THE RECIPROCITY FORMULA FOR DEDEKIND SUMS.*

By L. J. Mordell.

Let p, q be two positive integers without a common divisor. It is well known that

$$(1) \qquad \sum_{x=1}^{p-1} [qx/p] + \sum_{y=1}^{q-1} [py/q] = (p-1)(q-1),$$

where $[z]$ denotes the integer part of z. There exists another result of this type discovered by Dedekind in discussing the linear transformation of the modular function $\log \eta(\omega)$, one form of which is

$$(2) \qquad q\sum_{x=1}^{p-1} x[qx/p] + p\sum_{y=1}^{q-1} y[py/q] = \tfrac{1}{12}(p-1)(q-1)(8pq-p-q-1).$$

Rademacher [2] made a detailed study of this result and has published some five proofs. One is a joint proof with Whiteman, and the last has only just appeared. Some of them are arithmetical in character and quite simple. Another proof has just been given by Rédei [3], and a generalization by Apostol [1]. I notice, however, an entirely different way of considering the subject which is no less simple and relates the result to more general and obvious ones.

Let us consider the sum

$$(3) \qquad S = \sum_{K} (qx + py)$$

extended over the integer sets (x, y) or say the lattice points P lying in the region K defined by

$$0 < x < p, \quad 0 < y < q, \quad qx + py < pq,$$

and so if O, A, B are the points $(0,0)$, $(p,0)$, $(0,q)$ respectively, K is the open triangle OAB. We call K' the open triangle ACB where C is the point (p,q). We have a 1-1 correspondence between the lattice points $P(x,y)$ in K and $P'(x',y')$ in K' given by

$$x + x' = p, \qquad y + y' = q.$$

Since K and K' together contain $(p-1)(q-1)$ lattice points, K contains exactly $\frac{1}{2}(p-1)(q-1)$ lattice points.

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It is well known and obvious that the formula (1) states that the number of lattice points in $K+K'$ is the sum of those in the open triangles OAC and OBC.

In (3), we note that

$$\sum_K x = \sum_{x=1}^{p-1} x[q-qx/p] = \sum_{x=1}^{p-1}(p-x)[qx/p]$$

since $[q-qx/p]$ values of y correspond to given x. Hence

$$\begin{aligned} S &= q\sum_{x=1}^{p-1}(p-x)[qx/p] + p\sum_{y=1}^{q-1}(q-y)[py/q] \\ &= pq\Big(\sum_{x=1}^{p-1}[qx/p] + \sum_{y=1}^{q-1}[py/q]\Big) - \sum_{x=1}^{p-1} qx[qx/p] - \sum_{y=1}^{q-1} py[py/q]. \end{aligned}$$

Then from (1) and (2), we have

$$S = pq(p-1)(q-1) - \tfrac{2}{3}pq(p-1)(q-1) + \tfrac{1}{12}(p-1)(q-1)(p+q+1),$$

and so

$$(4) \qquad S = \tfrac{1}{3}pq(p-1)(q-1) + \tfrac{1}{12}(p-1)(q-1)(p+q+1).$$

Thus the proof of (2) is reduced to the evaluation of the sum (3) as given in (4).

We solve now the more general problem of evaluating

$$(5) \qquad T = \sum_K f(qx+py)$$

where f is an arbitrary polynomial and the summation is extended over the lattice points in K.

Write $\xi = qx+py$ for lattice points x, y in K so that ξ is not divisible by p or q and $0 < \xi < pq$. Then the numbers $pq-\xi$ cannot be represented in this way. For

$$\text{if } pq-\xi = qX+pY, \text{ then } pq = q(x+X)+p(y+Y),$$

and so $x+X \equiv 0 \pmod{p}$. Then $x+X=p$ and similarly $y+Y=q$, and this is clearly impossible. The number of ξ is $\frac{1}{2}(p-1)(q-1)$ and so the ξ and $pq-\xi$ are $(p-1)(q-1)$ in number, and so together they are precisely the integers X not divisible by p or q in the interval $0<X<pq$.

