ON CERTAIN CRINKLY CURVES*

BY

ELIAKIM HASTINGS MOORE

Introduction. – In any field of geometric investigation the curves fall roughly into two classes, constituted respectively of the curves ordinarily investigated and of the other curves; these unusual curves are in positive designation the crinkly curves.

In this paper we are to investigate by interplaying graphic and analytic methods (in I) the continuous surface-filling xy-curves: $x = \varphi(t)$, $y = \psi(t)$: of PEANO and HILBERT and (in II) the continuous tangentless yt-curve: $y = \psi(t)$: connected with PEANO's curve. We define the various curves K as point-for-point limit-curves for $n = \infty$ of certain curves K_n $(n = 1, 2, 3, \cdots)$; these curves K_n are broken-line curves derivable each from the preceding by processes simple and such that the (nodal) extremities of the various n-links of K_n persist as corresponding points and also nodes of the K_{n+1}; thus, the nodes of K_n are points of K; the set of all these nodes (for all n's) is on K everywhere dense. The curves K_n are continuous and approach their point-for-point limit-curve K uniformly; K is accordingly continuous, a conclusion however which is geometrically evident. From the continuity of K and the presence of the set of nodes the properties of K follow in such a way as to appeal vividly to the geometric imagination. Indeed the yt-curve from the simplicity of its geometric definition and from the intuitive clearness of its properties appears to be fit to replace the classical WEIERSTRASS curve as the standard example of continuous curves having no tangents, since, further, we develop closer knowledge of its progressive- and regressive-tangential properties (II §§ 8, 11).

The basal notions of this paper were communicated to Chicago colleagues in February and March, 1899. – Part II has certain relations of content with the interesting paper by STEINITZ, *Stetigkeit und Differentialquotienten*, Mathematische Annalen, vol. 52, pp. 58–69, May 1899. These relations are indicated in the foot-note of II §7. STEINITZ determines a class of continuous functions having for no argument a derivative; he does not broach the question of progressive and regressive derivatives. – [Jan. 17, 1900. Part II has relations of method, but neither of origin nor of content, with the memoir of

* Presented to the Society, August 25, 1899, at the Columbus meeting. Received for publication December 18, 1899.

BRODÉN: *Beiträge zur Theorie der stetigen Functionen einer reellen Veränderlichen*, Journal für Mathematik, vol. 118, pp. 1–60, 1897. As a generalization of the basal remark of BRODÉN (l. c., p. 1) I notice the theorem that every continuous curve in δ dimensions, ($\delta = 2, 3, \cdots$ or $\aleph_0$) is the point-for-point limit-curve for $n = \infty$ of a sequence of inscribed broken-line curves K_n $(n = 1, 2, \cdots)$, the sequence of nodes of every K_n corresponding to a sequence of increasing values of the parameter-of-continuity of K, the nodes of every K_n being nodes of K_{n+1}, and the complete set of nodal arguments being everywhere dense on the set of all arguments of points of K.]

I

Continuous surface-filling curves.

1. Defining (as usual) as a real continuous plane curve the locus of points (x, y) whose coördinates are single-valued real continuous functions

$$x = \varphi(t), \quad y = \psi(t)$$

of the real variable t, PEANO,* 1890, by arithmetic † process determined two functions of this kind for the range $0 \leqq t \leqq 1$ such that the corresponding continuous curve C fills the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. We give below (§ 7) a geometric determination of Peano's curve.

2. This interesting phenomenon of continuous surface-filling curves HILBERT ‡ in 1891 made luminous to the geometric imagination in the following way:

For every positive integer n the line $0 \leqq t \leqq 1$ is divided into 4^n intervals I_n of length 4^{-n} and the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ is divided into 4^n squares S_n of length 2^{-n}. A 1—1 correspondence is effected between the 4^n intervals I_n and the 4^n squares S_n $(n = 1, 2, 3, \cdots)$ satisfying the two conditions: (1) to two adjacent intervals I_n correspond two adjacent squares S_n; (2) to the four intervals I_{n+1} of an interval I_n correspond the four squares S_{n+1} of the corresponding square S_n. Then to an infinite sequence $\{I_n\}$ of intervals I_n $(n = 1, 2, 3, \cdots)$ in which every interval I_n includes the succeeding interval I_{n+1} corresponds such a sequence $\{S_n\}$ of squares S_n $(n = 1, 2, 3, \cdots)$. In accordance with the geometric axioms of the continuity of the line and the plane (or, if

---

* PEANO, *Sur une courbe, qui remplit toute une aire plane:* Mathematische Annalen, vol. 36, pp. 157–160, 1890.

† The analytic formulas for the Peano functions given by CESÀRO (*Sur la représentation analytiques des régions, et des courbes qui les remplissent:* (Bulletin des Sciences mathématiques, 2d ser., vol. 21, pp. 257–266, 1897) involve the arithmetic function $[u] = E(u)$. (Cesàro's formulas are in error for $t = 1$.)

‡ HILBERT, *Ueber die stetige Abbildung einer Linie auf ein Flächenstück:* Mathematische Annalen, vol. 38, pp. 459–460, 1891.

our geometric phrasings have solely analytic meaning, by a fundamental theorem concerning the system of real numbers), the sequence $\{I_n\}$ determines a point $T\{I_n\}$ of the line lying on its every interval I_n, and the sequence $\{S_n\}$ a point $Z\{S_n\}$ of the square lying on its every square S_n. Such points T and Z are set in correspondence.

This correspondence indeed determines for every point T of the line a definite point Z of the square, and in such a way that as T describes the line Z describes a continuous curve filling the square. – Proof: (α) A point T of the line which is the extremity of no interval determines uniquely a sequence $\{I_n\}$ to which it belongs and hence a point Z belonging to the corresponding sequence $\{S_n\}$ The same is true for the points $T = (0)$, (1). A point T which is common to two adjacent intervals I'_n, I''_n for some definite value of n, $n = \nu$, is common to two adjacent intervals I'_n, I''_n for every $n \geqq \nu$, and belongs to two sequences $\{I'_n\}$, $\{I''_n\}$; the corresponding points $Z\{S'_n\}$, $Z\{S''_n\}$ are, however, identical, since the squares S'_n, S''_n are adjacent for every $n \geqq \nu$. Thus to every T corresponds one Z. (β) Similarly, every Z of the square belongs to one, two, or four sequences $\{S_n\}$ and corresponds to one, two, three, or four points T of the line. (γ) This dependence of $Z = (x, y)$ upon $T = (t)$ determines x and y as single valued functions

$$x = \varphi(t), \quad y = \psi(t) \tag{1}$$

of t for the range $0 \leqq t \leqq 1$. These functions are continuous, since $\Big($by condition (1)$\Big)$ for every n

$$|\varphi(t_1) - \varphi(t_2)| \leqq 2 \cdot 2^{-n}, \quad |\psi(t_1) - \psi(t_2)| \leqq 2 \cdot 2^{-n} \tag{2}$$

for every pair of arguments t_1, t_2 of the range $0 \leqq t \leqq 1$ such that

$$|t_1 - t_2| \leqq 4^{-n}. \tag{3}$$

The relation (2) indeed shows that $\varphi(t)$, $\psi(t)$ are uniformly continuous functions of t for the range $0 \leqq t \leqq 1$.

