THEOREM 5. *If $k = q-1$, or $k = (q-1)/2$, then $f_k(z)$ has no zeros inside the unit circle. If $k = 0$, then $f_0(z)$ has the algebraic zero $z = 0$, and all its possible other zeros are transcendental. In all other cases, the zeros of $f_k(z)$ are algebraic n mbers, and there are an infinity of them inside the unit circle.*

In a similar way, the generating function of integers with more than one missing digit, or with a missing sequence of digits can be investigated.

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LATTICE POINTS IN A TETRAHEDRON AND GENERALIZED DEDEKIND SUMS

By L. J. MORDELL

Let p, q be two positive integers prime to each other. One form of the reciprocity formula for the so-called Dedekind sums is given by the

THEOREM

$$\sum_{x=1}^{p} \frac{x}{p}\left(\left(\frac{qx}{p}\right)\right) + \sum_{x=1}^{q} \frac{x}{q}\left(\left(\frac{px}{q}\right)\right) = \frac{1}{12}\left(\frac{p}{q}+\frac{q}{p}+\frac{1}{pq}-3\right). \quad (1)$$

Here $((X)) = X-[X]-\frac{1}{2}$, X not an integer,
$= 0$ X an integer.

Various proofs have been given by Rademacher, Rademacher and Whiteman, Re'dei and myself. For references see[1]. I have shown that the theorem is the particular case $f(x) = x$ in the evaluation of

$$\sum f\left(\frac{x}{p}+\frac{y}{q}\right), \quad (2)$$

where f is a polynomial and the summation is extended over those integer sets (x, y), i.e. lattice points, lying in the triangle

$$0 < x < p,\ 0 < y < q,\ \frac{x}{p}+\frac{y}{q} < 1. \quad (3)$$

This method suggests the extension of the formula (1) to a set of n positive integers $p, q, r, s, \ldots$ no two of which have a common factor. The results, however, now take a different form. Take $n = 3$ and write

$$S_3(p, q, r) = \sum_{x=1}^{p-1} \frac{x}{p}\left(\left(\frac{qrx}{p}\right)\right) + \sum_{x=1}^{q-1} \frac{x}{q}\left(\left(\frac{rpx}{q}\right)\right)$$
$$+ \sum_{x=1}^{r-1} \frac{x}{r}\left(\left(\frac{pqx}{r}\right)\right). \quad (4)$$

Denote by $N_3(p, q, r)$ the number of lattice points in the tetrahedron

$$0 \leqslant x < p,\ 0 \leqslant y < q,\ 0 \leqslant z < r,\ 0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 1. \quad (5)$$

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Write for shortness S_3, N_3. Then we have the

THEOREM.

$$S_3+N_3 = \tfrac{1}{6}pqr+\tfrac{1}{4}\sum qr+\tfrac{1}{4}\sum p+\frac{1}{12}\sum\frac{qr}{p}+\frac{1}{12\,pqr}-2, \qquad (6)$$

the summation referring to p, q, r.

A formula will also be found for $n=4$.

1. The method of proof shows that for $n>4$, the formula for S_n will depend upon the number of lattice points in sections of an n-dimensional tetrahedron defined by

$$\lambda < \Sigma x/p < \lambda+1$$

for a number of values of $\lambda \leqslant n-1$.

The function $((X))$ has some well-known properties. It is an odd periodic function of X, and so

$$((-X)) = -((X)),\ ((X+1)) = ((X)).$$

Also

$$((X))+((X+1/p))+\ldots+((X+(p-1)/p)) = ((pX)). \qquad (7)$$

2. We require a formula for N_3. One is given by

$$2N_3 = \sum_{x,y,z}{}' \left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right]-1\right)\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right]-2\right), \qquad (8)$$

where the summation is taken over

$$0 \leqslant x < p,\ 0 \leqslant y < q,\ 0 \leqslant z < r, \qquad (9)$$

and the accent denotes the omission of the term $x=y=z=0$.

For from (9), $0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 3$, and those lattice points for which

$$0 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 1,$$

contribute each 2 to the sum, and each of those for which

$$1 < \frac{x}{p}+\frac{y}{q}+\frac{z}{r} < 3$$

contribute zero. There are no lattice points with

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 1 \text{ or } 2.$$

We write (8) as

$$2N_3 = \sum_{x,y,z}{}' \left(E-3/2-((E))\right)\left(E-5/2-((E))\right), \qquad (10)$$

where $E = \frac{x}{p}+\frac{y}{q}+\frac{z}{r}$. Hence

$$2N_3 = A+B+C,$$

say, where

$$A = \sum_{x,y,z}{}' (E-3/2)(E-5/2),\quad B = -2\sum_{x,y,z}{}' (E-2)((E)),$$
$$C = \sum_{x,y,z}{}' ((E))^2, \qquad (11)$$

summed over (9).

