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On the Index of Toeplitz Operators of Several Complex Variables

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Abstract. Toeplitz operators on strictly pseudo-convex boundaries of complex domains are defined; they behave like pseudo-differential operators. An extension of the Atiyah-Singer formula is proved for elliptic systems of such operators.

Let H^2 be the space of holomorphic functions on the unit disc $D \subset \mathbb{C}$, with square integrable boundary value $\varphi = \sum_0^\infty a_n z^n$ $(\Sigma |a_n|^2 < \infty)$. If f is a continuous function on the circle ∂D, the Toeplitz operator T_f on H^2 is defined by $T_f(\varphi) = S(f\varphi)$ where S is the orthogonal projector $L^2(\partial D) \to H^2$. It is well known that if f is invertible, T_f is a Fredholm operator with index the opposite of the winding number:

$$\mathrm{Index}(T_f) = \frac{-1}{2i\pi} \int_{\partial D} f^{-1}\, df.$$

This result was extended by Venugopalkrishna [23] to the unit ball in $\mathbb{C}^n$. Here we extend the result of Venugopalkrisha to arbitrary strictly pseudoconvex domains. We will use a somewhat wider definition of Toeplitz operators; our operators are of the form $T_Q: \varphi \to SQ(\varphi)$ where Q is an arbitrary pseudo-differential operator on the boundary $\partial\Omega$ of a bounded complex domain Ω (assumed to be strictly pseudo-convex), and S is the orthogonal projector (Szegö projector) $L^2(\partial\Omega) \to H^2(\partial\Omega)$, the space of L^2 functions on $\partial\Omega$ which have a holomorphic extension in Ω. These operators seem interesting in themselves; they behave essentially in the same way as pseudo-differential operators, and will hopefully allow one to apply pseudo-differential techniques to new interesting situations (cf. [9]). They are described in §1.

A Toeplitz operator T_Q has a symbol, which is a homogeneous function (or a vector bundle homomorphism) on a half line bundle $\Sigma^+ \subset T^*\partial\Omega$ on $\partial\Omega$. If the symbol $\sigma(T_Q)$ is invertible, T_Q is a Fredholm operator whose index is given by an

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extension of Atiyah-Singer index formula. This is described in § 2, and proved in § 3.

In fact we prove index theorems in two situations which cannot quite be reduced one to the other. The first (Theorem 1) applies to systems (matrices) of Toeplitz operators acting on $(H^2(\partial\Omega))^N$; for this we need that $\partial\Omega$ be compact and strictly pseudo-convex, but Ω itself may be an arbitrary analytic space, with singularities so long as these do not meet $\partial\Omega(\Omega \cup \partial\Omega$ must be compact, at least if $\dim_{\mathbb{C}} \Omega = 2)$. A similar formula for boundaries of strictly pseudo-convex domains in $\mathbb{C}^n$ was announced by Dynin [12].

Theorem 1 was announced in lectures at the Nordic Summer School of Mathematics in Sweden, in July 1975. The proof given there was in spirit very close to the proof in this paper, mixing the given operator with the $\bar{\partial}_b$-complex so as to reduce to the index formula of Atiyah and Singer for elliptic operators on the boundary. However it was technically rather disagreable, and there was still a gap in the proof of Theorem 2, so I postponed the publication.

The second formula (Theorem 2) applies to Toeplitz operators acting on holomorphic sections of (distinct) holomorphic vector bundles on Ω; in this case we require Ω to be a Stein manifold. As it is, Theorem 2 should contain the Atiyah-Singer formula as a special case, taking Ω to be a complex tubular neighborhood of an arbitrary compact real analytic manifold (we will only give a short indication on this in § 3, and will return to this question elsewhere).

However, the proof of the index theorems in this paper is not independent of the Atiyah-Singer index theorem. Theorem 2 is reduced to Theorem 1 in § 3.b, and Theorem 1 is reduced to the Atiyah-Singer theorem (on the boundary $\partial\Omega$) in § 3.a, by means of a construction which related to the embedding constructions of § 3.b and of [2]. The invariance by embedding of the index may be somewhat more natural in the complex context of § 3.b than in the pseudo-differential context of [2]. On the other hand I have no direct proof in this complex context of the "excision property", which is an important property of the index, and is very natural and easy to prove in the context of peudo-differential operators. The index Theorem 1 and the embedding property can be extended to arbitrary contact manifolds; this yields an independent proof of the index theorem, but this whole construction is rather closely related to that of [2] anyway and does not bring anything essentially new.

We adopt the usual notations for functions and distributions. All manifolds are assumed to be C^∞ and paracompact. Unless otherwise specified, pseudo-differential operators and Fourier integral operators are supposed to be regular (or "classical") i.e. in any set of local coordinates the total symbol has an asymptotic expansion:

$$p(x,\xi) \sim \int_{j=0}^{\infty} p_{m-j}(x,\xi)$$

where p_{m-j} is C^∞ for $\xi \neq 0$, homogeneous of degree $m-j$ with respect to ξ (j is a positive ($\geqq 0$) integer); unless otherwise specified, the degree m is an integer (but it could be any complex number). In § 3 we will also use (irregular) pseudo-differential operators of type $\frac{1}{2}$, i.e. for the total symbol we have locally estimates of the form

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta} p|(x,\xi)|\right| \leqq \text{constant}\,(1+|\xi|)^{m+\frac{1}{2}|\alpha|-\frac{1}{2}|\beta|}.$$

If P, Q are pseudo-differential operators, we write $P \sim Q$ if $P-Q$ is of degree $-\infty$ (i.e. the total symbol is od rapid decrease when $\xi \to \infty$, or $P-Q$ has a C^{∞} Schwartz-kernel). The equivalence $P \sim Q$ still makes sense microlocally, i.e. in open cones of the cotangent bundle.

This article was written during a one year stay at the department of Mathematics in Princeton University, and I wish to take the opportunity to express my thanks for this invitation. My thanks also go to Hörmander and Anderson, who gave me the opportunity to present a first version of this index theorem at the Nordic Summer School of Mathematics in July 1975, and finally especially to Hörmander for his thorough reading of the manuscript and his remarks, which helped to improve the presentation.

§ 1. Toeplitz Operators

a) Definition

Let W be a (reduced) complex analytic space, and $\Omega \subset W$ a relatively compact open set with C^{∞}, strictly pseudo-convex boundary $\partial\Omega$. We require that W be smooth near $\partial\Omega$; we allow singularities inside Ω, but for what follows, these can always have been blown up by Hironaka's theorem. A typical example is $W=$ a complex cone in $\mathbb{C}^n$, smooth outside of 0, $\Omega=$ the intersection with the unit ball. We suppose Ω defined by an inequality $r<0$, where r is a real C^{∞} function with $dr \neq 0$ along $\partial\Omega$; the strict pseudo-convexity means that in local coordinates $z_1 \ldots z_n$, we have

$$\Sigma \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} v_j \bar{v}_k > 0 \quad \text{if } v=(v_1 \ldots v_n) \neq 0, \qquad \Sigma \frac{\partial r}{\partial z_j} v_j = 0$$

(or, more intrinsically, $\langle \partial\bar{\partial} r, v \wedge \bar{v}\rangle > 0$ if v is a holomorphic vector tangent to $\partial\Omega$, $v \neq 0$).

We will denote by α the differential form

$$(1.1)\qquad \alpha = \frac{1}{2i}(\partial r - \bar{\partial} r)_{|\partial\Omega}$$

(where as usual $d=\partial+\bar{\partial}$ is the decomposition of the exterior derivative in holomorphic and antiholomorphic parts – also denoted $d=d'+d''$). We further denote

$$(1.2)\qquad \Sigma^{+} = \text{the half line bundle generated by } \alpha \text{ in } T^*\partial\Omega.$$

The fact that the Leviform is nondegenerate reflects in the fact that α is a contact form (i.e. $\alpha \wedge (d\alpha)^{n-1}$ vanishes nowhere on $d\Omega$, if $n=\dim_{\mathbb{C}}\Omega$, so $\dim_{\mathbb{R}}\partial\Omega = 2n-1$), or equivalently, that Σ^{+} is a symplectic submanifold of $T^*\partial\Omega$.

We equip $\partial\Omega$ with a positive measure with C^∞ positive density, so that L^2-norms are well defined. We will denote by $H^s(\partial\Omega)$ the Sobolev space of (generalized) functions with s L^2 derivatives. As usual this is defined by duality of s is a negative integer, and by interpolation, or locally by Fourier transformation, if s is not an integer. We will also denote by $O^s(\partial\Omega)$ the subspace of $H^s(\partial\Omega)$ of functions which extend as holomorphic functions in a neighborhood of $\partial\Omega$ in Ω; this is the same as $H^s(\partial\Omega)\cap\operatorname{Ker}\bar\partial_b$, where $\bar\partial_b$ is the induced Cauchy-Riemann system[1]. Occasionally we will denote by $O^\infty(\partial\Omega)$ the space of C^∞ solutions of $\bar\partial_b$ (these extend as holomorphic functions which are C^∞ up to the boundary on $\bar\Omega=\Omega\cup\partial\Omega$), and by $O^{-\infty}(\partial\Omega)$ the space of distribution solutions (such a distribution f has a holomorphic extension F near $\partial\Omega$, of moderate growth along $\partial\Omega$; the fact that f is the boundary value meaning that f is the limit of the functions $F_{|r=-\varepsilon}$ when $\varepsilon\to+0$ in the distribution sense). If $\partial\Omega$ is real analytic, we also define $O^\omega(\partial\Omega)$, the space of analytic solutions of $\bar\partial_b$ (these have a holomorphic extension in a neighborhood of $\partial\Omega$ in W), and $O^{-\omega}(\partial\Omega)$, the space of hyperfunction solutions (these extend as holomorphic functions in Ω, with no restriction on the growth near $\partial\Omega$).

The space $O^0(\partial\Omega)$ is usually denoted H^2, but we will keep the notation above to avoid confusion with the Sobolev spaces, or with cohomology groups.

Let S be the orthogonal projector (Szegö projector): $L^2(\partial\Omega)\to O^0(\partial\Omega)$.

(1.3) **Definition.** *Let Q be a pseudo-differential operator of degree m on $\partial\Omega$. Then the Toeplitz operator T_Q: $O^m(\partial\Omega)\to O^0(\partial\Omega)$ is defined by $T_Q(\varphi)=S(Q\varphi)$.*

We will usually identify T_Q with the operator SQS, although the second factor S is of course superfluous when we restrict to O^m.

