DIFFERENTIAL AND ALGEBRAIC TOPOLOGY

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Section 1 : THE TOPOLOGY OF EUCLIDEAN SPACE

We begin these notes with a brief review of point set topology. We shall, in the sequel, always be concerned with "nice" spaces (metric spaces). Thus, we shall not deal here at all with pathological examples but shall restrict ourselves, whenever convenient, to metric spaces.

A topological space, (X,τ), is a set, X, together with a collection, τ, of subsets of X called open subsets of X. The collection is required to satisfy the following four axioms :

1) X is an open subset of X.
2) The empty set, ϕ, is an open subset of X.
3) An arbitrary union of open subsets is an open subset of X.
4) A finite intersection of open subsets is an open subset.

The main example that we have in mind in this chapter in Euclidean space, $\mathbb{R}^n$. Its underlying set consists of all n-tuples, $(x_1,\cdots, x_n)$, of real numbers. The topology is defined in terms of the usual distance function in $\mathbb{R}^n$:

$$d(x,y) = {}^{+}\sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2} .$$

We define the open ball of radius r centered about a point p to be :

$$B_r(p) = \{y \in \mathbb{R}^n \mid d(y,p) < r\}.$$

A set U in $\mathbb{R}^n$ is defined to be open if and only if for each $p \in U$ there is $\varepsilon > 0$ so that $B_\varepsilon(p) \subset U$. One checks easily that this collection of subsets satisfies the four axioms.

If X is a topological space, and if $A \subset X$ is a subset, then A inherits a topology from X. The open sets of A are all intersections, $A \cap U$, where U is an open subset of X. As an example, if we let $\mathbb{R}^k \subset \mathbb{R}^n$ be the subset of all n-tuples of the form $\{(x_1, \cdots, x_k, 0, \cdots, 0)\}$, then the topology that $\mathbb{R}^k$ inherits from $\mathbb{R}^n$ is identical to the topology defined abstractly, as above, for $\mathbb{R}^k$.

If X and Y are topological spaces, then $f : X \to Y$ is a <u>continuous</u> function if and only for every open set $U \subset Y$, the set $f^{-1}(U) \subset X$ is open. (Recall that $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$.) A <u>homeomorphism</u> from X to Y is a continuous bijection $f : X \to Y$ whose inverse $f^{-1} : Y \to X$ is also continuous.

As we have seen, the topology of $\mathbb{R}^n$ is defined in terms of the Pythagorean distance function. Abstracting the basic properties of this distance function leads to the concept of a metric space. Many of the basic properties of $\mathbb{R}^n$ are shared by all metric spaces.

<u>Definition</u> : Let X be a topological space. A metric is a continuous function $d : X \times X \to \mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$ such that :

1) $d(x,y) = d(y,x)$,

2) $d(x,y) = 0$ if and only if $x = y$, and

3) $d(x,y) + d(y,z) \leq d(x,z)$.

A word is necessary about the topology on $X \times X$. It is the so called product topology. In general, if A and B are topological spaces, then $A \times B$ receives a natural topology - the product topology. A set $V \subset A \times B$ is open if and only if for every $p \in V$, there are open sets, U_A of A and U_B of B, such that $p \in (U_A \times U_B) \subset V$. It is an easy exercise to show that the topology on $\mathbb{R}^n$ agrees with the (n-fold) product topology when we consider $\mathbb{R}^n$ as $\mathbb{R}\times\cdots\times\mathbb{R}$ (n-times).

The metric on $\mathbb{R}^n$ is, of course, the Pythagorean distance. We also denote $d(x,0)$ by $\|x\|$.

If X is a metric space and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of X, then we say that $\{x_n\}$ converges to x, or $\{x_n\} \to x$, if and only if $\lim_{n\to\infty} d(x_n,x) = 0$. (As an exercise, give the definition of convergence in an arbitrary topological space .) Clearly, a sequence can converge to at most one point in a metric space and need not converge to any point at all.

Lemma 1.1 : Let X and Y be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if whenever a sequence $\{x_n\}$ converges to x in X the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y.

<u>Proof</u> : Suppose that there is a sequence $\{x_n\}$ in X which converges to x but that $\{f(x_n)\}$ does not converge to $f(x)$. This means that there is an open ball, $B_\varepsilon(f(x))$ and a sequence of natural numbers n_k approaching $+\infty$ so that $f(x_{n_k}) \notin B_\varepsilon(f(x))$. Hence, $f^{-1}(B_\varepsilon(f(x)))$ contains x but does not contain any x_n. Since $\{x_{n_k}\} \to x$, this implies that no open ball, $B_\delta(x)$, is contained in $f^{-1}(B_\varepsilon(x))$. This shows that $f^{-1}(B_\varepsilon(f(x)))$ is not open, and consequently that f is not continuous. Conversely, suppose that whenever $\{x_n\} \to x$ then $\{f(x_n)\} \to f(x)$. Suppose in addition that f is not continuous. From these assumptions we will derive a contradiction. If f is not continuous, then there is an open set $U \subset Y$ so that $f^{-1}(U) \subset X$ is not open. Thus, there is $x \in f^{-1}(U)$ such that there is no open ball of the form $B_\delta(x)$ contained in $f^{-1}(U)$. Thus, for every $n > 0$, there is a point $x_n \in (X - f^{-1}(U))$ such that $d(x_n,x) < \frac{1}{n}$. The sequence $\{x_n\}$ converges to x. Since $x_n \notin f^{-1}(U)$, we have $f(x_n) \notin U$. Thus, $\{f(x_n)\}$ does not converge to $f(x)$. This is the sought after contradiction which shows that if $\{x_n\} \to x$ implies $\{f(x_n)\} \to f(x)$, then f is continuous.

<u>Examples</u> : 1) Any map $f : \mathbb{R}^n \to \mathbb{R}$ which is given by polynomials in the coordinates $(x_1, \cdots, x_n)$ is continuous. Hence the following maps are continuous :

$$(x_1, \cdots, x_n) \longmapsto \prod_{i=1}^{n} x_i$$

$$(x_1, \cdots, x_n) \longmapsto \sum_{i=1}^{n} x_i$$

$$(x_1, \cdots, x_n) \longmapsto ||x||^2 = \sum_{i=1}^{n} x_i^2$$

2) Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Any such mapping is identified with an $(m \times n)$-matrix, (α_{ij}). (Recall that when we identify linear maps with matrices we write elements in $\mathbb{R}^n$ and $\mathbb{R}^m$ as column vectors.) A matrix gives a linear map via matrix multiplication on the left :

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & & \alpha_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} \alpha_{1i} x_i \\ \vdots \\ \sum_{i=1}^{n} \alpha_{mi} x_i \end{pmatrix}$$

This correspondence identifies $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{m \cdot n}$. We use this identification to define a topology on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Thus, an open set of linear maps is one with the following property. Given any ϕ in the set with matrix representative (ϕ_{ij}) there is $\varepsilon > 0$ so that every ψ with $|\psi_{ij} - \phi_{ij}| < \varepsilon$ for every pair (i,j) is in the set. Tautologically, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ becomes homeomorphic to $\mathbb{R}^{n \cdot m}$ via this identification. Consider the evaluation map $e : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^m$ given by $e(\phi, x) = \phi(x)$. If we give $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^n$ the product topology, then e becomes continuous. The reason is that, in the coordinates $\{\alpha_{ij}, x_i\}$, e is given by quadratic polynomials.

3) The map $(x_1, \cdots, x_n) \longmapsto \dfrac{1}{x_1 \cdot \cdots \cdot x_n}$ is a continuous function on $\mathbb{R}^n - \{(x_1, \cdots, x_n) \mid \prod_{i=1}^{n} x_i = 0\}$.

4) Let $\mathbb{C}$ be the complex plane with variable z. Any complex polynomial, $p(z)$, defines a continuous function from $\mathbb{C}$ to $\mathbb{C}$, $z \mapsto p(z)$.

Definition : Let X be a topological space and $A \subset X$ a subspace. A is closed if and only if $(X - A)$ is open.

Theorem 1.2 : A set $X \subset \mathbb{R}^n$ is closed if and only if it contains all its limit points, i.e., if and only if whenever $\{x_n\} \subset X$ converges to $p \in \mathbb{R}^n$, $p \in X$.

The proof is straightforward and is left as an exercise. As a consequence, if $X \subset \mathbb{R}^n$, then its closure, $\overline{X}$, (i.e., the smallest closed subset of $\mathbb{R}^n$ containing X) is obtained by adjoining all limit points of X to X.

Exercise : Show that $\overline{B_\varepsilon(x)} = \{y \in \mathbb{R}^n \mid d(x,y) \leq \varepsilon\}$.

Definition A topological space X is connected if and only if it can not be written as $A \cup B$ with A and B both open and non-empty and $A \cap B = \phi$.

Lemma 1.3 : Let $X \subset \mathbb{R}^1$ be non-empty. It is connected if and only if whenever $r, s \in X$ with $r \leq s$, then the interval $[r,s] \subset X$.

Note : The subsets of $\mathbb{R}^1$ satisfying these properties are :
1) points, 2) intervals (closed, open, or half-open), 3) half-rays (in either direction and open or closed), 4) $\mathbb{R}^1$.

Proof : Let us show that the condition is necessary. Suppose, to the contrary, that X is connected, $r < t < s$, and $r, s \in X$ but $t \notin X$. Let $A = X \cap (-\infty, t)$ and $B = X \cap (t, \infty)$. Clearly, $X = A \cup B$ and A and B are open, disjoint, and non-empty. This is a contradiction. Conversely, suppose that whenever $r < s$ and $r, s \in X$, then $[r,s] \subset X$, but X is not connected. Say $X = A \cup B$ with A and B open, disjoint and non-empty. Take $r \in A$ and $s \in B$. For simplicity let us assume $r < s$. Consider $A' \subset [r,s]$ and $B' \subset [r,s]$ given by $A' = A \cap [r,s]$; $B' \cap [r,s]$. Clearly, $[r,s] = A' \cup B'$ and A' and B' are open, non-empty and disjoint. Let $\Omega = \{x \in [r,s] \mid [r,x] \subset A'\}$. Clearly, $r \in \Omega$ and s is an upper bound for Ω. Let $t \in [r,s]$ be the least upper bound for Ω. We claim that $t \notin A'$. For if $t \in A'$, then the interval $(t - \varepsilon, t + \varepsilon) \subset A$ for some $\varepsilon > 0$. Thus, either $t = s$, contradicting the fact that $s \in B'$, or t is not an upper bound for Ω. Thus $t \in B'$. Clearly, $t \neq r$. Hence, $(t - \delta, t] \subset B'$ for some $\delta > 0$. Since every $t' < t$ is contained in A' this implies that $A' \cap B' \neq \phi$. This contradiction establishes the sufficiency of the condition.

Definition : If X is a topological space, then an open cover of X is a collection of open sets of X, $\{U_\alpha\}_{\alpha \in I}$, so that $\bigcup_{\alpha \in I} U_\alpha = X$. A topological space X is compact if every open cover $\{U_\alpha\}$ has a finite sub-collection $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ which is also an open cover. We call such a sub-collection a finite sub-cover.

Theorem (Heine-Borel) 1.4 : $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded (bounded means $X \subset B_R(0)$ for some $R < \infty$).

<u>Lemma 1.5</u> : <u>Let Y be a compact metric space and $X \subset Y$ a subspace. Then X is compact if and only if X is closed.</u>

<u>Proof</u> : Suppose $X \subset Y$ is closed. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Then $\{U_\alpha \cup (Y - X)\}_{\alpha \in I}$ is an open cover of Y. ($U_\alpha \cup (Y - X)$ has complement $(X - U_\alpha)$ which is closed in X and hence closed in Y.) Let $\{U_{\alpha_1} \cup (Y - X), \cdots, U_{\alpha_n} \cup (Y - X)\}$ be a finite sub-cover of this open cover of Y. Then $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ is a finite sub-cover of the original cover of X.

Conversely, suppose $X \subset Y$ is compact and let $\{p_n\}$ be a sequence of points in X converging to $y \in Y$. Suppose $y \notin X$. We claim that for any $k \geq 0$, $\{p_n\}_{n=k}^{\infty} \cup \{p\}$ is a closed subspace of Y. This is because $\{p_n\}_{n=k}^{\infty} \cup \{p\}$ contains all its limit points. Thus, $\{\bigcup_{n \geq k} p_n\}$ is a closed subset of X. Hence, $U_k = X - \{\bigcup_{n \geq k} p_n\}$ is an open set in X. Clearly, the open cover $\{U_k\}_{k=1}^{\infty}$ of X has no finite sub-cover.

<u>Lemma 1.6</u> : <u>If X and Y are compact spaces, then $X \times Y$ is compact.</u>

(Here of course, $X \times Y$ is given the product topology.)

<u>Proof</u> : Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $X \times Y$. For each point $y \in Y$, there is a finite collection $\{U_{\alpha_i}(y)\}_{i=1}^{n_y}$ which cover $X \times \{y\}$ (since $X \times \{y\}$ is compact). We claim that there is an open set $V_y \subset Y$ containing y so that $X \times V_y \subset (\bigcup_{i=1}^{n_y} U_{\alpha_i}(y))$. The reason is that for each $\{x \times y\} \in X \times \{y\}$, there are open sets $W_x \subset X$ and $Z_x \subset Y$ so that $\{x \times y\} \in W_x \times Z_x \subset (\bigcup_{i=1}^{n_y} U_{\alpha_i}(y))$. This gives an open

cover $\{W_x \times Z_x\}_{x\in X}$ of $X \times \{y\}$. Let $\{W_{x_1} \times Z_{x_1}, \cdots, W_{x_n} \times Z_{x_n}\}$ be a finite sub-cover. Clearly, $X \times (\bigcap_{j=1}^{n} Z_{x_j}) \subset (\bigcup_{i=1}^{n_y} U_{\alpha_i(y)})$ and $\bigcup_{j=1}^{n} Z_{x_j}$ is an open subset of Y containing y.

Let us recapitulate what we have accomplished so far. Given any open cover $\{U_\alpha\}_{\alpha\in I}$ of $X \times Y$ we have found for each $y \in Y$:

a) a finite collection $\{U_{\alpha_i(y)}\}_{i=1}^{n_y}$, and

b) an open set $Z_y \subset Y$, containing y, so that

$$X \times Z_y \subset (\bigcup_{i=1}^{n_y} U_{\alpha_i(y)}).$$

The $\{Z_y\}_{y\in Y}$ form an open cover of Y. Let $\{Z_{y_1}, \cdots, Z_{y_s}\}$ be a finite sub-cover. Then

$$\{\{U_{\alpha_i(y_j)}\}_{i=1}^{n_{y_j}}\}_{j=1}^{s}$$

is a finite sub cover of $\{U_\alpha\}_{\alpha\in I}$.

<u>Note</u> : The statement that an arbitrary product of compact spaces is compact (Tychonoff's Theorem) is equivalent to the axiom of choice.

<u>Proposition 1.7</u> : <u>A closed interval</u> $[a,b] \subset \mathbb{R}^1$ <u>is compact</u>.

<u>Proof</u> : Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of $[a,b]$. Consider $\Omega = \{x \in [a,b] \mid [a,x]$ is covered by finitely many of the $\{U_\alpha\}\}$.

Clearly, $a \in \Omega$. We claim that Ω is both open and closed in $[a,b]$. If so, then, since $[a,b]$ is connected and $\Omega \neq \phi$, it will follow that $\Omega = [a,b]$, and consequently, that $[a,b]$ is compact. We first show that Ω is closed. Suppose $\{x_n\} \in \Omega$ and $x_n \to x \in [a,b]$. Then, there is a U_{α_0} so that $x \in U_{\alpha_0}$. This implies that for some $N > 0$, the interval $[x_N, x] \subset U_{\alpha_0}$. Since $X_N \in \Omega$, there is a finite cover $\{U_{\alpha_1}, \cdots, U_{\alpha_T}\}$ of $[a, x_N]$. Then $\{U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_T}\}$ covers $[a,x]$. On the other hand, if $x \in \Omega$, then $[a,x]$ has a finite cover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$. Then there is $\varepsilon > 0$ so that $((x-\varepsilon, x+\varepsilon) \cap [a,b]) \subset U_{\alpha_1}$.

Thus, $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ covers $[a,y]$ for every $y \in ((x-\varepsilon, x+\varepsilon) \cap [a,b])$. Hence, $((x-\varepsilon, x+\varepsilon) \cap [a,b]) \subset \Omega$. This shows that Ω is open.

Let us use these three lemmas to prove the Heine-Borel theorem. If $X \subset \mathbb{R}^n$ is compact, then $X \subset B_R(0)$ for some $R > 0$ (else the open cover $\{X \cap B_R(0)\}_{R>0}$ would have no finite refinement). Thus, X is bounded, and hence contained in a cube $(\times_{i=1}^{n} [a,b])$. By Lemma 1.6 and 1.7, this cube is compact. By Lemma 1.5, $X \subset (\times_{i=1}^{n} [a,b])$ must be closed.

Conversely, if X is bounded, then there is an interval $[a,b]$ so that $X \subset (\times_{i=1}^{n} [a,b])$.

If $X \subset \mathbb{R}^n$ is closed, then $X \subset (\times_{i=1}^{n} [a,b])$ is closed. By Lemma 1.5, X is the compact.