Hence

$$A+15/4 = \sum_{x,y,z} (E-3/2)(E-5/2).$$

Multiplying out and summing Σx^2, Σxy, Σx, we have, Σ now denoting $\sum_{p,q,r}$,

$$A+\frac{15}{4} = \sum\frac{(p-1)(p)(2p-1)}{6p^2}qr+\sum\frac{2}{4pq}(p)(p-1)(q)(q-1)r$$
$$-4/2\,\Sigma(p)\,(p-1)q/p+15\,pqr/4 = \Sigma\tfrac{1}{3}pqr-\Sigma\tfrac{1}{2}qr+\Sigma\tfrac{1}{6}qr/p$$
$$+\Sigma\tfrac{1}{2}pqr-\Sigma\tfrac{1}{2}(p+q)r+\Sigma\tfrac{1}{2}(r)-2\Sigma pqr+2\Sigma qr+15/4pqr,$$

and so

$$A = \tfrac{1}{4}pqr+\tfrac{1}{2}\Sigma qr+\tfrac{1}{2}\Sigma p+\tfrac{1}{6}\Sigma qr/p-15/4. \qquad (12)$$

Clearly we can include $x=y=z=0$ in the summations for B and C in (11).

Next for B. From (7) on summing for y, z in turn, we have

$$\sum_{x,y,z}\frac{x}{p}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)\right) = \sum_{x,y}\frac{x}{p}\left(\left(\frac{rx}{p}+\frac{ry}{q}\right)\right) = \sum_{x}\frac{x}{p}\left(\left(\frac{qrx}{p}\right)\right)$$

since ry runs through a complete set of residues mod q.

Also
$$\sum_{x,y,z}\left(\left(\frac{x}{p}+\frac{y}{q}+\frac{z}{r}\right)\right) = \sum_{x=0}^{pqr-1}\left(\left(\frac{x}{pqr}\right)\right),$$

since $qrx+rpy+pqz$ runs through a complete set of residues mod pqr. The sum is

$$\sum_{x=1}^{pqr-1}\left(\frac{x}{pqr}-\frac{1}{2}\right)=0.$$

Hence $B=-2\,S_3$. (13)

Finally

$$C=\sum_{x=0}^{pqr-1}\left(\left(\frac{x}{pqr}\right)\right)^2=\sum_{x=1}^{pqr-1}\left(\frac{x}{pqr}-\frac{1}{2}\right)^2$$

$$=\frac{1}{6p^2q^2r^2}(pqr-1)(pqr)(2pqr-1)-\frac{1}{2pqr}(pqr-1)(pqr)+\frac{1}{4}(pqr-1)$$

$$=\frac{(pqr-1)(2pqr-1)}{6pqr}-\frac{(pqr-1)}{2}+\frac{1}{4}(pqr-1)$$

$$=\frac{1}{12}pqr-\frac{1}{4}+\frac{1}{6pqr}.$$

Hence we have the required formula

$$2\,N_3=\frac{1}{3}pqr+\frac{1}{2}\Sigma qr+\frac{1}{2}\Sigma p+\frac{1}{6}\Sigma\frac{qr}{p}+\frac{1}{6pqr}-4-2\,S_3,$$

3. For the four dimensional result, let N_4 denote the number of lattice points in

$$0\leqslant x<p,\ 0\leqslant y<q,\ 0\leqslant z<r,\ 0\leqslant w<s,$$

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1,$$

where no two of p, q, r, s have a common factor. Write

$$S_4=\sum_{p,q,r,s,}\ \sum_{x=1}^{p-1}\frac{x}{p}\left(\left(\frac{qrsx}{p}\right)\right).$$

We consider now

$$S=\sum_{x,y,z,w}{}'\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}\right]-1\right)\left(\left[\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}\right]-2\right),$$

where the summation is taken over the lattice points L given by

$$0\leqslant x<p,\ 0\leqslant y<q,\ 0\leqslant z<r,\ 0\leqslant w<s$$

with the exclusion of $x=y=z=w=0$.

Here $$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<4.$$

The points L with

$$1<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<3$$

contribute zero to S, while each of the points L with

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1$$

say N_4 in number, and each of those with

$$3<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<4$$

say, N' in number, contribute 2 to S. Hence

$$S=2\,(N_4+N').$$

Let N'' be the number of the points L satisfying

$$0<\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+\frac{w}{s}<1,\ xyzw=0.$$

Then excluding these, we have a 1-1 correspondence between the remaining ones in N_4 and those in N' given by

$$x+x'=p,\ y+y'=q,\ z+z'=r,\ w+w'=s.$$

Hence $N_4=N'+N''$ and $S=4N_4-2N''$.