The definition can be immediately extended to operators on holomorphic sections of holomorphic vector bundles E, F defined in a neighborhood of $\bar\Omega=\Omega\cup\partial\Omega$ in W (we equip E and F with C^∞ hermitian norms to define the L^2 norms and the Szegö projectors S_E, S_F).

If Q is of degree m, T_Q is in fact continuous $O^s\to O^{s-m}$ for any $s\in\mathbb{R}$ (because the Szegö projector is in fact continuous $O^s\to O^s$ for any s (cf. [10]). Also Toeplitz operators are pseudo-local, i.e. they diminish singular supports, because this is true of pseudo-differential operators and of S (cf. [10]). Then one can localize, or microlocalize, Toeplitz operators mod. C^∞ operators, as is done for pseudo-differential operators.

b) Microlocal Structure of Toeplitz Operators

Let (x, y), $x\in\mathbb{R}^n$, $y\in\mathbb{R}^{n-1}$ denote the variable in $\mathbb{R}^{2n-1}$, and let (ξ,η) be the dual variable. We identify $T^*\mathbb{R}^n$ with the symplectic cone $\Sigma_0^+\subset T^*\mathbb{R}^{2n-1}$, defined by $y=\eta=0$. We set

$$D_j=\frac{\partial}{\partial y_j}+y_j|D_x| \qquad (j=1,\dots,n-1).$$

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[1] The range of the restriction map $O^s(\bar\Omega)\to O^{s-\frac12}(\partial\Omega)$ is closed, of finite codimension (independent of s), so for most of what follows one could replace $O^s(\partial\Omega)$ by $O^{s+\frac12}(\bar\Omega)$

Let H_0 be the Hermite operator (cf. [8]): $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n-1})$:

$$H_0 \varphi = (2\pi)^{-n} \int e^{ix\cdot\xi - \frac{1}{2}y^2|\xi|} \left(\frac{|\xi|}{2\pi}\right)^{\frac{n-1}{4}} \hat{\varphi}(\xi)\, d\xi$$

where we have set $y^2 = \Sigma y_j^2$, and $\hat{\varphi}$ is the Fourier transform.

Then it follows from [8, 10] that we have the following microlocal description of the Szegö projector S[2]: for any $z_0 \in \partial\Omega$, there exists a canonical map Φ from a conic open set $U \subset T^*\mathbb{R}^{2n-1} \setminus 0$ to a conic neighborhood V of $(z_0, \alpha(z_0)) \in \Sigma^+ \subset T^*\partial\Omega \setminus 0$, whose restriction defines a symplectic isomorphism $\chi: \Sigma_0^+ \cap U \to \Sigma^+ \cap V$; there exists an elliptic Fourier integral operator F (defined in V, mod. C^∞ operators) associated with Φ, which transforms the left ideal of pseudo-diffential operators generated by the D_j into the left ideal generated by the components of $\bar{\partial}_b$.

We set

(1.4) $A \sim H_0^* F^* F H_0$ (this is an elliptic positive pseudo-differential operator)

$H \sim F H_0 A^{-1/2}$ (this is a Fourier integral operator with complex phase (cf. [19])).

Now for any pseudo-differential operator Q on $\partial\Omega$, we have

(1.6) $T_Q = SQS \sim HPH^*$ near z_0 with $P \sim H^*QH \sim H^*T_Q H$.

P is a pseudo-differential operator on $\mathbb{R}^n$ (defined mod. C^∞ operators near (x_0, ξ_0), and regular i.e. its total symbol has an asymptotic expansion $p \sim \Sigma p_{m-j}(x, \xi)$ where p_{m-j} is homogeneous of integral degree $m-j$).

In fact the map $T_Q \mapsto P \sim H^* T_Q H$ is onto, for we have $\sigma(P) = \sigma(Q) \circ \chi$, so for any P there exists Q such that $P - H^* T_Q H$ is of degree $\leqq \deg P - 1$, and by successive approximations we see that there also exists Q with $P \sim H^* T_Q H$. It follows immediately that the set of Toeplitz operators is closed under composition – i.e. it is an algebra – and that $T_Q \to P \sim H^*QH$ is an isomorphism:

(1.7) *Scholie. The Toeplitz operators form an algebra of pseudolocal operators. This algebra is locally (and* mod. C^∞ *operators) isomorphic to the algebra of pseudo-differential operators in n real variables.*

Now any microlocal construction for pseudo-differential operators may be lifted (locally by means of H and χ) to Toeplitz operators. In particular we define the symbol:

(1.8) $\sigma_m(T_Q) = \sigma_m(Q)_{|\Sigma^+}$.

This is multiplicative ($\sigma_{m+m'}(T_Q T_{Q'}) = \sigma_m(T_Q)\sigma_{m'}(T_{Q'})$), and $\sigma_m(T_Q) = 0$ means that T_Q is really of degree $\leqq m-1$ (in fact if $\sigma_m(Q) = 0$ on Σ^+, there exist system A, B

[2] In [10] we suppose that Ω is a bounded open set in $\mathbb{C}^n$, but the only manner in which this used is in the Kohn estimates for the $\bar{\partial}$-Neumann problem, and these still hold with the hypotheses of § 1.a. If $n > 2$ one can even relax the compactness assumption on $\bar{\Omega}$, and the bundles E, F only need to be defined near $\partial\Omega$ ($\partial\Omega$ still must be compect) (cf. [7]). If $n = 2$ however, the assumptions above cannot be weakened

and Q', of degree $\leq m-1$, such that $Q \sim A\bar{\partial}_b + \bar{\partial}_b^* B + Q'$ near Σ^+, so $T_Q \sim T_{Q'}$). Further we have

$$(1.9)\quad \sigma_{m+m'-1}([T_Q, T_{Q'}]) = \frac{1}{i}\{\sigma_m(T_Q), \sigma_{m'}(T_{Q'})\}_{\Sigma^+}$$

where $\{\ \}_{\Sigma^+}$ is the Poisson bracket on Σ^+.

Also if $T_Q = SQS$ is a Toeplitz operator, the adjoint is $SQ^*S = T_Q^*$: it is a Toeplitz operator with symbol $\sigma(T_Q)^*$.

We may still define Toeplitz operators acting on holomorphic sections of holomorphic vector bundles E, F; if T_Q is such an operator, $\sigma_m(T_Q)$ is a C^∞ bundle homomorphism between the pull-backs of E and F to Σ^+ (it is also homogenous of degree m in the fibers of Σ^+).

We will say that a Toeplitz operator T_Q of degree m is elliptic if $\sigma_m(T_Q)$ is invertible. Then T_Q has a parametrix, i.e. there exists a Toeplitz operator $T_{Q'}$ of degree $-m$ (with $\sigma(T_{Q'}) = \sigma(T_Q)^{-1}$) such that $T_Q T_{Q'} \sim \mathrm{Id}$, and $T_{Q'} T_Q \sim \mathrm{Id}$. It follows immediately that T_Q has an index i.e. Ker T_Q is of finite rank, and the range is closed and of finite codimension.

For the index theorem, it will be convenient to consider also elliptic complexes of Toeplitz operators: let $E_j (r_0 \leq j \leq r_1)$ be holomorphic vector bundles on $\bar{\Omega}$ and let

$$d\colon 0 \to O(\partial\Omega, E_{r_0}) \xrightarrow{d_{r_0}} O(\partial\Omega, E_{r_0+1}) \xrightarrow{d_{r_0+1}} \dots O(\partial\Omega, E_{r_1}) \to 0$$

be a complex of Toeplitz operators (i.e. the d_j are Toeplitz operators, and $d_{j+1} d_j = 0$). We say that d is elliptic if its symbol is exact; this is so if and only if $d + d^*$ is elliptic, as an operator from $O(\partial\Omega, \bigoplus E_{2j})$ to $O(\partial\Omega, \bigoplus E_{2j+1})$. Then the homology is finite dimensional, and there exists a sequence $A = (A_j)$ of Toeplitz operators such that $dA + Ad \sim \mathrm{Id}$ (exactly as for elliptic complexes of pseudo-differential operators). We may even choose A so that $A^2 = 0$, dA is the orthogonal projector on the range of d, Ad is the orthogonal projector on the range of d^* (which is then equal to the range of A, or to the orthocomplement of Ker$d$), and $\mathrm{Id} - dA - Ad$ is the orthogonal projector on the complement, which is isomorphic to the homology (Hodge theory for elliptic complexes of Toeplitz operators). Let us finally recall that the Euler characteristic $\chi(d) = \Sigma(-1)^j \cdot \dim H^j(d)$ is then equal to the index of $d + d^*$ (acting from $\bigoplus E_{2j}$ to $\bigoplus E_{2j+1}$).

c) Examples

(1.11) Let f be a C^∞ function on $\partial\Omega$ (more generally a C^∞ bundle homomorphism). Then $T_f = SfS$ (resp. $S_F f S_E$) is a Toeplitz operator with symbol f. This is the natural extension of Toeplitz operators on the circle ∂D.

Clearly for any elliptic Toeplitz operator T_Q on sections of holomorphic vector bundles, there exists an elliptic Toeplitz operator T_A with self adjoint symbol (e.g. $\sigma(T_A) = |\xi|^m \mathrm{Id}$) such that $\sigma(T_Q) = \sigma(T_A T_f)$ for some C^∞ bundle homomorphism f. Now the index of an elliptic Toeplitz operator only depends on its symbol (since this determines the operator up to compact perturbations) so we have Index $(T_A)=0$, and for the index theorem operators of the form T_f are enough. We will use this in § 3.b (in § 3.a, we will rather use Toeplitz operators of degree 1).

(1.12) Let $P(z, D_z)$ be a differential operator with holomorphic coefficients. Then P induces a Toeplitz operator on holomorphic functions, with symbol

$$\sigma_m(T_Q)(z, \alpha(z)) = \sigma_m(P)\left(z, \frac{1}{i}\,\partial r\right).$$

This follows from the fact that Toeplitz operators form an algebra, and that for any holomorphic vector field X on $\bar{\Omega}$, there exists a C^∞ vector field X' defined near $\partial\Omega$, tangent to $\partial\Omega$, and such that $X - X'$ is antiholomorphic, so that X and $T_{X'}$ have the same effect on holomorphic functions. Also we have

$$\alpha = \frac{1}{2i}(\partial r - \bar{\partial} r)_{|\partial\Omega} = \frac{1}{i}\,\partial r_{|\partial\Omega},$$

so

$$\sigma(X')(z, \alpha(z)) = i\langle X', \alpha(z)\rangle = i\left\langle X, \frac{1}{i}\,\partial r\right\rangle.$$

The fact that $P(z, D_z)$ acting on holomorphic functions on Ω has an index if $\sigma(P)\left(z, \frac{1}{i}\,\partial r\right)$ is invertible was noticed by Bony and Schapira in [5]; if $\deg P > 0$, this index may be non-trivial, and is given by the theorem of § 2.