<u>Corollary 1.8</u> : <u>Let</u> $f : X \to \mathbb{R}^1$ <u>be a continuous function, and suppose that</u> X <u>is compact. There are numbers</u> m <u>and</u> M <u>so that</u> $m \leq f(x) \leq M$ <u>for all</u> $x \in X$. <u>Let</u> $\overline{m}$ <u>be the least upper bound for all such</u> m, <u>and</u> $\overline{M}$ <u>be the greatest lower bound for all such</u> M. <u>There</u> are points $x, y \in X$ so that $f(x) = \overline{m}$ and $f(y) = \overline{M}$.

<u>Proof</u> : Consider $f(X) \subset \mathbb{R}^1$. It is a simple lemma (Exercise 3) that the image under a continuous function of a compact space is compact. Any compact set in $\mathbb{R}^1$ is closed and bounded. Being closed it contains its greatest lower bound, $\overline{m}$, and its least upper bound, $\overline{M}$.

Two of the basic facts about compact spaces are given in the next propositions.

<u>Proposition 1.9</u> : <u>Let</u> X <u>be a compact space and</u> $\{x_i\}_{i=1}^{\infty}$ <u>a consequence of points. Then, there is a subsequence</u> $\{x_{n_j}\}_{j=1}^{\infty}$ <u>which converges to</u> $x \in X$.

<u>Proof</u> : If $\{x_i\}_{i=1}^{\infty}$ has no convergent subsequences, then $\{\bigcup_{i=N}^{\infty} x_i\}$ is a closed subset for each $N \geq 1$. Thus $X - \{\bigcup_{i=N}^{\infty} x_i\} = U_N$ is open, and the open covering $\{U_N\}_{N=1}^{\infty}$ has no finite sub-cover. Hence, X is not compact.

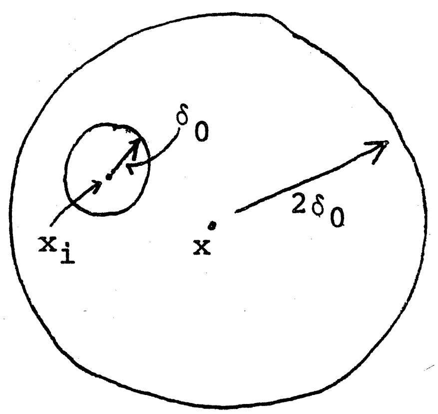
<u>Proposition 1.10</u> : (Uniform Continuity) : <u>Let</u> X <u>be a compact metric</u>

space, and let $f : X \to \mathbb{R}^1$ be a continuous function. Given $\varepsilon > 0$ there is $\delta > 0$ so that $|f(x_1) - f(x_0)| < \varepsilon$ whenever $d(x_1, x_0) < \delta$.

Proff : Given $\varepsilon > 0$ and $x \in X$ let $\overline{\delta}_x$ be the least upper bound for the set $\{\delta \; |f(y) - f(x)| < \varepsilon$ for $y \in B_\delta(x)\}$. ($\overline{\delta}_x$ could conceivably be $+\infty$.) We claim that there is $\overline{\delta} > 0$ so that $\overline{\delta}_x > \overline{\delta}$ for all $x \in X$. If we can show this, then the result will follow. If we suppose, to the contrary, that no such $\overline{\delta}$ exists, then there is a sequence $\{x_i\}_{i=1}^{\infty}$ so that $\overline{\delta}_{x_i} \to 0$. By taking a subsequence, if necessary, we can assume that $\{x_i\} \to x$. There is δ_0 so that $|f(y) - f(x)| < \frac{\varepsilon}{2}$ if $y \in B_{2\delta_0}(x)$. There is $N > 0$ so that $d(x_i, x) < \delta_0$ for $i \geq N$. By the triangle inequality

$B_{\delta_0}(x_i) \subset B_{2\delta}(x)$

for $i \geq N$:



Thus for any $i \geq N$, $|f(y) - f(x)| < \frac{\varepsilon}{2}$ for all $y \in B_{\delta_0}(x_i)$ and $|f(x_i) - f(x)| < \frac{\varepsilon}{2}$. Consequently, $|f(y) - f(x_i)| < \varepsilon$ for all $y \in B_{\delta_0}(x_i)$ and all $i \geq N$. This proves that $\overline{\delta}_{x_i} \geq \delta_0$ and gives a contradiction.

<u>Exercises</u> : 1) Let X be a topological space. Define what it means for a sequence $\{x_n\}_{n=1}^{\infty}$ in X to converge to $x \in X$.

2) Show that if X is Hausdorff*, then a sequence $\{x_n\}_{n=1}^{\infty}$ can converge to at most one point of X.

3) Let X be a compact space and $f : X \to Y$ a continuous mapping. Show that $f(X) \subset Y$ is a compact space.

4) Suppose Y is Hausdorff and $A \subset Y$ is a compact subspace. Show that A is closed in Y.

5) Show that $\{(x_1, \cdots, x_n) \mid \sum_{i=1}^{h} a_i x_i^2 = 1\}$ is compact and non-empty if and only if every a_i is positive.

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*A Hausdorff space is one in which any two distinct points x snd y are contained in disjoint open sets U_x and U_y.

§2. The Differential Structure of Euclidean Space

The topology of R^n adds much structure but there is even more--the differential structure. In this section we shall study this aspect of the situation. Let $U \subset R^n$ be an open set. Recall that a function $f: U \to R$ is <u>differentiable at</u> $p \in U$, if and only if there is a linear function $L: R^n \to R$ such that:

$$\lim_{\|h\| \to 0} \left(\frac{|f(p+h) - f(p) - L(h)|}{\|h\|} \right) = 0.$$

The linear function, if it exists, is easily seen to be unique. It is called the <u>differential of</u> f <u>at</u> p and is denoted $Df_p: R^n \to R$. If $f: U \to R$ is differentiable at every point of U, then we have $Df: U \to \mathcal{L}(R^n, R)$.

We say that f is C^1 (differentiable of the first class) on U if and only if $Df: U \to \mathcal{L}(R^n, R)$ is continuous. One could continue in this fashion defining C^2, C^3, etc., but the resulting definitions are somewhat clumsy. There is an alternate definition of C^1 which generalizes more easily.

<u>Lemma 2.1</u>: <u>Let</u> $U \subset R^n$ <u>be an open set and</u> $f: U \to R$. <u>Then</u> f <u>is</u> C^1 <u>if and only if the</u> n <u>partial derivatives</u> $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ <u>exist at every point of</u> U <u>and are continuous functions</u>.

(N.B. It is not the case, in general, that f is differentiable at p if all the partial derivatives exist at p, nor is it the case that f is differentiable at every point of U if the partial derivatives exist at every point of U.)

Proof: If $f\colon U \to R$ is differentiable at $p \in U$, then $\frac{\partial f}{\partial x_i}(p)$ exists and the $(1\times n)$-matrix $(\frac{\partial f}{\partial x_1}(p),\ldots,\frac{\partial f}{\partial x_n}(p))$ represents $Df(p)\colon R^n \to R$. (Recall that we are thinking of points of R^n as column vectors, and we are letting $(1\times n)$-matrices act by left multiplication.) We must show, conversely, that if $\{\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\}$ exist and are continuous near p, then f is differentiable at p. For $h \in R^n$, $h = (h_1,\ldots h_n)$, let $h^i = (h_1,\ldots,h_i,0,\ldots,0)$, and let $\gamma_i(t) = p+th^i+(1-t)h^{i-1}$, for $0 \leq t \leq 1$. Then, $f\circ\gamma_i$ is differentiable and $f\circ\gamma_i'(t) = h_i\cdot\frac{\partial f}{\partial x_i}(p+th^i+(1-t)h^{i-1})$. Thus,

$$f(p+h^i) - f(p+h^{i-1}) = \int_0^1 h_i\cdot\frac{\partial f}{\partial x_i}(p+th^i+(1-t)h^{i-1})dt,$$

and consequently,

$$f(p+h) - f(p) = \sum_{i=1}^{n} \int_0^1 h_i\cdot\frac{\partial f}{\partial x_i}(p+th^i+(1-t)h^{i-1})dt.$$

Given $\varepsilon > 0$, there is a $\delta > 0$ so that

$$\left|\frac{\partial f}{\partial x_i}(p+q) - \frac{\partial f}{\partial x_i}(p)\right| < \varepsilon/n \quad \text{if } \|q\| < \delta.$$

If we pick h so that $\|h\| < \delta$, then $\|th^i+(1-t)h^{i-1}\| < \delta$ for all t between 0 and 1. Hence,

$$|f(p+h) - f(p) - \sum_{i=1}^{n} h_i \cdot \frac{\partial f}{\partial x_i}(p)|$$
$$\leq \sum_{i=1}^{n} |h_i| \cdot |\int_0^1 \frac{\partial f}{\partial x_i}(p+th^i+(1-t)h^{i-1}) - \frac{\partial f}{\partial x_i}(p)dt|$$
$$\leq \sum_{i=1}^{n} |h_i| \cdot (\varepsilon/n) \leq \|h\| \cdot \varepsilon.$$

Since ε was chosen arbitrarily, this proves that

$$\lim_{\|h\|\to 0} \left(\frac{|f(p+h) - f(p) - \Sigma_{i=1}^{n} h_i \cdot \frac{\partial f}{\partial x_i}(p)|}{\|h\|}\right) = 0.$$

Thus, f is differentiable at p, and $Df(p)$ is represented by the $(1\times n)$-matrix $(\frac{\partial f}{\partial x_1}(p),\ldots,\frac{\partial f}{\partial x_n}(p))$.

<u>Definition</u>: A map $f\colon U \to R^1$ defined on an open subspace, U, of R^n is said to be a <u>C^1-function</u> if and only if the n partial derivatives--$\frac{\partial f}{\partial x_1}(p),\ldots,\frac{\partial f}{\partial x_n}(p)$--exist and are continuous throughout U. The map is said to be C^r, $r \geq 1$, if and only if all partial derivatives of order r,

$$\frac{\partial^r f}{\partial x_{i_1} \ldots \partial x_{i_r}}(p),$$

exist and are continuous throughout U. If partial derivatives of all orders exist and are continuous, then we say that f is C^∞. It is common notation to denote continuous maps by C^0. However, when we say C^r, we shall always implicitly be assuming $1 \leq r \leq +\infty$.

<u>Examples</u>: 1) All polynomial, exponential and logarithm maps are C^∞ where defined:

a) $(x_1,\ldots,x_n) \longmapsto e^{x_1 \cdots x_n}$ is C^∞ on all of R^n.

b) $(x_1,\ldots,x_n) \longmapsto \dfrac{x_1 \cdots\cdot x_{n-1}}{x_n}$ is C^∞ on $\{(x_1,\ldots,x_n) \mid x_n \neq 0\}$.

c) $(x_1,\ldots,x_n) \longmapsto \log(x_1+\ldots+ x_n)$ is C^∞ on

$$\{(x_1,\ldots,x_n) \mid \Sigma_{i=1}^n x_i > 0\},$$

d) $$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

Clearly, f is C^∞ away from $(0,0)$. We claim that $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$. (This follows from the fact that $f(t,0) = f(0,t) = 0$.) However, f is not differentiable at $(0,0)$ since it is not continuous at this point. (See exercise 3 below.)

<u>Definition</u>: Let $U \subset R^n$ be an open set and $f\colon U \to R^k$ be a function: $f(x) = (f_1(x),\ldots,f_k(x))$. We say that f is C^r if and only if each of the f_i are of class C^r. In particular, f is differentiable at $p \in U$ if and only if there is a linear function $Df_p\colon R^n \to R^k$ so that

$$\lim_{\|h\|\to 0} \left(\frac{\|f(p+h) - f(p) - Df_p(h)\|}{\|h\|} \right) = 0,$$

and $f\colon U \to R^k$ is C^1 if and only if $Df\colon U \to \mathcal{L}(R^n, R^k)$ is continuous.

Proposition 2.2: *Let* $U \subset R^n$ *and* $V \subset R^k$ *be open sets. Suppose* $f\colon U \to V$ *and* $g\colon V \to R^\ell$ *are* C^r, *then*

$$g\circ f\colon U \longrightarrow R^\ell \text{ is } C^r, \text{ and } D(g\circ f)_x = Dg_{f(x)} \circ Df_x.$$

(This is the chain rule in several variables. For a proof, consult any text on advanced calculus.)

Definition: Let U and V be open sets in R^n and let $\Phi\colon U \to V$ be a C^r-map. Φ is a *C^r-diffeomorphism* if and only if

1) Φ is a homeomorphism, and
2) $\Phi^{-1}\colon V \to U$ is C^r.

Exercises: 1) Show that $f\colon R^1 \to R^1$ defined by $f(t) = t^3$ is a C^∞-map and a homeomorphism but that f is not a C^1-diffeomorphism.

2) Show that

$$f(t) = \begin{cases} e^{-1/t^2} & t > 0 \\ 0 & t \le 0 \end{cases}$$

is a C^∞-function on R^1. Show that there does not exist a convergent power series $\Sigma_{i=0}^{\infty} a_i t^i$ which represents $f(t)$ in any neighborhood of 0.

3) Show that if U is an open set in R^n and $f: U \to R$ is differentiable at $p \in U$, then f is continuous at p.

4) Suppose that U is an open set of R^n and V is an open set of R^m, and suppose that there are C^1-maps $f: U \to V$ and $g: V \to U$ so that $f \circ g = Id_V$ and $g \circ f = Id_U$. Show $n = m$. (Hint: Use the chain rule.)

5) For each $r \geq 1$, give an example of a function which is C^r but not C^{r+1}.

6) Give an example of two open sets $U, V \subset R^n$ which are not diffeomorphic.

7) Show that any power series is the Taylor series of some C^∞-function at the origin in $\mathbb{R}^1$.

§3. Inverse and Implicit Function Theorems

In this section we shall begin the study of the zeroes of sets of differentiable functions. Some care must be taken to ensure that the zero set has the right dimension and is "smooth". Of course, the study of more general solution sets is interesting. It is much more complicated and should be taken up after one has familairity with the easiest case. There is another reason for concentrating on the "smooth" case and that is that it is generic.

For example, if one considers an open set $U \subset R^n$ and the space of all C^∞-functions from U to R^1 (with a suitable topology), then those functions whose solution set is "smooth" form an open and dense subset.

Let us begin with a few examples which show what can go wrong:

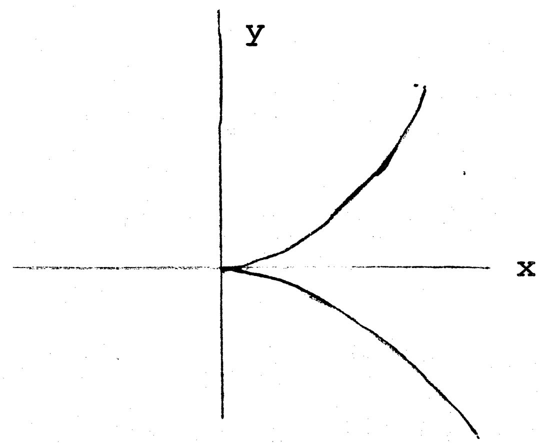
1) $\Sigma_{i=1}^{n} x_1^2 = 0$. Here, even though we have put only one condition on n-variables, the result is a single point.

2) $x_1^2 + x_2^2 = 0$ in R^3. Again, the dimension is "wrong". One condition in R^3 should leave us with a 2-dimensional solution set but here the solution set is the curve $\{(0,0,x_3)\}$.

3) $x_1 \cdot x_2 = 0$ in R^2. Here the dimension is correct, but something bad is happening at the origin where the solution

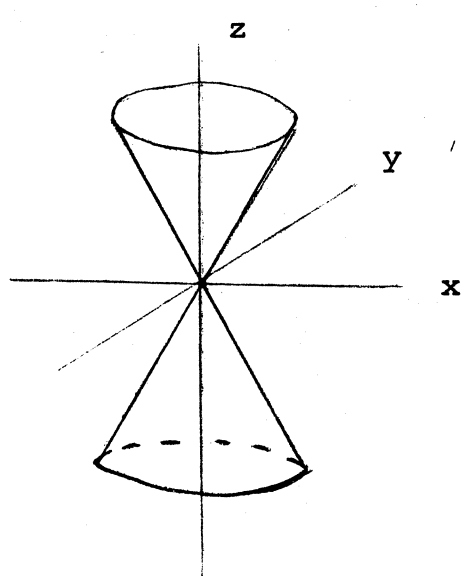
set is two lines crossing.

4) $x^3 = y^2$.



Here, the solution set has the correct dimension, 1, but it has a cusp (i.e., a non-smooth point) at the origin.