3. One finds that the fundamental correspondence subject to the conditions (1) (2) between the I_n, S_n $(n = 1, 2, 3, \cdots)$ involves 8 elements of indetermination for each successive n introduced, – or 2 elements (after the initial 8), if in condition (1) the initial and final intervals I_n of the line are counted as adjacent intervals – and that it becomes uniquely determinate by the stipulation of the sequences of squares corresponding to the sequences of initial and final intervals. In particular, Hilbert gives the correspondence determined by the stipulations

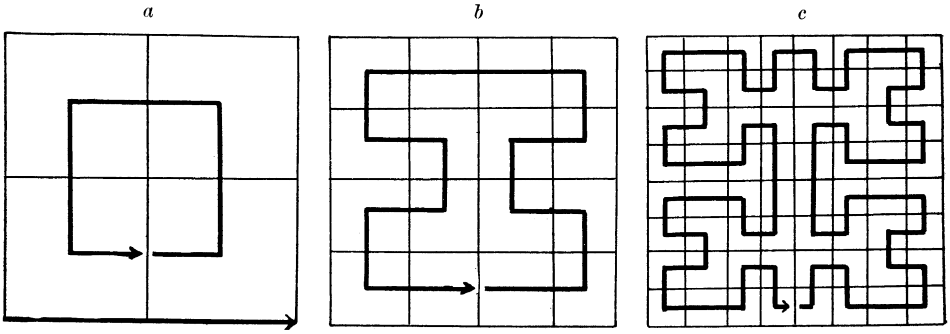
$$\big(\varphi(0), \ \psi(0)\big) = (0, 0), \ \big(\varphi(1), \ \psi(1)\big) = (1, 0), \tag{4}$$

while the figure 1 below is for one of the two determined by the stipulations

(5) $$\varphi(0) = \varphi(1) = \tfrac{1}{2}\,, \qquad \psi(0) = \psi(1) = 0\,.$$

4. For every particular n to indicate the sequence of the 4^n squares S^n corresponding to the natural sequence of the 4^n intervals I_n, HILBERT draws a broken line passing from the center of the initial square S_n through the centers of the squares in sequence to the center of the final square S_n. – *Let us see how to use this broken line to obtain a still more vivid geometric picture of these continuous surface-filling curves.* We prolong the line across the initial and the

FIGURE 1



Lines C_0 and C_1 Line C_2 Line C_3

final squares. This (completed) broken line C_n having in every square S_n a segment of length 2^{-n} has the length 2^n. We regard the original line C_0 as say the x-side of the original square S_0 and the lines C_n $(n = 0, 1, 2, \cdots)$ as derived each from the preceding by *uniform* stretching (doubling) and suitable "breaking" and locating. Then every point

$$T = T_0 = (t) = (t\,,\ 0) = (x_0\,,\ y_0)$$

of C_0 determines a definite point

$$T_n = (x_n\,,\ y_n) = \Big(\varphi_n(t)\,,\ \psi_n(t)\Big)$$

of C_n $(n = 0, 1, 2, 3, \cdots)$, and the sequence $\{T_n\}$ of points T_n $(n = 0, 1, 2, 3, \cdots)$ has as limit the point Z of the surface-filling curve C,

(6) $$Z = (x\,,\ y) = \Big(\varphi(t),\ \psi(t)\Big)\,,$$
$$\varphi(t) = \mathop{\mathrm{L}}_{n=\infty} \varphi_n(t)\,, \quad \psi(t) = \mathop{\mathrm{L}}_{n=\infty} \psi_n(t)\,.$$

THEOREM.—*The continuous surface-filling curve C is the point-for-point limit-curve for $n = \infty$ of the broken-line continuous curves C_n $(n = 1, 2, 3, \cdots)$ derived individually from the original line by uniform stretching and suitable breaking and locating.*

I notice further that

$$(7) \qquad |\varphi_{n_1}(t_1) - \varphi_{n_2}(t_2)| \leqq 2 \cdot 2^{-n}, \quad |\psi_{n_1}(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot 2^{-n}$$

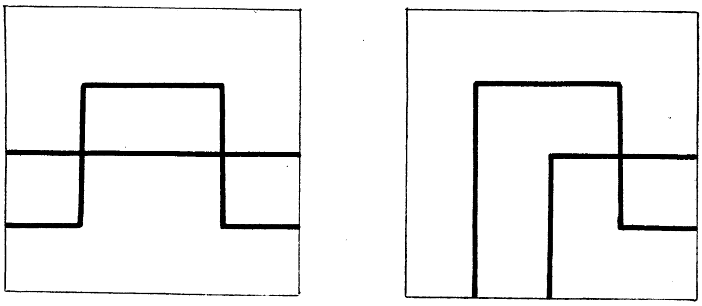
for all integers n_1, n_2, n $(\geqq 0)$ and numbers t_1, t_2 $(\geqq 0, \leqq 1)$ such that

$$n_1 \geqq n, \quad n_2 \geqq n, \quad |t_1 - t_2| \leqq 4^{-n}.$$

5. The squares S_n are crossed* by the lines C_n, C_{n+1} in essentially only two ways.

FIGURE 2

a b



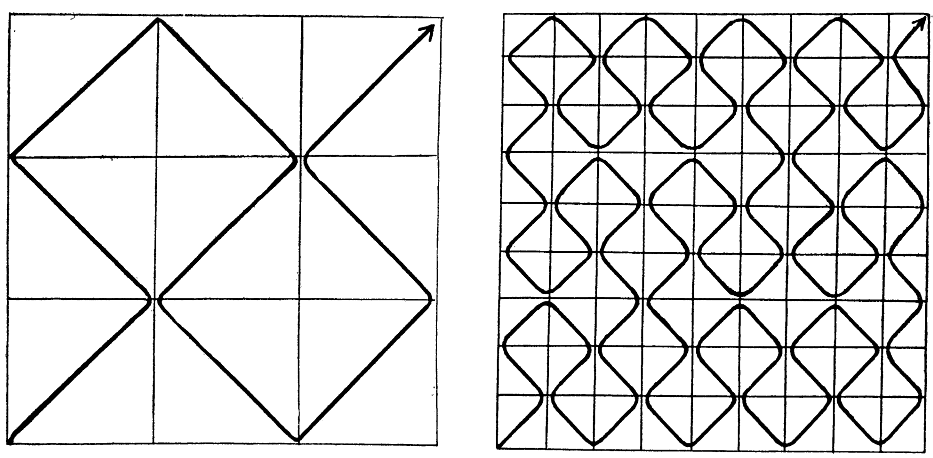
The lines C_n, C_{n+1} are seen to intersect at and only at corresponding points $T_n(t) = T_{n+1}(t)$, which, furthermore, are distinct from their respective corresponding points $T_{n+2}(t)$. Thus on no C_n is there one point $T_n(t)$ in its limit-position $Z(t)$.

6. No simple modification of these approximation curves C_n which preserves their essential relations to the correspondence between the I_n and the S_n and their derivation each from the preceding by uniform stretching and which moreover introduces on every C_n in its every S_n at least one point $T_n(t)$ in its limit-position suggests itself. However without essential change we may replace HILBERT's basal integer 2 by any *odd* integer ω (> 1) and then find a correspondence between the ω^{2n} intervals I_n and the ω^{2n} squares S_n $(n = 1, 2, 3, \cdots)$ such that the broken lines C_n (may be chosen to) cross the squares S_n on diagonals and that then the extremities of these diagonals are points T_n in final position. The curves $C^{(\omega)}$ so obtained are exactly PEANO's surface-filling curves.