(1.13) For any real $s > -\frac{1}{2}$, let B_s be the orthogonal projector on holomorphic functions for the Hilbert norm

$$\int_\Omega |f|^2 |r|^{2s} d\mu$$

(where μ is any positive measure on Ω, with C^∞ positive density near $\partial\Omega$). Then if f is a C^∞ function on $\bar{\Omega}$, the operator $\varphi \to B_s(f\varphi)$ is a Toeplitz operator with symbol $f|\partial\Omega$. This result extends immediately to the case where f is a C^∞ vector bundle homorphism. We will use it in § 3.b. It is proved in the following appendix.

d) Appendix

Let Λ be an elliptic positive pseudo-differential operator on $\partial\Omega$. We define the Hilbert space H^Λ as the completion of $C^\infty(\partial\Omega)$ for the Hilbert norm

$$\|f\|_\Lambda^2 = (\Lambda f | f) = \int_{\partial\Omega} \Lambda f \bar{f}.$$

We then define $O^\Lambda = H^\Lambda \cap \operatorname{Ker} \bar{\partial}_b$, and the orthogonal projector $S_\Lambda \colon H^\Lambda \to O^\Lambda$.

In the formulas below, all adjoints are taken with respect to L^2-norms; hence the factor Λ, which corresponds to taking adjoints with respect to the norm of H^Λ.

With the notations of § 1.6 for F, H_0, A, H, we set

$$\text{(1.4) bis}\quad \begin{aligned} A_\Lambda &\sim H_0^* F^* \Lambda F H_0 \sim A^{1/2} H^* \Lambda H A^{1/2} \\ H_\Lambda &\sim F H_0 A_\Lambda^{-1/2} \quad \sim H A^{1/2} A_\Lambda^{-1/2} \end{aligned}$$

so that, mod. C^∞ operators, H_Λ is an isometry from $L^2(\mathbb{R}^n)$ to O^Λ:

$$\text{(1.5) bis}\quad \begin{aligned} \mathrm{Id} &\sim H_\Lambda^* \Lambda H_\Lambda \\ S_\Lambda &\sim H_\Lambda H_\Lambda^* \Lambda \end{aligned}$$

Now for any pseudo-differential operator Q on $\partial\Omega$, let T_Q^Λ be the restriction to holomorphic functions of $S_\Lambda Q$ (or $S_\Lambda Q S$): locally we have

$$S_\Lambda Q S \sim H P' H^*$$

with

$$P' \sim H^* S_\Lambda Q H \sim A^{1/2} A_\Lambda^{-1} A^{1/2} H^* \Lambda Q H.$$

so it follows that T_Q^Λ is a Toeplitz operator, and since

$$\sigma(A^{1/2} A_\Lambda^{-1} A^{1/2} H^* \Lambda H) = 1 \quad \text{we have} \quad \sigma(T_Q^\Lambda) = \sigma(T_Q) = \sigma(Q)_{|\Sigma^+}.$$

To investigate the operator $B_s f$ of example (1.13), we will suppose that Ω is smooth, and that the measure μ has a positive C^∞ density; but by restricting to a neighborhood of $\partial\Omega$ and replacing $=$ by $\sim$ in the formulas below, we see that the results allow singularities inside of Ω. We will use the Poisson potential K which solves the Dirichlet problem:

$$\varphi = K\varphi_0 \Leftrightarrow \Delta''\varphi = \bar\partial^* \bar\partial \varphi = 0, \qquad \varphi_{|\partial\Omega} = \varphi_0$$

(we choose a C^∞ hermitian metric on $\bar\Omega$ to define the adjoint $\bar\partial^*$). This is governed by the symbolic calculus of [6].

Now for any $s > -\frac{1}{2}$, $\Lambda = \Lambda_s = K^* |r|^{2s} K$ is an elliptic positive pseudo-differential operator of degree $-2s-1$ on $\partial\Omega$, with symbol

$$\sigma(\Lambda_s) = (2s)!\,(2|\xi|)^{-2s-1} \left(\frac{\partial r}{\partial n}\right)^{2s}$$

where $\frac{\partial}{\partial n}$ is the unit exterior normal vector on $\partial\Omega$; this is seen as follows: first on the half space $x_m > 0$ in $\mathbb{R}^m$, and with K defined by

$$F' K \varphi = e^{-x_m|\xi|} \hat\varphi \qquad (F' \text{ the partial Fourier transform with respect to } x_1, \dots, x_{m-1})$$

we have $\widehat{\Lambda_s \varphi} = g(\xi)\hat\varphi$ with

$$g(\xi) = \int_0^\infty e^{-2x_m|\xi|} x_m^{2s}\, dx_m = (2s)!\,(2|\xi|)^{-2s-1}.$$

The general case follows easily by the technique of [6], the computation above giving the principal symbol. (Actually [6] only gives a complete proof when $2s$ is an integer, $|r|^{2s}K$ being then a Poisson operator of degree $-2s$ (or $K^*|r|^{2s}$ a trace operator of degree $-2s-1$); but the case where $2s$ is not an integer can be treated in the same manner, with no new difficulty).

Thus K defines an isometry from H_Λ to $L^2_s \cap \operatorname{Ker} \Delta''$, where L^2_s is the completion of $C^\infty(\bar{\Omega})$ for the Hilbert norm $\int_\Omega |f|^2 |r|^{2s} d\mu$, with inverse $\Lambda^{-1} K^* |r|^{2s}$. So if B_s is the orthogonal projector on holomorphic functions in L^2_s, K transforms S_Λ into the restriction of B_s to harmonic functions:

$$B_s K = K S_\Lambda, \quad \text{or} \quad B_s = K S_\Lambda \Lambda^{-1} K^* |r|^{2s}$$

(we have chosen $\Delta'' = \bar{\partial}^* \bar{\partial}$ rather than the usual Laplace operator so that $\operatorname{Ker} \Delta''$ contains all holomorphic functions).

Now if f is a C^∞ function on $\bar{\Omega}$, we have

$$B_s f K \varphi = K T_Q^\Lambda \varphi$$

with

$$T_Q^\Lambda = S_\Lambda \Lambda^{-1} K^* |r|^{2s} f K$$

As above, $Q = \Lambda^{-1} K^* |r|^{2s} f K$ is a pseudo-differential operator of degree 0, and since we have $\sigma(\Lambda^{-1} K^* |r|^{2s} K) = 1$, it follows that $\sigma(Q) = f|\partial\Omega$. So the restriction T_Q^Λ of $S_\Lambda Q$ to holomorphic functions is a Toeplitz operator with symbol $f|\partial\Omega$. The proof above applies with essentially no modification when f is a C^∞ vector bundle homomorphism.

For a description of B_s when $\partial\Omega$ is real analytic, we refer to Kashiwara [18].

§ 2. The Index Formula

a) Review of K Theory

We only give here a brief description of what is needed, and refer to [1, 2] for further details.

Let X be a (paracompact) locally compact space, $Y \subset X$ a subspace. The elements of $K_c(X, Y)$ (the relative group with compact supports) are equivalence classes of complexes of vector bundles:

$$(2.1) \quad 0 \to E_{r_0} \xrightarrow{d} E_{r_0+1} \xrightarrow{d} \dots E_{r_1} \to 0$$

on X, where the differential d (collection of vector bundle homomorphisms with $d^2 = 0$) is exact on Y and outside some compact subset of X. The complex (2.1) defines in fact the same element of $K_c(X, Y)$ as the 2-step complex

$$d + d^* : \bigoplus E_{2j} \to \bigoplus E_{2j+1}$$

We also write $K_c(X)$ if Y is empty.

We set $K_c^1(X) = K_c(X \times \mathbb{R})$, and identify this with the group of homotopy classes of maps $a: X \to \bigcup_{n \geq 1} GL_n(\mathbb{C})$, with $a = \mathrm{Id}$ outside some compact subset – the element of $K_c(X \times \mathbb{R})$ corresponding to a being defined by the 2-step complex

$$X \times \mathbb{R} \times \mathbb{C}^n \xrightarrow{\tilde{a}} X \times \mathbb{R} \times \mathbb{C}^n$$

where $\tilde{a}$ is any continuous function: $X \times \mathbb{R} \to L(\mathbb{C}^n)$ such that $\tilde{a}(x,t) = a(x)$ if t is large, and $\tilde{a}(x,t) = \mathrm{Id}$ if t is small or x lies outside of a sufficiently large compact subset.

If U is an open subset of X, i the injection $U \hookrightarrow X$, there is a canonical extension map

(2.2) $i_! = K_c(U) \to K_c(X)$.

This can be defined as follows: any element $\xi \in K_c(U)$ can in fact be defined by a 2-step complex

$$\alpha: E \to U \times \mathbb{C}^N.$$

Then we may define an extension $\tilde{E}$ of E to X by pasting E with $(x - A) \times \mathbb{C}^N$ outside a compact set $A \subset\subset U$ by means of α, and $i_! \xi$ is then defined by the 2-step complex $\tilde{\alpha}: \tilde{E} \to X \times \mathbb{C}^N$ with $\tilde{\alpha} = \alpha$ in U, $\tilde{\alpha} = \mathrm{Id}_{\mathbb{C}^n}$ outside U.

Similarly one defines $i_!$: $K_c^1(U) \to K_c^1(X)$ (an element of $K_c^1(U)$ is represented by a continuous function $a: U \to GL_n(\mathbb{C})$, we have $a = \mathrm{Id}$ outside some compact subset of U, and we extend a by Id outside U).

b) The Koszul Complex and the Bott Isomorphism

Let $N \xrightarrow{\pi} X$ be a complex vector bundle on X. Then Koszul complex is the complex k on N:

(2.3) $\ldots \to E_{-n} \to E_{-n+1} \to \ldots E_0 \to 0$

with $E_{-j} = \Lambda^j \pi^* N'$ (the pull back of the j-th exterior power of the dual bundle N'), and differential $d = i(z)$, the interior product by z at the point $z \in N$.

Although k is not exact on the zero section, multiplication by k: $\xi \mapsto k \otimes \xi$ defines a homomorphism $\beta: K_c \to K_c(N)$, and it follows from the Bott periodicity theorem that this is an isomorphism. Similarly one defines $\beta: K_c^1(X) \xrightarrow{\sim} K_c^1(N)$.