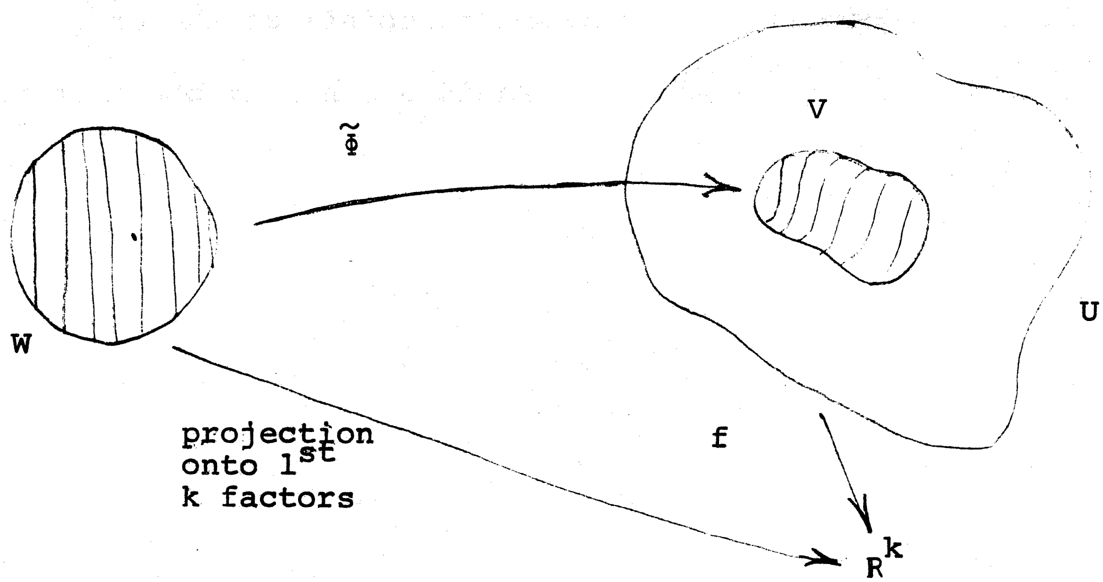
5) $x^2 + y^2 - z^2 = 0$. Again a singular point at the origin.



There is a natural and simple condition which rules out all these pathologies and ensures that the solution set locally looks like a Euclidean space of the "proper" dimension. It is given by the Implicit Function Theorem.

Theorem 3.1: (Implicit Function Theorem) Let $U \subset R^n$ be an open set and $f: U \to R^k$ be a C^r-function. Suppose for $p \in f^{-1}(0)$ the differential $Df(p): R^n \to R^k$ is onto (i.e., of rank k). Then there is a C^r-diffeomorphism $\tilde{\Phi}: W \to V \subset U$, with W and V open in R^n and $p \in V$, so that $f\circ\tilde{\Phi}: W \to R^k$ sends $(x_1,\ldots,x_n)$ to $(x_1,\ldots,x_k)$..

Explanation:



(The lines represent the level sets of projection and f respectively.) Thus, $\tilde{\Phi}$ transforms the subspace $\{(x_1,\ldots,x_n) \in W \mid x_1 = x_1^0,\ldots,x_k = x_k^0\}$ of W to the level set $f^{-1}(x_1^0,\ldots,x_k^0) \cap V$. In particular, let Φ be $\tilde{\Phi}$ restricted to $\{0\} \times R^{n-k} \subset R^k \times R^{n-k} = R^n$. Then Φ: $(\{0\} \times R^{n-k}) \cap W \to f^{-1}(0) \cap V$ gives a C^r-map which is a homeomorphism between an open set in R^{n-k} and an open subset of the level set $f^{-1}(0)$ containing p. We say that any identification induced in this manner is a system of local coordinates for $f^{-1}(0)$ near p. The coordinates are functions $\varphi_1,\ldots,\varphi_{n-k}$ defined on $f^{-1}(0) \cap V$, $\varphi_i = x_{i+k} \circ \Phi^{-1}$ which are homeomorphic (via Φ^{-1}) to the standard coordinate functions on an open set of R^{n-k}. In particular, an open set of $f^{-1}(0)$ containing p is homeomorphic to an open set in R^{n-k}. Of course, this coordinate system depends on choices that we make. Thus, what is important is not a particular coordinate system but rather the <u>existence</u> of one (and hence many) such systems. It is also important to understand how different coordinate systems mesh or match. The result is that their difference is a C^r-diffeomorphism.

<u>Proposition 3.2</u>: <u>Let</u> $U \subset R^n$ <u>be an open set and</u> $f\colon U \to R^k$ <u>a</u> C^r-<u>function. Suppose that for every</u> $p \in f^{-1}(0)$, <u>the differential</u> $Df_p\colon R^n \to R^k$ <u>is onto. Let</u> $\Phi_i\colon W_i \cap R^{n-k} \to f^{-1}(0) \cap V_i$ <u>be two</u> C^r-<u>coordinate systems</u> $(i = 0,1)$ <u>as</u>

<u>constructed above. Then let</u> $\Phi_i^{-1}(V_0 \cap V_1 \cap f^{-1}(0)) \subset W_i$ <u>be</u> Z_i <u>for</u> $i = 0$ <u>and</u> 1. <u>The map</u>

$$Z_0 \xrightarrow{\Phi_0} V_0 \cap V_1 \cap f^{-1}(0) \xrightarrow{\Phi_1^{-1}} Z_1$$

<u>is a</u> C^r<u>-diffeomorphism between open sets in</u> R^{n-k}.

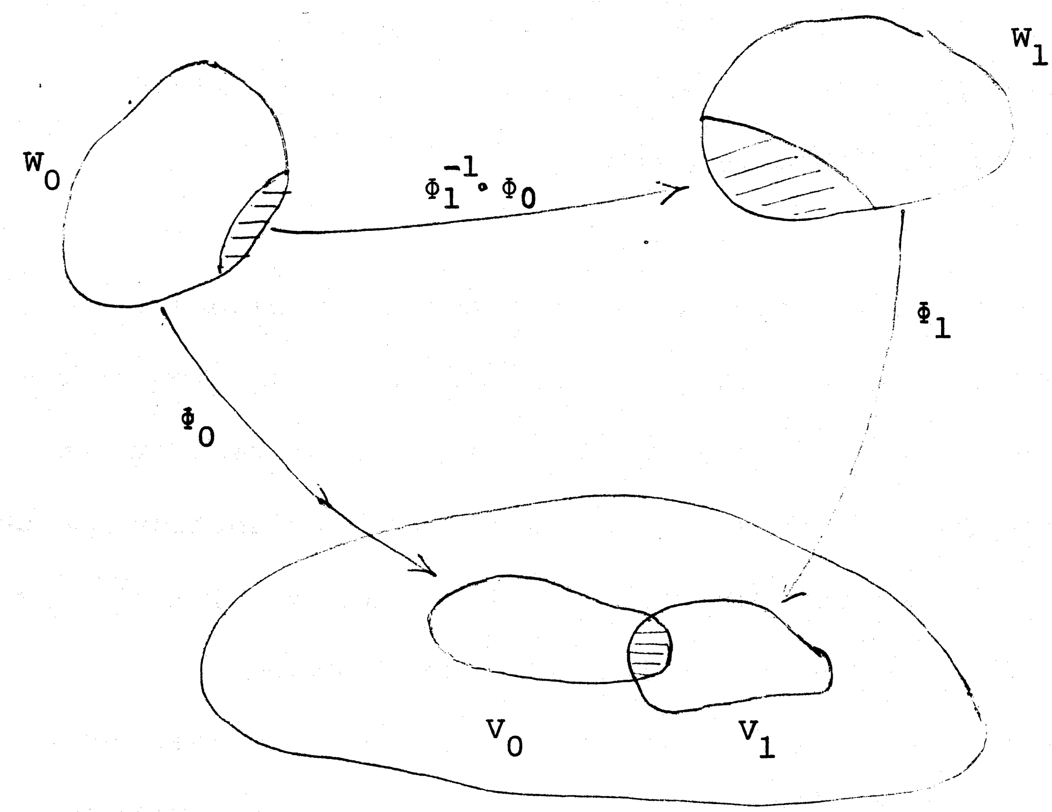
<u>Proof</u>: By construction Φ_0 and Φ_1 come via restriction from C^r-diffeomorphisms

$$\tilde{\Phi}_0 : \tilde{W}_0 \longrightarrow V_0$$

and

$$\tilde{\Phi}_1 : \tilde{W}_1 \longrightarrow V_1 .$$

Hence, $\tilde{\Phi}_1^{-1} \cdot \tilde{\Phi}_0 : \tilde{\Phi}_0^{-1}(V_1 \cap V_0) \to \tilde{\Phi}_1^{-1}(V_1 \cap V_0)$, is a C^r-diffeomorphism between open sets in R^n. Restricting $\tilde{\Phi}_1^{-1} \circ \tilde{\Phi}_0$ gives the map $\Phi_1^{-1} \circ \Phi_0 : \Phi_0^{-1}(V_1 \cap V_0 \cap f^{-1}(0)) \to \Phi_1^{-1}(V_1 \cap V_0 \cap f^{-1}(0))$. Hence, the latter map is a C^r-homeomorphism. Since the same argument works for $\Phi_0^{-1} \cdot \Phi_1$ we see that $\Phi_1^{-1} \circ \Phi_0$ is actually a C^r-diffeomorphism.



The change of coordinates map $\Phi_1^{-1} \circ \Phi_0$, where defined, is called the overlap function. The above proposition says that the overlap functions for the local coordinates on $f^{-1}(0)$ are diffeomorphisms of the same class of differentiability as f.

<u>Definition</u>: We say that $M \subset R^n$ is a <u>C^r-manifold of dimension (n-k)</u> if for every $p \in M$ there are:

1) an open set $U \subset R^n$ containing p,

2) an open set $W \subset R^n$, and

3) a C^r-diffeomorphism Φ: $W \to U$ so that

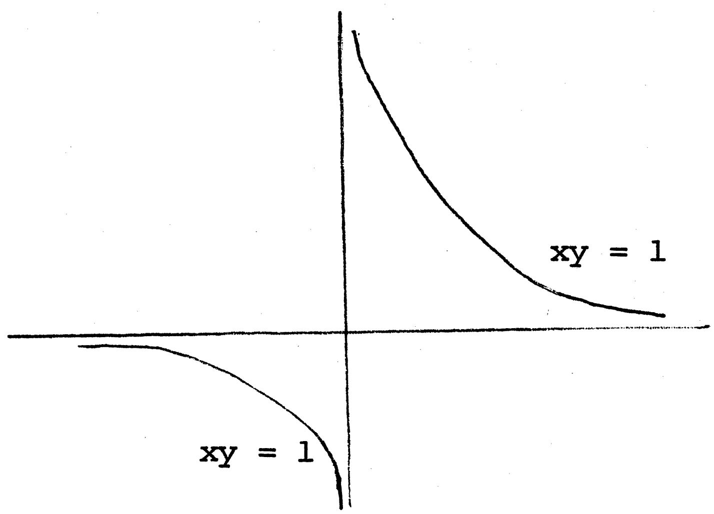
$$\Phi^{-1}(M \cap U) = (\{0\} \times \mathbb{R}^{n-k}) \cap W.$$

Thus, if $M = f^{-1}(0)$, where $f: R^n \to R^k$ is a C^r-function with $Df(p)$ of rank k for every $p \in M$, then M is a C^r-manifold of dimension $(n-k)$.

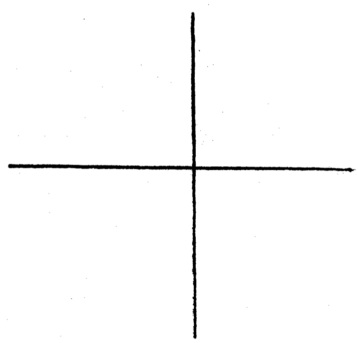
The converse is true locally. That is to say, if $M^{n-k} \subset R^n$ is a C^r-manifold of dimension $(n-k)$ and if $p \in M$, then there is an open set $U \subset R^n$ and a C^r-function $f: U \to R^k$ so that $U \cap M = f^{-1}(0)$ and so that Df is of rank k at every point of $U \cap M$. To construct such a function one begins with a C^r-diffeomorphism Φ: $W \to U$ so that $\Phi(\{0\} \times R^{n-k}) = M \cap U$. The map $U \xrightarrow{\Phi^{-1}} W \xrightarrow{\rho} R^k$ is the required function ($\rho(x_1,\ldots,x_n) = (x_1,\ldots,x_k)$). It is not true that every manifold $M^{n-k} \subset R^n$ can be given <u>globally</u> by a function $f: R^n \to R^k$ with Df of rank k at every point of M. We shall see examples of this later.

<u>Examples</u>: 1) $f(x) = (\Sigma_{i=1}^n x_i^2) - 1$ defines a C^∞-manifold of dimension $(n-1)$. The reason is that $Df(x_1,\ldots,x_1)$ $= (2x_1,\ldots,2x_n)$, and hence $Df(p)$ has rank 1 for all $p \neq 0$. This manifold is the $(n-1)$-dimensional sphere, S^{n-1}.

2) $xy - 1 = 0$.



This equation defines a C^{∞}-manifold. Note that $xy = 0$ fails to satisfy the differential condition at (0,0), and that indeed $xy = 0$ is not a manifold near the point (0,0):



3) Let $f\colon R^{n-k} \to R^{k}$ be any C^{r}-function. The graph of f, $\Gamma(f)$, which is the set of all pairs $\{(x,f(x)) \in R^{n-k} \times R^{k} = R^{n}\}$, is a C^{r}-manifold. The defining equation for $\Gamma(f)$ is

$$\{f(x_1,\ldots,x_{n-k}) - (x_{n-k+1},\ldots,x_n) = 0\}.$$

(Check that the differential has rank k at every point.)

4) Let ζ and z be complex variables. The equation

$$\zeta^2 = z^3 - 1$$

defines a subset of $\mathbb{C}^2 = R^4$. If we write $\zeta = x + iw$ and $z = u + iv$, then the above complex equation becomes 2 real equations :

$$\begin{cases} x^2 - w^2 = u^3 - 3uv^2 \\ 2xw = 3u^2v - v^3 \end{cases} .$$

These equations define a C^∞-manifold of dimension 2 in R^4. (Again, check that the differential is rank 2.)

5) If $p(x_1,\ldots,x_n) = \Sigma_{i=1}^n (x_i/a_i)^2$ ($a_i \neq 0$ for all i), then $p(x_1,\ldots,x_n) = 1$ defines a C^∞-manifold called an ellipsoid.

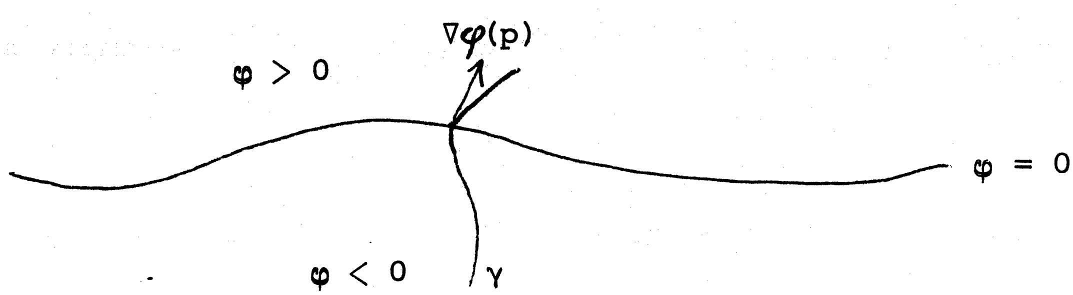
6) Let $(n\times n)$-matrices be identified with R^{n^2}. The invertible matrices, $GL(n,R)$, form an open subspace of R^{n^2} given by det $\neq 0$. Inside $GL(n,R)$ we have the matrices of trace 1. This is a manifold of dimension $(n^2 - 1)$. Its defining equation is {trace = 1} (i.e., $x_{11} + x_{22} + \ldots + x_{nn} = 1$). The orthogonal group $O(n)$ is a manifold of dimension $n(n-1)/2$. Its defining equations are:

$$\left(\sum_{j=1}^{n} x_{ij} \cdot x_{kj} = \delta_{ik}\right) \quad (1 \leq i \leq k \leq n).$$

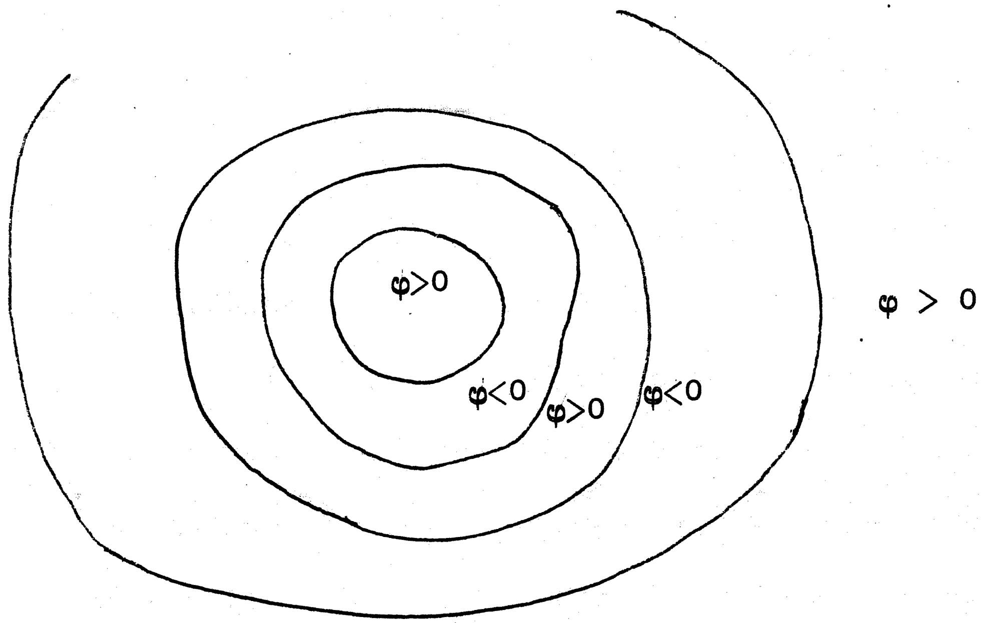
It is an easy exercise to show that the map $F\colon R^{n^2} \to R^{(n+1)n/2}$

defined by $F(x_{ij}) = \{\Sigma_{j=1}^{n} x_{ij}\cdot x_{kj} - \delta_{ik}\}$ $(1 \le i \le k \le n)$ has DF of rank $(n)(n+1)/2$ everywhere along $F^{-1}(0)$.

