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Research note

The evolution of the concept of homeomorphism

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Abstract

Topology, or analysis situs, has often been regarded as the study of those properties of point sets (in Euclidean space or in abstract spaces) that are invariant under "homeomorphisms." Besides the modern concept of homeomorphism, at least three other concepts were used in this context during the late 19th and early 20th centuries, and regarded (by various mathematicians) as characterizing topology: deformations, diffeomorphisms, and continuous bijections. Poincaré, in particular, characterized analysis situs in terms of deformations in 1892 but in terms of diffeomorphisms in 1895. Eventually Kuratowski showed in 1921 that in the plane there can be a continuous bijection of P onto Q, and of Q onto P, without P and Q being homeomorphic.

Zusammenfassung

Topologie—oder Analysis Situs—wurde oft als Studium solcher Eigenschaften von Punktmengen (im Euklidischen Raum oder in abstrakten Räumen) angesehen, die invariant unter Homöomorphismen sind. Außer dem modernen Konzept des Homöomorphismus wurden während des späten 19. und frühen 20. Jahrhunderts noch mindestens drei Konzepte benutzt und (von mehreren Mathematikern) als charakteristisch für die Topologie angesehen: Deformationen, Diffeomorphismen, und stetige Bijektionen. Poincaré insbesondere charakterisierte Analysis Situs durch Deformationen in 1892 aber dann durch Diffeomorphismen in 1895. Schließlich zeigte Kuratowski in 1921 dass es in der Ebene stetige Bijektionen von P auf Q und von Q auf P geben kann, ohne dass P und Q homöomorph sind.

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1. Introduction

Late in the 20th century, topology (which early in the century, as during the 19th century, had usually been called *analysis situs*) was often presented in textbooks as the study of those properties invariant under "homeomorphisms." The aim of this note is to investigate how the concept of homeomorphism arose and evolved. According to the modern definition, a homeomorphism between two topological spaces X and Y is a one–one function f from X onto Y such that f is continuous and the inverse of f is also continuous. We shall see that, at different times and by different authors, at least four distinct concepts were identified in Euclidean spaces with those mappings under which topological properties were invariant. These were the ideas of deformation, of diffeomorphism, and of one–one continuous

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mapping and the modern idea of homeomorphism. Even in Euclidean spaces, homeomorphisms are more general than deformations (which allow bending and stretching) and diffeomorphisms (where the mappings must be differentiable), while one–one continuous functions are more general than homeomorphisms. All four of these concepts proved useful as topology developed. But it took decades for mathematicians to learn to distinguish clearly between them. In the end, it was the concept of homeomorphism that served in topology in the same way that the concept of isomorphism did in algebra for groups or fields.

Lützen has argued that "in the history of mathematics and science it is often insufficient to consider how concepts are defined; one needs also to consider how they are used. This will often lead to a different and more complex story" [2003, 157]. We have kept this perspective in mind while attempting to unravel the tangled threads of the various concepts related to homeomorphism. We shall see that for certain authors it sheds additional light to look not only at the definition but at how the concept was used, whereas for other authors there remain ambiguities even after this is done.

2. Homeomorphisms and diffeomorphisms

The evolution of the concept of "homeomorphism" was essentially complete by 1935 when Pavel Aleksandrov (Paul Alexandroff) at the University of Moscow and Heinz Hopf at the Eidgenossische Technische Hochschule in Zurich published their justly famous book *Topologie*, aiming to unify the two major branches of topology, the algebraic and the set-theoretic. They took as their fundamental undefined concept "topological space," based on the closure axioms of Kazimierz Kuratowski [1922].[1] And they defined a homeomorphism between topological spaces in the way that is now standard: "A one–one continuous mapping f of a space X into a space Y is called a *topological mapping* or a *homeomorphism* (between X and $f(X) = Y' \subseteq Y$) if the inverse of f is a continuous mapping of Y' to X. Two spaces... are called *homeomorphic* if they can each be mapped topologically onto each other" [Aleksandrov and Hopf, 1935, 52].[2]

Concerning the origins of topology, Aleksandrov and Hopf wrote: "We must regard Poincaré and Cantor as the *immediate* founders of topology" [1935, 5]. So the reader might think that he should read the works of Poincaré and Cantor if he wished to find the origin of the concept of homeomorphism. However, the reader would then find that Cantor's published works contain nothing at all about homeomorphisms, and very little about continuous functions, except for his inadequate proof [1879] of the proposition:

(1) There is no one–one continuous function from a continuous manifold of dimension n to a continuous manifold of dimension m if $m < n$.

