pi_{4i}(G/PL)$. From [6], q_i is the order of the sub-group $b(P_{4i}) \subset \Gamma_{4i-1}$ of homotopy spheres which bound parallelizable manifolds. This number q_i has been calculated and has the form $2^{2i-2} a_i$(odd) where $a_i = 1$ or 2. On the other hand, $\langle s_i, h(\gamma_i)\rangle = \frac{1}{2}(2i)! \cdot a_i$. Thus, since $(2i)! = 2^{2i-\alpha(i)}$, we obtain the result.

The final step in our calculation is to identify the part of $\lambda^*(k_{4i})$ contained in the torsion part of $H^*(G/O)$. (Here we are using a basis for G/O dual to the basis for $H_*(G/O)$, coming from projection of the basis for $H_*(G)$ used in [10], [14].) This we do in the next section.

§4. The image of k_{4i} and the proof of G

Consider the map $SO \to SG$. In mod 2 homology, this induces an injection $H_*(SO, Z_2) = E(e_1, e_2 \ldots) \to H_*(SG) = E(e_1 \ldots e_n \ldots) \otimes P$, where P is a polynomial algebra. In the Bockstein spectral sequence of SO, we have $E^2 = E^\infty = E(p_3, p_7, \ldots, p_{4i-1})$, where p is the primitive. However, since $H_*(SG, Q) \equiv 0$, we know that the p_{4i-1} are in the images of higher differentials in $E^*(SG)$. Indeed, there is a polynomial algebra $P(A_4, A_8, \ldots, A_{4i}, \ldots) \subset P$ so $H_*(SO) \otimes P(A_4 \ldots A_{4i} \ldots)$ is a closed sub-differential module in $E^*(SG)$. In particular, these A_{4i} in $H_*(G/O)$ generate the torsion-free part of the homology.

Definition 4.1. An element y in $H_*(X, Z_{2^i})$ is called proper if $2^i y = 0$ but $2^{i-1}(y) \neq 0$.

Lemma 4.2. Let $\mathcal{P}_i$ denote the Pontrjagin squaring operator $H_r(G, Z_{2^{i-1}}) \to H_{2r}(G, Z_{2^i})$. Then A_{4i} is the mod 2 restriction of $\mathcal{P}_2^{(*)}(B_{2i})$, and these $\mathcal{P}_2^{(*)}(B_{2i})$, together with elements $\mathcal{P}_2^{(0)}(\alpha)$, generate the set of proper Z_4-elements in $H_*(SG, Z_4)$ under $*$ and $\cdot$, where α is a proper Z_2 class.

Similarly, we have

Lemma 4.3. $\mathcal{P}_i \ldots [\mathcal{P}_2^*(B_{2j})]$, together with iterated Pontrjagin squares $\mathcal{P}_i \ldots \mathcal{P}_2(\alpha)$ for α a proper Z_2-class, generate the set of proper Z_{2^i}-elements.

Now, from the results of [3],† we see that the generators k_{4i} are primitive on proper Z_{2^i}-classes with respect to both composition and loop sum for $i > 2$. Hence the only time that $\langle k_{4i}, y\rangle$ is non-zero on a proper Z_{2^i}-class is when y is $\mathcal{P}_i \ldots [\mathcal{P}_2^*(B_{2j})]$ or $\mathcal{P}_i \ldots \mathcal{P}_2(\alpha)$. On the other hand, the fact that k_{4i} is a suspension shows that it evaluates 0 on classes of the second type. Hence it can be non-zero only on $\mathcal{P}_i \ldots [\mathcal{P}_2^*(B_{2j})]$. But these classes project into the torsion-free part of $H_*(G/O, Z_{2^i})$. This, together with 3.4 and the remarks given in the introduction, completes the proof of Theorem G.

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† The language of [3] might lead the reader to think that the evaluation formula for $(*)^*(k)$, which is needed above, works only for Z_2-coefficients, but a little reflection will convince that it is actually valid as stated for all coefficients Z_{2^i}.

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