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THE RECIPROCITY FORMULA FOR DEDEKIND SUMS.*

By L. J. MORDELL.

Let p, q be two positive integers without a common divisor. It is well known that

$$(1) \quad \sum_{x=1}^{p-1} [qx/p] + \sum_{y=1}^{q-1} [py/q] = (p-1)(q-1),$$

where $[z]$ denotes the integer part of z . There exists another result of this type discovered by Dedekind in discussing the linear transformation of the modular function $\log \eta(\omega)$, one form of which is

$$(2) \quad q \sum_{x=1}^{p-1} x[qx/p] + p \sum_{y=1}^{q-1} y[py/q] = \frac{1}{12}(p-1)(q-1)(8pq - p - q - 1).$$

Rademacher [2] made a detailed study of this result and has published some five proofs. One is a joint proof with Whiteman, and the last has only just appeared. Some of them are arithmetical in character and quite simple. Another proof has just been given by Rédei [3], and a generalization by Apostol [1]. I notice, however, an entirely different way of considering the subject which is no less simple and relates the result to more general and obvious ones.

Let us consider the sum

$$(3) \quad S = \sum_K (qx + py)$$

extended over the integer sets (x, y) or say the lattice points P lying in the region K defined by

$$0 < x < p, \quad 0 < y < q, \quad qx + py < pq,$$

and so if O, A, B are the points $(0, 0), (p, 0), (0, q)$ respectively, K is the open triangle OAB . We call K' the open triangle ACB where C is the point (p, q) . We have a 1-1 correspondence between the lattice points $P(x, y)$ in K and $P'(x', y')$ in K' given by

$$x + x' = p, \quad y + y' = q.$$

Since K and K' together contain $(p-1)(q-1)$ lattice points, K contains exactly $\frac{1}{2}(p-1)(q-1)$ lattice points.

* Received December 5, 1950.

It is well known and obvious that the formula (1) states that the number of lattice points in $K + K'$ is the sum of those in the open triangles OAC and OBC .

In (3), we note that

$$\sum_K x = \sum_{x=1}^{p-1} x[q - qx/p] = \sum_{x=1}^{p-1} (p-x)[qx/p]$$

since $[q - qx/p]$ values of y correspond to given x . Hence

$$\begin{aligned} S &= q \sum_{x=1}^{p-1} (p-x)[qx/p] + p \sum_{y=1}^{q-1} (q-y)[py/q] \\ &= pq \left(\sum_{x=1}^{p-1} [qx/p] + \sum_{y=1}^{q-1} [py/q] \right) - \sum_{x=1}^{p-1} qx[qx/p] - \sum_{y=1}^{q-1} py[py/q]. \end{aligned}$$

Then from (1) and (2), we have

$$S = pq(p-1)(q-1) - \frac{2}{3}pq(p-1)(q-1) + \frac{1}{12}(p-1)(q-1)(p+q+1),$$

and so

$$(4) \quad S = \frac{1}{3}pq(p-1)(q-1) + \frac{1}{12}(p-1)(q-1)(p+q+1).$$

Thus the proof of (2) is reduced to the evaluation of the sum (3) as given in (4).

We solve now the more general problem of evaluating

$$(5) \quad T = \sum_K f(qx + py)$$

where f is an arbitrary polynomial and the summation is extended over the lattice points in K .

Write $\xi = qx + py$ for lattice points x, y in K so that ξ is not divisible by p or q and $0 < \xi < pq$. Then the numbers $pq - \xi$ cannot be represented in this way. For

$$\text{if } pq - \xi = qX + pY, \text{ then } pq = q(x + X) + p(y + Y),$$

and so $x + X \equiv 0 \pmod{p}$. Then $x + X = p$ and similarly $y + Y = q$, and this is clearly impossible. The number of ξ is $\frac{1}{2}(p-1)(q-1)$ and so the ξ and $pq - \xi$ are $(p-1)(q-1)$ in number, and so together they are precisely the integers X not divisible by p or q in the interval $0 < X < pq$.