But by showing that there is a one–one mapping of n-dimensional Euclidean space onto a line segment, he had established that the number of coordinates does not determine the dimension of the space [1878]. Both E. Netto [1878] and E. Jürgens [1898] believed Cantor's mapping to show that Riemann's 1854 claim—that an n-dimensional manifold is determined by n coordinates—was mistaken. In this context Johnson [1979, 127] writes insightfully about Riemann:

> What we find conspicuously lacking in Riemann's work is the notion of a topological mapping. For modern mathematicians topology is inseparable from homeomorphisms. Riemann never contemplated these in his programme of analysis situs.

Cantor's 1878 article led to numerous attempts to prove (1). Jacob Lüroth [1878] established (1) for $m = 1$ and $m = 2$. Other less successful attempts were made to prove the general case of (1), and eventually a rigorous proof was published by Brouwer [1911].[3] But Cantor's article and those stimulated by it had the effect of making one–one

[1] They showed that Hausdorff's axioms for a topological space were equivalent to Kuratowski's, provided that one omitted Hausdorff's [1914] requirement that any two distinct points are contained in disjoint open sets [Aleksandrov and Hopf, 1935, 43].

[2] Even here it is essential to insist that the homeomorphism is between X and Y', not between X and Y. For if X is the open interval $(0, 1)$ and Y is the closed interval $[0, 1]$, there is a homeomorphism of X with a subset of Y and likewise a homeomorphism of Y with a subset of X, but there is no homeomorphism of X onto Y.

[3] A detailed historical treatment can be found in Dauben [1975] and Johnson [1979, 1981].

continuous functions important in *analysis situs*. Even Brouwer (probably influenced by Schoenflies) thought that the relevant functions for *analysis situs* were those that are one–one and continuous, for he wrote that "we shall treat in this paper an arbitrary surface, which in the sense of analysis situs is equivalent to a sphere, in other words which is a continuous one–one image of a sphere" [1909, 788]. We shall see below how this perspective played a role in the work of von Dyck, Hurwitz, and Schoenflies.

As for Poincaré, there is a particularly interesting discovery to be made. Although Poincaré coined the word "homeomorphism" in [1895], he meant by it something quite different from and more restricted than what is meant nowadays. It is not only that Poincaré worked in n-dimensional Euclidean space (rather than in an abstract space) but that his "homeomorphism" had strong requirements of differentiability and smoothness that have nothing directly to do with topology:

> Let us consider a substitution that transforms $x_1, x_2, \dots, x_n$ into $x'_1, x'_2, \dots, x'_n$, which I subject only to the following conditions:
>
> (4) $$x'_i = \phi_i(x_1, x_2, \dots, x_n) \quad (i = 1, 2, \dots, n).$$
>
> In a certain domain the functions ϕ_i are single-valued, bounded, and continuous. They have continuous derivatives and their functional determinant [Jacobian] does not become zero.
>
> ...
>
> It is clear that the set of substitutions which satisfies these conditions forms a group, and that this group is one of the most general which can be imagined. The science whose object is the study of this group and of some other analogous groups has received the name of *analysis situs*. [1895, 198]

Observe that here Poincaré does not think of *analysis situs* as having to do with the invariants of homeomorphisms in the modern sense, since homeomorphisms need not be differentiable. Then he considered two varieties V and V' such that V was a subset of some domain D:

> Let us assume that a point $x_1, x_2, \dots, x_n$ of the variety V can be made to correspond to a point $x'_1, x'_2, \dots, x'_n$ of the variety V' in such way that we have
>
> (5) $$x'_k = \psi_k(x_1, x_2, \dots, x_n) \quad (k = 1, 2, \dots, n).$$
>
> ...
>
> I assume that, *in the domain D*, the functions ψ_k are bounded, continuous, and single-valued, that they have continuous derivatives, and that their functional determinant does not become zero.
>
> Solving the equations (5), we find that
>
> (6) $$x_k = \psi'_k(x'_1, x'_2, \dots, x'_n) \quad (k = 1, 2, \dots, n).$$

He then considered the domain D' defined by inequalities corresponding to those which defined D:

> and I suppose that, *in the domain D'*, the functions ψ'_k are bounded, continuous, and single-valued, that they have continuous derivatives and that their functional determinant does not become zero.
>
> It follows from these hypotheses that to each point of V corresponds a single point of V' and inversely; to each variety W included in V corresponds a variety W' of the same number of dimensions, included in V'.... .
>
> If all these conditions are fulfilled, we will say that the varieties V and V' are equivalent from the viewpoint of *analysis situs*, or, more briefly, that they are *homeomorphic*, i.e. of the same form.
>
> I can also say that two more complicated figures, composed of some number of varieties, are homeomorphic when we can pass from one to the other by a transformation of the form (5).
>
> Thus two polygons with the same number of sides are homeomorphic; two polyhedra whose faces are equal in number, have the same number of sides, and are laid out similarly, will be homeomorphic, etc. [1895, 199]

This key passage makes it clear that for Poincaré a "homeomorphism" is not a homeomorphism in the modern sense, but rather a diffeomorphism—a much more restricted concept even for continuous manifolds. Two polygons with the same number of sides are diffeomorphic (as Poincaré in effect remarks above). By contrast, any two polygons are homeomorphic (in the modern sense) whether the number of sides is the same or different. Moreover, he repeatedly used his differentiability assumptions in his theorems later in that article [1895, 202, 219].

The editors of Poincaré's collected works took the unusual step of supplying their readers with a lexicon, giving the terminology used in his topology article and the corresponding terminology as used by French mathematicians circa 1950. In their lexicon those editors translated Poincaré's use of homeomorphism by the modern term "diffeomorphism" [Poincaré, 1953, 185].

By contrast, in the chapter "Topologie" in Dieudonné's book *Abrégé d'histoire des mathématiques*, Hirsch remarked that

> by transformation or *topological* mapping we do not mean the mappings considered in topology (which are continuous mappings) but rather the particular case of bijective and bicontinuous mappings which were later (undoubtedly for the first time by Poincaré) called *homeomorphisms*. This notion was introduced by Möbius in 1860 under the name of *elementary correlation*. . . .
>
> Taken in the broad sense, topology is the study of those properties invariant under homeomorphisms. [1978, 380]

Given the above quotation from Poincaré, we conclude that Hirsch was mistaken about what Poincaré meant by homeomorphism when he coined that term in [1895].

The story is richer yet, for there was an evolution in Poincaré's understanding of which mappings are those giving the invariants of *analysis situs*. In his earliest article on *analysis situs* in 1892, he asked "if the Betti numbers suffice to determine a closed surface from the point of view of *analysis situs*, i.e. if, given two closed surfaces which have the same Betti numbers, we can always pass from one to the other by a continuous deformation" [1892, 189–190]. Thus he appears to have used deformations as those mappings which make two varieties indistinguishable from the viewpoint of *analysis situs*, rather than diffeomorphisms as he did in 1895. But in 1900 (if we look at how he is using the term) he has altered the meaning that he gives to his term "homeomorphism," since he now says that a polygon is "homeomorphic" to a circle [1900, 351]—in contrast to what he wrote in 1895, where only polygons with the same number of sides were homeomorphic (in our terms, diffeomorphic) to each other. So in 1900 Poincaré may be using the term homeomorphism in the modern sense, something that he was clearly not doing in 1895. Nevertheless, he did not state anywhere that in [1900] he was using the term homeomorphism with a meaning different from that which he had originated in [1895].

3. Homeomorphisms and D-homeomorphisms

Since Poincaré's 1895 article is the origin of the word "homeomorphism" but not of the *concept* of homeomorphism, how did that concept originate?

At the time that Poincaré wrote, there was a second and broader concept of homeomorphism in use. This second concept was clearly stated by Walther von Dyck in an 1888 article on *analysis situs*: "Absolute properties [in *analysis situs*] can also be characterized as those, for whose agreement on two manifolds, it is necessary and sufficient that there exists a one–one continuous function [umkehrbar eindeutiger stetiger Beziehung] between all elements of the two manifolds" [1888, 457].

For the rest of the present note, a one–one continuous function from one subset of Euclidean space onto another will be called a "D-homeomorphism," after von Dyck, since a D-homeomorphism is a broader concept (both for Euclidean spaces and more generally) than a homeomorphism. But more than three decades were to pass before D-homeomorphisms were clearly distinguished from homeomorphisms. In practice, von Dyck considered curves that were given by an equation (presumably algebraic), and he explicitly applied both continuous deformations and differentiability [von Dyck, 1888, 466–467].[4]

[4] It is possible, as a referee has suggested, that von Dyck did not distinguish between two ways of reading the German phrase "umkehrbar eindeutiger stetiger Beziehung." The first way, which to me seems the most natural, is to interpret von Dyck's phrase as "(umkehrbar eindeutiger) und stetiger," i.e., as one–one and continuous. The second way is to interpret it as "umkehrbar (eindeutiger und stetiger)," i.e., as both reversibly single-valued and reversibly continuous. Whether or not von Dyck did fail to distinguish between these two readings, later mathematicians, beginning with Hurwitz [1898] and then Schoenflies [1900], construed the phrase in the first way and not the second. That it was important to distinguish between the two readings only became clear in 1924, as we shall see below.