Remark. Taking adjoints in (2.3) one gets the complex of vector bundles

$$\beta:\ 0 \to E'_0 \to E'_1 \to \ldots,$$

with $E'_j = \pi^* \Lambda^j \bar{N}$, $\bar{N}$ the complex conjugate of N, and with differential $d = e(\bar{z})$, the exterior multiplication by $\bar{z}$ at the point $z \in N$ – i.w. the exterior complex of $\bar{N}$. Of course this gives the same map $K_c(X) \to K_c(N)$. In §3 both the Koszul complex (§3.b) and its adjoint (§3.a) will occur, as symbols of complexes of

Toeplitz operators. In [2] it is the conjugate complex, i.e. the exterior complex of N rather than $\bar{N}$, that is used to define the Bott isomorphism $K_c(X) \to K_c(N)$. However, for the index theorem of [2] the Bott isomorphism is only used when N is the complexification of a real vector bundle, hence isomorphic to $\bar{N}$, and it makes no difference whether N or $\bar{N}$ is used to define the Bott isomorphism. (For the index of Toeplitz operators, or for the Riemann-Roch formula, N is not always isomorphic to $\bar{N}$, and it is the Koszul complex above, or equivalently the exterior complex of $\bar{N}$, that has to be used.)

In particular the dimension gives an isomorphism $K(\text{point}) \xrightarrow{\sim} \mathbb{Z}$, and combining this with the Bott periodicity map, we get an isomorphism

(2.4) $\chi_{\mathbb{C}^n}: K_c(\mathbb{C}^n) \xrightarrow{\sim} \mathbb{Z}$

(where the positive generator corresponds to the Koszul complex).

c) Positive Complex Structures

Let E be a real symplectic vector space, with symplectic form σ. We denote by $\{\ \}$ the inverse symplectic form on the dual E^*: $\{f,g\}=\sigma(\alpha^{-1}f,\alpha^{-1}g)$ if α: $E \to E^*$ is defined by $\sigma(x,y)=\langle \alpha x, y\rangle$. (One also usually defines $H_f = -\alpha^{-1} f$, so $\{f,g\}=\langle H_f, g\rangle$).

We recall that a complex structure on E is an automorphism $J \in GL(E)$ such that $J^2 = -\mathrm{Id}$, or equivalently a decomposition $\mathbb{C} \otimes E = E' \oplus E''$, with $E'' = \bar{E}'$ ($E' = \mathrm{Ker}(J-i)$, $E'' = \mathrm{Ker}(J+i)$). A linear form $f \in \mathbb{C} \otimes E^*$ is holomorphic if $fJ = if$ (i.e. f vanishes on E''). A complex structure on E is compatible with the symplectic structure if σ is the imaginary part of a hermitian form (σ is "of type (1, 1)"), or equivalently if E' (or E'') is isotropic, or $\{f,g\}=0$ for any holomorphic forms f, g.

(2.5) **Definition.** *A complex structure is positive* ($\gg 0$) *if it is compatible and if one of the following equivalent conditions holds:*

(i) *Let* $z_1 \dots z_n$ *be a basis of holomorphic forms. Then* $\{z_j, z_k\}=0$ *and* $\frac{1}{i}\{\bar{z}_j, z_k\}$ *is a hermitian* $\gg 0$ *matrix.*

(i) bis $\sigma = \frac{i}{2} \Sigma a_{jk} dz_j \wedge d\bar{z}_k$, *with* (a_{jk}) *hermitian* $\gg 0$.

(i) ter $\sigma = \frac{i}{2} \partial\bar{\partial} h$, *with* h *a hermitian (or quadratic)* $\gg 0$ *form.*

(ii) $\sigma = -\mathrm{Im}\, h$, *with* h *a hermitian* $\gg 0$ *form.*

(ii) bis $\sigma(Jx, Jy)=\sigma(x,y)$ *for all* x, y, *and* $\sigma(x, Jx)>0$ *if* $x \neq 0$.

(iii) ter $i\sigma(\bar{v}, v)>0$ *if* $v \in E' = \mathrm{Ker}(J-i) \subset \mathbb{C} \otimes E$, $v \neq 0$ (*and* $\sigma=0$ *on* E').

(We will also say that the symplectic structure is $\gg 0$ with respect to the complex structure.)

(2.6) *Examples.* On $E = T^*\mathbb{R}^n$, with canonical form $\sigma = \Sigma d\xi_j \wedge dx_j$ (ξ being the vertical component), we have a canonical complex structure defined by the condition that $\xi+ix$ is holomorphic: $T^*\mathbb{R}^n\to\mathbb{C}^n$; this is $\gg 0$. (On the dual bundle $T\mathbb{R}^n$, with coordinates (x, y), y being vertical, it is usual to take the dual complex structure, for which $x+iy$ is holomorphic).

On $\mathbb{C}^n$, the canonical symplectic (or Kaehler) form

$$\sigma=\frac{i}{2}\partial\bar{\partial}|z|^2=\frac{i}{2}\Sigma\, dz_j\wedge d\bar{z}_j \quad \text{is } \gg 0.$$

(2.7) Let Q be any $\gg 0$ quadratic form on E. Then (cf. [8]) there exists a unique positive complex structure on E for which Q is hermitian: if we still denote by $Q(x, y)$ the associated scalar product, and define A by $\sigma(x,y)=Q(Ax,y)$, E' is spanned by the eigenvectors of A corresponding to eigenvalues with positive imaginary parts, and $J=A|A|^{-1}$, with $|A|=(A^*A)^{1/2}=(-A^2)^{1/2}$, the positive square root with respect to Q. This depends smoothly on Q.

Conversely if N is a complex vector space, and h a positive (non-degenerate) hermitian form, $-\operatorname{Im} h$ is a $\gg 0$ symplectic form on N.

(2.8) If now N is a real symplectic vector bundle with a paracompact base X, there exists a $\gg 0$ complex structure on N, and this is unique up to homotopy equivalence, so the Bott isomorphism β: $K_c(X)\xrightarrow{\sim} K_c(N)$ is well defined.

d) The Homomorphism $i_!$ in K-Theory

Let X and Y be two almost complex manifolds (i.e. the tangent bundles TX, TY are equipped with complex structures, not necessarily integrable) and $i: X\hookrightarrow Y$ a complex embedding. Then the normal tangent bundle $N=N_X(Y)$ inherits a complex structure. If we compose the Bott periodicity map β: $K_c(X)\xrightarrow{\sim} K_c(N)$ with the isomorphism of N on a tubular neighborhood U of X in Y (the only requirement being that the tangent map on the zero section be compatible with the complex structures, since this defines the map $N\to U$ up to homotopy equivalence) then with the extension map (2.2): $K_c(U)\to K_c(Y)$, we get a canonically defined homomorphism

(2.9) $\quad i_!: K_c(X)\to K_c(Y)$.

This is functorial (transitive), i.e. $(ij)_!=i_!j_!$ if we have two successive complex embeddings i, j.

If now X and Y are almost symplectic manifolds (i.e. TX, TY are equipped with symplectic forms σ_X, σ_T, but we do not require $d\sigma_X=0$, $d\sigma_T=0$) and $i: X\hookrightarrow Y$ is a symplectic embedding, there is as in (2.9) a canonical homomorphism

(2.9) bis $\quad i_!: K_c(X)\to K_c(Y)$

defined via the $\gg 0$ complex structures.

e) Contact Manifolds and Symplectic Cones

Let Y be a $2n-1$ real manifold. An oriented contact structure on Y is a differential form $\omega \in \Omega'(Y)$ such that $\omega \wedge (d\omega)^{n-1}$ vanishes nowhere, two forms ω, ω' defining the same structure if $\omega' = \lambda\omega$ with $\lambda \in C^\infty$, $\lambda > 0$.

If $\omega \in \Omega'(Y)$ is a 1-form with no singularities ($\omega \neq 0$ everywhere), it defines a half-line bundle $\Sigma^+ \subset T^*Y \setminus 0$, and it is equivalent to say that ω is a contact form, or that Σ^+ is symplectic (as a submanifold of T^*Y).

Let now Σ be a symplectic cone, i.e. a symplectic manifold which is a principal bundle under the multiplicative group $\mathbb{R}_+^\times$ of positive real numbers, and whose symplectic form σ is homogeneous of degree 1. We identify Σ with $Y \times \mathbb{R}_+$ (with $Y = \Sigma/\mathbb{R}_+^\times$ the basis), and σ may be written in a unique manner

$$\sigma = r\omega' + dr \wedge \omega$$

where ω, ω' are pull backs of forms on Y. The condition $d\sigma = r d\omega' + dr \wedge (\omega' - d\omega)$ then implies $\sigma = d(r\omega)(d\omega' = 0,\ \omega' = d\omega)$, so there exists a contact form ω on Y such that $\sigma = d(r\omega)$, and this is of course unique. Thus we have an equivalence of categories between symplectic cones and oriented contact manifolds.

Let now Σ^+ be a symplectic cone, Y the basis (which is a contact manifold). We may choose C^∞ functions $g_1 \dots g_N$ on Y, which define an embedding $Y \to \mathbb{R}^N$, then C^∞ functions $f_1 \dots f_N$ on Σ^+, homogeneous of degree 1, such that the contact form of Σ^+ is

$$r\omega = \Sigma f_j dg_j$$

Then the functions $\xi_j = f_j$, $x_j = g_j$ define a symplectic embedding: $\Sigma^+ \to T^*\mathbb{R}^N \setminus 0$. (One may similarly construct a symplectic embedding $X \to T^*\mathbb{R}^N$ for any symplectic manifold X whose symplectic form is exact.)

f) The Index Character

(2.10) **Proposition.** *There exists a unique manner of assigning to each almost symplectic manifold X a homomorphism $\chi_X\colon K_c(X) \to \mathbb{Z}$ in such a way that*

(i) $\chi_{(\text{point})}$ *is the dimension map*

(ii) $\chi_X = \chi_Y \circ i_!$ *for any symplectic embedding $i\colon X \hookrightarrow Y$*

(iii) χ_X *depends continuously on the symplectic structure (i.e. it is a homotopy invariant)*[3].

In this statement, we may replace "almost symplectic" by "almost complex".