* For the particular case of § 3, the lines C_n are closed and enclose surfaces of area $2^{-1} - 4^{-n}$ $(n = 1, 2, 3, \cdots)$, so that the limit for $n = \infty$ of the area of the surface enclosed by C_n is 2^{-1}, while the point-for-point closed limit-curve C fills the whole unit-square.— The lines C_n are symmetrical with respect to the lines $2x - 1 = 0$, $2y - 1 = 0$.

7. *Geometric determination of Peano's curve* $C^{(\omega)}$ (with figures for the $C^{(3)}$). ω is an odd integer > 1. We stretch the original line $0 \leqq t \leqq 1$ uniformly in the ratio $\sqrt{2} : 1$ and locate it as $C_0^{(\omega)}$ or say C_0 in the diagonal joining the vertices $(x, y) = (0, 0), (1, 1)$ of the initial square S_0. Without changing the extremities of C_0 we uniformly stretch it in the ratio $\omega : 1$ and break and locate it as C_1 (figure 3) traversing on their diagonals the ω^2 squares S_1 of S_0 in such

FIGURE 3



a Line $C_1^{(3)}$ *b* Line $C_2^{(3)}$

wise that the S_1 of every vertical column are traversed sequentially. Then a similar treatment of these diagonals of the ω^2 S_1 changes C_1 into C_2. And so on. –

It is convenient to term the diagonals of the C_n *links* (or n-links) and their extremities *nodes* (or n-nodes). The line C_n lies completely on the next line C_{n+1}, but the only points $T_n(t) = T_{n+1}(t)$ are the nodes and the middle points of the links of the C_n.

8. In notations analogous to those of §§ 2, 4 one has first the general relations:

$$(1) \qquad |\varphi(t_1) - \varphi(t_2)| \leqq 2 \cdot \omega^{-n}, \qquad |\psi(t_1) - \psi(t_2)| \leqq 2 \cdot \omega^{-n},$$

$$(2) \qquad |\varphi(t_1) - \varphi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}, \qquad |\psi(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n},$$

$$(3) \qquad |\varphi_{n_1}(t_1) - \varphi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}, \qquad |\psi_{n_1}(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}$$

for all integers $n_1, n_2, n\ (\geqq 0)$ and numbers $t_1, t_2\ (\geqq 0, \leqq 1)$ such that

$$n_1 \geqq n, \qquad n_2 \geqq n, \qquad |t_1 - t_2| \leqq \omega^{-2n};$$

(4) $$\varphi(t) = \mathop{\mathbf{L}}_{n=\infty} \varphi_n(t), \qquad \psi(t) = \mathop{\mathbf{L}}_{n=\infty} \psi_n(t) \qquad (0 \leqq t \leqq 1).$$

The uniform continuity of $\varphi(t)$, $\psi(t)$ for $0 \leqq t \leqq 1$ is implied by (1).

One has next the relations connected with $t = \varphi(t) = \psi(t) = 0$:

(5) $$\omega \cdot \varphi_{n+1}(t) = \varphi_n(\omega^2 t), \qquad \omega \cdot \psi_{n+1}(t) = \psi_n(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-2}),$$

(6) $$\omega \cdot \varphi(t) = \varphi(\omega^2 t), \qquad \omega \cdot \psi(t) = \psi(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-2}),$$

(7) $$\omega \cdot \varphi_n(t) = \psi_n(\omega t), \qquad \psi_{n+1}(t) = \varphi_n(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}),$$

(8) $$\omega \cdot \varphi(t) = \psi(\omega t), \qquad \psi(t) = \varphi(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}),$$

where n is any integer $\geqq 0$. The relations (7, 8) are derivable from the formulas of §§ 9, 10 and also from the geometrical considerations of II § 5.

One has further the relations connected with $t = \varphi(t) = \psi(t) = \frac{1}{2}$:

(9) $$\varphi_n(\tfrac{1}{2} + u) + \varphi_n(\tfrac{1}{2} - u) = 1, \quad \psi_n(\tfrac{1}{2} + u) + \psi_n(\tfrac{1}{2} - u) = 1 \quad (|u| \leqq \tfrac{1}{2}),$$

(10) $$\varphi(\tfrac{1}{2} + u) + \varphi(\tfrac{1}{2} - u) = 1, \quad \psi(\tfrac{1}{2} + u) + \psi(\tfrac{1}{2} - u) = 1 \quad (|u| \leqq \tfrac{1}{2}),$$

(11) $$\begin{aligned} \omega\,\{\varphi_{n+1}(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\varphi_n(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \\ \omega\,\{\psi_{n+1}(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\psi_n(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \end{aligned} \qquad (|u| \leqq \tfrac{1}{2}\omega^{-2}),$$

(12) $$\begin{aligned} \omega\,\{\varphi(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\varphi(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \\ \omega\,\{\psi(\tfrac{1}{2} + u) - \tfrac{1}{2}\} &= (-1)^{\frac{\omega-1}{2}}\{\psi(\tfrac{1}{2} + \omega^2 u) - \tfrac{1}{2}\} \end{aligned} \qquad (|u| \leqq \tfrac{1}{2}\omega^{-2}),$$

where n is any integer $\geqq 0$. These geometrically evident relations are easily derivable from the formulas of §§ 9, 10.

9. The C_n has $\omega^{2n} + 1$ n-nodes $(x_{n\,k}, y_{n\,k})$ with the n-nodal arguments $t = t_{n\,k}$, where

(13) $$t_{n\,k} = k\omega^{-2n}, \qquad x_{n\,k} = \varphi_n(t_{n\,k}), \qquad y_{n\,k} = \psi_n(t_{n\,k})$$
$$(k = 0, 1, 2, \cdots, \omega^{2n}).$$

We have

(14) $$\varphi_n(t) = \varphi(t), \qquad \psi_n(t) = \psi(t)$$

for n-nodal arguments t, and further

(15) $$x_{n\,k+1} - x_{n\,k} = \varepsilon_{n\,k}\,\omega^{-n}, \quad y_{n\,k+1} - y_{n\,k} = \eta_{n\,k}\,\omega^{-n}$$
$$(k=0,\,1,\,\cdots,\,\omega^{2n}-1),$$

where the $\varepsilon_{n\,k}$, $\eta_{n\,k}$ are signs to be determined. We write k in the scale of ω:

(16) $$k = \{a_1a_2a_3\cdots a_{2n-1}\,a_{2n}\} = \sum_{i=1}^{i=2n} a_i\omega^{2n-i},$$

where the a_i are integers $0 \leqq a_i < \omega$. For $n = 1$ we have at once

(17) $$\varepsilon_{1\,k} = (-1)^{a_2}, \quad \eta_{1\,k} = (-1)^{a_1},$$

and so, for the general n, in view of the sequential derivation of the C_{m+1} from the C_m $(m = 1,\,2,\,\cdots,\,n-1)$,

(18) $$\varepsilon_{n\,k} = (-1)^{\sum_{l=1}^{l=n} a_{2l}}, \quad \eta_{n\,k} = (-1)^{\sum_{l=1}^{l=n} a_{2l-1}}.$$