It is a plausible conjecture that many of those who used D-homeomorphisms (something which continued to occur at least until 1924) implicitly assumed that every D-homeomorphism is a homeomorphism, i.e., that the inverse function is continuous. Such a result was necessary, in particular, if the inverse of a D-homeomorphism was to be a D-homeomorphism. And such a result would have been true if they had only considered, in n-dimensional Euclidean space, sets which are both closed and bounded. But prior to Camille Jordan's work [1893], discussed in Section 4 below, this condition was never made explicit. As we shall see when discussing Hurwitz [1898], there was a good deal of murkiness around D-homeomorphisms.

"It should be emphasized," von Dyck added in a footnote, "that we are concerned with *continuous relations between continuous manifolds*, i.e., with those relations by which neighboring elements are again sent to neighboring elements" [1888, 457]. Here he claimed that functions between manifolds of different dimension, such as that in [Cantor, 1878], are necessarily discontinuous. Ironically, Peano [1890] would show—in the same journal where von Dyck's article appeared—that von Dyck's claim was radically mistaken, since there is a continuous mapping of a one-dimensional manifold onto a two-dimensional manifold.

Von Dyck's footnote sheds some light on a still earlier notion related to that of D-homeomorphism, namely Möbius's notion of elementary relationship ("elementar Verwandtschaft"). For von Dyck's "relations by which neighboring elements are again sent to neighboring elements" is certainly a very similar idea to Möbius's "elementary relationships." In his article of 1863, Möbius wrote:

> Two geometric figures are said to be *elementarily related* to each other if each infinitesimal element (in all dimensions) of the one figure corresponds to a similar element of the other in such a way that if any two bounding elements of the one figure are contiguous, then the same is true of the corresponding elements in the other; or, what expresses the same thing, if a point of the one figure corresponds to a point of the other, then any two infinitely close points of the one figure correspond to infinitely close points of the other.
>
> In this regard, a curve can only be elementarily related to a curve, a surface only to a surface, and a solid body only to a solid body. [1863, 435]

Nothing in Möbius's article settles the question of whether he was thinking of his "elementary relationships" as homeomorphisms or as diffeomorphisms or as something different from both, such as deformations. All of his diagrams dealt with smooth two-dimensional manifolds. His use of infinitesimals suggests that he regarded his relations as differentiable. (In any case, at the time he wrote, examples of continuous nowhere differentiable functions were known only to a few mathematicians.)

The theorems stated in Möbius's article would all remain true whether he was thinking of his elementary relationships as homeomorphisms, diffeomorphisms, or deformations. A typical such theorem was that a curve could be elementarily related only to a curve, a surface to a surface, and a solid body to a solid body. Another theorem was that two surfaces in a plane are elementarily related if and only if they have the same finite number of boundary curves [1863, 439].

Möbius's article of 1863 was part of a manuscript that he had submitted to a competition held by the Paris Academy of Sciences. Those participating in this competition, for the "Grand Prix de Mathématiques" in 1861, were asked to make a major contribution to some aspect of the geometric theory of polyhedra. J.-C. Pont has pointed out that Möbius's submission contained the first reference to these elementary relationships as a kind of rubber-sheet geometry. For Möbius wrote (as quoted by Pont, 1974, 97): "If, for example, we imagine the surface of a sphere as perfectly flexible and elastic, then all the possible forms which one can give to it by bending and stretching (without tearing it) are elementarily related to each other. The surface of each Eulerian polyhedron is elementarily related to a sphere." In this context of bending and stretching, Möbius was thinking of deformations.

Nevertheless, the popular image of rubber-sheet geometry is misleading for (modern) topology. Rubber-sheet geometry is the geometry of deformations, which are more restrictive as functions than homeomorphisms. In particular, any two knots are homeomorphic, but distinct knots cannot be deformed into each other. The distinction between a deformation and a homeomorphism only became clear long after Möbius wrote. However, the concept that Möbius named an elementary relationship was more likely that of deformation than that of homeomorphism.