A manifold $M^{k-1} \subset \mathbb{R}^n$ defined by one equation $\varphi(x) = 0$ (with $D\varphi$ of rank 1 at every point of $\varphi^{-1}(0)$) is called a hypersurface. The fact that $D\varphi(p) \neq 0$ for every $p \in M$ means that $\nabla\varphi(p) \neq 0$ for every $p \in M$. If we take a C^1-curve γ: $(-\delta,\delta) \to R^n$ with $\gamma(0) = p$ and $\gamma'(0) = \nabla\varphi(p)$, then $\varphi\circ\gamma(0) = 0$ and $\varphi\circ\gamma'(0) = \|\nabla\varphi(p)\|^2 > 0$. Hence, there is $\epsilon > 0$ so that $\varphi(\gamma(t)) < 0$ for $-\epsilon < t < 0$ and $\varphi(\gamma(t)) > 0$ for $0 < t < \epsilon$. Thus γ crosses from the region where φ is negative to the region where φ is positive.



Thus on one side of M, φ is positive and on the other it is negative. Of course, M^{n-1} does not have to be connected so that there can be several regions where φ is positive and negative. For example, let $\varphi(x,y) = (x^2+y^2-1)(x^2+y^2-2)(x^2+y^2-3)(x^2+y^2-4)$. Then, $M = \varphi^{-1}(0)$ is four circles of radii 1, $\sqrt{2}$, $\sqrt{3}$, and 2.



<u>Corollary 3.3</u>: <u>Suppose</u> U <u>is a connected open set in</u> R^n <u>and</u> $\varphi: U \to R^1$ <u>is a</u> C^1<u>-function with</u> $D\varphi(p)$ <u>of rank</u> 1 <u>for all</u> $p \in \varphi^{-1}(0)$. <u>Then</u> $U - \varphi^{-1}(0)$ <u>has at least two connected components</u>.

<u>Proof</u>: $\varphi > 0$ and $\varphi < 0$ are open, disjoint, non-empty subsets of U.

<u>Examples</u>: 1) The equations:

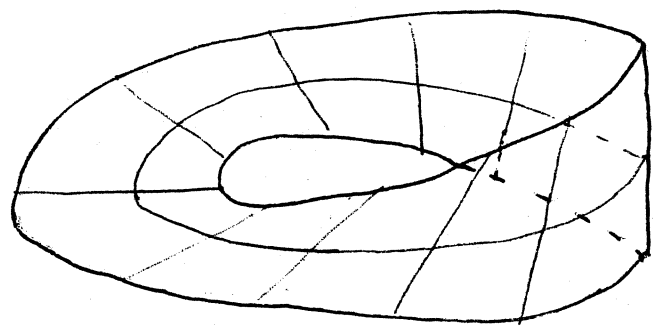
$$\begin{cases} x^2 + y^2 = R^2 \\ z = 0 \end{cases}$$

define a circle of radius R in R^3, S^1_R. Let U be the open set of $\{p \in R^3 | d(p,S^1_R) < \epsilon\}$. Here, $d(p,S^1_R) = \min_{x \in S^1_R} d(p,x)$.

(We choose $\epsilon < R/2$.) Inside U we have

$$\{((R+t\sin(\tfrac{\theta}{2}))\cos\theta, (R+t\sin(\tfrac{\theta}{2})\sin\theta, t\cos(\tfrac{\theta}{2})) \mid 0<|t|<\varepsilon \text{ and } 0\le\theta\le 2\pi\}.$$

It is the Möbius band.

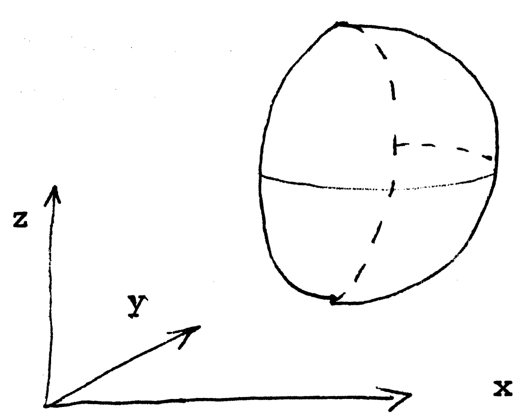


We claim that it is a 2 dimensional manifold inside U, but it is not given by one equation since it has only one "side", i.e. $U - M^2$ is connected. In fact, there is no open set V, $M \subset V \subset U$, and C^∞-function $f\colon V \to R^1$ so that rank $Df(m) = 1$ for every $m \in M$ and so that $M = f^{-1}(0)$.

One way to define manifolds in open sets of R^n is to adjoint inequalities to the equalities. For example:

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x > 0 \end{cases}$$

defines a hemisphere which is a hypersurface in the open set $\{(x,y,z) \mid x > 0\}$.



Thus a general hypersurface in an open set would be given by :

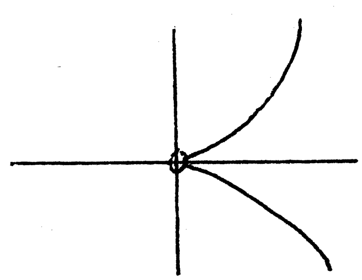
$$\begin{cases} f(x_1,\ldots,x_n) = 0 \\ \varphi_1(x_1,\ldots,x_n) > 0 \\ \vdots \\ \varphi_k(x_1,\ldots,x_n) > 0 \end{cases}$$

with the proviso that $\nabla f(p) \neq 0$ for all p such that $f(p) = 0$ and $\varphi_i(p) > 0$.

2)

$$\begin{cases} y^2 = x^3 \\ x^2 + y^2 > 0 \end{cases}$$

defines a C^∞-manifold in $R^2 - \{(0,0)\}$.



In effect, we have removed the singular point of the cusp.

Exercises: 1) Suppose $f(z_1,\ldots,z_n) = 0$ is a complex analytic function defined in an open set $U \subset \mathbb{C}^n$. If

$(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n})$ are never all zero, then show that f defines a C^∞-manifold of dimension $(2n - 2)$.

2) Let $p(x,y) = ax^2 + bxy + cy^2$ be a quadratic polynomial. Show that $p(x,y) = 1$ defines a C^∞-manifold.

3) Show that the Möbius band as described in example 2 is a 2-dimensional manifold in U, i.e. show that for each $p \in M$ there is an open set W containing p and a C^∞-function $f: W \to R^1$ so that $f^{-1}(0) = M \cap W$ and so that $Df(x)$ is rank 1 for every $x \in M \cap W$.

4) Show that

$$\begin{cases} f(x_1,\ldots,x_n) = 0 \\ \|\nabla f(x_1,\ldots,x_n)\|^2 > 0 \end{cases}$$

defines a hypersurface in the open set $\{(x_1,\ldots,x_n) | Df(x_1,\ldots,x_n) \neq 0\}$.

From the given formulation, the name "Implicit Function Theorem" seems somewhat mysterious. There is, however, another (slightly stronger) formulation which explains the name more clearly. Suppose $U \subset R^n$ is an open set and $f: U \to R^k$ is a given C^r-function. We say that the level set $f^{-1}(y_1,\ldots,y_k)$ implicitly defines $x_{n-k+1},\ldots,x_n$ as functions of $(x_1,\ldots,x_{n-k})$ if and only if there are C^r-functions on an open set W in R^{n-k}, $g_{n-k+1},\ldots,g_n$: $W \to R^1$, so that

$$f^{-1}(y_1,\ldots,y_k)$$

$$= \{(x_1,\ldots,x_{n-k},g_{n-k+1}(x_1,\ldots,x_{n-k}),\ldots,g_n(x_1,\ldots,x_{n-k}))\,|\,(x_1,\ldots,x_{n-k})\in W\}.$$

This means that the level set $f^{-1}(y_1,\ldots,y_k)$ is actually the graph of a function g: $W \to R^k$, $g = (g_{n-k+1},\ldots,g_n)$. With this in mind we give the reformulation of the Implicit Function Theorem.

Theorem 3.4: Let $U \subset R^n$ be an open set, and let f: $U \to R^k$ be a C^r-function with $f(p) = 0$ and with

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right) \qquad (i = 1,\ldots,k;\ j = n-k+1,\ldots,n)$$

an invertible $(k \times k)$-matrix. Then there is an open set $V \subset U$ with $p \in V$, an open set $W \subset R^n$, and a C^r-diffeomorphism Φ: $W \to V$,

$$\Phi(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-k},\Phi_{n-k+1}(x_1,\ldots,x_n),\ldots,\Phi_n(x_1,\ldots,x_n)),$$

so that $f\circ\Phi(x_1,\ldots,x_n) = (x_{n-k+1},\ldots,x_n)$.

Note that if we fix $(y_1,\ldots,y_k) \in R^k$ and define $g_i(x_1,\ldots,x_{n-k})$ to be $\Phi_i\ (x_1,\ldots,x_{n-k},y_1,\ldots,y_k)$, then the graph of $g = (g_{n-k+1},\ldots,g_n)$ is equal to $f^{-1}(y_1,\ldots,y_k)$.

Hence, Φ describes, all at once, every level set $\{f^{-1}(y_1,\ldots,y_k)\} \cap V$ as the graph of a function

$$\Phi(x_1,\ldots,x_{n-k},y_1,\ldots,y_k)$$
$$= (\Phi_{n-k+1}(x_1,\ldots,x_{n-k},y_1,\ldots,y_k),\ldots,\Phi_n(x_1,\ldots,x_{n-k},y_1,\ldots,y_k)).$$

This version of the theorem is stronger than the first version since it restricts the type of C^r-diffeomorphism which is allowed. To see that the second version actually implies the first, note that if we have $f\colon U \to R^k$ with $Df(p)$ of rank k, then there are k coordinates (which after renumbering we can assume to be $(x_{n-k+1},\ldots,x_n)$) so that

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right) \qquad (i = 1,\ldots,k;\ j = n-k+1,\ldots,n)$$

is invertible.

<u>Example</u>: $\Sigma_{i=1}^{n} x_i^2 = 1$, near the point $(0,0,\ldots,0,1)$, implicitly defines x_n as a function of $(x_1,\ldots,x_{n-1})$, namely $x_n = +\sqrt{1-x_1^2-x_2^2 -\ldots- x_{n-1}^2}$. Near $(0,0,\ldots,0,-1)$, x_n is implicitly defined as $-\sqrt{1 - x_1^2 -\ldots- x_{n-1}^2}$.

We shall deduce the Implicit Function Theorem from a special case $(k = n)$ which is called the Inverse Function Theorem.

Theorem 3.5 (Inverse Function Theorem): Let $U \subset R^n$ be an open set and let $f\colon U \to R^n$ be a C^r-map with Df_p invertible. There are open sets V and W in R^n, with $x \in V \subset U$, so that $f|V\colon V \to W$ is a C^r-diffeomorphism. In particular, $f|V\colon V \to W$ has a C^r-inverse $f^{-1}\colon W \to V$.

Proof that Inverse Function Theorem $\Rightarrow$ Implicit Function Theorem:

Let $f\colon U \to R^k$ be a C^r-map on an open subset of R^n and suppose

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right) \qquad (i = 1,\ldots,k;\ j = n-k+1,\ldots,n)$$

is invertible. Define $F\colon U \to R^{n-k} \times R^k$ by

$$F(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-k}, f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n))$$

This map, F, is C^r and DF is given by the matrix

$$\begin{array}{cc}
 & \begin{array}{cc} \qquad (n-k) \qquad & \qquad k \qquad \end{array} \\
\begin{array}{c} (n-k) \\ \\ \\ \\ k \end{array} &
\left(\begin{array}{c|c}
\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{matrix} & 0 \\ \hline
\multicolumn{2}{c}{\begin{matrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{matrix}}
\end{array}\right)
\end{array}$$

Hence, DF_x is invertible if and only if

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right) \qquad (i = 1,\ldots,k;\ j = n-k+1,\ldots n)$$

is invertible.

Thus $DF(p)$ is invertible. Consequently, there are open sets $V \subset U$, with $x \in V$, and $W \subset R^{n-k} \times R^k$ so that $F|V\colon V \to W$ is a C^r-diffeomorphism. Consider $F^{-1}\colon W \to V$. Clearly $F^{-1}(y_1,\ldots,y_n)$

$= (y_1,\ldots,y_{n-k},F^{-1}_{n-k+1}(y_1,\ldots,y_n),\ldots,F^{-1}_n(y_1,\ldots,y_n))$ and $f\circ F^{-1}(y_1,\ldots,y_n) = (y_{n-k+1},\ldots,y_n)$. Hence, F^{-1} is the diffeomorphism required by the Implicit Function Theorem (strong version).

<u>Proof of the Inverse Function Theorem:</u>

<u>Lemma 3.6</u>: <u>Let</u> U <u>and</u> W <u>be open sets in</u> R^n <u>and</u> $f\colon U \to W$ <u>be a</u> C^r-<u>homeomorphism with</u> $f^{-1}\colon W \to U$ <u>differentiable at every point</u> $p \in W$. <u>Then</u> f^{-1} <u>is a</u> C^r <u>map, i.e.</u> f <u>is a</u> C^r-<u>diffeomorphism</u>.

<u>Proof</u>: Let $\text{Auto}(R^n) \subset \mathcal{L}(R^n,R^n)$ be the open subset of invertible linear maps. ($\text{Auto}(R^n)$ is open since it is defined by the condition {determinant $\neq 0$}.) Consider the map $\omega\colon \text{Auto}(R^n) \to \text{Auto}(R^n)$ given by sending an invertible linear map to its inverse. This is a C^∞-

diffeomorphism. The reason is that, in terms of matrix entries, the map ω is given by

$$(\alpha_{ij}) \overset{\omega}{\longmapsto} \frac{1}{\det(\alpha_{ij})}((-1)^{i}M_{ij})$$

where M_{ij} is the determinant of the $(n-1) \times (n-1)$-matrix obtained by deleting the i^{th}-row and the j^{th}-column of (α_{ij}). (Recall that the determinant of a matrix is a polynomial in the entries.) Let $f\colon U \to W$ be a C^r-homeomorphism with $f^{-1}\colon W \to U$ differentiable at every point of W. By the chain rule $D(f^{-1})_{f(p)} = \omega(Df_p)$. Thus $D(f^{-1})\colon W \to \mathcal{L}(R^n,R^n)$ is given by the composition

$$W \xrightarrow{f^{-1}} U \xrightarrow{Df} \mathrm{Auto}(R^n) \xrightarrow{\omega} \mathrm{Auto}(R^n).$$

By assumption f is a C^r-homeomorphism, $r \geq 1$. Hence f^{-1} and Df are continuous. Since ω is C^∞, it follows that $D(f^{-1})\colon W \to \mathcal{L}(R^n,R^n)$ is continuous, i.e., f^{-1} is C^1. Suppose that we have shown that f^{-1} is C^s, $1 \leq s < r$. Then $D(f) = \omega \circ Df \circ f^{-1}$ is C^s. This means that f is C^{s+1}. This proves, inductively, that f^{-1} is actually C^r, and consequently that f is a C^r-diffeomorphism.

<u>Lemma 3.7</u>: <u>Let</u> $L\colon R^n \to R^n$ <u>be a linear map. Define</u> $m(L) = \min_{x\in S^{n-1}} \|L(x)\|$ <u>and</u> $M(L) = \max_{x\in S^{n-1}} \|L(x)\|$. <u>Then,</u>

$0 \leq m \leq M$ <u>and</u> $m\|x\| \leq \|L(x)\| \leq M\|x\|$ <u>for all</u> $x \in R^n$.

L <u>is invertible if and only if</u> $0 < m(L)$. <u>The functions</u> m, $M: \mathcal{L}(R^n, R^n) \to R^1$ <u>are continuous</u>.

(Note: By Lemma 1 this minimum and this maximum exist since S^{n-1} is compact.)

<u>Proof</u>: Since L is linear $L(x) = \|x\| \cdot L(x/\|x\|)$ for any $x \neq 0$. Thus $m(L) \cdot \|x\| \leq \|L(x)\| \leq M(L) \cdot \|x\|$ for any $x \in R^n$. Also $m(L) \neq 0$ if and only if $L(x) = 0$ implies $x = 0$. This means that $m(L) \neq 0$ if and only if $L: R^n \to R^n$ is injective. But any injective linear map from R^n to R^n is an isomorphism.