Setting for any t $(0 \leqq t \leqq 1)$

(19) $$t = t_{n\,k} + \tau_n = k\omega^{-2n} + \tau_n$$

where $0 \leqq \tau_n \leqq \omega^{-2n}$, we have

(20) $$\varphi(t) = \varphi(t_{n\,k}) + \varepsilon_{n\,k}\,\varphi(\tau_n), \quad \psi(t) = \psi(t_{n\,k}) + \eta_{n\,k}\,\psi(\tau_n),$$

(21) $$\varphi_n(t) = \varphi(t_{n\,k}) + \varepsilon_{n\,k}\,\tau_n\omega^n, \quad \psi_n(t) = \psi(t_{n\,k}) + \eta_{n\,k}\,\tau_n\omega^n.$$

10. *Arithmetic-analytic determination of Peano's curve* $C^{(\omega)}$. – For $t = 1$

(22) $$\varphi(1) = 1, \quad \psi(1) = 1.$$

The arguments t, $0 \leqq t < 1$, are written uniquely in the scale of ω,

(23) $$t = \{.\,a_1a_2\cdots a_{2n-1}a_{2n}\cdots\} = \sum_{i=1}^{i=\infty} a_i\omega^{-i},$$

where the a_i are integers $0 \leqq a_i < \omega$ and where in no case is ultimately every a_i $\omega - 1$. For the 1-nodes with $t = \{.\,a_1a_2\dot{0}\} = \{.\,a_1a_2\}$ one has directly

(24) $$\varphi(\{.\,a_1a_2\}) = x_{1\,k} = x_{1\,\{a_1a_2\}} = \left(a_1 + \tfrac{1}{2}\{1-(-1)^{a_2}\}\right)\omega^{-1},$$

(25) $$\psi(\{.\,a_1a_2\}) = y_{1\,k} = y_{1\,\{a_1a_2\}} = \tfrac{1}{2}\{1-(-1)^{a_1}\} + (-1)^{a_1}a_2\omega^{-1},$$

formulas in accord with § 9 (15, 16, 18). Setting

$$(26)\qquad \varepsilon_0(t) = \eta_0(t) = 1\,,\quad \varepsilon_n(t) = (-1)^{\sum_{l=1}^{l=n} a_{2l}}\,,\quad \eta_n(t) = (-1)^{\sum_{l=1}^{l=n} a_{2l-1}}$$

$$(n = 1,\ 2,\ 3,\ \cdots)$$

and using several formulas of §§ 8, 9, 10, one has

$$(27)\qquad \varphi(t) = \sum_{\nu=0}^{\nu=\infty} \varphi(\{\,.\,a_{2\nu+1}a_{2\nu+2}\}) \cdot \varepsilon_\nu(t) \cdot \omega^{-\nu}\,,$$

$$(0 \leqq t < 1)$$

$$(28)\qquad \psi(t) = \sum_{\nu=0}^{\nu=\infty} \psi(\{\,.\,a_{2\nu+1}a_{2\nu+2}\}) \cdot \eta_\nu(t) \cdot \omega^{-\nu}\,.$$

11. *A continuous representation of the t-ray $(t \geqq 0)$ upon the xy-quadrant* $(x \geqq 0\,,\ y \geqq 0)$. – For a simple extension of Peano's curve $C^{(\omega)}$ over the first xy-quadrant corresponding to the extension of the interval $t = 0 \cdots 1$ over the positive t-ray, wherein we take the relations (6) of § 8 as permanent, we have as definitions for $\varphi(t)\,,\ \psi(t)\ \ (t > 1)$

$$(29)\qquad \varphi(t) = \omega^e \varphi(\omega^{-2e}t)\,,\qquad \psi(t) = \omega^e \psi(\omega^{-2e}t)$$

where e is an (any) integer such that $\omega^{2e} \geqq t$. We thus have the desired single-valued continuous functions $\varphi(t)$, $\psi(t)$ for $t \geqq 0$. – The functions $\varphi_n(t)$, $\psi_n(t)$ are likewise extended. The relations (1–4, 5–8) of § 8 and (29) of § 11 hold permanently for all t, t_1, t_2, $e \geqq 0$. – One readily follows these extensions graphically. For instance, to the interval $0 \leqq t \leqq \omega^{2e}$ corresponds the square $0 \leqq x \leqq \omega^e$, $0 \leqq y \leqq \omega^e$.

12. *A continuous representation of the complete t-line upon the complete xy-plane.* – Taking the relations (12) of § 8 as permanent we have as definitions for $\varphi(\tfrac{1}{2} + u)$, $\psi(\tfrac{1}{2} + u)$ $(|u| > \tfrac{1}{2})$

$$(30)\qquad \begin{aligned} \varphi(\tfrac{1}{2} + u) - \tfrac{1}{2} &= (-1)^{\frac{(\omega-1)e}{2}} \omega^e \{\varphi(\tfrac{1}{2} + \omega^{-2e}u) - \tfrac{1}{2}\}\,, \\ \psi(\tfrac{1}{2} + u) - \tfrac{1}{2} &= (-1)^{\frac{(\omega-1)e}{2}} \omega^e \{\psi(\tfrac{1}{2} + \omega^{-2e}u) - \tfrac{1}{2}\} \end{aligned}$$

where e is an (any) integer such that $\omega^{2e} \geqq |u|$. We thus have the desired single-valued continuous functions $\varphi(t)$, $\psi(t)$ for all real values of the argument t. To the interval of length ω^{2e} and center $t = \tfrac{1}{2}$ corresponds the xy-square of side ω^e and center $(x,\ y) = (\tfrac{1}{2}\,,\ \tfrac{1}{2})$. – The functions $\varphi_n(t)$, $\psi_n(t)$ are likewise extended. The relations (1–4, 9–12) of § 8 and (30) of § 12 hold permanently for all t, t_1, t_2, e.

II

Continuous curves having no tangents.

1. The surface-filling xy-curves C

$$x = \varphi(t), \quad y = \psi(t)$$

of PEANO and of HILBERT give rise (as stated by each of them) to continuous xt-, yt-curves X, Y

$$x = \varphi(t), \quad y = \psi(t)$$

having no tangents. – We shall consider the curves $X^{(\omega)}$, $Y^{(\omega)}$ associated with Peano's curve $C^{(\omega)}$ (I §§ 7–10), on the basis however of direct geometric definitions.

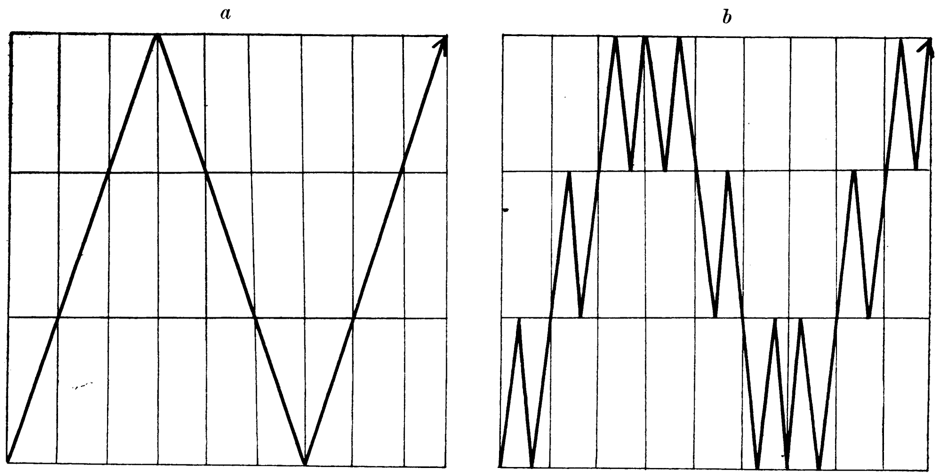
2. We have at once $Y^{(\omega)}$ defined as the point-for-point limit curve for $n = \infty$ of the sequence of (broken-line) curves $Y_n^{(\omega)}$ $(n = 0, 1, 2, \cdots)$

$$y = \psi_n(t)$$

connected with the respective curves $C_n^{(\omega)}$.