The first explicit reference to deformations appears to date from the same period as Möbius's article. For in 1866, in an article entitled "On the deformation of surfaces," Jordan wrote:

> One of the best known problems of geometry is the following:

> To find necessary and sufficient conditions so that two surfaces, or parts of surfaces, which are flexible but not extensible can be applied one to the other without tearing or pasting.
>
> One may propose an analogous problem by supposing that, on the contrary, the surfaces are arbitrarily extensible. [1866, 105]

Whereas Möbius solved this problem for surfaces in a plane, Jordan solved it more generally for surfaces in three dimensions. Might Jordan have read Möbius's 1861 submission to the Paris Academy of Sciences, or Möbius's 1863 article, and been influenced in this way to study deformations? In the case of Möbius's published article this is possible, but quite uncertain, since Jordan nowhere referred to any other author in his 1866 article. It is quite unlikely that Jordan saw Möbius's unpublished 1861 submission at the Academy, since Jordan only completed his doctorate that same year.

In this context we wish to examine some of Felix Klein's thoughts on *analysis situs*. In his *Erlanger Programm* of 1872, and again in its English translation of 1893, he wrote: "In the so-called analysis situs we try to find what remains unchanged under transformations resulting from a combination of infinitesimal distortions" [1893, 235]. This way of expressing the matter is close to that used earlier by Möbius, whose collected works Klein edited in 1886. In his second article on non-Euclidean geometry, Klein described the transformation group of *analysis situs* in a similar way: "Here the group consists of those space-transformations which are called deformations and which are composed from infinitesimal real space-transformations" [1873, 121]. Klein, like other authors of the 1870s, does not appear to have distinguished between deformations and homeomorphisms. Moreover, it is unclear what role differentiation played for his functions. Did it play the same important role as in [Poincaré, 1895]? Perhaps, but this is by no means certain. What is certain is that in 1908 Klein referred to *analysis situs* in terms of D-homeomorphisms, since he wrote then: "We inquire as to the properties of geometric figures which remain unchanged under these most general one–one continuous transformations.... The totality of properties which we find in the treatment of this question makes up the field that is called analysis situs" [1908, 105].

The ambiguities which permeated *analysis situs* during the late 19th and early 20th centuries can be made clearer by considering the paper which Adolf Hurwitz gave at Zurich in 1897 to the International Congress of Mathematicians. In that paper, written from the standpoint of analysis and entitled "On recent developments in the general theory of analytic functions," he wrote:

> Let the closed point-set Q be the continuous image of the closed point-set P, but at the same time let the relation between the points of these two sets be a one–one function. (Then obviously the point-set P is also a continuous image of the point-set Q.)
>
> Two point sets, which can be related in this way by a one–one continuous function, I wish to call "equivalent" and to partition point-sets, by means of this concept of equivalence, into classes. In this way two closed point-sets will be reckoned in the same class (or not) according to whether or not they are equivalent. As has often been remarked, this partition of point-sets into classes forms the most general foundation of *analysis situs*. The task of analysis situs is to seek the invariants of the individual classes of point-sets. [1898, 102]

Unfortunately, when Hurwitz claimed that "the point-set P is also obviously a continuous image of the point-set Q," he was wrong. There are simple point sets P and Q that provide a counterexample. Let P be the ray consisting of the x-axis to the right of the origin and including the origin, and let Q be the unit circle centered at the origin. Both P and Q are closed sets in the plane. Now P is homeomorphic to the half-open interval $[0, 2\pi)$, and the function $g(x) = (\sin x, \cos x)$ maps the interval $[0, 2\pi)$ one–one and continuously onto Q. Thus Q is the one–one continuous image of P. But P cannot be the continuous image of Q, since Q is compact while P is not.

Presumably Hurwitz was aware that his claim is not true for arbitrary point sets in Euclidean space, and that is why he restricted it to closed sets. Yet his claim is not correct there either. It would have been correct if he had restricted himself to point sets that are both closed and bounded (i.e., compact). At the time he wrote, however, the role of compactness in analysis had not yet been made clear. Analysts such as Weierstrass often extended the complex plane, for example, by adjoining a point at infinity, which then made that plane compact. But they usually regarded this as a mere convenience to avoid exceptions in expressing theorems, rather than a fundamental difference.