Proof. The conditions (i) and (ii) determine χ_X uniquely if $\chi = \mathbb{C}^n$ or $T^*\mathbb{R}^n$, with its canonical symplectic (or complex) structure (χ_X is then the Bott isomorphism

[3] This map is in fact well known topologically. See for instance M.F. Atiyah and F. Hirzebruch, Riemann-Roch theorem for differentiable manifolds, Bull. Amer. Math. Soc. 65, 276–281 (1959)

(2.4)), hence also when X can be imbedded symplectically in $T^*\mathbb{R}^n$, i.e. its symplectic form is exact.

In the general case, T^*X inherits a symplectic structure from that of X, which is uniquely determined up to homotopy by the condition that it induces the given structure on X (identified with the zero section), and the dual structure on each fiber. For instance we may choose a connection on T^*X, i.e. an isomorphism $T(T^*X) \simeq \Pi^*(T^*X) \oplus \Pi^*(TX)$, the first factor being the vertical component of $T(T^*X)$, and we define the symplectic form by

$$\sigma_1((u, v), (u', v')) = \sigma_X(\alpha^{-1} u, \alpha^{-1} u') + \sigma_X(v, v')$$

where α: $TX \to T^*X$ is the isomorphism defined by σ_X: $\sigma_X(v, v') = \langle \alpha v, v' \rangle$.

Now this symplectic form σ, is homotopic to the canonical symplectic form σ_0 of T^*X (as cotangent bundle): for instance we may always impose on the connection above that the horizontal space $\Pi^*(TX)$ be isotropic for σ_0 (in general we have $\sigma_0((u, v), (u', v')) = \langle u, v' \rangle - \langle u', v \rangle + \langle \beta v, v' \rangle$ where β is a skew symmetric map: $\Pi^*(TX) \to \Pi^*(T^*X)$, and we define the new horizontal component as the set of vectors $\left(-\frac{\beta}{2} v, v\right)$ in the old decomposition). We may then set for $0 \leqq t \leqq 1$

$$\sigma_t = \cos \frac{t\pi}{2} \sigma_0 + \sin \frac{t\pi}{2} \sigma_1$$

$\Bigg($the matrix of σ_1 with respect to σ_0 is $A = \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}$ and we have $A^2 = -\mathrm{Id}$, so σ_t is always of maximal rank$\Bigg)$. Condition (iii) then determines χ_X uniquely.

The existence of χ_X then follows immediately from the transitivity of $i_!$.

(2.11) **Definition.** *The index character* χ_X *of almost symplectic or almost complex manifolds is the homomorphism* χ_X: $K_c(X) \to \mathbb{Z}$ *defined by proposition* (2.10).

The index character χ_Y^1 *of oriented contact manifolds is the homomorphism* $\chi_Y^1 = \chi_{\Sigma^+} \circ \partial$: $K_c^1(Y) \to \mathbb{Z}$, *where* $\Sigma^+ \simeq Y \times \mathbb{R}_+$ *is the associated symplectic cone, and* ∂: $K_c^1(Y) \xrightarrow{\sim} K_c(\Sigma^+)$ *the canonical isomorphism.*

g) The Index Theorem

Let Ω, $\partial\Omega$ be as in § 1.a (whose notations we keep). Then $\alpha = \frac{1}{2i} (\partial r - \bar{\partial} r)_{|\partial\Omega}$ is a contact form on $\partial\Omega$, for which the associated cone is Σ^+. Let T_Q be an $N \times N$ elliptic matrix of (scalar) Toeplitz operators on $\partial\Omega$, with symbol $q = \sigma(T_Q)$. Then $q \circ \alpha$ is invertible on $\partial\Omega$, and defines an element $[q]' \in K^1(\partial\Omega)$.

Theorem 1. *Notations being as above, we have* $\mathrm{Index}(T_Q) = \chi_{\partial\Omega}^1([q]')$.

Let now E and F be holomorphic vector bundles on $\bar{\Omega}$, and let T_Q be an elliptic Toeplitz operator from E to F, with symbol $q = \sigma(T_Q)$. Then $q \circ \alpha$ is a C^∞ vector bundle isomorphism $E_{|\partial\Omega} \to F_{|\partial\Omega}$, so it defines an element $[q] \in K(\Omega, \partial\Omega) = K_c(\Omega)$.

Theorem 2. *We suppose that Ω is a (smooth) Stein manifold. Then, notations being as above, we have* $\operatorname{Index}(T_Q)=\chi_\Omega([q])$.

Naturally, these formulas can also be written in terms of cohomology; for this we refer to [3]. In particular the formula of Theorem 2 becomes

$$\operatorname{Index}(T_Q)=\chi_\Omega([q])=\langle \operatorname{ch}[q]\,\tau(\Omega),[\Omega]\rangle$$

where $\tau(\Omega)$ is the Todd class of Ω (for its complex structure), and $\operatorname{ch}[q]$ the Chern character.

If Ω is an open set in $\mathbb{C}^n$, the Todd class vanishes and the formula of Theorem 1 reduces to

$$\operatorname{Index}(T_Q)=-\frac{(n-1)!}{(2n-1)!\,(2i\pi)^n}\int_{\partial\Omega}\operatorname{Tr}(q^{-1}dq)^{2n-1}\qquad \text{(cf. [12, 13, 17])}$$

where $\mathbb{C}^n$ has its usual orientation and $\partial\Omega$ is oriented as a boundary.

h) Appendix

For the sake of completeness, and although this will implicitly follow from § 3, we check here that the formulas of Theorem 1 and 2 agree when they both apply: we have a canonical map $\partial\colon K^1(\partial\Omega)\to K(\Omega,\partial\Omega)=K_c(\Omega)$, and must check $\chi_{\partial\Omega}=\chi_\Omega\circ\partial$.

Notations being as in § 1.a let Φ be an isomorphism of Σ^+ on a tubular neighborhood U of $\partial\Omega$, so that Σ^+ points outwards (this condition determines Φ uniquely up to homotopy equivalence). Then ∂ is the composition of the canonical isomorphism $K^1(\partial\Omega)\to K_c(\Sigma^+)$, of the push forward Φ_*, and of the extension map $K_c(U)\to K_c(\Omega)$. We then have to prove that the pull back of the complex structure on U to Σ^+ is homotopic to a positive structure, and it is enough to check this on the 1st order jet of Φ along $\partial\Omega$.

As in § 1.a, we suppose Ω defined by $r<0$. We may suppose that r is strictly pseudo-convex near $\partial\Omega$, and then equip a neighborhood (U) of $\partial\Omega$ with the positive symplectic (Kaehler) form $\sigma_r=i\partial\bar\partial r=d\left(\frac{1}{2i}(\partial r-\bar\partial r)\right)$. Let $\frac{\partial}{\partial n}$ be the outward unit normal vector of $\partial\Omega$, so $J\frac{\partial}{\partial n}$ is tangent to $\partial\Omega$.

We have $\alpha=\frac{1}{2i}(\partial r-\bar\partial r)_{|\partial\Omega}$, and may obviously suppose $\Phi(z,\alpha(z))=z$. Then at least the restriction of Φ to the section $\alpha(\partial\Omega)$ of Σ^+ agrees with the symplectic forms, and since $\sigma_r\left(\frac{\partial}{\partial n},J\frac{\partial}{\partial n}\right)>0$, we only have to check $\sigma_{\Sigma^+}\left(\rho\frac{\partial}{\partial\rho},J\frac{\partial}{\partial n}\right)>0$, where $\rho\frac{\partial}{\partial\rho}$ is the radial vector $\left(\rho\frac{\partial}{\partial\rho}=\Sigma\xi_j\frac{\partial}{\partial\xi_j}\text{ in local coordinates on }\partial\Omega\right)$. Now this sign condition only depends on the 2nd order jet of $\partial\Omega$ at any given point; we may always choose local coordinates in which $\partial\Omega$ has a contact of order $\geqq 2$ with the unit sphere $z\bar z=1$ of $\mathbb{C}^n$, and since the result is true for the unit sphere, it is true in general.

(If Ω is the unit ball in $\mathbb{C}^n$, we choose $r=\frac{1}{2}(z\bar{z}-1)$, so

$$i\partial\bar{\partial}r=\tfrac{1}{2}d\left(\frac{1}{2i}(\bar{z}\cdot dz-z\cdot d\bar{z})\right)=\frac{i}{2}\Sigma dz_j\wedge d\bar{z}_j$$

is the canonical symplectic form of $\mathbb{C}^n$, and is homogeneous of degree 2; then the map Φ: $\left(z,\frac{t}{4i}(\bar{z}\partial z-z\partial\bar{z})_{|\partial\Omega}\right)\mapsto t^2 z$ from Σ^+ to $\mathbb{C}^n\setminus 0$ is symplectic, since it transforms the canonical 1-form of Σ^+ (i.e. the identity map: $\Sigma^+\to T^*\partial\Omega$) into $\left.\frac{1}{4i}(\bar{z}\cdot dz-z\cdot d\bar{z})\right)$.

§ 3. Proof of the Index Theorem

a) Proof of Theorem 1

Let Ω, $\partial\Omega$ be as in § 1.a, and let Q be an $N\times N$ matrix of pseudo-differential operators on $\partial\Omega$, elliptic on Σ^+. Then the Toeplitz operator T_Q has an index, which only depends on the symbol $\sigma(T_Q)=\sigma(Q)_{|\Sigma^+}$. As was remarked in § 1.c, we may suppose that Q is of degree 1 (since any Toeplitz operator with symbol $|\xi|$ Id has index 0). We may also modify $\sigma(Q)$ arbitrarily outside Σ^+, and we will suppose $\sigma(Q)=|\xi|\,\mathrm{Id}$ on $\Sigma^-=-\Sigma^+$.

Let $\bar{\partial}_b$ denote the induced Cauchy-Riemann complex. This is a complex of first order differential operators:

$$0\to C^\infty(\partial\Omega,\Lambda^0E'')\xrightarrow{\bar{\partial}_b}C^\infty(\partial\Omega,\Lambda^1E'')\xrightarrow{\bar{\partial}_b}\dots C^\infty(\partial\Omega,\Lambda^{n-1}E'')\to 0$$

where E'' is the dual of the sub-vector bundle of $C\otimes T\partial\Omega$ spanned by anti-holomorphic vectors tangent to $\partial\Omega$, or equivalently the quotient $E''=\mathbb{C}\otimes T^*\partial\Omega/F'$ where F' is spanned by all differentials of type 1, 0: $\partial f_{|\partial\Omega}$ (with $f\in C^\infty(\bar{\Omega})$). There is a natural projection $\xi\mapsto\xi''$ from $T^*\partial\Omega$ to E'', and the symbol of $\bar{\partial}_b$ is

$\sigma(\bar{\partial}_b)=ie(\xi'')$, the exterior multiplication by $i\xi''$.