3. *Geometric determination of the curves* $Y^{(\omega)}$ (with figures for the $Y^{(3)}$): ω is an odd integer > 1. In the ty-plane we have the fundamental rectangle R_0 $(0 \leqq t \leqq 1, \ 0 \leqq y \leqq 1)$ of dimensions 1×1. We subdivide this into ω^3 rectangles R_1 of dimensions $\omega^{-2} \times \omega^{-1}$, and further into ω^6 rectangles R_2 of

FIGURE 4



Line $Y_1^{(3)}$ Line $Y_2^{(3)}$

dimensions $\omega^{-4} \times \omega^{-2}$, and in general into ω^{3n} rectangles R_n of dimensions $\omega^{-2n} \times \omega^{-n}$ $(n = 0, 1, 2, \cdots)$. A diagonal of a R_n makes with the t-axis an angle whose tangent, the *slope* of the diagonal, is $+\ \omega^n$ or $-\ \omega^n$.

The unit-segment $0 \leqq t \leqq 1$ of the t-axis is uniformly stretched and located with the designation Y_0 as the diagonal $(0, 0)(1, 1)$ of R_0. Without change of extremities Y_0 is uniformly stretched, broken, and located as $Y_1^{(\omega)}$ or Y_1 (figure 4) traversing on their diagonals certain ω^2 rectangles R_1; these ω^2 R_1 are one from every vertical column of ωR_1, and such that Y_1 preserves its direction through every node not on the boundaries $y = 0$, $y = 1$ of R_0. A similar treatment of the ω^2 links of the broken line Y_1, diagonals of the ω^2 rectangles R_1, yields the broken line Y_2 composed of ω^4 links diagonals of certain ω^4 R_2. And so on.

The point $(t, y) = (t_0, 0)$ of the t-axis by this process determines for every integer n a point $V_n = (t, y) = \big(t_0, \psi_n(t_0)\big)$. One has

$$(1) \qquad |\psi_{n_1}(t_1) - \psi_{n_2}(t_2)| \leqq 2 \cdot \omega^{-n}$$

for all integers n_1, n_2, n $(\geqq 0)$ and numbers t_1, t_2 $(\geqq 0, \leqq 1)$ such that

$$n_1 \geqq n, \qquad n_2 \geqq n, \qquad |t_1 - t_2| \leqq \omega^{-2n}.$$

The sequence $\{V_n\}$ of points V_n $(n = 0, 1, 2, \cdots)$ has for $n = \infty$ the limit-point $V = \big(t, \psi(t)\big)$,

$$(2) \qquad V = \mathop{\mathrm{L}}_{n=\infty} V_n, \qquad \psi(t) = \mathop{\mathrm{L}}_{n=\infty} \psi_n(t).$$

One verifies either geometrically or analytically the (notationally implied) relations of the curves and the functions * here introduced with those of part I. In particular, the point $V_n(t)$ of Y_n lies on an n-link of slope $\eta_n(t) \cdot \omega^n$, where, if $t = \{\cdot a_1 a_2 \cdots a_i \cdots\}$ (I § 10),

$$(3) \qquad \eta_n(t) = (-1)^{\sum_{l=1}^{l=n} a_{2l-1}}.$$

Only the node-points, $t = \{.a_1 \cdots a_{2n}\dot{0}\} = \{.a_1 \cdots a_{2n}\}$, lie on two links; of these two links the one with slope $\eta_n(t) \cdot \omega^n$ is the right link; the left link has the slope $\pm \eta_n(t) \cdot \omega^n$ according as $a_{2n} \gtreqless 0$.

4. *Geometric determination of the curve* $X^{(\omega)}$ (with figures for the $X^{(3)}$).— This determination is sufficiently indicated by the graphs of figure 5, the line

---

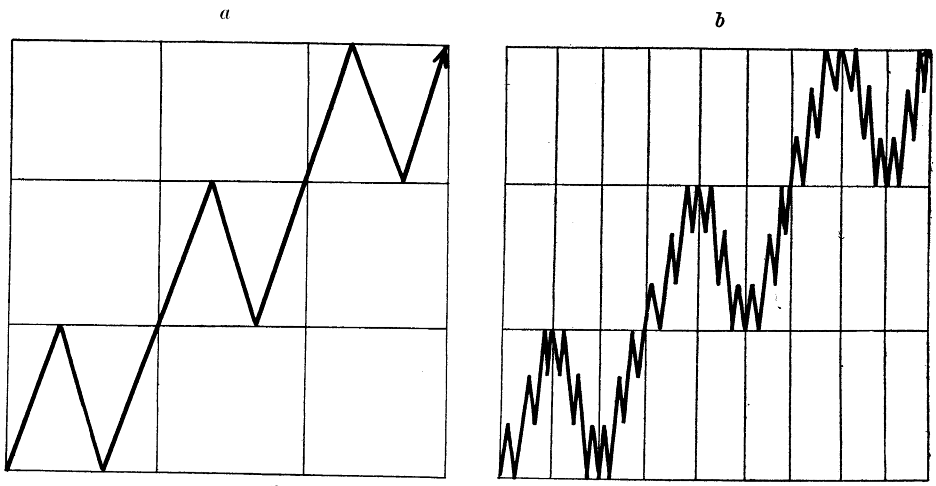
* One notes the formula

$$\psi_1(t) = (-1)^{E(\omega t)}\big(\omega t - E(\omega t)\big) + \tfrac{1}{2}\big(1 - (-1)^{E(\omega t)}\big) \qquad (0 \leqq t \leqq 1),$$

where for real arguments u $E(u)$ denotes the largest integer not greater than u.

$X_0^{(3)}$ being the diagonal of the fundamental square. The functions $\varphi_n(t)$, $\varphi(t)$ $(n = 0, 1, 2, \cdots)$ are introduced with obvious meanings.