Let us show that $m: \mathcal{L}(R^n, R^n) \to R^1$ is continuous. (The argument for M is similar.) Let $\{L_i\} \to L$ in $\mathcal{L}(R^n, R^n)$. Choose $x \in S^{n-1}$ so that $\|L(x)\| = m(L)$, and choose $\epsilon > 0$. We know that $\{\|L_i(x)\|\} \to \|L(x)\|$ and that $m(L_i) \leq \|L_i(x)\|$. From this it follows that $m(L_i) - \epsilon \leq m(L)$ for all sufficiently large i. Let us show, conversely, that $m(L_i) + \epsilon \geq m(L)$ for all sufficiently large i. These two inequalities together imply that $\lim_{i \to \infty} m(L_i) = m(L)$. This, of course, implies that m is continuous. We prove the second inequality by contradiction. If it does not hold, then there are integers $n_1 < n_2 < \ldots$ such that $m(L_{n_k}) + \epsilon \leq m(L)$ for all k. Choose $x_{n_k} \in S^{n-1}$ so that $\|L_{n_k}(x_{n_k})\| = m(L_{n_k})$. By taking a subsequence, if necessary, we can assume that $\{x_{n_k}\} \to x$ in S^{n-1} (since S^{n-1} is compact). The evaluation map $\mu: \mathcal{L}(R^n, R^n) \times R^n \to R^n$ is continuous. Hence $\lim_{k\to\infty} L_{n_k}(x_{n_k}) = L(x)$, and consequently,

$\lim_{k\to\infty} m(L_{n_k}) = \lim_{k\to\infty} \|L_{n_k}(x_{n_k})\| = \|L(x)\| \geq m(L)$. This is impossible since $m(L_{n_k}) + \epsilon \leq m(L)$ for all k. This contradiction shows that $m(L_i) + \epsilon > m(L)$ for all sufficiently large i.

<u>Lemma 3.8</u>: <u>Let</u> $U \subset R^n$ <u>be an open set, and let</u> $f: U \to R^k$ <u>be a</u> C^1<u>-function. Define</u> $\varphi: U \times U \to R^1$ <u>by</u>

$$\varphi(x,y) = \begin{cases} \dfrac{\|f(x)-f(y)-Df_y(x-y)\|}{\|x-y\|} & \text{for } x \neq y \\ 0 & \text{for } x = y. \end{cases}$$

<u>Then</u> φ <u>is a continuous function</u>.

<u>Proof</u>: Clearly φ is continuous everywhere except possibly along the diagonal $\{x=y\}$. To show that it is continuous there we must show that given $p \in U$ and $\epsilon > 0$ there is $\delta > 0$ so that

$$\|f(x) - f(y) - Df_y(x-y)\| \leq \epsilon \cdot \|x-y\|$$

for all $x,y \in B_\delta(p)$. Choose $\delta > 0$ so that $B_\delta(p) \subset U$ and so that $|\frac{\partial f}{\partial x_i}(q) - \frac{\partial f}{\partial x_i}(p)| < \frac{\epsilon}{2n}$ for all $q \in B_\delta(p)$. (Here we use the fact that f is C^1.) It follows that for any points $q,q' \in B_\delta(p)$

$$|\frac{\partial f}{\partial x_i}(q') - \frac{\partial f}{\partial x_i}(q)| < \frac{\epsilon}{n}.$$

Recall that if $x - y = (h_1,\ldots,h_n)$, then

$$f(x) - f(y) = \int_0^1 \sum_{i=1}^n h_i \cdot \frac{\partial f}{\partial x_i}(y + t(x-y))dt.$$

Thus,

$$f(x) - f(y) - Df_y(h) = \int_0^1 \sum_{i=1}^n h_i \cdot (\frac{\partial f}{\partial x_i}(y+t(x-y)) - \frac{\partial f}{\partial x_i}(y))dt;$$

and hence

$$\|f(x) - f(y) - Df_y(h)\| \le \sum_{i=1}^n |h_i| \cdot \frac{\epsilon}{n} \le \epsilon \cdot \|h\|$$

for any $x,y \in B_\delta(p)$.

<u>Proof of the Inverse Function Theorem</u>: Suppose that we have a C^r-function $f: U \to R^n$ with Df_p invertible. We shall find a smaller open set $V \subset U$, with $p \in V$, so that:

1) $f|V$ is 1-1,

2) $f(V) \subset R^n$ is open,

3) $f^{-1}: f(V) \to V$ is continuous, and

4) $f^{-1}: f(V) \to V$ is differentiable at every point $g \in f(V)$.

By Lemma 3.6 this will prove that $f: V \to f(V)$ is a C^r-diffeomorphism.

Since Df_p is invertible, $m(Df_p) > 0$. Since f is C^1 and m,M are continuous, there are constants $0 < a < A$ and an open set $V_1 \subset U$ so that $a \leq m(Df_q)$ and $A \geq M(Df_q)$ for all $q \in V_1$. Since the function $\varphi: U \times U \to \mathbb{R}^1$ of Lemma 3.8 is continuous, there is $\delta > 0$ so that $\overline{B_\delta(p)} \subset V_1$ and

$$(\dagger) \qquad \|f(x') - f(x) - Df_x(x'-x)\| \leq \frac{a}{2}\|x' - x\|$$

for all x and x' in $\overline{B_\delta(p)}$. Since $a\|x' - x\| \leq \|Df_x(x'-x)\| \leq A\|x' - x\|$, we see that

$$(*) \qquad \frac{a}{2}\|x' - x\| \leq \|f(x') - f(x)\| \leq (A + \frac{a}{2})\|x' - x\|$$

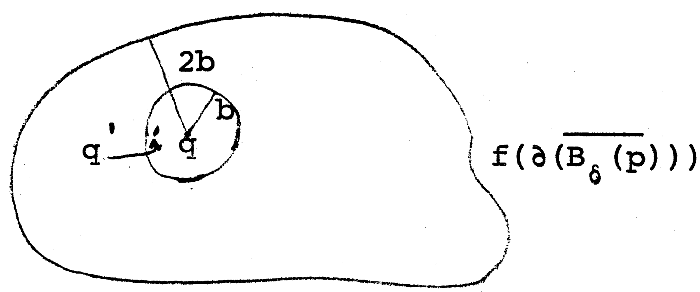
for all x and x' in $\overline{B_\delta(p)}$. Let V be $B_\delta(p)$. Then from the first inequality in $(*)$ it follows immediately that $f|\overline{V}$ is one-to-one, and that $f^{-1}: f(V) \to V$ is continuous.

Next, we claim that $f(V) \subset \mathbb{R}^n$ is an open set. Let $q = f(v)$ for some $v \in V$. Let b be one half the distance from q to $f(\partial(\overline{B_\delta(p)}))$. Note that $b > 0$, i.e., $q \notin f(\partial(\overline{B_\delta(p)}))$, since $f|\overline{B_\delta(p)}$ is one-to-one and $q = f(v)$ for some $v \in B_\delta(p)$.

Claim: $B_b(q) \subset f(V)$.

If we can establish this claim, then we shall have proven that $f(V) \subset R^n$ is open.

Proof of Claim: Let $q' \in B_b(q)$. Define $\psi: \overline{B_\delta(p)} \to R^1$ by $\psi(x) = \|f(x) - q'\|^2$. Clearly, $\psi(v) < b^2$, whereas for any $x' \in \partial(\overline{B_\delta(p)})$, $\psi(x') > b^2$ by the triangle inequality:



Since $\overline{B_\delta(p)}$ is compact, ψ achieves its minimum at some point $x \in \overline{B_\delta(p)}$. Since $\psi(v) < \psi(x')$ for any $x' \in \partial(\overline{B_\delta(p)})$, the point x must be in $B_\delta(p) = V$. At such a minimum, $D\psi_x(h) = 0$ for all h in R^n. By the chain rule, $D\psi_x(h) = 2(f(x) - q')\cdot Df_x(h)$. Since Df_x is invertible, we conclude that $f(x) - q' = 0$, i.e. that $f(x) = q'$. This proves the claim.

Lastly, we must show that $f^{-1}\colon f(V) \to V$ is differentiable. Let $y \in f(V)$ and $y + h \in f(V)$ with $f(x) = y$ and $f(x') = y + h$. Given $\epsilon > 0$ there is $\mu > 0$ so that if $\|x'-x\| < \mu$, then $\|h - Df_x(x'-x)\| < \epsilon \cdot \|x'-x\|$. Of course, $\|x'-x\| < \frac{2\|h\|}{a}$. Thus, if $\|h\| < \frac{a}{2}\mu$, then

$$\begin{aligned}\|(Df_x)^{-1}(h-Df_x(x'-x))\| &\leq M((Df_x)^{-1}\|h - Df_x(x'-x)\| \\ &< M((Df_x)^{-1}) \cdot \epsilon \cdot \|x'-x\| \\ &\leq M((Df_x)^{-1}) \cdot \epsilon \cdot \frac{2}{a}\|h\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}(Df_x)^{-1}(h-Df_x(x'-x)) &= -(x'-x-(Df_x)^{-1}(h)) \\ &= -(f^{-1}(y+h)-f^{-1}(y)-(Df_x)^{-1}(h).\end{aligned}$$

Putting these two statements together proves that

$$\lim_{\|h\| \to 0} \frac{\|f^{-1}(y+h) - f^{-1}(y) - (Df_{f^{-1}(y)})^{-1}(h)\|}{\|h\|} = 0$$

This proves that f^{-1} is differentiable and that $D(f^{-1})_{f(x)} = (Df_x)^{-1}$. This completes the proof of the Inverse Function Theorem.

§4. Manifolds--The Abstract Definition.

The definition of manifolds as the level sets of certain functions defined on open sets of R^n has the disadvantage of carrying much excess baggage along. For most considerations, the fact that M is a subset of R^n or that certain functions define it is totally irrelevant and, in fact, only obscures the central issue. What is important is that M be a space with systems of local coordinates which differ by C^r-diffeomorphism. In this section, we emphasize this more abstract point of view by giving a second definition of a manifold. We will also compare the two definitions.

Definition: A pre-C^r-manifold of dimension n is a triple--a topological space X, an open cover $\{U_\alpha\}_{\alpha\in I}$ of X, and homeomorphisms $\varphi_\alpha\colon V_\alpha \to U_\alpha$, where V_α is an open subset of R^n--which satisfies the following axioms:

1) X is a Hausdorff, metrizable space.
2) $\varphi_\beta^{-1}\cdot\varphi_\alpha\colon \varphi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \varphi_\beta^{-1}(U_\alpha \cap U_\beta)$ is a C^r-diffeomorphism between open sets in R^n for all $\alpha,\beta \in I$.

Clearly, condition 2) posits the existence of local coordinate systems which overlap in a C^r-manner. Axiom 1 requires amplification. First of all, it is not a consequence of all the other assumptions (i.e., that X is locally homeomorphic to R^n). For example, take two copies of R^1, $R^1 \times \{a\}$ and $R^1 \times \{b\}$, and identify $\{x \times a\}$ with $\{x \times b\}$ for all $x \neq 0$. This produces a line with a double origin:

which is locally homeomorphic to R^1 but which is not Hausdorff. The condition that X be metrizable once it is Hausdorff is equivalent to X's being paracompact. This means that any open covering has a countable, locally finite refinement. This will be assured if X is covered by countably many open sets U_α which are homeomorphic to open subsets of R^n. The standard "nonmetrizable manifold" is the "long line". Let Ω be the first uncountable cardinal and consider the set of cardinals $S = \{\alpha | \alpha < \Omega\}$. This is an uncountable set but each element in S has at most countably many predecessors. Let $W = S \coprod_{\alpha \in S} \alpha \times T$ where T is the open interval $(0,1)$. Define an ordering on W by:

1) restricted to $S \subset W$, the ordering is the usual one,

2) $\alpha \times t < \alpha$ for any $t \in T$

3) $\alpha < \beta \times t$ if $\alpha \leq \beta$,

4) $\alpha \times t < \beta$ if $\alpha < \beta$,

5) $\alpha \times t < \beta \times s$ if $\alpha < \beta$ or if $\alpha = \beta$ and $t < s$.

Once given an order, define open intervals (v,u) for $v,u \in W$ to be $\{a \in W | v < a < u\}$. An arbitrary open set is a union of open intervals. This defines a topology on W which makes it the "long line", L. One establishes the following:

1) L is Hausdorff and locally Euclidean.

2) Any sequence in L has a convergent subsequence.

3) If L were a metric space, then, for each integer n, it would have a finite set X_n so that every $p \in L$ is within distance $\frac{1}{n}$ to some point in X_n.

4) If L were a metric space, then it would have a countable, dense subset.

5) L has no countable, dense subset.

These examples show the necessity of assuming Hausdorff and metrizable; but the main condition of interest is the existence of local coordinates which overlap in a C^r manner.

The homeomorphisms $\varphi_\alpha \colon V_\alpha \to U_\alpha$, where $V_\alpha \subset R^n$ set and $U_\alpha \subset M$ are open, are called <u>charts</u>. We think of such a chart as giving coordinates $(x_1,\dots,x_n)$ valid in U_α. Actually, if $(x_1,\dots,x_n)$ are the usual coordinates on V_α, then the induced coordinates on U_α are $(x_1 \circ \varphi_\alpha^{-1},\dots,x_n \circ \varphi_\alpha^{-1})$. (Here, we are viewing x_i as a function $x_i \colon V_\alpha \to R$.) A collection of charts $\{U_\alpha,\varphi_\alpha,V_\alpha\}_{\alpha\in I}$ which cover M, i.e., so that $\bigcup_{\alpha\in I} U_\alpha = M$, is called a C^r-atlas. An atlas $\{U_\alpha,\varphi_\alpha,V_\alpha\}_{\alpha\in I}$

determines a unique maximal C^r-atlas consisting of all homeomorphisms, $\varphi\colon V \to U$ from open sets in R^n to open sets in M, with the property that $\varphi_\alpha^{-1}\circ\varphi\colon \varphi^{-1}(U \cap U_\alpha) \to \varphi_\alpha^{-1}(U \cap U_\alpha)$ is a C^r-diffeomorphism for all $\alpha \in I$. Any element in the maximal atlas defines C^r-coordinates on some open subset of M.

<u>Definition</u>: A C^r-<u>manifold</u> is a Hausdorff, metrizable space M and a maximal C^r-atlas for M.

Of course, any pre-C^r-manifold determines a C^r-manifold but many different pre-C^r-manifolds can determine the same one.

Note that if M is a C^r-manifold and $U \subset M$ is an open set, then U itself inherits the structure of a C^r-manifold. Thus any open subset of R^n is a C^∞-manifold.

If $(X,\{U_\alpha,\varphi_\alpha,V_\alpha\})_{\alpha\in I}$ is a pre-C^r-manifold and $f\colon X \to R$ is a continuous function, then f is said to be class C^s for any $s \leq r$ provided that $f\circ\varphi_\alpha\colon V_\alpha \to R$ is of class C^s for every $\alpha \in A$. Of course, if we check the condition that $f\circ\varphi_\alpha$ is C^s for all φ_α forming an atlas, then it follows for all the φ_α in the maximal atlas that they generate. Similarly, we define $f\colon X \to R^k$ to be C^s if all its coordinate functions are C^s. If $(X,\{U_\alpha,\varphi_\alpha,V_\alpha\})$ and

$(Y, \{U'_\alpha, \varphi'_\alpha, V'_\alpha\})$ are pre-C^r-manifolds, then $f\colon X \to Y$ is of class C^s for any $s \leq r$, provided that the composition:

$$\varphi_\alpha^{-1}(f^{-1}(U_{\beta'})) \xrightarrow{\varphi_\alpha} f^{-1}(U_{\beta'}) \xrightarrow{f} U_{\beta'} \xrightarrow{\varphi_{\beta'}^{-1}} V_{\beta'}$$

is of class C^s for all pairs (α, β'). The map $f\colon X \to Y$ is a C^r-diffeomorphism if f is a homeomorphism and both f and f^{-1} are C^r- functions. Two pre-C^r-manifold structures on a space M define the same C^r-manifold structure if and only if the identity $Id_M\colon M \to M$ is a C^r-diffeomorphism from one structure to the other.

Example: Give R^1 the usual structure as a C^∞-manifold. Use the homeomorphism $t \mapsto t^3$ to define a different C^∞-manifold structure. In the second structure $\varphi\colon U \to R^1$ is a C^∞-mapping on $U \subset R^1$ if and only if $\varphi(t^3)$ defines a C^∞-mapping in t. Call this new structure R'. These two structures are different since $\sqrt[3]{t}$ is C^∞ on R' but not on R^1. These manifolds are, however, C^∞-diffeomorphic. In fact $\rho\colon R' \to R^1$ given by $\rho(t) = \sqrt[3]{t}$ is a C^∞-diffeomorphism. It turns out that in higher dimensions one can find two different C^∞-manifold structures on a topological space which are not even diffeomorphic. The lowest dimensional example of this is S^7 where there are 28 distinct differentiable structures.

In our definition of an atlas we required each V_α to be an open set in the same dimensional Euclidean space. If M is connected, then this requirement is superfluous; it actually follows from the other axioms. To see this, let $\{U_\alpha, \varphi_\alpha, V_\alpha\}_{\alpha \in I}$ be a C^r-atlas except for this condition and let $\bigcup_{\alpha \in I} U_\alpha$ be connected. If $U_\alpha \cap U_\beta \neq \emptyset$, then V_α and V_β must be open sets in the same dimensional Euclidean space. The reason is that $\varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\varphi_\beta^{-1}(U_\alpha \cap U_\beta)$ are diffeomorphic and hence by exercise 4 of section 2 are open sets in the same dimensional Euclidean space. Define $W_n \subset M$ to be $\bigcup_{\{\alpha | \dim V_\alpha = n\}} U_\alpha$. Clearly, $\bigcup_{n \geq 0} W_n = M$, and each W_n is open. By the above discussion, $W_n \cap W_m = \emptyset$ if $n \neq m$. If M is connected, then all the W_n except 1 must be empty. If M is not connected, then its various components can have different dimensions (if we drop the requirement that all the V_α be of the same dimension). If every component has dimension n, then we say that M is of dimension n.