$\bar{\partial}_b$ is elliptic except on $\Sigma^+\cup\Sigma^-$, and we see that $\sigma(\bar{\partial}_b)$ is the dual of the Koszul complex for the symplectic embedding $\Sigma^+\hookrightarrow T^*\partial\Omega$ (the fact that it corresponds to a $\gg 0$ complex structure follows from the fact that the Levi form is $\gg 0$). We have chosen Q with $\sigma(Q)=|\xi|\,\mathrm{Id}$ on $\Sigma^-=-\Sigma^+$, so although $\bar{\partial}_b$ is not elliptic on Σ^-, the element $[\sigma(Q)\otimes\sigma(\bar{\partial}_b)]\in K_c(T^*\partial\Omega)$ defined by the elliptic complex $\sigma(Q)\otimes\sigma(\bar{\partial}_b)$ is precisely $i_![\sigma(T_Q)]$, where $[\sigma(T_Q)]$ is the element of $K_c(\Sigma^+)\simeq K^1(\partial\Omega)$ defined by $\sigma(T_Q)$[4].

[4] Let (E',d') and (E'',d'') be two complexes of vector bundles:

$$0\to E'_{r0}\xrightarrow{d'}E'_{r0+1}\xrightarrow{d'}\dots E'_{r_1}\to 0$$
$$0\to E''_{r0}\xrightarrow{d''}E''_{r0+1}\xrightarrow{d''}\dots E''_{r_1}\to 0.$$

The product complex (E,d) is defined as usual by $E_j=\bigoplus_{p+q=j}E'_p\otimes E''_q$, $d=d'\otimes\mathrm{Id}+(-1)^p\,\mathrm{Id}\otimes d''$. We denote it for short by $d'\otimes d''$. Here in particular $\sigma(Q)\otimes\sigma(\bar{\partial}_b)$ is the complex defined by

$$E_j=\mathbb{C}^N\otimes\Lambda^jE''\oplus\mathbb{C}^N\otimes\Lambda^{j-1}E'',\quad d_j=\begin{pmatrix}\mathrm{Id}\otimes\sigma(\bar{\partial}_b) & 0\\ \sigma(Q)\otimes\mathrm{Id} & -\mathrm{Id}\otimes\sigma(\bar{\partial}_b)\end{pmatrix}$$

We will show that T_Q has the same index as any complex of pseudo-differential operators on $\partial\Omega$ with symbol $\sigma(Q)\otimes\sigma(\bar\partial_b)$: Theorem 1 will then follow from the index formula of Atiyah and Singer [1].

Let us recall (cf. [8]) that the homology of $\bar\partial_b$ is of finite rank, except in degree 0, $n-1$, and that there exists an operator A, which is in fact a pseudo-differential operator of type $\frac{1}{2}$ and degree $-\frac{1}{2}$ such that

(3.1) $\bar\partial_b A$ is the orthogonal projector on the range of $\bar\partial_b$,
$A\bar\partial_b$ is the orthogonal projector on the range of $\bar\partial_b^*$,

$\mathrm{Id}-\bar\partial_b A - A\bar\partial_b = \vec{S} = \bigoplus_0^{n-1} S_j$ is the orthogonal projector on the homology, identified with $\mathrm{Ker}(\bar\partial_b+\bar\partial_b^*)$. Then the S_j are of finite rank if $j\neq 0$, $n-1$; $S_0=S$ is the Szegö projector, and S_{n-1} is analoguous to the Szegö projector, but supported by Σ^-.

(3.2) **Lemma.** *There exists a system Q_0 of pseudo-differential operators of degree 1, acting on sections of $\mathbb{C}^N\otimes\bigoplus_0^{n-1}\Lambda^j E''$, such that $\sigma(Q_0)=\sigma(Q)\otimes\mathrm{Id}$ and $Q_0\bar\partial_b=\bar\partial_b Q_0$.*

Proof. Let Q_1 be any pseudo-differential system with symbol $\sigma(Q)\otimes\mathrm{Id}$ (respecting the graduation $\bigoplus_j \mathbb{C}^N\otimes\Lambda^j E''$). We set

$$Q_0=\vec{S}Q_1\vec{S}+AQ_1\bar\partial_b+\bar\partial_b A Q_1\bar\partial_b A$$

(for short we write $\vec{S}$, A, $\bar\partial_b$ instead of $\mathrm{Id}_{\mathbb{C}^N}\otimes\vec{S}$, etc. ...). Q_0 is a pseudo-differential operator of degree 0 and type $\frac{1}{2}$. It commutes with $\bar\partial_b$ since we have $\bar\partial_b\vec{S}=\bar\partial_b\bar\partial_b A=A\bar\partial_b\bar\partial_b=0$, and $\bar\partial_b A\bar\partial_b=\bar\partial_b$, hence

$$\bar\partial_b Q_0=\bar\partial_b A Q_1\bar\partial_b, \quad \text{and also}$$
$$Q_0\bar\partial_b=\bar\partial_b A Q_1\bar\partial_b A\bar\partial_b=\bar\partial_b A Q_1\bar\partial_b.$$

On the other hand the symbol $\sigma(Q_1)=\sigma(Q)\otimes\mathrm{Id}$ commutes with the symbols of $\bar\partial_b$, $\vec{S}$ and $\bar\partial_b A$ (rather $\mathrm{Id}_{\mathbb{C}^N}\otimes\bar\partial_b$, etc. ...). Since $\vec{S}$ and $\bar\partial_b A$ are both pseudo-differential operators of degree 0 and type $\frac{1}{2}$, their commutators with Q_1 are of degree $\leqq\frac{1}{2}$; also $[Q_1,\bar\partial_b]$ is of degree $\leqq 1$ so $A[Q_1,\bar\partial_b]$ is of degree $\leqq\frac{1}{2}$. Now we have $\vec{S}=\vec{S}^2$ and $\bar\partial_b A=(\bar\partial_b A)^2$ so $\vec{S}^2+A\bar\partial_b+(\bar\partial_b A)^2=\mathrm{Id}$, hence

$$\begin{aligned}Q_0-Q_1&=\vec{S}Q_1\vec{S}+AQ_1\bar\partial_b+\bar\partial_b AQ_1\bar\partial_b A-\vec{S}^2Q_1-A\bar\partial_b Q_1-(\bar\partial_b A)^2Q_1\\&=\vec{S}[Q_1,\vec{S}]+A[Q_1,\bar\partial_b]+\bar\partial_b A[Q_1,\bar\partial_b A]\end{aligned}$$

and it follows that Q_0-Q_1 is of degree $\leqq\frac{1}{2}$, and $\sigma(Q_0)=\sigma(Q_1)=\sigma(Q)\otimes\mathrm{Id}$ as required (the part of Q_0 which is really of type $\frac{1}{2}$ is in fact of degree $\leqq\frac{1}{2}$, and does not contribute to the index).

Now Q_0 is a homomorphism of complexes, and the double complex it defines is elliptic with symbol $\sigma(Q)\otimes\sigma(\bar\partial_b)$. Its Euler characteristic is the alternating sum of the indices of the restrictions of Q_0 to the homology of $\bar\partial_b$:

$$\mathrm{Index}(Q_0\otimes\bar\partial_b)=\Sigma(-1)^j\,\mathrm{Index}(S_jQ_0S_j).$$

If $j \neq 0$, $n-1$, S_j is of finite rank so $\mathrm{Index}(S_j Q_0 S_j)=0$. $S_{n-1} Q_0 S_{n-1}$ behaves exactly as a Toeplitz operator, except that it is supported by Σ^-, and since its symbol $|\xi| \mathrm{Id}_{|\Sigma^-}$ is self adjoint, its index is 0. Finally there just remains $S_0 Q_0 S_0$, which is a Toeplitz operator with same symbol hence same index as T_Q.

b) Embedding in a Complex Vector Bundle

Let W be a complex Stein manifold, and $\Omega \subset W$ an open subset as in § 1.a. We suppose Ω defined by an inequality $r<0$, where r is strictly plurisubharmonic near $\bar{\Omega}=\Omega \cup \partial\Omega$ (i.e. $i\partial\bar{\partial}r \gg 0$ near $\bar{\Omega}$). Let $N \xrightarrow{\pi} W$ be a holomorphic vector bundle, and let $\|n\|^2$ be a hermitian norm on N which is plurisubharmonic as a function on the total space of N (such a norm exists – at least above a neighborhood of $\bar{\Omega}$ because N is a direct factor of a trivial bundle near $\bar{\Omega}$ since W is Stein). Finally let $\tilde{\Omega} \subset N$ be defined by the inequality $r+\frac{1}{\varepsilon}\|n\|^2<0$ ($\varepsilon>0$): it is relatively compact, and $\partial\tilde{\Omega}$ is strictly pseudo-convex; taking ε small will make $\tilde{\Omega}$ arbitrarily close to the zero section. We denote by $\tilde{\Sigma}^+$ the analogue of Σ^+ for $\tilde{\Omega}$.

Let E and F be two holomorphic vector bundles on W, and f: $E \to F$ a C^∞ bundle-homomorphism. We denote by T_f the corresponding Toeplitz operator (as was remarked in § 1.c, it is enough to consider operators of this form for the index theorem). We denote by $\tilde{E}$, $\tilde{F}$ and $\tilde{f}$: $\tilde{E} \to \tilde{F}$ the pull backs on N.

Let k be the Konzul complex of N:

$$k\colon 0 \to E_{-q} = \Lambda^q \tilde{N}' \to \dots E_{-j} = \Lambda^j \tilde{N}' \to \dots E_0 \to 0$$

where $\tilde{N}'$ is the pull back of the dual bundle of N, q the dimension of its fibers. The differential is $i(z)$, the interior product by z at the point $z \in N$. Then $k_E = \mathrm{Id}_{\tilde{E}} \otimes k$, $k_F = \mathrm{Id}_{\tilde{F}} \otimes k$ are the Koszul complexes of E and F. The corresponding Toeplitz operators on $\tilde{\Omega}$ actually form a complex – i.e. $T_{i(z)} T_{i(z)}=0$ because $i(z)$ is holomorphic, so $T_{i(z)} T_{i(z)} = S\,i(z) S\,i(z) S = S(i(z))^2 S = 0$.

(3.3) **Proposition.** *There exists a complex T_Q of Toeplitz operators on $\tilde{\Omega}$, with symbol $\tilde{f} \otimes k$, and this has the same index as T_f if T_f is elliptic.*

Since the element $[\tilde{f} \otimes k] \in K(\tilde{\Omega}, \partial\tilde{\Omega})$ is precisely (by definition) the image by $i_!$ of $[f] \in K(\Omega, \partial\Omega)$, Theorem 2 on Ω will follow from Theorem 2 on $\tilde{\Omega}$.