Stimulated by Hurwitz's paper, Arthur Schoenflies began in 1899 to publish a series of articles on *analysis situs*. The first of them was presented by Hilbert to the Göttingen Academy of Science in January 1900. In it Schoenflies wrote about D-homeomorphisms:

> Up to the present, the investigation of one–one continuous mappings of point sets has been involved almost exclusively with proving that certain mappings are impossible, namely those between domains of different dimensions. However, it appears to me more correct to put the matter more positively and to seek directly those properties of point sets which are preserved by the above named [one–one continuous] mappings. The problem described in this way is none other than that first posed by Herr Hurwitz in his Zurich lecture "to determine all the kinds of equivalent point-sets, to partition them into classes, and to seek the invariants of these classes."... To this problem, which has been little investigated thus far, I intend to make a contribution by proving that *the one–one continuous image of the surface of a square is again a simply connected surface*, namely that it consists of the totality of all points which are interior to some closed curve. [1900, 282]

In his later article on the relationship of set theory to geometry and to the theory of functions, Schoenflies stated that, in the plane, the following are invariant under one–one continuous mappings: limit point, closed set, perfect set, dense set, nowhere dense set, connected set, and the dimension of a set [1906, 558].[5] He continued to view *analysis situs* as concerned with the invariants under one–one continuous mappings when he published his book [1908, 149] on the development of the theory of point sets.

Intriguingly, Ludovic Zoretti (at the University of Caen) wrote in *Encyclopédie des sciences mathématiques pures et appliquées* that "for A. Schoenflies analysis situs is the study of those properties of figures (or point-sets) which are preserved by all one–one bicontinuous transformations" [1912, 167]. That is, Zoretti required that the inverse function also be continuous. Zoretti cites the same page of [Schoenflies, 1908] that we do, but interprets it differently. No doubt Zoretti was aware that Schoenflies needed to require the inverse function to be continuous, and assumed that this was what Schoenflies must have meant. By contrast, when A. Rosenthal revised Zoretti's article for the German version of the same encyclopedia, *Encyklopädie der mathematischen Wissenschaften*, he rewrote that passage, mentioning Klein and Hurwitz rather than Schoenflies:

> For F. Klein [1872] and A. Hurwitz [1898] analysis situs is the study of those properties of figures (or point-sets) which are preserved by all one–one continuous mappings.
>
> It is altogether preferable to require here invariance with respect to one–one continuous mappings whose inverses are *also* continuous. However, this distinction is only relevant for those figures which are not closed and bounded. [Rosenthal, in Zoretti and Rosenthal, 1924, 1013]

Thus Rosenthal, unlike Klein and Hurwitz, understood topological invariance in the modern sense. Moreover, he distinguished explicitly between homeomorphisms and D-homeomorphisms, and was one of the first to do so.

In 1930 in the same *Encyklopädie*, Heinrich Tietze and Leopold Vietoris devoted an article to the interrelations between the different branches of topology. There they defined a "topological mapping" or "homeomorphism" in n-dimensional Euclidean space to be a one–one continuous function whose inverse is continuous; and a property was said to be "topological" if it was preserved by topological mappings:

> Among the most important tasks of analysis situs is, for the collection of all point-sets (or within a fixed subcollection of them) to *characterize completely the individual point-sets through a suitable system of topological properties* such that agreeing for these properties is not only necessary but also sufficient for the homeomorphism of two sets. In a few cases this problem is solved.... In other cases there remains unsolved even the simpler problem of giving a procedure whereby it can be determined whether two sets in a given collection are homeomorphic or not (the homeomorphism problem). [Tietze and Vietoris, 1930, 146–147]

As it happens, the homeomorphism problem remains unsolved to this day.

There is evidence that even Felix Hausdorff, a decade before he introduced his version of topological spaces, shared the erroneous view of Hurwitz [1898] and Schoenflies that topology is about the invariants of D-homeomorphisms. For in an unpublished manuscript composed circa 1904–1905, Hausdorff wrote about transformations in two or three dimensions: "The study of these invariant properties—invariant under the group of one–one continuous point-transformations [eindeutigen stetigen Punkttransformationen] forms that branch of geometry which is called topology or analysis situs" [Hausdorff, in Epple et al., 2002, 683].

[5] In fact, however, Schoenflies was mistaken, since a one–one continuous mapping could transform a closed set in the plane (e.g., a ray) into one that was not closed (e.g., a half-open interval).

As late as 1924 some eminent algebraic topologists continued to use the older and inadequate definition of homeomorphism, i.e., of D-homeomorphism. Among those who did so was Solomon Lefschetz. In his textbook *L'analysis situs et la géométrie algébrique* Lefschetz wrote: "Two varieties are *homeomorphic* if there is a one–one continuous function between them, without exception.... The goal of *analysis situs* is to study those properties of a figure which are preserved when a figure is replaced by another homeomorphic to it" [1924, 1]. However, six years later Lefschetz adopted the modern definition of homeomorphism [1930, 3], and he retained this definition in later works [1942, 7].