When dealing with C^∞-manifolds and C^∞-maps, we shall use the words differentiable manifold or differentiable function. When we are dealing with C^r-manifolds, we shall say C^r explicitly. Though much of what we shall do for C^∞-manifolds can be carred through for C^r-manifolds $r \geq 1$, we shall usually adopt the easier course of restricting to the C^∞-case.

<u>Exercises:</u> 1) Show that the ellipse $\{\Sigma_{i=1}^{n}(\frac{x_i}{a_i})^2 = 1\}$ and S^{n-1} are diffeomorphic.

2) Given a C^∞-atlas for S^{n-1}.

3) Given a C^∞-atlas for $\{(\zeta,z) \in \mathbb{C}^2 \mid \zeta^2 = z^3 - 1\}$.

§5: Examples of Differentiable Manifolds

One general way to construct new manifolds from old ones is by taking quotients of certain group actions. Even if the original manifold comes equipped with defining equations in R^n, the quotient manifold may have no such natural description. Thus, when taking quotients, it is much easier to work with abstract manifolds.

Let G be a group and give G the discrete topology. An action of G on X is a continuous map

$$\varphi: G \times X \longrightarrow X$$

such that $\varphi(gh,x) = \varphi(g,\varphi(h,x))$, $\varphi(e,x) = x$. It follows immediately that $\varphi(g,\ ): X \to X$ is a homeomorphism whose inverse is $\varphi(g^{-1},\ ): X \to X$. This leads to an alternate description of an action of G on X: An action of G on X is a group homomorphism from G to Homeo(X), the group (under composition) of homeomorphisms of X. We often denote the homeomorphism associated to g by $x \mapsto g\cdot x$. An action is free if $g\cdot x = x$ from some $x \in X$ implies that $g = e$. An action is properly discontinuous if for every $x \in X$ there is an open set $U \subset X$ containing x, so that $g\cdot U \cap U = \emptyset$ for all but a finite number of elements $g \in G$. In an action, the <u>stabilizer</u> of a point $x \in X$ is the subgroup $\{g \in G | gx = x\}$. Thus, an action is free if and only if the stabilizer of every point is the identity subgroup.

Lemma 5.1: *If G acts properly discontinuously on X, then the stabilizer of every point is finite. Furthermore, if X is a Hausdorff space, then given $x \in X$ there is an open set $U \subset X$ containing x, so that $U \cap gU \neq \emptyset$ only for g in the stabilizer of x.*

Proof: The first assertion is clear. As for the second, consider $x \in X$ and $g \in G$ such that $gx \neq x$. Since X is Hausdorff, there are open sets V_x and V_{gx} containing x and gx respectively, such that $V_x \cap V_{gx} = \emptyset$. Consider $U = V_x \cap g^{-1}(V_{gx})$. This is an open set containing x, and $U \cap gU = \emptyset$. Now suppose G acts properly discontinuously on X. Choose U so that $U \cap gU \neq \emptyset$ for only finitely many $g \in G$, say $\{g_1,\ldots,g_T\}$. For each g_i which does not stabilize x, we choose U_i, an open set containing x, so that $U_i \cap g_iU_i = \emptyset$. The intersection

$$U \cap \Big(\bigcap_{\{i | g_ix \neq x\}} U_i \Big)$$

is the required open set.

Corollary 5.2: *If X is Hausdorff and G acts freely and* properly discontinuously on X, then for each $x \in X$ there is an open set U, containing x, so that $U \cap gU = \emptyset$ for all $g \neq e$.

Given an action $G \times X \to X$, we define the quotient space X/G. As a set, it is the equivalence classes under the relation $x \sim gx$ for all $x \in X$ and $g \in G$. (These classes are called the orbits of G, and the quotient space is the orbit space.) The topology on X/G is the quotient topology for the map $\pi: X \to X/G$. This means that $U \subset X/G$ is open if and only if $\pi^{-1}(U) \subset X$ is open.

If M is a C^r-manifold, then an action of G on M is a C^r-action if the homeomorphism induced by each $g \in G$ is a C^r-diffeomorphism.

Theorem 5.3: Let M be a C^r-manifold and let $G \times M \to M$ be a free, properly discontinuous, C^r-action. Then M/G naturally inherits the structure of a C^r-manifold so that $\pi: M \to M/G$ is a C^r-function.

Proof: We define an atlas for M/G. For each $x \in M$, choose $\varphi_\alpha: V_\alpha \to U_\alpha$, a C^r-chart with $x \in U_\alpha$, so that $U_\alpha \cap gU_\alpha = \emptyset$ for all $g \neq e$ in G. Consider $\pi(U_\alpha) \subset M/G$. It is an open set since its preimage is $\bigcup_{g \in G} gU_\alpha$. Furthermore, $\pi|U_\alpha: U_\alpha \to \pi(U_\alpha)$ is a homeomorphism. We take

$\pi \circ \varphi_\alpha$: $V_\alpha \to \pi(U_\alpha)$ to be a chart near $[x] \in M/G$. One checks that the overlap functions for this atlas are C^r, and hence that this atlas defines a C^r-manifold structure on M/G. Clearly, π: $M \to M/G$ is a C^r-map whose differential at $x \in M$, calculated in local C^r-coordinates, is of maximal rank.

Examples: 1) Let $\mathbb{Z}$ (the integers) act on R by translation $n \cdot r = n+r$. Clearly, this is a free, properly discontinuous, C^∞ action on R. The quotient manifold $R/\mathbb{Z}$ is C^∞-diffeomorphic to the circle: $S^1 = \{(x,y) | x^2 + y^2 = 1\}$. The C^∞-diffeomorphism

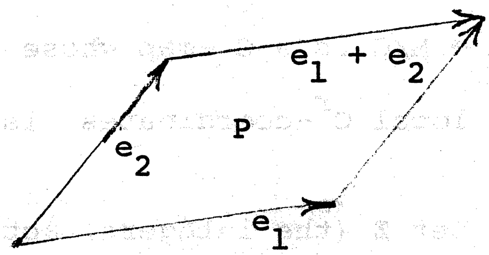
$$\varphi: R/\mathbb{Z} \longrightarrow S^1$$

is defined by

$$\varphi(r) = (\cos(2\pi r), \sin(2\pi r)).$$

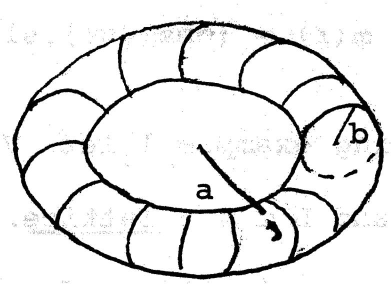
2) Generalizing example 1, let V be an n-dimensional real vector space and $L \subset V$ a lattice. This means L is all integral linear combinations of a basis $(e_1, \ldots, e_n)$ for V. We let L act on V by translation $\ell \cdot v = \ell + v$. This is clearly a free, C^∞-action. We claim that, in addition, it is properly discontinuous. To show this it suffices to show that there is an open set U, containing 0, so that $\ell \cdot U \cap \ell' \cdot U = \emptyset$ if $\ell \neq \ell'$ are lattice elements. The open set is all $\{\Sigma_{i=1}^n \alpha_i r_i \,\big|\, |\alpha_i| < \frac{1}{2}\}$. The quotient V/L is an

n-dimensional torus. As a specific example, let $n = 2$ and let L be spanned by e_1 and e_2.



Then R^2/L is obtained from the parallelogram P by identifying opposite sides. It is diffeomorphic to the following subset of R^3:

$$\{((a+b\cos\psi)\cos\theta, (a+b\cos\psi)\sin\theta, b\sin\psi) \mid 0 \leq \theta, \psi \leq 2\pi\}.$$



Notice that V/L is always a compact manifold since the compact set $\{\Sigma_{i=1}^{n} \alpha_i e_i \mid 0 \leq a_i \leq 1\}$ in V maps onto V/L.

3) There is a free action of the cyclic group of order 2, $\{e, \gamma \mid \gamma^2 = e\}$, on the sphere $S^{n-1} \subset R^n$. It is defined by $\gamma(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$. It is easily seen to be free, properly discontinuous, and C^∞. The quotient C^∞-manifold is called RP^{n-1}, real projective (n-1) space.

It is identified with the space of lines in R^n through the origin. Each line meets S^{n-1} in a pair of antipodal points, and hence each line determines a point in RP^{n-1} and vice versa. Since S^{n-1} is compact, so is RP^{n-1}. There is another way to think of RP^{n-1}. That is as a compactification of R^{n-1}. We define a mapping $R^{n-1} \to RP^{n-1}$ as follows. Let $n(x) = +\sqrt{\frac{1}{\|x\|^2+1}}$. Then send

$(x_1,\ldots,x_{n-1}) \mapsto [n(x)\cdot(x_1,\ldots,x_{n-1},1)]$. This is the obvious map from R^{n-1} to the hemisphere

$$\begin{cases} \sum_{i=1}^{n} y_i^2 = 1 \\ y_n > 0 \end{cases}$$

followed by the projection to RP^{n-1}. This map is a C^∞-diffeomorphism from R^{n-1} onto an open set in RP^{n-1}. The complement of the image is $\{[x_1,\ldots,x_{n-1},0]\}$, and hence the complement is an $RP^{n-2} \subset RP^{n-1}$. This copy of RP^{n-2} is said to be the "lines at ∞" in the compactification of R^{n-1}.

There is a complex version of this, $\mathbb{C}P^{n-1}$, complex projective space. It is the space of complex lines through the origin in $\mathbb{C}^n$. There is a map $\mathbb{C}^n - \{0\} \xrightarrow{\pi} \mathbb{C}P^{n-1}$ which associates to a non-zero point in $\mathbb{C}^n$ the unique complex line through it and the origin. $\mathbb{C}P^{n-1}$ has the quotient topology under this map. Thus points in $\mathbb{C}P^{n-1}$ are described by "homogeneous coordinates" $[z_1,\dots,z_n]$, not all zero, where $[z_1,\dots,z_n] = [\lambda z_1,\dots,\lambda z_n]$ for any $\lambda \in \mathbb{C} - \{0\}$. To show that $\mathbb{CP}^{n-1}$ is a C^∞-manifold, we describe an atlas for it. Let $U_i \subset \mathbb{CP}^{n-1}$ be all points represented by homogeneous coordinates $[z_1,\dots,z_n]$ with $z_i \neq 0$. Define

$$\varphi_i\colon \mathbb{C}^{n-1} \longrightarrow U_i$$

by $\varphi_i(\zeta_1,\dots,\zeta_{n-1}) = [\zeta_1,\zeta_2,\dots,\zeta_{i-1},1,\zeta_i,\dots,\zeta_{n-1}]$. Define $\psi_i\colon U_i \to \mathbb{C}^{n-1}$ by

$$\psi_i([z_1,\dots,z_n]) = \left(\frac{z_1}{z_i},\dots,\frac{z_{i-1}}{z_i},\frac{z_{i+1}}{z_i},\dots,\frac{z_n}{z_i}\right).$$

One checks easily that φ_i and ψ_i are well-defined, and that they are inverses. Thus, φ_i is a homeomorphism. Clearly $\bigcup\limits_{i=1}^{n} U_i = \mathbb{CP}^{n-1}$. Lastly, we claim that the overlap functions are C^∞. Let $i < j$ and consider $\varphi_i^{-1}(U_i \cap U_j) \subset \mathbb{C}^{n-1}$. It is all $(\zeta_1,\dots,\zeta_{n-1})$ such that $\zeta_{j-1} \neq 0$. Similarly $\varphi_j^{-1}(U_i \cap U_j)$ is all $(\zeta_1,\dots,\zeta_{n-1})$ with $\zeta_i \neq 0$. The map

$$\varphi_j^{-1}\circ\varphi_i\colon \varphi_i^{-1}(U_i \cap U_j) \longrightarrow \varphi_j^{-1}(U_i \cap U_j)$$

sends

$$(\zeta_1,\dots,\zeta_{n-1})$$

to

$$\left(\frac{\zeta_1}{\zeta_{j-1}},\ldots,\frac{\zeta_{i-1}}{\zeta_{j-1}},\frac{1}{\zeta_{j-1}},\frac{\zeta_i}{\zeta_{j-1}},\ldots,\frac{\zeta_{j-2}}{\zeta_{j-1}},\frac{\zeta_j}{\zeta_{j-1}},\ldots,\frac{\zeta_{n-1}}{\zeta_{j-1}}\right).$$

This is a diffeomorphism from $\{\zeta_{j-1} \neq 0\}$ to $\{\zeta_i \neq 0\}$. Thus, these charts define a C^∞-atlas, and consequently, a C^∞-manifold structure on $\mathbb{C}P^{n-1}$. As before, $\mathbb{C}P^{n-1}$ is a compactification of $\mathbb{C}^{n-1}$ obtained by adding the "complex lines at ∞" which form the complement of $\mathbb{C}^{n-1} \subset \mathbb{C}P^{n-1}$ (which is $\mathbb{C}P^{n-2}$).

Let M be a C^r-manifold of dimension n, and let $\{U_\alpha,\varphi_\alpha,V_\alpha\}$ be a C^r-atlas for it. Suppose that we have C^r-functions $f_\alpha\colon U_\alpha \to R^k$ so that

1) $(f_\alpha^{-1}(0)) \cap U_\beta = (f_\beta^{-1}(0)) \cap U_\alpha$, and

2) $Df_\alpha(p)$ is of rank k for all $p \in f_\alpha^{-1}(0)$.

Then, $\bigcup_\alpha (f_\alpha^{-1}(0)) \subset M$ is a manifold of dimension $(n-k)$.

As an example of this, let us extend the manifold M given by $\{\zeta^2 = z^3 - 1\}$ in $\mathbb{C}^2$ to a manifold $\bar{M} \subset \mathbb{C}P^2$. $\bar{M}$ will be the compactification of M. The first step is to add a third complex variable t to (ζ,z) and use $[\zeta,z,t]$ as homogeneous coordinates in $\mathbb{C}P^2$ with $\mathbb{C}^2$ being $[\zeta,z,1]$. Next make the equation homogeneous, i.e., replace it by

$$\zeta^2 t = z^3 - t^3.$$

Such a homogeneous equation has solution set in $\mathbb{C}^3$ consisting of a union of complex lines. For if $a^2c = b^3 - c^3$, then $(\lambda a)^2(\lambda c) = (\lambda b)^3 - (\lambda c)^3$ for all $\lambda \in \mathbb{C}$. Let $\bar{M} \subset \mathbb{C}P^2$ be the set of points corresponding to the lines in the solution set, i.e., $[a,b,c] \in \bar{M}$ if and only if $a^2c = b^3 - c^3$. If we consider $\bar{M} \cap \mathbb{C}^2$, then we have $\{[\zeta,z,1] \mid \zeta^2 = z^3 - 1\}$. Thus $\bar{M} \cap \mathbb{C}^2 = M$. Points at ∞, i.e., points in $\bar{M} - M$, have homogenous coordinates $[\zeta,z,0]$. For such a point to be in $\bar{M}$, it is necessary for z to be 0. Thus, there is only one point at ∞, $[1,0,0] = [\zeta,0,0]$. Let us consider the coordinates $(X,Y) = (z/\zeta, t/\zeta)$ in the open set $U = \{[\zeta,z,t] \mid \zeta \neq 0\}$. Clearly, $\bar{M} \cap U$ is given by the equation $Y = X^3 - Y^3$. Since the partial derivatives of this equation are $(3X^2, -3Y^2 - 1)$, they do not both vanish at any point of $\bar{M} \cap U$. Thus, $\bar{M}$ is a C^∞-manifold. Being a closed subset of $\mathbb{C}P^2$, it is compact.

It is true for any manifold $M \subset \mathbb{C}^2$ given by one polynomial equation $p(\zeta,z) = 0$ that we can form the compactification $\bar{M}$ of M in $\mathbb{C}P^2$. It will always be the case that $\bar{M} - M$ is a finite set of points. Often however, $\bar{M}$ will not be a differentiable manifold, i.e., it will have a singularity at one of its points at ∞.

A <u>Lie Group</u> is a C^∞-manifold G with a group multiplication so that the map $G \times G \to G$ defined by

$(g,h) \mapsto g^{-1}h$ is a C^∞-map.

Examples: 1) Auto(R^n), the space of linear automorphisms of R^n, is the open subset of $\mathcal{L}(R^n,R^n)$ given by the condition {determinant $\neq 0$}. Consequently, Auto(R^n) is a C^∞-manifold. The group law is given by composition of automorphisms. In terms of matrices, it is matrix multiplication. Thus $(g,h) \to g^{-1}h$ is a C^∞-mapping. This Lie group is also called the general linear group and is denoted GL(n,R).

2) R^n is a Lie group with the group law being translation.

3) The set of upper triangular real matrices with 1's down the diagonal:

$$\begin{pmatrix} 1 & * & * & & * \\ & 1 & * & & * \\ & & 1 & & \\ & & & \ddots & \\ & 0 & & & * \\ & & & & 1 \end{pmatrix}$$

is a Lie group under composition. It is a nilpotent Lie group.