To prove the proposition, we first need to examine the complex of Toeplitz operators corresponding to the Koszul complex a little more closely. Let $x_0 \in \partial\Omega \subset N$, and let $z_1, \dots, z_q$ be a basis of linear forms on N near x_0.

(3.4) **Lemma.** The matrix $\frac{1}{i}\{\bar{z}_j, z_k\}_{\tilde{\Sigma}^+}$ $(1 \leq j, k \leq q)$ is $\gg 0$.

Proof of the Lemma: We may choose holomorphic functions $z_{q+1} \dots z_n$ on N (defined in a neighborhood of x_0) so that $z_1 \dots z_n$ are local coordinates near x_0, and so that $\frac{\partial}{\partial z_n}$ is transversal to $\partial\tilde{\Omega}$ at x_0. We will prove that the matrix

$\frac{1}{i}\{\bar{z}_j, z_k\}_{\tilde{\Sigma}^+}$ $(1 \leqq j, k \leqq n-1)$ is $\gg 0$. For this we notice that there exists a basis of holomorphic vectors $Z_1 \dots Z_{n-1}$ tangent to $\partial\tilde{\Omega}$, with symbols $\zeta_1 \dots \zeta_{n-1}$, such that $Z_j(z_k) = \frac{1}{i}\{\zeta_j, z_k\}_{T^*\partial\tilde{\Omega}} = \delta_{jk}$ (the Kronecker symbol)

$$\left(\text{e.g. } Z_j = \frac{\partial}{\partial z_j} - \frac{\partial X}{\partial z_j}\left(\frac{\partial \rho}{\partial z_n}\right)^{-1}\frac{\partial}{\partial z_n}, \text{ where } \rho = r + \frac{1}{\varepsilon}\|n\|^2\right).$$

So we have

$$\{z_j, z_k\} = \{z_j, \bar{z}_k\} = \{\zeta_j, \zeta_k\} = \{\zeta_j, \bar{z}_k\} = 0 \qquad \text{on } \tilde{\Sigma}^+$$

(where the Poisson brackets are taken on $T^*\partial\tilde{\Omega}$), and

(3.5) $\quad c_{jk} = \frac{1}{i}\{\bar{\zeta}_j, \zeta_k\} \qquad$ is a hermitian $\gg 0$ matrix on $\tilde{\Sigma}^+$.[5]

Our assertion now is that we have $\left(\frac{1}{i}\{\bar{z}_j, z_k\}_{\tilde{\Sigma}^+}\right) = (c_{kj})^{-1}$, so it is elliptic, $\gg 0$, of degree -1.

To prove this, we notice that $\tilde{\Sigma}^+$ is defined by the equations $\zeta_j = \bar{\zeta}_j = 0$ $(j = 1 \dots n-1)$, *so for any functions* f, g on $T^*\partial\tilde{\Omega}$, we have $\{f, g\}_{\tilde{\Sigma}^+} = \{f', g'\}_{|\tilde{\Sigma}^+}$, where $f = f'$ and $g = g'$ on $\tilde{\Sigma}^+$, and the 1st order jets of f', g' along $\tilde{\Sigma}^+$ are determined by the condition that they commute with the ζ_j, $\bar{\zeta}_j$.

In particular we have $\frac{1}{i}\{\bar{z}_j, z_k\}_{\tilde{\Sigma}^+} = \frac{1}{i}\{\bar{z}'_j, z'_k\}_{|\tilde{\Sigma}^+}$, with

$$z'_k = z_k + \Sigma b_{kl}\bar{\zeta}_l,$$

where the $b_{kl|\tilde{\Sigma}^+}$ are determined by the condition

$$\frac{1}{i}\{\zeta_j, z'_k\}_{|\tilde{\Sigma}^+} = \delta_{jk} - \Sigma b_{kl}c_{lj} = 0 \quad \text{i.e.} \quad (b_{kl}) = (c_{kl})^{-1} \qquad \text{on } \tilde{\Sigma}^+.$$

Now from $\frac{1}{i}\{\zeta_j, z_k\} = \delta_{jk}$ follows $\frac{1}{i}\{\bar{\zeta}_j, \bar{z}_k\} = -\delta_{jk}$, $\quad$ so

$$\begin{aligned}\frac{1}{i}\{\bar{z}_j, z_k\}_{\tilde{\Sigma}^+} &= \frac{1}{i}\{\bar{z}_j + \Sigma\bar{b}_{jl}\zeta_l, z_k + \Sigma b_{kl}\bar{\zeta}_l\}'_{\tilde{\Sigma}^+} \\ &= \bar{b}_{jk} + b_{kj} - \sum_{l,m}\bar{b}_{jl}c_{ml}b_{km} = b_{kj} = (c_{jk})^{-1} \gg 0.\end{aligned}$$

This proves the assertion above, and the lemma.

5 This matrix is $\ll 0$ on $\tilde{\Sigma}^- = -\tilde{\Sigma}^+$ (cf. also [10] to check the signs). An easy way to remember the sign rule is that we have $\sigma([A^*, A]) = \frac{1}{i}\{\bar{a}, a\} \geqq 0$ if A is hypoelliptic, and on $\tilde{\Sigma}^+$ it is ∂_b which is hypoelliptic and $\bar{\partial}_b$ which is not

It follows now from Lemma (3.4) and from [8] that the complex of Toeplitz operators defined by the Koszul complex (or rather its adjoint) behaves microlocally exactly as $\bar{\partial}_b$ (except for the fact that in the microlocal model as described in § 1.b, the dimensions n and $n-1$ should be replaced by the complex dimensions of Ω and of the fibers of N respectively); the same holds for the complexes k_E and k_F. In particular there exist Toeplitz operators A_E, A_F which do for k_E, k_F the same as A for $\bar{\partial}_b$ in § 3.a (these are not quite Toeplitz operators in the sense of Definition 1.3, but are of the form SQS with Q of type $\frac{1}{2}$ and degree $\frac{1}{2}$):

$$d_E A_E + A_E d_E = \mathrm{Id} - H_E, \qquad d_F A_F + A_F d_F = \mathrm{Id} - H_F$$

where H_E, H_F are the orthogonal projectors on the homology of k_E, k_F. Now the construction of § 3.a can be exactly reproduced – namely we first choose a Toeplitz operator T_{Q_1} (from $\tilde{E} \otimes \Lambda \tilde{N}'$) with symbol $\tilde{f} \otimes \mathrm{Id}$, and replace T_{Q_1} by

$$T_{Q_0} = H_F T_{Q_1} H_E + d_F T_{Q_1} A_E + A_F d_F T_{Q_1} A_E d_E$$

which is a Toeplitz operator of symbol $\tilde{f} \otimes \mathrm{Id}$ which commutes with the differentials $d_E d_F$ (as in § 3.a it is really of type $\frac{1}{2}$, but the part of type $\frac{1}{2}$ is of degree $\leqq -\frac{1}{2}$ and does not contribute to the index).

Then T_{Q_0} defines a double complex T_Q with symbol $\tilde{f} \otimes k$, whose index is the alternating sum of the indices of the restrictions of T_{Q_0} to the homology groups of d_E, d_F. Here in fact the picture is a little simpler than in § 3.a because the Koszul complex $k_E k_F$ only have homology in degree 0, and this is isomorphic to the space of holomorphic sections of E or F on Ω. In fact let us choose on $\partial\tilde{\Omega}$ the measure $d\tilde{\sigma}$ defined by

$$\int_{\partial\tilde{\Omega}} \varphi(x,n)\, d\tilde{\sigma}(x,n) = \int_{\Omega} |r|^{-\frac{1}{2}}\, d\mu(x) \int_{\text{sphere}} \varphi(x,n)\, d\sigma_x(n)$$

where $d\mu$ is the given measure on Ω (with C^∞, positive density) and for each $x \in \Omega$, $d\sigma_x$ is the canonical (rotation invariant) measure on the sphere of radius $\sqrt{\varepsilon |r(x)|}$ of the fiber of N at x, so that the resulting measure $d\tilde{\sigma}$ has a smooth positive density on $\partial\tilde{\Omega}$. Then the ranges of the orthogonal projectors H_E, H_F on the homology of k_E, k_F consist exactly of the sections which are constant along the fibers, and we have

$$\mathrm{Index}\, T_Q = \mathrm{Index}\, H_F T_Q H_E = \mathrm{Index}\, H_F \tilde{f} H_E.$$

Now we may still interpret the ranges of H_E, H_F as sitting inside the space of sections of E or F on Ω; the norm is not the L^2-norm on $\partial\Omega$, but rather

$$\|\varphi\|^2 = \int_{\partial\tilde{\Omega}} |\tilde{\varphi}|^2\, d\tilde{\sigma} = c_q \varepsilon^{\frac{2q-1}{2}} \int_{\Omega} |\varphi|^2 |r|^{q-1}\, d\mu$$

$\left(\text{with } c_q = \frac{2\pi^q}{(q-1)!}, \text{ the volume of the unit sphere in } \mathbb{C}^q\right)$. Thus $H_F \tilde{f} H_E$ is not quite the Toeplitz operator $T_f = S_F f S_E$, but it follows from § 1.c, example (1.13) and appendix that it is still a Toeplitz operator with symbol f, so it has the same index as T_f. This ends the proof.

c) Embedding in $\mathbb{C}^n$

Let W be a Stein manifold, and Ω, $\partial\Omega$ as above. We may embed W in a numeric space $\mathbb{C}^n$. Let N be the normal tangent bundle of this embedding: the exact sequence

$$0 \to TW \to W \times \mathbb{C}^n = T\mathbb{C}^n|_W \to N \to 0$$

is split, since W is a Stein manifold. So we may realize N as a holomorphic subbundle of $W \times \mathbb{C}^n$, transversal to TW. Now the map $(x, n) \in N \mapsto x + n \in \mathbb{C}^n$ has an invertible derivative on W (identified with the zero section), so it defines an isomorphism of a neighborhood of W in N to a neighborhood of W in $\mathbb{C}^n$ (complex tubular neighborhood). We may then identify $\tilde{\Omega}$ with its image in $\mathbb{C}^n$ (if it is sufficiently close to the zero section, i.e. if the number ε in its definition is small). Now if E, F are holomorphic vector bundles on Ω, and $f: E \to F$ a C^∞ bundle homomorphism, invertible on $\partial\Omega$, and if i is the inclusion $\Omega \hookrightarrow \mathbb{C}^n$, $i_![f]$ is as we have remarked the element of $K_c(\tilde{\Omega})$ defined by $f \otimes k$. So Theorem 2 will follow if we prove it for open sets in $\mathbb{C}^n$.