FIGURE 5



Line $X_1^{(3)}$ Line $X_2^{(3)}$

5. We have

(4) $$\omega \cdot \varphi_{n+1}(t) = \varphi_n(\omega^2 t), \quad \omega \cdot \varphi(t) = \varphi(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-2}),$$

(5) $$\omega \cdot \psi_{n+1}(t) = \psi_n(\omega^2 t), \quad \omega \cdot \psi(t) = \psi(\omega^2 t) \qquad (0 \leqq t \leqq \omega^{-2}).$$

The curves Y_{n+1}, Y lie within the rectangle $(0, 0)$ $(\omega^{-1}, 0)$ $(\omega^{-1}, 1)$ $(0, 1)$ just as do the curves X_n, X within the rectangle $(0, 0)$ $(1, 0)$ $(1, 1)$ $(0, 1)$. Hence

(6) $$\psi_{n+1}(t) = \varphi_n(\omega t), \quad \psi(t) = \varphi(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}).$$

Similarly the curves X_n, X lie within the rectangle $(0, 0)$ $(\omega^{-1}, 0)$ $(\omega^{-1}, \omega^{-1})$ $(0, \omega^{-1})$ just as do the curves Y_n, Y within the rectangle $(0, 0)$ $(1, 0)$ $(1, 1)$ $(0, 1)$, and so

(7) $$\omega \cdot \varphi_n(t) = \psi_n(\omega t), \quad \omega \cdot \varphi(t) = \psi(\omega t) \qquad (0 \leqq t \leqq \omega^{-1}).$$

Further from considerations of central symmetry in rectangles with center at $(\frac{1}{2}, \frac{1}{2})$ one has the relations (9–12) of I § 8, of which we need (10):

(8) $$\varphi(t) + \varphi(1-t) = 1, \quad \psi(t) + \psi(1-t) = 1 \qquad (0 \leqq t \leqq 1).$$

6. *The complete curves* X, Y; *the continuous functions* $\varphi(t)$, $\psi(t)$ *of the*

unrestricted real argument t.– We have as definitions of $\varphi(t)$, $\psi(t)$ for $t > 1$, $t < 0$:

$$(9) \qquad \varphi(t) = \omega^e \varphi(\omega^{-2e} t), \quad \psi(t) = \omega^e \psi(\omega^{-2e} t) \qquad (t > 1)$$

where e is an (any) integer such that $\omega^{2e} \geqq t$,

$$(10) \qquad \varphi(t) = -\varphi(-t), \quad \psi(t) = -\psi(-t) \qquad (t < 0).$$

Similar definitions for the $\varphi_n(t)$, $\psi_n(t)$ being given, the relations 1, 2, 4, 5, 6, 7, 9, 10, hold for all values of t, t_1, t_2, e. One easily follows geometrically these extended curves as limit-curves. [These curves X, Y yield a curve C representing the complete t-line continuously upon the first and third xy-quadrants. Cf. I § 11.]

7. THEOREM.—The continuous curve Y has at no point V a tangent; the continuous function $y = \psi(t)$ has for no argument $t = t_0$ a derivative.– Instead of this* we consider the closer theorem of § 8. We desire results comparable (cf. § 11) with those of WEIERSTRASS for his continuous function without derivative.

8. THEOREM.†—The continuous curve Y with the equation

$$y = \psi(t)$$

has the following properties:

---

* The proof of this theorem is very immediate.— Setting

$$s(t, u) = \frac{\psi(t) - \psi(u)}{t - u}$$

we have for three arguments $t_1 < t < t_2$ $s(t_1, t_2)$ intermediate in value between $s(t, t_1)$ and $s(t, t_2)$. This remark, apparent geometrically or from the formula

$$s(t_1, t_2) = \frac{t - t_1}{t_2 - t_1} s(t, t_1) + \frac{t_2 - t}{t_2 - t_1} s(t, t_2),$$

due to THOMAE (DU BOIS REYMOND, Mathematische Annalen, vol. 16, p. 121), shows that if $\psi(t)$ has a derivative $\psi'(t)$ then $\psi'(t)$ is the limit necessarily existent of $s(t_1, t_2)$ on the set $t_1 < t < t_2$ for $t_1 = t$, $t_2 = t$.— In our case one finds easily in every vicinity of (every) t *nodal* values t_1', t_2' $(t_1' < t < t_2')$ such that $s(t_1', t_2')$ is numerically as large as may be wished and others t_1'', t_2'' $(t_1'' < t < t_2'')$ such that $s(t_1'', t_2'')$ is zero or of opposite sign from $s(t_1', t_2')$, so that indeed $\psi'(t)$ exists for no t.

This proof is suggested by that of STEINITZ (loc. cit., p. 65).— Indeed our function $\psi(t)$ or $\psi(t; \omega)$ is (for $0 \leqq t \leqq 1$) exactly Steinitz's function $f(t; \delta_1, \cdots, \delta_k, \cdots, \delta_{\omega^2})$ (l. c., p. 64) where $\delta_k = \delta_{(g-1)\omega+h} = (-1)^g \omega^{-1}$ $(g, h = 1, 2, \cdots, \omega)$. It falls under the class of functions $f(t; \delta_1, \cdots, \delta_m)$ recognized by STEINITZ (l. c., p. 67, § 6) as having for no t a finite derivative, but not under his class of functions having for no t a finite or definitely infinite derivative (l. c., § 7). However if in defining this latter class (l. c., p. 68) one stipulates merely that every $\Delta_{1,\iota}'$ shall be *either* of opposite sign from the corresponding $\Delta_{1,\iota}$ *or* zero one has a wider class possessing the desired property and including our functions $\psi(t; \omega)$.— The functions $f(t; \delta_1, \cdots, \delta_m)$ may easily be studied by the graphical methods of this paper.

† The parallel analytic statements as to the function $\psi(t)$ are omitted.

(α) Y has at no point V a progressive or a regressive non-vertical tangent;

(β) Y has at no point V both a progressive and a regressive vertical tangent;

(γ) Y has at and only at certain points $V = V(t)$ vertical progressive or regressive tangents:– the parameter t of such a point $V(t)$ being written in the scale of ω

$$t = \pm\,\omega^{2e} t_0, \quad t_0 \doteq \{.\, a_1 a_2 \cdots a_{2n-1} a_{2n} \cdots\} \quad \begin{pmatrix} e \text{ an (any) integer} \\ 0 \leqq t_0 < 1 \end{pmatrix},$$

there is no integer v such that for $i \geqq v$ a_i is permanently $\omega - 1$ or permanently 0, but there is an integer u such that for $j \geqq u$ either (γ_1) a_{2j-1} is permanently $\omega - 1$ and a_{2j} not permanently $\omega - 1$, or (γ_2) a_{2j-1} is permanently 0 and a_{2j} not permanently 0, and further μ_j denoting the (largest) number ($\geqq 0$) of the digits a_{2m} ($m = j+1, j+2, \cdots$) immediately succeeding a_{2j} and in value in the respective cases $\omega - 1$, 0, the relation

$$\mathop{\mathbf{L}}_{j=\infty} (j - \mu_j) = +\infty$$

holds; then, $\eta(t)$ having the meaning $\eta_u(t_0)$, *i. e.*,

$$\eta(t) = (-1)^{\sum_{l=1}^{l=u} a_{2l-1}},$$

Y has at $V(t)$ a vertical tangent of slope $\eta(t)\cdot\infty$, which is (γ_1) progressive or regressive, (γ_2) regressive or progressive, according as t is positive or negative.

(δ) The points $V(t)$ of the curve Y separate into five sets: the sets of points at which Y has (δ_1) neither a progressive nor a regressive tangent; (δ_2, δ_3) no regressive but a progressive tangent of slope $+\infty$, $-\infty$; (δ_4, δ_5) no progressive but a regressive tangent of slope $+\infty$, $-\infty$. Each of these sets is everywhere dense on the curve Y.