4) $O(n) \subset GL(n,R)$, the orthogonal group, is a Lie group. It is the space of matrices (α_{ij}) whose columns, thought of as vectors in R^n, all have length 1 and which are mutually perpendicular. The group law is again matrix

multiplication. A more abstract definition of $O(n)$ is the subgroup of $\mathrm{Auto}(R^n)$ consisting of those automorphisms which preserve lengths of vectors in R^n and angles between vectors.

5) $SO(n)$ is the subgroup of $O(n)$ consisting of those matrices in $O(n)$ of determinant 1. Alternatively, it consists of those elements of $O(n)$ which preserve the orientation of R^n. Similarly, $SL(n,\mathbb{R})$ is the subgroup of $GL(n,\mathbb{R})$ consisting of those matrices of determinant 1.

Definition: If G is a Lie group and $\Gamma \subset G$ is a subgroup, then Γ is a discrete subgroup if and only if there is an open set $U \subset G$ so that $\gamma U \cap \gamma' U = \emptyset$ for all γ and γ' distinct elements of Γ.

Theorem 5.4: *If $\Gamma \subset G$ is a subgroup, then Γ acts on G via $\gamma \cdot g = \gamma g$. If $\Gamma \subset G$ is a discrete subgroup, then this action is free and properly discontinuous. Hence G/Γ is a differentiable manifold.*

Proof: This is immediate from the definitions.

Examples: 1) Let $P_n \subset R$ be a regular n-gon centered at the origin:

Consider the group of rigid motions of this figure, $S(P_n)$. It is a group of order 2n and a discrete subgroup of $O(2)$. The quotient $O(2)/S(P_n)$ turns out to be diffeomorphic to the circle.

2) Let $D \subset R^3$ be a regular dodecahedron in R^3 centered at the origin (12 pentagonal faces.) Consider the group of rotations (i.e. elements in $SO(3)$) which when applied to D bring it back to itself, Γ_D. This is a group of order 60. (In fact, it is A_5.) The quotient $SO(3)/\Gamma_D$ is a C^∞-manifold of dimension 3 first discovered by Poincaré. It is a counter-example to one of his early conjectures about manifolds.

There are many more examples of this type. In fact, there is active research today centered on such manifolds.

Exercises: 1) Show that $\mathbb{C}P^{n-1}$ is compact.

2) Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.

3) Show that RP^1 is diffeomorphic to S^1.

4) Show that $SO(n)$ is a connected Lie group.

5) Show that Γ_D in example 2 above has order 60.

6) If G is a Lie group and $\Gamma \subset G$ is a discrete subgroup, then Γ is said to be uniform if and only if there is a compact set $K \subset G$ so that $\bigcup_{\gamma \in \Gamma} \gamma K = G$. Show that Γ is uniform if and only if G/Γ is compact.

7) Show that $\mathbb{R}P^n$ can be described as follows. Its underlying set is $\mathbb{R}^n \amalg \mathbb{R}P^{n-1}$. The open subsets for the topology are all $U \amalg V \subset \mathbb{R}^n \amalg \mathbb{R}P^{n-1}$ such that:

a) $U \subset \mathbb{R}^n$ is open.

b) If $p \in V$, then there is an open set W_0 of $\mathbb{R}P^{n-1}$ with $p \in W_0 \subset V$, and $R > 0$, so that if $W = \{x \in \mathbb{R}^n | x$ is contained in a line in $W_0\}$, then $W \cap (\mathbb{R}^n - B_R(0))$ is contained in U.

§6. Further Notes and Generalizations.

Note that a C^r-manifold defines a C^s-manifold for all $1 \leq s < r$, but that different C^r-manifolds can define the same C^s-manifold.

One can define many other types of manifolds simply by restricting the overlap functions which one allows. Thus, consider $\mathcal{P}$ a subset of homeomorphisms between all pairs of open sets in R^n. $\mathcal{P}$ must satisfy the following axioms:

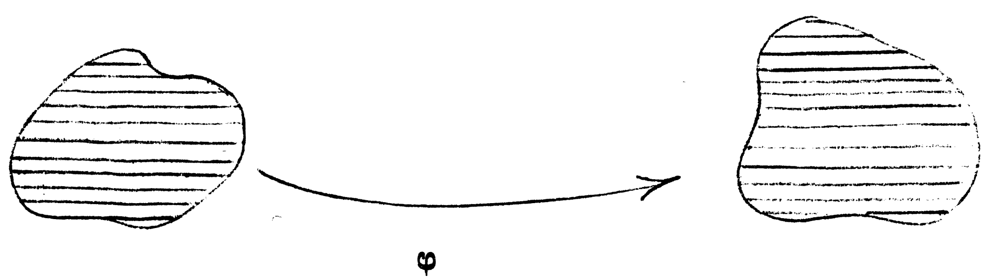
1) If $f\colon U \to V$ is in $\mathcal{P}$ and $W \subset V$ is an open set, then $f|W\colon W \to f(W)$ is in $\mathcal{P}$.
2) If $U = \cup U_\alpha$, and $f\colon U \to V$ is a homeomorphism with $f|U_\alpha\colon U_\alpha \to f(U_\alpha)$ in $\mathcal{P}$, then f is in $\mathcal{P}$.
3) $Id_U\colon U \to U$ is in $\mathcal{P}$.
4) Compositions of elements in $\mathcal{P}$ are in $\mathcal{P}$.
5) Inverses of elements in $\mathcal{P}$ are in $\mathcal{P}$.

<u>Note</u>: If $U \subset R^n$ is open and $f\colon U \to f(U) \subset R^n$ is a homeomorphism, then $f(U)$ is automatically open in R^n. We have proved this for C^1-diffeomorphisms. The proof for arbitrary homomorphisms is much more involved. Later in the course we shall give a proof using homology.

Given such a collection $\mathcal{P}$, we define $\mathcal{P}$-atlases and

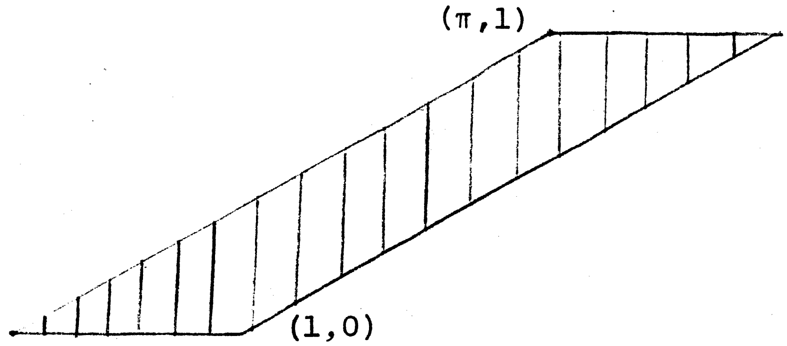
$\mathcal{P}$-manifolds as before. If we take $\mathcal{P}$ to be all homeomorphisms, the result is topological manifolds. If we take $\mathcal{P}$ to be C^r-diffeomorphisms, then the result is C^r-manifolds. If we take $\mathcal{P}$ to be real analytic diffeomorphisms, then the result is real analytic manifolds. If $n = 2k$ and we give R^n the structure of $\mathbb{C}^k$ and take $\mathcal{P}$ to be the complex analytic diffeomorphisms, then the resulting manifolds are complex analytic manifolds. $\mathbb{C}P^{n-1}$ is a complex analytic manifold.

As a different type of example consider R^n as $R^k \times R^{n-k}$ with coordinates (x,y) and take diffeomorphisms of the form $\varphi(x,y) = (\varphi_1(x,y),\varphi_2(y))$. These are precisely those diffeomorphisms which preserve the family of k-dimensional subspaces given by $\{y = \text{constant}\}$:



The resulting manifolds are C^∞-manifolds with <u>a codimension</u> $(n-k)$ <u>foliation</u>. Thus, M^n is written as a union of manifolds of dimension k, called the leaves of the foliation, which locally look like the family of $R^k \times \{\text{constant}\} \subset R^n$.

<u>Example</u>: Let $L \subset R^2$ be the lattice generated by $(1,0)$ and $(\pi,1)$:



The vertical foliation is preserved by the translations in L. Thus, the torus is foliated. Each leaf is a copy of R^1 and is everywhere dense (since π is irrational).

Sometimes it is hard to tell whether or not two manifolds of the same dimension are diffeomorphic. For example S^2, $\mathbb{C}P^1$, and the ellipse $\{x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ are all diffeomorphic. But RP^2, S^2 and R^2/L are all different. Of course, to show that two manifolds are diffeomorphic one constructs an explicit diffeomorphism between them. The usual way to show that two manifolds are not diffeomorphic is to find some numerical (or algebraic) invariant which is associated to each manifold which takes a different value on the two manifolds in question. (The word invariant here means that the thing associated to diffeomorphic manifolds is the same.) Much of this course will be concerned with defining suitable invariants. As a first example of such an invariant (at least for connected manifolds), we have the dimension.

We shall also need the concept of a manifold with boundary. Let $\mathbb{H}^n \subset R^n$ be the half space $\{(x_1,\ldots,x_n | x_n \geq 0\}$. If $U \subset \mathbb{H}^n$ is an open subspace of $\mathbb{H}^n$ define ∂U to be $U \cap \{(x_1,\ldots,x_n)|x_n = 0\} = U \cap \partial\mathbb{H}^n$.

<u>Lemma 6.1</u>: <u>If</u> U <u>and</u> V <u>are open sets in</u> $\mathbb{H}^n$ <u>and if</u> $f: U \to V$ <u>is a</u> C^1<u>-diffeomorphism, then</u> $f(\partial U) = \partial V$.

(A C^1-diffeomorphism between open sets in $\mathbb{H}^n$ is a map which is the restriction of a C^1-diffeomorphism between open sets of R^n to their intersections with $\mathbb{H}^n$.)

<u>Proof</u>: Suppose that there is a point $p \in \partial U$ so that $f(p) \notin \partial V$. By restricting to smaller open sets we can assume that $\partial V = \emptyset$ and $\partial U \neq \emptyset$. If $\partial V = \emptyset$, then V is open in R^n and hence U must be open in R^n (by the Inverse Function Theorem applied to f^{-1}). But this implies that $\partial U = \emptyset$. This contradiction proves that $f(\partial U) \subset \partial V$. Likewise, $f^{-1}(\partial V) \subset \partial U$. This means $f(\partial U) = \partial V$.

<u>Definition</u>: A C^r-<u>manifold with boundary</u> is a Hausdorff, metrizable space with an atlas $\{U_\alpha,\varphi_\alpha,V_\alpha\}_{\alpha\in I}$ where each V_α is an open subset of $\mathbb{H}^n$ and where the overlap functions $\varphi_\alpha^{-1}\circ\varphi_\beta$ are C^r-diffeomorphisms. If M is a C^r-manifold with boundary, then ∂M, the boundary of M, is $\bigcup_\alpha \partial U_\alpha$. Using

Lemma 6.1 one sees that ∂M is a C^r-manifold (without boundary) of dimension $(n-1)$.

A C^r-manifold with boundary whose boundary is empty is naturally identified with an ordinary C^r-manifold. If M is a C^r-manifold with boundary, then $\operatorname{int} M = M - \partial M$ is a C^r-manifold without boundary.

Algebraic varieties are defined similarly, in spirit, to the way manifolds are defined. Let $V \subset \mathbb{R}^n$ be defined by polynomials $\{p_1 = 0,\ldots,p_k = 0\}$. It is said to be an affine algebraic variety. A rational function on V is the restriction to V of a quotient of polynomials:

$$\frac{a(x_1,\ldots,x_n)}{b(x_1,\ldots,x_n)} \quad .$$

Strictly speaking such a rational function gives a continuous, real-valued function only on

$$V \cap \{(x_1,\ldots,x_n) \mid b(x_1,\ldots,x_n) = 0\}.$$

A rational map between affine varieties $V \to W$, $V \subset R^n$, $W \subset R^m$, is a function from $V \cap \{x \mid b_1 \neq 0, b_2 \neq 0,\ldots,b_m \neq 0\}$ to W of the form $v \mapsto (a_1(v)/b_1(v),\ldots,a_m(v)/b_m(v))$ with the $\{a_i\}$ and $\{b_i\}$ polynomials. A real algebraic variety is a Hausdorff space X with an open cover $\{X_\alpha\}$ and homeomorphisms $\varphi_\alpha\colon Y_\alpha \to X_\alpha$, with Y_α an affine variety, so that all overlap functions are rational maps.

This defines a real algebraic variety. An algebraic variety over any field, k, e.g. $\mathbb{C}$, $\mathbb{Z}/p$, $\mathbb{Q}[i]$, is defined analogously.

<u>Exercises</u>: 1) Let $p(x,y)$ be a complex polynomial in two (complex) variables. Suppose that for each (x_0,y_0) such that $p(x_0,y_0) = 0$, either $\partial p/\partial x(x_0,y_0)$ or $\partial p/\partial y(x_0,y_0)$ is non-zero show that $p^{-1}(0) \subset \mathbb{C}^2$ is a complex analytic manifold.

2) Let $\mathbb{C}P^n$ be complex projective n space. Suppose given a homogeneous polynomial $p(z_0,\ldots,z_n)$. Let $X \subset \mathbb{C}P^n$ be the solution set, i.e. $X = \{[z_0,\ldots,z_n] \mid p(z_0,\ldots,z_n) = 0\}$. Show that if for every $x \in X$ there is i such that $\partial p/\partial z_i(x) \neq 0$, then $X \subset \mathbb{C}P^n$ is a compact complex analytic manifold of <u>real</u> dimension $(2n-2)$, i.e. of complex dimension $(n-1)$.

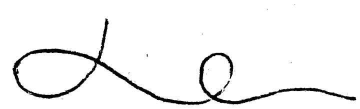
3) If M is an n-dimensional manifold show that $\operatorname{int} M = M - \partial M$ is a n-dimensional manifold and that ∂M is an $(n-1)$-dimensional manifold.

§7. Maps between Manifolds.

If X^n and Y^m are C^r-manifolds, then a function $f: X \to Y$ is C^r if for every pair of charts in the C^r-atlases for X and Y, $\varphi_\alpha: V_\alpha \to U_\alpha \subset X$ and $\psi_\beta: V_\beta \to W_\beta \subset Y$, the composition $\psi_\beta^{-1} \circ f \circ \varphi_\alpha$ is C^r. Of course, if $f|U$ is C^r in one pair of charts, then it will be C^r in any pair of charts since the change of coordinate functions are themselves C^r. Things like the matrix entries for Df_p will vary as we change the coordinates, and thus they have no intrinsic meaning. There is, however, a quantity associated with matrix representatives for Df_p which is invariant as we change coordinates. This is the rank of Df_p thought of as a linear map, from R^n to R^m. For as we change coordinates in X^n and Y^m, the resulting change in the matrix representing Df_p is given as follows. Suppose that in one pair of local coordinate systems for X near p and Y near $f(p)$, Df_p is given by (α_{ij}). Let us change coordinates in X and Y with (λ_{ij}) being the differential of the change of coordinates of $f(p) \in Y$ and (μ_{ij}) being the differential of the change of coordinates at $p \in X$. Then, in the new systems Df_p is represented by $(\lambda_{ij})(\alpha_{ij})(\mu_{ij})$. Since (λ_{ij}) is an invertible $(m\times m)$-matrix and (μ_{ij}) is an invertible $(n\times n)$-matrix, this alteration does not affect the rank (which is the dimension of the image).

Thus, if $f: X \to Y$ is a C^r-map, then associated to every $p \in X$ is an integer, $rk(Df_p)$. This is not necessarily a continuous function, but it is lower semi-continuous. A C^r-map $f: X^n \to Y^m$ is said to be an <u>immersion</u> if $rk(Df_p) = n$ for all $p \in X$. It is said to be a <u>submersion</u> if $rk(Df_p) = m$ for all $p \in X$.

<u>Examples</u>: The following are immersions of R^1 into R^2:

1) 

2) $$x \mapsto \left(\cos\left(\frac{2\pi}{1+e^x}\right),\ \sin\left(\frac{2\pi}{1+e^x}\right)\right)$$

3) The mapping $R^1 \to R^2/L$ defined by $x \mapsto [(x,0)]$ is an immersion of R^1 into the torus R^2/L. If $L \cap (R^1 \times \{0\}) \neq \emptyset$, then the image is a circle. If $L \cap (R^1 \times \{0\}) = \emptyset$, then the image is a copy of R^1 dense in R^2/L. In fact, in this case the image of R^1 is one of the leaves of the foliation of Section 6.

4) The map $R^2 \to R^1$ given by $(x,y) \mapsto x + y$ is a submersion.

5) The map $R^2/Z^2 \to R/Z$ given by $[(x,y)] \mapsto [x]$ is a submersion.

6) The map $R^n \xrightarrow{\pi} R^n/L$, $x \mapsto [x]$, is both a submersion and an immersion.

If M^n is a C^r-manifold, then a subspace $X \subset M$ is a k-dimensional C^r-submanifold if for every $p \in M$, there is a C^r-coordinate system $(x_1,\ldots,x_n)$ valid on an open set U containing p so that $X \cap U = U \cap \{x_{k+1} = 0,\ldots,x_n = 0\}$. If $f\colon X \mapsto M$ is a one-to-one immersion whose image $f(X) \subset M$ is a C^r-submanifold, then f is a C^r-embedding (or an embedding for short). Note that our original definition of a C^r-manifold yields a C^r-submanifold of R^n. The following theorem makes clear the relation of immersions to embeddings.