There is, however, such a representation for the numbers $2pq - \xi$. Write $\xi' = qx' + py'$ for lattice points x', y' in K' so that $pq < \xi' < 2pq$. We have now a 1-1 correspondence between the representation of ξ and ξ' given by

$$x' + x = p, \quad y' + y = q, \quad \xi' + \xi = 2pq.$$

We prove two fundamental formulae. The first is

$$(7) \quad \sum_{y=1}^{q-1} \sum_{x=1}^{p-1} f(qx + py) = \sum_K f(\xi) + \sum_K f(2pq - \xi).$$

The left hand side consists of two parts corresponding to $qx + py < pq$, and to $pq < qx + py < 2pq$. The first is $\sum_K f(\xi)$, and on putting $x = p - x'$, $y = q - y'$, the second is $\sum_K f(2pq - \xi)$.

We prove next that

$$(8) \quad \sum_{X=1}^{pq-1} f(X) - \sum_{X=1}^{p-1} f(qX) - \sum_{X=1}^{q-1} f(pX) = \sum_K f(\xi) + \sum_K f(pq - \xi).$$

The left hand side of this is $\sum f(Y)$ extended over the numbers Y in $0 < Y < pq - 1$ and not divisible by p or q . These numbers Y can be written as ξ or $pq - \xi$ since the numbers Y in $0 < Y < pq - 1$ which cannot be represented by ξ are given by $pq - \xi$. This gives (8).

The two equations (7), (8) determine $\sum_K f(\xi)$ for any polynomial $f(\xi)$. Take $f(\xi) = \xi^2$, then (7) becomes

$$\sum_{y=1}^{q-1} \sum_{x=1}^{p-1} (qx + py)^2 = \sum_K \xi^2 + \sum_K (2pq - \xi)^2$$

and so

$$\begin{aligned} & \frac{1}{6}q^2(q-1)p(p-1)(2p-1) + \frac{1}{6}p^2(p-1)q(q-1)(2q-1) \\ & \quad + \frac{1}{2}p^2q^2(p-1)(q-1) \\ & = 2 \sum_K \xi^2 - 4pq \sum_K \xi + 2p^2q^2(p-1)(q-1), \end{aligned}$$

since K contains $\frac{1}{2}(p-1)(q-1)$ lattice points.

Hence

$$(9) \quad \begin{aligned} & \frac{1}{6}q(q-1)(p-1)(2p-1) + \frac{1}{6}p(p-1)(q-1)(2q-1) \\ & \quad - \frac{3}{2}pq(p-1)(q-1) = 2/(pq) \sum_K \xi^2 - 4 \sum_K \xi. \end{aligned}$$

Next (8) becomes

$$\sum_{X=1}^{pq-1} X^2 - q^2 \sum_{X=1}^{p-1} X^2 - p^2 \sum_{X=1}^{q-1} X^2 = \sum_K \xi^2 + \sum_K (pq - \xi)^2$$

or

$$\begin{aligned} & \frac{1}{6}pq(pq-1)(2pq-1) - \frac{1}{6}q^2p(p-1)(2p-1) \\ & \quad - \frac{1}{6}p^2q(q-1)(2q-1) \\ & = 2 \sum_K \xi^2 - 2pq \sum_K \xi + \frac{1}{2}p^2q^2(p-1)(q-1). \end{aligned}$$

This becomes

$$\begin{aligned}
 (10) \quad & \frac{1}{6}(pq-1)(2pq-1) - \frac{1}{6}q(p-1)(2p-1) - \frac{1}{6}p(q-1)(2q-1) \\
 & - \frac{1}{2}pq(p-1)(q-1) \\
 & = 2/(pq) \sum_K \xi^2 - 2 \sum_K \xi.
 \end{aligned}$$

These two equations (9), (10) determine $\sum_K \xi$, $\sum_K \xi^2$ and give the result (4). Thus (9) is

$$(11) \quad \frac{1}{6}(p-1)(q-1)(-5pq-p-q) = 2/(pq) \sum_K \xi^2 - 4 \sum_K \xi,$$

and (10) is

$$(12) \quad \frac{1}{6}(p-1)(q-1)(-pq+1) = 2/(pq) \sum_K \xi^2 - 2 \sum_K \xi.$$

In fact the left hand side of (10) vanishes when $p=1$ or $q=1$, so we can write it as $\frac{1}{6}(p-1)(q-1)(apq+b(p+q)+c)$, where a, b, c are constants.