There is, however, such a representation for the numbers $2pq-\xi$. Write $\xi' = qx'+py'$ for lattice points x', y' in K' so that $pq < \xi' < 2pq$. We have now a 1-1 correspondence between the representation of ξ and ξ' given by

$$x'+x=p, \qquad y'+y=q, \qquad \xi'+\xi = 2pq.$$

We prove two fundamental formulae. The first is

$$(7)\qquad \sum_{y=1}^{q-1}\sum_{x=1}^{p-1} f(qx+py) = \sum_K f(\xi) + \sum_K f(2pq-\xi).$$

The left hand side consists of two parts corresponding to $qx+py<pq$, and to $pq<qx+py<2pq$. The first is $\sum_K f(\xi)$, and on putting $x=p-x'$, $y=q-y'$, the second is $\sum_K f(2pq-\xi)$.

We prove next that

$$(8)\qquad \sum_{X=1}^{pq-1} f(X) - \sum_{X=1}^{p-1} f(qX) - \sum_{X=1}^{q-1} f(pX) = \sum_K f(\xi) + \sum_K f(pq-\xi).$$

The left hand side of this is $\sum f(Y)$ extended over the numbers Y in $0<Y<pq-1$ and not divisible by p or q. These numbers Y can be written as ξ or $pq-\xi$ since the numbers Y in $0<Y<pq-1$ which cannot be represented by ξ are given by $pq-\xi$. This gives (8).

The two equations (7), (8) determine $\sum_K f(\xi)$ for any polynomial $f(\xi)$. Take $f(\xi)=\xi^2$, then (7) becomes

$$\sum_{y=1}^{q-1}\sum_{x=1}^{p-1}(qx+py)^2 = \sum_K \xi^2 + \sum_K (2pq-\xi)^2$$

and so

$$\begin{aligned}\tfrac{1}{6}q^2(q-1)p(p-1)(2p-1) + \tfrac{1}{6}p^2(p-1)q(q-1)(2q-1)\\ + \tfrac{1}{2}p^2q^2(p-1)(q-1)\\ = 2\sum_K \xi^2 - 4pq\sum_K \xi + 2p^2q^2(p-1)(q-1),\end{aligned}$$

since K contains $\tfrac{1}{2}(p-1)(q-1)$ lattice points.

Hence

$$\begin{aligned}(9)\qquad \tfrac{1}{6}q(q-1)(p-1)(2p-1) + \tfrac{1}{6}p(p-1)(q-1)(2q-1)\\ - \tfrac{3}{2}pq(p-1)(q-1) = 2/(pq)\sum_K \xi^2 - 4\sum_K \xi.\end{aligned}$$

Next (8) becomes

$$\sum_{X=1}^{pq-1} X^2 - q^2\sum_{X=1}^{p-1} X^2 - p^2\sum_{X=1}^{q-1} X^2 = \sum_K \xi^2 + \sum_K (pq-\xi)^2$$

or

$$\begin{aligned}\tfrac{1}{6}pq(pq-1)(2pq-1) - \tfrac{1}{6}q^2p(p-1)(2p-1)\\ - \tfrac{1}{6}p^2q(q-1)(2q-1)\\ = 2\sum_K \xi^2 - 2pq\sum_K \xi + \tfrac{1}{2}p^2q^2(p-1)(q-1).\end{aligned}$$

This becomes

$$(10)\quad \tfrac{1}{6}(pq-1)(2pq-1)-\tfrac{1}{6}q(p-1)(2p-1)-\tfrac{1}{6}p(q-1)(2q-1)$$
$$-\tfrac{1}{2}pq(p-1)(q-1)$$
$$=2/(pq)\sum_K \xi^2-2\sum_K \xi.$$

These two equations (9), (10) determine $\sum_K \xi$, $\sum_K \xi^2$ and give the result (4). Thus (9) is

$$(11)\quad \tfrac{1}{6}(p-1)(q-1)(-5pq-p-q)=2/(pq)\sum_K \xi^2-4\sum_K \xi,$$

and (10) is

$$(12)\quad \tfrac{1}{6}(p-1)(q-1)(-pq+1)=2/(pq)\sum_K \xi^2-2\sum_K \xi.$$

In fact the left hand side of (10) vanishes when $p=1$ or $q=1$, so we can write it as $\tfrac{1}{6}(p-1)(q-1)(apq+b(p+q)+c)$, where a, b, c are constants.