In their 1934 textbook of algebraic topology, Herbert Seifert and William Threlfall (Dresden) wrote that "topology has to do with those properties of geometric figures that are unaltered by *topological mappings*, i.e., one–one functions such that they and their inverses are continuous" [1934, 1]. Thus Seifert and Threlfall accepted the modern definition of homeomorphism. From about 1930, the modern definition has been dominant both in general topology and in algebraic topology.

4. Homeomorphisms in abstract spaces

Already by 1906, spaces more abstract than n-dimensional Euclidean space had been proposed. In his doctoral dissertation, Maurice Fréchet had been the first to introduce metric spaces (though under another name) as well as his still more general L-spaces, which were based on an abstract notion of the limit of an infinite sequence of points [1906]. In this context, Fréchet carried over from analysis the concept of a continuous function defined in terms of sequences. That is, in any of his abstract spaces S, a function $f : S \to S$ was said to be continuous if, whenever a sequence of points b_n converged to the point b in the space, the sequence $f(b_n)$ converged to $f(b)$ in the space [1906, 7].

Although in 1906 Fréchet did not discuss homeomorphisms between two of his L-spaces, he did so four years later in an article in *Mathematische Annalen* on the concept of topological dimension. Given two sets E_1 and E_2, each part of an L-space, he wrote "that *they are the image of each other*, or that they are *homeomorphic*, if there exists between them a one–one correspondence which is bicontinuous [une correspondance biunivoque qui est bicontinue]" [1910, 146]. And he was quite explicit that such a function was bicontinuous if and only if it and its inverse were continuous. This was the first time that the concept of homeomorphism was formulated in a more general context than n-dimensional Euclidean space.[6]

When Hausdorff formulated the concept of a Hausdorff topological space in his 1914 book, he generalized the concept of a continuous function to such spaces. His definition was explicitly motivated by the epsilon–delta definition of continuity of a real function due to Weierstrass. Since Hausdorff's axioms for a topological space were in terms of neighborhoods, his new generalized definition of continuous function was also formulated in terms of neighborhoods:

> We take this as the general definition of continuity, whereby A and B are only assumed to be topological spaces in which the neighborhood axioms are satisfied.
>
> **Definition.** The function $y = f(x)$ is said to be continuous at the point a if to each neighborhood V_b of the point $b = f(a)$ there exists a neighborhood U_a of the point a, whose image is a subset of V_b: $f[U_a] \subseteq V_b$. [1914, 359]

He emphasized that U_a must be a subset of A and V_b a subset of B.

Hausdorff's insight was apparent when he showed that the continuity of $f : A \to B$ at a point p is equivalent to a condition stated in terms of interior points, and also equivalent to another condition in terms of limit points. Describing as "continuous" the function $f : A \to B$ between the topological spaces A and B if f is continuous at every point of A, he showed that this is equivalent to the now classic formulation: $f : A \to B$ is continuous if and only if the inverse image of any set open in B is open in A. And likewise, he proved that $f : A \to B$ is continuous if and only if the inverse image of any set closed in B is closed in A [1914, 361].

He then isolated the essential relationship between homeomorphisms and one–one continuous functions by giving the condition under which a one–one continuous function is actually a homeomorphism, although he did so without ever using the term homeomorphism: "If B is the one–one continuous image of a set A that is compact-in-itself,[7]

[6] Fréchet adopted the term "homeomorphism" from his teacher Hadamard, rather than directly from Poincaré.

[7] A set B is "compact-in-itself" if every infinite subset A of B has a limit point that belongs to B [Hausdorff, 1914, 264]. This concept originated with Fréchet.

then A is also the continuous image of B" [1914, 365]. This now standard argument relied on his earlier theorem that the continuous image of a set that is compact-in-itself is compact-in-itself, and also that a set that is compact-in-itself must be closed (since a topological space, for Hausdorff, must separate points by open sets). He did not give any counterexample for the case when A is not compact-in-itself.

It is surprising, under these circumstances, that nowhere in his 1914 book does Hausdorff define the concept of homeomorphism, and nowhere in it does he use the word "homeomorphism" or an equivalent term. Only in the second edition of the book in 1927 did he define the concept of homeomorphism. There he gave the definition in a way that suggests that he may have gotten it from Fréchet's article of 1910. Hausdorff was thinking of a set A and a set B, together with a function $\phi: A \to B$ and its inverse function $\psi: B \to A$ (he explicitly allowed both ϕ and ψ to be many-valued). Then he wrote:

> If, however, both functions $y = \phi(x)$ and $x = \psi(y)$ are single-valued and continuous, then each of them will be called *reversibly continuous* or continuous from both sides or doubly continuous (fonction bicontinue). B is also called a *homeomorphic image* of $A\ldots$, and the one–one mapping between both sets is called a *homeomorphism*.... [1927, 195–196]

The context in which Hausdorff gave this definition was that of metric spaces, since topological spaces were only discussed, very briefly, later in the book [1927, 226–232].