Now $$N''=N_4''+N_3''+N_2''+N_1'',$$

where N''_r denotes the number of the lattice points L when exactly r of the variables equal zero. Clearly

$$N_3''=\sum_{p,q,r,s}(p-1),\ N_4''=0,$$

also

$$N_2''=\sum_{p,q,r,s}(p-1)(q-1)/2,$$

since there are $(p-1)(q-1)$ solutions of

$$0<x<p,\ 0<y<q,\ x/p+y/q<2$$

and there is a 1-1 correspondence between those in $x/p+y/q<1$ and those in $x/p+y/q>1$ given by $x+x'=p, y+y'=q$. Finally $N_1''=\sum_{p,q,r,s}N_3(p,q,r)$, and so $S=4\,N_4-2\sum_{p,q,r,s}(p-1)-\sum_{p,q,r,s}(p-1)(q-1)-2\sum_{p,q,r,s}N_3(p,q,r)$. (14)

Next we split S into three sums, say A', B', C', corresponding to A, B, C but now a fourth variable w/s also occurs. We find $A'+15/4$

$$= \sum \frac{(p-1)(p)(2p-1)qrs}{6p^2} + \sum \frac{2}{4pq}(p)(p-1)(q)(q-1)rs$$

$$-\frac{4}{2}\sum \frac{(p)(p-1)\,qrs}{p} + \frac{15}{4}pqrs = \sum \frac{(p-1)(2p-1)qrs}{6p}$$

$$+\Sigma \tfrac{1}{2}(p-1)(q-1)rs - 2\Sigma(p-1)qrs$$

$$+15\, pqrs/4 = pqrs/12 + \Sigma\, \tfrac{1}{2}rs + \Sigma qrs/(6p).$$

Next $$B' = -2S_4.$$

Finally

$$C' = \sum_{x=1}^{pqrs-1} (x-\tfrac{1}{2})^2 = \frac{1}{12}pqrs - \tfrac{1}{4} + \frac{1}{6pqrs}.$$

This gives

$$S = \frac{1}{6}pqrs + \tfrac{1}{2}\,\Sigma pq - 4 + \sum \frac{qrs}{p} + \frac{1}{6pqrs} - 2S_4.$$

Hence on substituting for N_3 from (6) in (14),

$$4N_4 + 2S_4 + 2\sum_{p,q,r,s} S_3(p, q, r) = 2\Sigma(p-1) + \Sigma(p-1)(q-1)$$

$$+\frac{1}{3}\Sigma pqr + \tfrac{1}{2}\Sigma(qr+rp+pq) + \tfrac{1}{2}\Sigma(p+q+r)$$

$$+\frac{1}{6}\sum\left(\frac{qr}{p}+\frac{rp}{q}+\frac{pq}{r}\right) + \frac{1}{6}\sum\frac{1}{pqr} - 16$$

$$+\frac{1}{6}pqrs + \tfrac{1}{2}\Sigma pq + \frac{1}{6}\sum\frac{qrs}{p} - 4 + \frac{1}{6pqrs}$$

$$= \frac{1}{6}pqrs + 2\tfrac{1}{2}\,\Sigma pq + \tfrac{1}{2}\,\Sigma p - 22 + \frac{1}{3}\,\Sigma pqr$$

$$+\frac{1}{6}\sum\left(\frac{qr}{p}+\frac{rp}{q}+\frac{pq}{r}\right) + \frac{1}{6}\sum\frac{qrs}{p} + \frac{1}{6}\sum\frac{1}{pqr} + \frac{1}{6pqrs}.$$

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A NOTE ON FERMAT'S CONJECTURE

By Paul Turan

1. The classical conjecture of Fermat, unpoved so far, asserts that the equation

$$x^q + y^q = z^q \tag{1.1}$$

is unsolvable in positive integers if q is a fixed integer $\geqslant 3$. The history of the (few) attempts, which mean a a progress however slight in this direction, is well known. In this paper, which does not claim to be in this category, we try to modify the problem in a rather plausible way and to give by some quite elementary considerations the first steps in this trend which can be certainly superseded by applying more powerful devices.

2. In what follows we restrict ourselves to the case when q denotes an odd prime; the general case can be reduced easily to it. Denoting by $R_q(N)$ the number of solutions of (1.1) with

$$1 \leqslant x \leqslant N, \quad 1 \leqslant y \leqslant N, \quad 1 \leqslant z \leqslant N, \quad (x, y) = 1, \tag{2.1}$$

the proof of Fermat's conjecture would result:

$$R_q(N) = 0 \qquad \text{for all } N \geqslant 1.$$

Now the question arises: what can be proved actually by way of an upper estimate for $R_q(N)$? I shall show the following

Theorem. *If $c_1, c_2, \ldots$ denote quantities depending only upon q, we have*

$$R_q(N) < c_1 N \log^{1+\frac{q}{(q-1}} N. \tag{2.2}$$

Replacing the second half of the argument by a well-known but somewhat lengthy one, (2.2) could be improved to

$$R_q(N) < c_2 N \log^{\frac{q}{q-1}} N. \tag{2.3}$$

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