d) End of Proof

Let finally $\Omega \subset \mathbb{C}^n$ be bounded, with strictly pseudo-convex boundary $\partial\Omega$; let E and F be holomorphic vector bundles defined in a neighborhood of $\bar{\Omega}$, and $f: E \to F$ a $\mathbb{C}^\infty$ homomorphism, invertible on $\partial\Omega$. Since $\bar{\Omega}$ has Stein neighborhoods, there exists a holomorphic vector bundle $F^\perp$ such that $F \oplus F^\perp$ is trivial (isomorphic to $\tilde{\Omega} \times \mathbb{C}^N$). We may replace f by $f \oplus \mathrm{Id}_{F^\perp}$ (this obviously does not change the index of T_f, nor the index character), so we are reduced to the case where $F = \bar{\Omega} \times \mathbb{C}^N$ is trivial. In this case E extends to the whole of $\mathbb{C}^n$ as a topological bundle (e.g. we can paste it with $(\mathbb{C}^n \setminus \Omega) \times \mathbb{C}^N$ by means of f on $\partial\Omega$), so it is trivial as a topological bundle, as any bundle on $\mathbb{C}^n$. Then it follows from Grauert's Theorem [14] that E is also trivial as a holomorphic bundle on $\bar{\Omega}$, so Theorem 2 follows from Theorem 1. This ends the proof.

Let us notice that it follows from the index formula that the index of Toeplitz operators satisfies the following "excision" property (cf. also [11]: if $\Omega \subset \Omega'$ are two open sets in $\mathbb{C}^n$ (or in some analytic manifold W), f' a C^∞ homomorphism between two holomorphic vector bundles on $\bar{\Omega}'$, f its restriction to $\bar{\Omega}$, and if f' is invertible on $\bar{\Omega}' \setminus \Omega$, then T_f and $T_{f'}$ have the same index. A direct proof of this would greatly simplify the proof of Theorem 1 (or Theorem 2) for open sets in $\mathbb{C}^n$, since we can always choose Ω' be to a large ball, in which case we have $K(\Omega', \partial\Omega') = \mathbb{Z}$ so to prove the index theorem, it is enough to check for one operator, for instance the complex of Toeplitz operators associated to the Koszul complex of a point of Ω', whose index is 1 (this is essentially the proof of [23]).

In general $K(\Omega, \partial\Omega)$ may be much larger than $\mathbb{Z}$; for example if Ω is the open set defined by $\Sigma(|z_j|^2 - 1)^2 < \varepsilon^2$ in $\mathbb{C}^n$, $\partial\Omega$ is C^∞ and strictly pseudo convex if ε is small, and $K(\Omega, \partial\Omega) = \mathbb{Z}^{2^{n-1}}$; so the index theorem does not reduce to the fact that the index of the Koszul complex of a point is 1.

3. Final Remarks

Let us notice that both Theorem 1 and Theorem 2 allow parameters $\lambda \in \Lambda$. The index should then be interpreted as an element of $K(\Lambda)$ as in [4]. Theorem 1 also allows the action of a compact group G, the index being interpreted as an element of $K_G(\cdot)=R(G)$, the ring of finite virtual representations of G, as in [2]; it is likely that the same is true for Theorem 2: the first three steps in the proof (§ 3.a, b, c) allow such a group action (this is obvious for the first two steps, and for the third one may always embed a neighborhood of $\bar{\Omega}$ in a finite G-vector space, equivariantly, then construct an equivariant tubular neighborhood); I do not know if Grauert's Theorem allows a compact group action.

Theorem 1 is of a quite stable nature. In it Ω may have singularities (so long as these do not meet $\partial\Omega$); $\partial\Omega$ must be compact but if $n=\dim_{\mathbb{C}}\Omega>2$, Ω itself needs not to be compact (cf. [7]) (if $n=2$, this restriction is necessary otherwise the Szegö projector might not be well behaved). In fact the only structure which really matters for the index theorem is the contact structure of $\partial\Omega$; we will show elsewhere that on any compact oriented contact manifold X there exists a projector S_X which has the same microlocal behaviour as the Szegö projector; the operators $S_X Q S_X$ (with Q a pseudo-differential operator) operating on the range of S_X are then analogues of Toeplitz operators, and the index of elliptic systems of such operators is still given by Theorem 1 (with essentially the same proof as in § 3.a).

For Theorem 2, the present proof requires that $\bar{\Omega}$ be contained in a Stein manifold (i.e. a closed, smooth, complex submanifold of $\mathbb{C}^n$). It is quite likely that it still holds when Ω has no singularities (it certainly does when $\partial\Omega$ is empty – in this case the index formula reduces to the Riemann-Roch theorem, which follows from the Atiyah-Singer index formula – cf. [3]). Further desirable extensions should include the case where Ω has singularities, and where the holomorphic bundles E and F are replaced by coherent sheaves. However as it is, Theorem 2 should already contain the Atiyah-Singer index formula. We give a brief indication here, and propose to come back elsewhere with more details to this question:

Let X be a compact real analytic manifold, and $P(x, D)$ an elliptic differential operator with analytic coefficients, acting from the sections of E to those of F, where E, F are two real analytic vector bundles on X (any C^∞ elliptic operator is homotopic to such an operator). Let $\tilde{X}$ be a complexification of X, and let B_ε be the tubular neighborhood defined by $d(z, X)<\varepsilon$ (for some hermitian metric on $\tilde{X}$). If ε is small enough, ∂B_ε is compact and strictly pseudo convex, and E, F, $P(x, D_x)$, $\operatorname{Ker} P$ and $\operatorname{Ker} P^*=\operatorname{Coker} P$ extend analytically to a neighborhood of $\bar{B}_\varepsilon$. As we have seen in § 1.c, the extension of P to $O(\partial B_\varepsilon)$ is a Toeplitz operator with symbol $p(z, v)$, where $p=\sigma(P)$, and $v=\frac{1}{i}\partial r$. If we take for defining function $r=\frac{1}{\varepsilon}(d(z, x)^2-\varepsilon^2)$, we have in local coordinates $z=x+iy=(z_1, \ldots, z_n)$

$$d(z, X)^2=\Sigma a_{jk} y_j y_k,$$

with $a_{jk}=a_{jk}(x, y)$ real analytic, $a=(a_{jk})\gg 0$. So

$$dr=\frac{2}{\varepsilon}(ay\cdot dy+O(y^2)) \qquad (a=(a_{jk}))$$

$$\partial r=\frac{1}{i\varepsilon}(ay\cdot dz+O(y^2))$$

and since $ay\cdot dy_{|\partial B_\varepsilon}=O(y^2)$, we finally get

$$v=-\left(\frac{ay}{\varepsilon}\right)\cdot dx+O(\varepsilon)=\xi_\varepsilon\cdot dx+O(\varepsilon).$$

Thus $p(z, v)$ is very close – hence homotopic – to $p(x, \xi_\varepsilon)$ with x the projection of z on X, $\xi_\varepsilon=-\frac{a\cdot y}{\varepsilon}$ (notice the sign).

It follows immediately that T_P is an elliptic Toeplitz operator, and that the index characters of P on X and of its Toeplitz extension T_P are the same; the analytic indices are also equal since the kernel and cokernel do not change if ε is small.

A similar proof can be given for analytic pseudo-differential operators.

A second method of proof is the following (this will be described in more details elsewhere); beginning with the approximation of v above, it is not hard to prove that there actually exists a contact isomorphism χ from S^*X to ∂B_ε, where S^*X is the cosphere bundle of X (close to the map $(x,\xi)\mapsto x-i\varepsilon\xi$ $(|\xi|=1)$). With this we may construct a Fourier integral operator $\mathscr{H}\colon C^\infty(X)\to O^\infty(\partial B_\varepsilon)$ such that $\mathscr{H}^*\mathscr{H}\sim\mathrm{Id}$, $\mathscr{H}\mathscr{H}^*\sim S$, and that the singular support of $\mathscr{H}\varphi$ (for any distribution φ on X) is the image by χ of the wavefront set of φ). $\mathscr{H}$ is then an approximate square root of the Szegö kernel, as in (1.4) (1.5), and it can be used to transport pseudo-differential operators on X into Toeplitz operators on ∂B_ε; we then get the index theorem for systems (matrices) of pseudo-differential operators as a consequence of Theorem 1. (At this point, the reduction of the general case to Theorem 2 by this method is not quite complete for one still needs to show that for the analogue $\mathscr{H}_E$ of $\mathscr{H}$ for a vector bundle E there is a canonical choice with index 0; that such a choice exists is plausible since the index formulas are identical).

e) Appendix

As a by product of § 3.b, we get the following microlocal model for $\bar\partial_b$ (or rather the adjoint complex) in the setting of Toeplitz operators: Ω is the unit ball in $\mathbb{C}^n (N=2n-1)$, and the adjoint of $\bar\partial_b$ is microlocally equivalent to the Koszul complex of $\mathbb{C}^n\subset\mathbb{C}^N$, whose differential is the interior product by $(z_1, \dots, z_n)$ on $\Lambda\mathbb{C}^n$. This is elliptic, except when $z_1=\dots z_n=0$. Using the Poincaré-Cartan formula

$$di(z)+i(z)d=\Theta(z),$$

the Lie derivative of $z=\sum_1^n z_j \frac{\partial}{\partial z_j}$, it is easy and elementary to describe the parametrix $\Big($the inverse of $\Theta(z)$ is given by $\omega=\Sigma a_{\alpha,J} z^\alpha dz_J \mapsto \Sigma \frac{a_{\alpha,J}}{|\alpha|+|J|} z^\alpha dz_J$ – where we have written $dz_J = dz_{i_1} \wedge \ldots \wedge dz_{i_k}$, $1 \leqq i_1 < \ldots i_k \leqq n$, and the coefficients $a_{\alpha,J}$ are functions of $z_{n+1}, \ldots, z_N$ – except in degree 0 where we must omit the constant term $(|\alpha|+|J|=0)\Big)$. Finally in this setting, the analogue of the Szegö projector (orthogonal projector on the homology) is the projector on functions independent of $z_1, \ldots, z_n$: $f \mapsto f(0, \ldots, 0, z_{n+1}, \ldots, z_N)$.

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