The property (δ) follows from (α, β, γ) immediately. At the point $V(t)$ the directed secant $\overline{V(t)\ V(t_1)}$ of the curve Y has the *slope* $s(t, t_1)$

$$s(t, t_1) = \frac{\psi(t_1) - \psi(t)}{t_1 - t}. \tag{11}$$

In case on the set $t_1 > t$ the slope $s(t, t_1)$ (of the progressive secant) has for $t_1 = t$ a limit either finite or definitely infinite ($+\infty$, $-\infty$),

$$\mathop{\mathbf{L}}_{\substack{t_1 > t \\ t_1 = t}} s(t, t_1) = \psi'_p(t), \tag{12}$$

then this limit $\psi'_p(t)$ is the slope of the progressive tangent at $V(t)$. The similar limit $\psi'_r(t)$ (if existent) of the slope of the regressive secant from $V(t)$ is the slope of the regressive tangent at $V(t)$.

The curve Y is odd, $\psi(-t) = -\psi(t)$; the regressive (progressive) secants at $V(t)$ $(t \leqq 0)$ have the same slopes as corresponding progressive (regressive) secants at $V(-t)$ $(-t \geqq 0)$.

The curve Y has the property $\psi(t) = \omega^e \psi(\omega^{-2e}t)$ (§ 6) and hence the relation

$$(13) \qquad s(t,\ t_1) = \omega^{-e}\, s(\omega^{-2e}t,\ \omega^{-2e}t_1)$$

between the slopes of the corresponding progressive or regressive secants at $V(t)$, $V(\omega^{-2e}t)$ where $t \geqq 1$, $0 \leqq \omega^{-2e}t \leqq 1$.

But further the property $\psi(t) + \psi(1-t) = 1$ yields the relation

$$(14) \qquad s(t,\ t_1) = s(1-t,\ 1-t_1) \qquad (0 < t < 1)$$

between the slopes of the regressive secants at $V(t)$ $(0 < t < 1)$ and those of the corresponding progressive secants at $V(1-t)$.

We see then easily that the theorem follows from that part of it relating to progressive tangents at points $V(t)$ $(0 \leqq t < 1)$.

9. *The proof* of the theorem* (§ 8) *as to progressive tangents to the curve* Y *at points* $V(t)$ $(0 \leqq t < 1)$. – The argument t $(0 \leqq t < 1)$ written in the scale of $\dot{\omega}$ (I § 10),

$$(15) \qquad t = \{\ .\ a_1 a_2 \cdots a_{2n-1}\, a_{2n} \cdots \},$$

is *nodal*, if ultimately every a_i is zero; otherwise it is say *ordinary*. The points $V(t)$ of the curve Y are correspondingly nodes or ordinary points.

The theorem for nodes. – The (general) node $V(t)$

$$t = \{\ .\ a_1 a_2 \cdots a_{2m-1} a_{2m}\} \qquad (a_{2m-1}a_{2m}) \neq (00)$$

appears as a node on the broken line Y_m; on $Y_n (n > m)$ the progressive link at $V(t)$, a progressive secant of Y, has the slope $s(t,\ t + \omega^{-2n}) = \eta_n(t) \cdot \omega^n$ (§ 3) while the secant-slope $s(t,\ t + 2\omega\omega^{-2n}) = 0$. Thus, *at no node has the curve* Y *a progressive tangent.*

The theorem for ordinary points. – The (general) ordinary point $V(t)$ has the argument t,

$$t = \{\ .\ a_1 a_2 \cdots a_{2m-1} a_{2m} \cdots \},$$

where there is no positive integer v such that either for every i $(i > v)$ $a_i = 0$ or for every i $(i > v)$ $a_i = \omega - 1$. We consider separately certain two supplementary classes of ordinary points.

(Class A). There is no integer u such that for every j $(j \geqq u)$ $a_{2j-1} = \omega - 1$,

* The reader is requested to have in vision the graphs suggested by the analytic phrasings of the text.

that is, there is an infinite sequence $j_1, j_2, \cdots, j_l, \cdots$ of increasing positive integers j_l such that a_{2j_l-1} has one of the values $0, 1, \cdots, \omega - 2$. – We say that a point $V(t)$ of Y corresponds to the n-link of Y_n on which its corresponding point $V_n(t)$ lies. An ordinary point corresponds thus to a definite n-link for every n. Then for every l our point $V(t)$ corresponds to one of the ω j_l-links progressively sequent to the node $V(t_l)$:

$$t_l = \{ . \, a_1 a_2 \cdots a_{2j_l-1} 0 \} .$$

These ω links are the diagonals of ω rectangles R_{j_l} and together form the diagonal of a rectangle $R^{(l)}$ of dimensions $\omega \cdot \omega^{-2j_l} \times \omega \cdot \omega^{-j_l}$; the nodal extremities of this diagonal are $V(t_l)$, $V(t_l')$; the next ω_{j_l}-links form the diagonal of a rectangle $R^{[l]}$ of the same dimensions; its nodal extremities are $V(t_l')$, $V(t_l'')$. Here we have

$$\begin{aligned} t_l' &= t_l + \omega\omega^{-2j_l}, & \psi(t_l') &= \psi(t_l) + \eta^{(l)}\omega\omega^{-j_l}, & \eta^{(l)} &= \eta_{j_l}(t_l), \\ t_l'' &= t_l' + \omega\omega^{-2j_l}, & \psi(t_l'') &= \psi(t_l), & \eta^{[l]} &= -\eta^{(l)}. \end{aligned}$$

Setting

$$t_l' - t = \theta_l \, \omega\omega^{-2j_l}, \qquad \psi(t_l') - \psi(t) = \theta_l' \eta^{(l)} \omega\omega^{-j_l},$$

where θ_l, θ_l' are certain two numbers such that

$$0 < \theta_l < 1, \qquad 0 \leqq \theta_l' \leqq 1,$$

we find for the progressive secant-slopes $s(t, t_l')$, $s(t, t_l'')$ the values

$$s(t, t_l') = \eta^{(l)} \omega^{j_l} \theta_l' / \theta_l, \qquad s(t, t_l'') = -\eta^{(l)} \omega^{j_l} (1 - \theta_l')/(1 + \theta_l).$$

These slopes to two certain points in every progressive vicinity of $V(t)$ are of opposite sign (or one may vanish); further their difference taken numerically increases indefinitely with l. Thus, *at no ordinary point of class A has the curve Y a progressive tangent.*

(Class B). There is an integer u such that for every j ($j \geqq u$) $a_{2j-1} = \omega - 1$. We have in this case (necessarily) an infinite sequence $j_1, j_2, \cdots, j_l, \cdots$ of increasing integers embracing of the integers $j > u$ all and only those for which $a_{2j} \neq \omega - 1$. For every j ($j > u$) we denote by μ_j the (largest) number ($\geqq 0$) of the digits a_{2m} ($m = j + 1, j + 2, j + 3, \cdots$) which immediately succeed a_{2j} and are each $\omega - 1$; thus $\mu_{j_l} = j_{l+1} - j_l - 1$. For the j's ($j \geqq u$) the point $V(t)$ corresponds to j-links of progressive slopes $\eta_j(t)\omega^j = \eta_u(t)\omega^j$ all of the same sign $\eta_u(t)$. By consideration of the slopes to certain points in every progressive vicinity of $V(t)$ we find on the sequence $\{\mu_j\}$ a necessary condition that the curve Y have at $V(t)$ a progressive tangent (necessarily vertical of slope $\eta_u(t)\cdot\infty$), and then we prove that this condition is sufficient.