Equating terms in $p^2q^2, p+q, 1$, clearly

$$\frac{a}{6} = \frac{2}{6} - \frac{1}{2}, \quad \frac{c}{6} = \frac{1}{6}, \quad \frac{b}{6} - \frac{c}{6} = -\frac{1}{6},$$

and so $a=-1, c=1, b=0$.

Hence from (11), (12)

$$(13) \quad 2 \sum_K \xi = \frac{1}{6}(p-1)(q-1)(4pq+p+q+1)$$

$$(14) \quad 2/(pq) \sum_K \xi^2 = \frac{1}{6}(p-1)(q-1)(3pq+p+q+2).$$

The result (13) is the required result (4).

It is also clear that on taking $f(\xi) = \xi^{2n}$ in (7), (8), the equations determine $\sum_K \xi^{2n}, \sum_K \xi^{2n-1}$ when we know the values of $\sum_K \xi^r, 0 \leq r \leq 2n-2$.

If, however, in (7) we replace the function $f(\xi)$ by $f(\xi-pq)$ and subtract from (8), we have

$$\begin{aligned}
 (15) \quad & \sum_K f(\xi) - \sum_K f(\xi-pq) \\
 & = \sum_{X=1}^{pq-1} f(X) - \sum_{X=1}^{q-1} f(pX) - \sum_{X=1}^{p-1} f(qX) - \sum_{y=1}^{q-1} \sum_{x=1}^{p-1} f(qx+py-pq).
 \end{aligned}$$

Write in the usual notation for the Bernouillian polynomial $B_n(x)$,

$$(16) \quad te^{tx}/(e^t - 1) = \sum_{n=0}^{\infty} B_n(x) t^n/n!,$$

so that

$$te^{tx} = \sum_{n=0}^{\infty} (B_n(x+1) - B_n(x)) t^n/n!, \quad B_n(x+1) - B_n(x) = nx^{n-1}.$$

If we take $f(X) = B_n(1 + X/(pq))$ in (15), we have the explicit formula for $\sum_K \xi^{n-1}$. This takes the shape

$$\begin{aligned} n \sum_K \xi^{n-1} &= B_n(pq) - p^{n-1}B_n(q) - q^{n-1}B_n(p) + (pq)^{n-1}B_n(1) \\ &\quad - \sum_{y=1}^{q-1} \sum_{x=1}^{p-1} B_n(x/p + y/q), \end{aligned}$$

on noting that

$$\sum_{r=0}^{p-1} B_n(x + r/p) = B_n(px)/p^{n-1}.$$

(as remarked to me by Professor Rademacher).

We can also find an explicit formula for

$$p^{n+1}q^{n+1} \sum_K B_n(x/p + y/q)$$

as a polynomial in p and q . One of a different type has been given by Apostol for odd n by using in a different way lattice points in a triangle.

As well known, $B_n(X) = (-1)^n B_n(1 - X)$, from (16). Hence when n is even, we have at once from (8),

$$2 \sum_K B_n(x/p + y/q) = \sum_{X=1}^{pq-1} B_n(X/pq) - \sum_{X=1}^{p-1} B_n(X/p) - \sum_{X=1}^{q-1} B_n(X/q).$$

To find the result for all n , write (15) as

$$\begin{aligned} (17) \quad & \sum_K f(\xi/pq) - \sum_K f(\xi/pq - 1) \\ &= \sum_{X=1}^{pq-1} f(X/pq) - \sum_{X=1}^{p-1} f(X/p) - \sum_{X=1}^{q-1} f(X/q) - \sum_{Y=1}^{q-1} \sum_{X=1}^{p-1} f(X/p + Y/q - 1). \end{aligned}$$

Write $B_n(X) = b_n X^n + b_{n-1} X^{n-1} + \dots + b_0$. Take

$$f(X-1) = (b_n/\{n+1\})B_{n+1}(X) + \dots + (b_0/1)B_1(X).$$

Then

$$\begin{aligned} f(\xi/pq) - f(\xi/pq - 1) &= b_n/\{n+1\} (B_{n+1}(\xi/pq + 1) - B_{n+1}(\xi/pq)) \\ &\quad + \dots = b_n(\xi/pq)^n + b_{n-1}(\xi/pq)^{n-1} + \dots = B_n(\xi/pq). \end{aligned}$$

Since we can put $\xi = qx + py$, the result is given by (17) on summing for X etc.

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