The French phrase "fonction bicontinue" that Hausdorff included in his original German text shows that he had in mind some French author as the source of the underlying idea that a function and its inverse are both continuous. Apparently this use of "bicontinue" originated with Fréchet [1910].

Hausdorff's 1914 results about topological spaces also shed light on the situation in n-dimensional Euclidean space. They made it clear that compactness was an important property, even in the plane, if a one–one continuous function was to be a homeomorphism, i.e., to have a continuous inverse.

Nevertheless, the key ideas were already present in Jordan's *Cours d'analyse* of 1893 for the case of n-dimensional Euclidean space, since in such a space the (later) concept of a set B being compact reduces to its being closed and bounded. By using the Bolzano–Weierstrass theorem, Jordan established that if A is a closed and bounded set and if f is a continuous one–one function from A onto B, then the inverse function of f from B to A is also continuous. In effect he showed that a continuous one–one function on a closed and bounded set in Euclidean space is a homeomorphism, although he did not use this word [1893, 53–54]. He gave no example to show that his theorem might be false if A failed to be closed or bounded.

Arboleda [1981, 346] found an undated draft in Fréchet's *Nachlass* in Paris in which Fréchet stated (but without mentioning Jordan) that if f is a continuous one–one function from a closed and bounded set A onto B, then the inverse function of f from B to A is also continuous. Moreover, going beyond Jordan, Fréchet pointed out that if A fails to be closed or fails to be bounded, then the inverse function of f is not necessarily continuous. He gave the following example in the Euclidean plane, where A, which was not closed, was the set $\{(x, 0)\colon\ 0 < x < 1\} \cup \{(1, 1)\}$ and f was such that $f(x, 0) = (1/x, 0)$ if $0 < x < 1$ and $f(1, 1) = (0, 0)$.

It is not known when such an example was first formulated and published, since Fréchet's example was undated and did not appear in print until 1981. The example given in Section 3 (above) of two closed sets P and Q in the plane and a one–one continuous function f of P onto Q that is not a homeomorphism, namely when P is a closed ray and Q is a circle, is a modification of an earlier example that is now standard in topology textbooks. This earlier example was that of a one–one continuous mapping of a half-open interval onto a circle, showing that such a mapping may fail to be a homeomorphism, and occurred in the second edition of a textbook by the British topologist M.H.A. Newman [1951, 70]. In the first edition he had proposed a different and more complicated example for the plane [1939, 58].[8]

Intriguingly, by 1921 the Polish school of topologists had already gone far beyond such an example and had also surpassed French and German authors in understanding the difference between homeomorphisms and D-homeomorphisms, even in Euclidean spaces. For in 1920 the Warsaw mathematician Waclaw Sierpinski proposed the following problem:

[8] A much simpler example than any of these was found in 2002 by my student David Zywina. He let A and B be on the Euclidean line, with $B = [0, 1]$ and $A = [0, 1) \cup \{2\}$ and with $f(x) = x$ for $0 \leqslant x < 1$ and $f(2) = 1$.

> If a set P of points is the one–one continuous image (but not necessarily the bicontinuous image) of a set Q, and if Q is the one–one continuous image of P, are the sets P and Q necessarily homeomorphic? [1920, 223]

It did not make sense even to pose this problem unless one had already understood that, in Euclidean space, a D-homeomorphism can fail to be a homeomorphism.

Kuratowski, then a young topologist who had just finished his doctorate, solved this problem in the negative in [1921] by defining, on the real line, subsets P and Q that satisfied Sierpinski's hypotheses but that were not homeomorphic. Moreover, Kuratowski gave a different example in the plane in order to show that P and Q need not be homeomorphic even if P and Q are required to be closed and connected (i.e., to be "continua" in the terminology of the time).

Both Sierpinski and Kuratowski wrote important topology textbooks, Sierpinski in Polish [1928], which was then translated into English [1934], and Kuratowski in French [1933]. In both of these books, homeomorphisms were given the modern meaning. Thus by the early 1930s, homeomorphisms in the modern sense were the standard within English, French, German, and Polish textbooks of topology.

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