Equating terms in p^2q^2, $p+q$, 1, clearly

$$\frac{a}{6}=\frac{2}{6}-\frac{1}{2},\ \frac{c}{6}=\frac{1}{6},\ \frac{b}{6}-\frac{c}{6}=-\frac{1}{6},$$

and so $a=-1$, $c=1$, $b=0$.

Hence from (11), (12)

$$(13)\quad 2\sum_K \xi=\tfrac{1}{6}(p-1)(q-1)(4pq+p+q+1)$$

$$(14)\quad 2/(pq)\sum_K \xi^2=\tfrac{1}{6}(p-1)(q-1)(3pq+p+q+2).$$

The result (13) is the required result (4).

It is also clear that on taking $f(\xi)=\xi^{2n}$ in (7), (8), the equations determine $\sum_K \xi^{2n}$, $\sum_K \xi^{2n-1}$ when we know the values of $\sum_K \xi^r$, $0\leqq r\leqq 2n-2$.

If, however, in (7) we replace the function $f(\xi)$ by $f(\xi-pq)$ and subtract from (8), we have

$$(15)\quad \sum_K f(\xi)-\sum_K f(\xi-pq)$$
$$=\sum_{X=1}^{pq-1} f(X)-\sum_{X=1}^{q-1} f(pX)-\sum_{X=1}^{p-1} f(qX)-\sum_{y=1}^{q-1}\sum_{x=1}^{p-1} f(qx+py-pq).$$

Write in the usual notation for the Bernouillian polynomial $B_n(x)$,

$$te^{tx}/(e^t-1)=\sum_{n=0}^{\infty}B_n(x)t^n/n!, \tag{16}$$

so that

$$te^{tx}=\sum_{n=0}^{\infty}(B_n(x+1)-B_n(x))t^n/n!, \qquad B_n(x+1)-B_n(x)=nx^{n-1}.$$

If we take $f(X)=B_n(1+X/(pq))$ in (15), we have the explicit formula for $\sum_K \xi^{n-1}$. This takes the shape

$$n\sum_K \xi^{n-1}=B_n(pq)-p^{n-1}B_n(q)-q^{n-1}B_n(p)+(pq)^{n-1}B_n(1)$$
$$-\sum_{y=1}^{q-1}\sum_{x=1}^{p-1}B_n(x/p+y/q),$$

on noting that

$$\sum_{r=0}^{p-1}B_n(x+r/p)=B_n(px)/p^{n-1}.$$

(as remarked to me by Professor Rademacher).

We can also find an explicit formula for

$$p^{n+1}q^{n+1}\sum_K B_n(x/p+y/q)$$

as a polynomial in p and q. One of a different type has been given by Apostol for odd n by using in a different way lattice points in a triangle.

As well known, $B_n(X)=(-1)^nB_n(1-X)$, from (16). Hence when n is even, we have at once from (8),

$$2\sum_K B_n(x/p+y/q)=\sum_{X=1}^{pq-1}B_n(X/pq)-\sum_{X=1}^{p-1}B_n(X/p)-\sum_{X=1}^{q-1}B_n(X/q).$$

To find the result for all n, write (15) as

$$\sum_K f(\xi/pq)-\sum_K f(\xi/pq-1) \tag{17}$$
$$=\sum_{X=1}^{pq-1}f(X/pq)-\sum_{X=1}^{p-1}f(X/p)-\sum_{X=1}^{q-1}f(X/q)-\sum_{Y=1}^{q-1}\sum_{X=1}^{p-1}f(X/p+Y/q-1).$$

Write $B_n(X)=b_nX^n+b_{n-1}X^{n-1}+\cdots+b_0$. Take

$$f(X-1)=(b_n/\{n+1\})B_{n+1}(X)+\cdots+(b_0/1)B_1(X).$$

Then

$$f(\xi/pq)-f(\xi/pq-1)=b_n/\{n+1\}(B_{n+1}(\xi/pq+1)-B_{n+1}(\xi/pq))$$
$$+\cdots=b_n(\xi/pq)^n+b_{n-1}(\xi/pq)^{n-1}+\cdots=B_n(\xi/pq).$$

Since we can put $\xi = qx + py$, the result is given by (17) on summing for X etc.

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