We set and have

$$\begin{array}{ll}
\eta = \eta_u(t), & t_l = \{\,.\, a_1 a_2 \cdots a_{2j_l-1} a_{2j_l}\} \qquad (a_{2j_l} \leqq \omega - 2), \\
t_l' = t_l + \omega^{-2j_l}, & \psi(t_l') = \psi(t_l) + \eta\omega^{-j_l}, \\
t_l'' = t_l' + \omega^{-2j_l}, & \psi(t_l'') = \psi(t_l') + \eta\omega^{-j_l}, \\
t_l''' = t_l' + \omega^{-2j_l}(1-\omega^{-1}), & \psi(t_l''') = \psi(t_l'), \\
t_l' = t + \theta_l\,\omega^{-2j_l}\,\omega^{-2\mu_{j_l}-1}, & \psi(t_l') = \psi(t) + \theta_l'\eta\omega^{-j_l}\,\omega^{-\mu_{j_l}},
\end{array}$$

where θ_l, θ_l' are certain two numbers such that

$$0 < \theta_l < 1, \quad \omega^{-1} \leqq \theta_l' \leqq 1.$$

We have then

$$s(t,\, t_l'') = \eta\omega^{j_l}(1 + \theta_l'\omega^{-\mu_j}) \,/\, (1 + \theta_l\omega^{-2\mu_{j_l}-1}),$$
$$s(t,\, t_l''') = \eta\omega^{j_l-\mu_{j_l}}\,\theta_l' \,/\, (1 - \omega^{-1} + \theta_l\omega^{-2\mu_{j_l}-1}).$$

From the form of $s(t,\, t_l'')$ one sees that *if the curve Y has at the ordinary point $V(t)$ of class B a progressive tangent its slope is $\eta\cdot\infty = \eta_u(t)\cdot\infty$*, *and* from the form of $s(t,\, t_l''')$ that *the μ_j have the property*

$$\underset{j=\infty}{\mathrm{L}}(j - \mu_j) = +\infty, \tag{16}$$

since $j - \mu_j \geqq j_l - \mu_{j_l}$ for all j's for which $j_l \leqq j < j_{l+1}$.

Conversely, *the curve Y has at the ordinary point $V(t)$ of class B a progressive tangent of slope $\eta_u(t)\cdot\infty = \eta\cdot\infty$ if the infinite sequence $\{\mu_j\}$ determined by t has the property (16).* We take the general argument t' where $t < t' < t + \omega^{-2u}$ in the form

$$t' = \{\,.\, a_1' a_2' \cdots a_{2m-1}' a_{2m}' \cdots\}$$

and have for a certain integer $l(t')$ or say l ($j_l > u$, $l > 1$)

$$a_i' = a_i, \quad a_{2j_l}' > a_{2j_l} \qquad (i = 1, 2, \ldots, 2j_l - 1),$$

so that indeed in terms of the earlier notation t_l' (with respect however to this particular l)

$$t' = t'_l + \theta(\omega - a_{2j_l} - 1)\omega^{-2j_l}, \quad \psi(t') = \psi(t'_l) + \theta'\eta(\omega - a_{2j_l} - 1)\omega^{-j_l},$$

where θ, θ' are certain two numbers depending each on t and t' such that

$$0 \leqq \theta < 1, \; 0 \leqq \theta' \leqq 1 .$$

We have then

$$s(t, t') = \eta\omega^{j_l - \mu_{j_l}}\{\theta'_l + \theta'(\omega - a_{2j_l} - 1)\omega^{\mu_{j_l}}\}/\{\theta_l\omega^{-2\mu_{j_l}-1} + \theta(\omega - a_{2j_l} - 1)\},$$

and further, since $\theta'_l \geqq \omega^{-1}$, $\theta_l > 0$, $\theta \geqq 0$, $\theta' \geqq 0$, and by hypothesis

$$\mathop{\mathrm{L}}_{l=\infty}(j_l - \mu_{j_l}) = +\infty,$$

we have the desired conclusion

$$\mathop{\mathrm{L}}_{\substack{t'>t\\ t'=t}} s(t, t') = \eta \;\infty = \eta_u(t)\cdot\infty .$$

The conclusions thus reached as to progressive tangents to the curve Y at points $V(t)$ $(0 \leqq t < 1)$ are those affirmed by the theorem of § 8. Hence that theorem is now fully proved.

10. *The curve* X. – By the use of the relation $\varphi(t) = \omega^{-1}\psi(\omega t)$ (§ 5) one easily translates the theorem of § 8 concerning the curve Y into an analogous one concerning the curve X.

11. *Comparison of tangential-properties of the curve Y and the Weierstrass curve W.* – WEIERSTRASS first exhibited (Cf. Crelle's Journal, vol. 79, p. 29, 1875; *Abhandlungen aus der Functionenlehre*, p. 97) a continuous function of the real variable having no derivative, viz.

$$f(t) = \sum_{n=0}^{\infty} b^n \cos(a^n t\pi),$$

where a is an odd integer, b is a positive number less than 1, and ab is greater than $1 + \frac{3}{2}\pi$. He proved that the yt-curve W, $y = f(t)$, has in the immediate vicinity of every point secants progressive and regressive of slope numerically as large as one will and of opposite signs. The curve W has then at no point a tangent and at no point a progressive or a regressive non-vertical tangent. So far as I know it has not been determined that at no point or at what points the curve W has (1) a progressive vertical tangent, (2) a regressive vertical tangent, (3) a progressive vertical and a regressive vertical tangent of opposite slope (cusp with vertical tangent). – We have thus in these regards a closer knowledge of our curve Y, and this is true in comparison also with the other continuous tangentless curves which have been exhibited.

12. The curve Y has within every interval for every given positive number G points of arguments t_1, t, t_2 ($t_1 < t < t_2$) such that $s(t, t_1)$, $s(t, t_2)$ are of opposite sign and numerically larger than G, for example, three successive nodes of Y_n, the n and the nodes being properly chosen (§ 3). Hence,* within every interval for every constant C finite or definitely infinite a point $V(t)$ exists such that in every vicinity of $V(t)$ the slopes $s(t, t_1)$ are dense at C.

13. We inquire whether there exists a zt-curve of the equation $z = \chi(t)$, where $\chi(t)$ is a single-valued continuous function of the real variable t, such that in every vicinity of every point (z, t) of the curve both the progressive and the regressive slopes $s(t, t_1)$ are dense at $+\infty$ and at $-\infty$. *No such curve exists*, since on every interval i every continuous function $\chi(t)$ assumes for some value $t = t'$ of the interval a maximum value and then at the point (z', t') of the curve $z = \chi(t)$ the secant-slopes $s(t', t_1)$ to points (z_1, t_1) of argument t_1 of the interval i, if progressive, are all negative or zero, and, if regressive, are all positive or zero.

THE UNIVERSITY OF CHICAGO.

* By an obvious generalization of a theorem due to KÖNIG (Cf. p. 12 of the memoir, *Über stetige Functionen, die innerhalb jedes Intervalls extreme Werte besitzen*, Monatshefte für Mathematik und Physik, vol. 1, pp. 7–12, 1890).