<u>Theorem 7.1</u>: <u>Let</u> $f\colon X \to Y$ <u>be a</u> C^r-<u>immersion. For</u> f <u>to be an embedding it is necessary and sufficient that</u> f <u>be one-to-one and closed</u>.

(A continuous map $f\colon A \to B$ is called <u>closed</u> if whenever $X \subset A$ is a closed set $f(X) \subset B$ is closed.) Before beginning the proof proper we need a lemma about any C^r-immersion.

<u>Lemma 7.2</u>: <u>Let</u> $f\colon X^n \to Y^m$ <u>be a</u> C^r-<u>immersion and let</u> $p \in X$. <u>There are open sets</u> $V \subset X$ <u>containing</u> p <u>and</u> $U \subset Y$ <u>containing</u> $f(p)$, <u>and</u> C^r-<u>coordinates</u> $(x_1,\ldots,x_n)$ <u>valid in</u> V <u>and</u> $(y_1,\ldots,y_m)$ <u>valid in</u> U, <u>so that</u> $f\colon V \to U$ <u>sends</u> $(x_1,\ldots,x_n)$ <u>to</u> $(x_1,\ldots,x_n,0,\ldots,0)$.

<u>Proof</u>: Choose $V \subset X$ containing p with coordinates $(x_1, \ldots, x_n)$ and $U \subset Y$ containing $f(p)$ with coordinates $(y_1, \ldots, y_m)$. We can assume that in these coordinates

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right) \qquad (i, j = 1, \ldots, n)$$

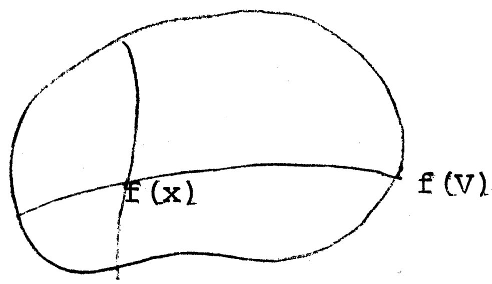
is non-singular. There is a map F, defined on an open subset of $(p \times 0) \in V \times R^{m-n}$ with values in U, given by $F(x, z_1, \ldots, z_{m-n}) = f(x) + (0, \ldots, 0, z_1, \ldots, z_{m-n})$. We see that F is C^r and that

$$DF = \begin{array}{c} n \\ \\ \\ \\ m-n \end{array} \left(\begin{array}{c|ccccc} \multicolumn{6}{c}{Df} \\ \hline & 1 & & & & 0 \\ & & 1 & & & \\ 0 & & & 1 & & \\ & & & & 1 & \\ & 0 & & & & 1 \end{array}\right)$$
$$\begin{array}{cc} \qquad\qquad\quad n & \qquad\qquad m-n \end{array}$$

Hence, $DF(p \times 0)$ is non-singular. Invoking the Inverse Function Theorem, we see that F, restricted to a small open set about $(p \times 0)$ in $V \times R^{m-n}$, is a C^r-diffeomorphism. We use the coordinates $(x_1, \ldots, x_n, z_1, \ldots, z_{m-n})$, pushed foward via F to an open set about $f(p)$. In these C^r-coordinates $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$.

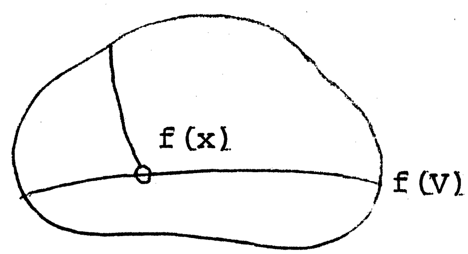
Of course, in a general immersion $U \cap f(X)$ can be bigger than $f(V)$. For example:

1) There can be another branch cutting through $f(V)$ at $f(x)$:

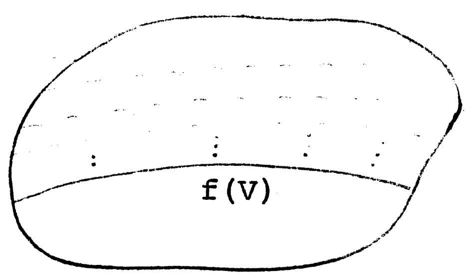


(in which case, f is not 1-1).

2) There can be another branch coming down to $f(x)$ but not touching it:



3) There can be a sequence of branches piling up onto $f(V)$:



(In case 2 and 3, f will not be closed.)

Proof of 7.1: Let us return to the proof of the theorem. By definition if f is an embedding, then it is one-to-one. Let us show that it must also be closed. If $f: X^n \to Y^m$ is an embedding and $p \in X$, then there are open sets $V \subset X$, containing p and U in Y containing $f(p)$ with C^r coordinates $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ so that $f\colon V \to U$ is given by $f(x_1,\dots,x_n) = (x_1,\dots,x_n,0,\dots,0)$. Thus, $f(V)$ is a closed subset of U and a C^r-manifold of dimension n. Since f is an embedding, there is another open set U' and C^r-coordinates $(y_1',\dots,y_m')$ so that $U' \cap f(X) = \{y_{n+1}' = 0,\dots,y_m' = 0\}$. We can assume that U' is an open ball in $(y_1',\dots,y_m')$-space and that $U' \subset U$. Restricting to $U \cap U'$ and replacing V with $f^{-1}(U \cap U')$ allows us to assume that $U = U'$. Clearly, $f(V) \subset f(X) \cap U$. Also, $f(X) \cap U$ is a closed subset of U and a connected C^r-manifold of dimension n. (It is connected since U' is an open ball in $(y_1',\dots,y_m')$ space and $f(X) = U' \cap \{y_{n+1}' = 0,\dots,y_m' = 0\}$.) The map $f\colon V \to f(X) \cap U$ is a local C^r-diffeomorphism and hence $f(V) \subset f(X) \cap U$ is an open subset. We have already seen that it is a closed subset. Since $f(X) \cap U$ is connected, it follows that $f(V) = f(X) \cap U$. Thus, if $f\colon X \to Y$ is an embedding and $q \in Y$, then there is an open set $U \subset Y$ containing q with C^r coordinates $(y_1,\dots,y_m)$ so that:

1) $f^{-1}(U) = V \subset X$ has C^r-coordinates $(x_1,\dots,x_n)$, and
2) $f\colon V \to U$ is given by $f(x_1,\dots,x_n) = (x_1,\dots,x_n,0,\dots,0)$.

If $C \subset X$ is closed, then we shall show that $Y - f(C)$ is open. For this let $y \in Y - f(C)$. Choose U open about

y as above. Since $C \cap f^{-1}(U)$ is closed and $f|f^{-1}(U) \to U \cap \{y_{n+1} = 0,\ldots,y_m = 0\}$ is a homeomoprhism $f(C) \cap U$ is a closed subset of $U \cap \{y_{n+1} =\ldots= y_m = 0\}$ and hence a closed subset of U. Hence, $U - (f(C) \cap U)$ is an open set in U, and consequently, an open set in Y which contains q and misses $f(C)$. This proves that $f(C)$ is closed.

Conversely, suppose that $f\colon X \to Y$ is a closed, one-to-one immersion. Let $q \in Y$. If $q \notin f(X)$, then, since $f(X) \subset Y$ is a closed subset, there is an open set U containing q and missing $f(X)$. Restricting U to be smaller, we can assume that on U we have C^r-coordinates $(y_1,\ldots,y_m)$ so that $\{y_{n+1} = 0,\ldots,y_m = 0\} \cap U = \emptyset$. This produces the required coordinate system about q. Now suppose $q = f(p)$. Since f is an immersion, there are C^r-coordinates, on open sets $V \subset X$ containing p and $U \subset Y$ containing q, so that $f\colon V \to U$ is given by $f(x_1,\ldots,x_n) = (x_1,\ldots,x_n,0,\ldots,0)$. Since f is closed, $f(X - V) \subset Y$ is a closed subset. Replace U by $U' = U \cap (Y - f(X-V))$. Because f is one-to-one, $f(V) \subset U'$, and, in fact, $f^{-1}(U') = V$. This gives the coordinates required near $f(p)$ on the open sets U' and V.

<u>Corollary 7.3</u>: <u>Let</u> $f\colon X^n \to Y^n$ <u>be a one-to-one immersion</u>. <u>It is an embedding if</u>:

1) X <u>is compact, or</u>

2) f <u>is proper (i.e.</u> $f^{-1}(K)$ <u>is compact whenever</u> K <u>is compact</u>).

<u>Proof</u>: If X is compact and $C \subset X$ is closed, then C is compact. As a result, $f(C) \subset Y$ is compact. Since Y is Hausdorff, $f(C)$ is closed. This proves that f is closed when X is compact.

Likewise, under the hypothesis that f is proper, it follows that f is closed. For suppose that f is proper and that $C \subset X$ is closed. Suppose that $\{y_i\}_{i=1}^{\infty} \in f(C)$ is a sequence that converges to $y \notin f(C)$. The set $(\bigcup_{i=1}^{\infty} y_i \cup y) \subset Y$ is compact. Consider $f^{-1}(\bigcup_{i=1}^{\infty} y_i \cup y) \cap C$. If f is proper, this must be compact. But there are points $x_i \in f^{-1}(y_i) \cap C$. If the sequence $\{x_i\}$ had a converge subsequence $\{x_{n_k}\} \to x \in C$, then $f(x) = y$. Since $y \notin f(C)$ this means that $\{x_i\}$ has no convergence subsequence, and hence that $f^{-1}(\bigcup_{i=1}^{\infty} y_i \cup y) \cap C$ is not compact. The contradiction establishes the fact that if f is proper, then it is closed.

<u>Exercises</u>: 1) Give an example of an immersion $f\colon X \to Y$ which is not an embedding but whose image is a submanifold.

2) We showed that a proper mapping between metric spaces was closed. Show that a finite-to-one, closed map between metric spaces is proper.

3) Let y and z be complex variables and $p(z)$ a polynomial without repeated roots. Show that $\{y^2 = p(z)\}$ defines a complex analytic manifold, V, and that $\pi: V \to \mathbb{C}$, $\pi(y,z) = z$ is an immersion $V \to \mathbb{C}$ at all (y,z) except those for which $p(z) = 0$.

4) Let A be a (2×2)-matrix of determinant 1, i.e., $A \in SL(2,\mathbb{R})$. Show that A determines a diffeomorphism of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ onto itself.

5) More generally, show that any $A \in SL(n,\mathbb{R})$ determines a diffeomorphism of $T^n = \mathbb{R}^n/\mathbb{Z}^n$ onto itself.

SUPPLEMENT TO CHAPTER I

Let $U \subset \mathbb{C}^1$ be an open set, and let $f : U \to \mathbb{C}^1$ be a complex valued function on U. It is said to be holomorphic if

$$\lim_{h\to 0} \left(\frac{f(z+h) - f(z)}{h}\right) = f'(z)$$

exists for all $z \in U$. (Here, h is a complex variable.) It turns out that if $f : U \to \mathbb{C}^1$ is holomorphic, then f is C^∞, and, in fact, complex analytic. This means that if $z \in U$ and $B_\varepsilon(z) \subset U$, then there is a power series, $\sum_{n=0}^{\infty} a_n(\zeta - z)^n$, which is absolutely convergent in $B_\varepsilon(z)$ and which represents f there, i.e. with

$$f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - z)^n .$$

At the end of Chapter III we shall give a proof of this (in the case when f is C^1) using Stokes' Theorem. If we identify $\mathbb{C}^1$ with $\mathbb{R}^2$ $(z = x + iy)$, then $f : U \to \mathbb{C}^1$ becomes $(R, I) : U \to \mathbb{R}^2$ $(f(z) = R(z) + i\,I(z)$. The existence of $f'(z)$ is equivalent to

1) R and I being differentiable at z, and

2) $\frac{\partial R}{\partial x}(z) = \frac{\partial I}{\partial y}(z)$; $\frac{\partial R}{\partial y}(z) = -\frac{\partial I}{\partial x}(z)$.

The two equations in 2) are called the Cauchy-Riemann equations. If they are satisfied, then $D(R, I)_z : \mathbb{R}^2 \to \mathbb{R}^2$ is the (2×2)-matrix representing complex multiplication by $f'(z)$.

If $U \subset \mathbb{C}^n$ is an open set, then the following are equivalent :

1) $f : U \to \mathbb{C}^1$ is holomorphic in each variable separately; i.e.,

$$\lim_{h\to 0} \left(\frac{f(z_1,\dots, z_i+h, z_{i+1},\dots, z_n) - f(z_1,\dots, z_n)}{h}\right) = \frac{\partial f}{\partial z_i}(z_1,\dots, z_n)$$

exists for all $(z_1,\dots, z_n)$ in U and all i, $1 \le i < n$.

2) $Df_z : \mathbb{C}^n \to \mathbb{C}$ exists for each $z \in U$ and is a complex linear mapping.

3) Near each $(z_1, \ldots, z_n)$ f is represented by an absolutely convergent power series .

$$f(\zeta_1, \ldots, \zeta_n) = \Sigma\, a_{i_1 \ldots i_n} (\zeta_1 - z_1)^{i_1} \ldots (\zeta_n - z_n)^{i_n}.$$

We shall not prove this theorem but it can be found in any book on several complex variables .

A function $U \xrightarrow{F} \mathbb{C}^k$ is said to be holomorphic (or complex analytic) if each of its coordinate functions are.

<u>Theorem</u> (Complex Analytic Inverse Function Theorem).

<u>Let</u> $U \subset \mathbb{C}^n$ <u>be an open set and</u> $F : U \to \mathbb{C}^n$ <u>be holomorphic</u> . If $DF_z : \mathbb{C}^n \to \mathbb{C}^n$ <u>is non-singular, then there are open sets</u> $U' \subset U$ <u>containing</u> z <u>and</u> $W \subset \mathbb{C}^n$, <u>so that</u> $F : U' \to W$ <u>has a holomorphic inverse</u>.

<u>Proof</u>: By the C^∞-Inverse Function Theorem, there are open sets $U' \subset U$ and $W \subset \mathbb{C}^n$ so that $F : U' \to W$ is a C^∞-diffeomorphism . It remains to show that $F^{-1} : W \to U'$ is complex analytic. For this we need only to show that for each $w \in W$ DF^{-1}_w is a complex linear map $\mathbb{C}^n \to \mathbb{C}^n$. Since $DF^{-1}_w = \left\{DF_{F^{-1}(w)}\right\}^{-1}$ and $DF_{F^{-1}(w)}$ is complex linear, it follows that DF^{-1}_w is complex linear.

<u>Corollary</u> (Real Analytic Inverse Function Theorem).

<u>Let</u> $U \subset \mathbb{R}^n$ <u>be open and</u> $F : U \to \mathbb{R}^n$ <u>be a real analytic function with</u> DF_p <u>invertible</u> . <u>Then, there are open sets</u> $U' \subset U$, $p \in U'$, <u>and</u> $W \subset \mathbb{R}^n$ <u>so that</u> $F : U' \to W$ <u>has a real analytic inverse</u> .

Proof: By restricting to a sufficiently small open set about p we can assume that F is given by an absolutely convergent power series.

$$F(x_1, \ldots, x_n) = \Sigma\, a_{i_1 \ldots i_n} (x_1 - p_1)^{i_1} \cdots (x_n - p_n)^{i_n}$$

where the $a_{i_1 \ldots i_n}$ are elements of $\mathbb{R}^n$. The exact same power series defines a function, $\tilde{F}$, on a neighborhood of p in $\mathbb{C}^n$ with values in $\mathbb{C}^n$ which is clearly complex analytic. Since $DF_p : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, so is $D\tilde{F}_p : \mathbb{C}^n \to \mathbb{C}^n$. Applying to complex analytic Inverse Function Theorem, we find a complex analytic inverse $\tilde{F}^{-1}$ for $\tilde{F}$ near p. Restricting $\tilde{F}^{-1}$ to the real points gives the required real analytic inverse to F near p.

Arguing as in Section 3, we can deduce from these results complex analytic and real analytic versions of the Implicit Function Theorem.

A complex analytic hypersurface in $\mathbb{C}^n$ is the solution set to an equation of the form

$$X^{n-1} = \{p(z_1, \ldots, z_n) = 0\}$$

where p is a (complex) polynomial. If $p(z) = 0$ but $\frac{\partial p}{\partial z_i}(z) \neq 0$, then there is a neighborhood of z in $\mathbb{C}^n$, U, so that $X \cap U$ is a complex manifold of (complex) dimension $(n-1)$. If $p(z) = 0$ and $\frac{\partial p}{\partial z_i}(z) = 0$ for all i, then z is said to be a singular point of X. It is an isolated singular point, if there is a neighborhood of z in X which contains no other singular point. One way to study an isolated singularity of a complex hypersurface is to study its link.

Theorem <u>Let</u> $\{p(z_1, \ldots, z_n) = 0\}$ <u>be a complex hypersurface with an isolated singularity at the origin. For all</u> $\varepsilon > 0$ <u>which are sufficiently small, the</u> ε-link <u>of the singularity</u>, i.e.,

$$S_\varepsilon^{2n-1} \cap \{p(z_1, \ldots, z_n) = 0\},$$

<u>is a smooth submanifold of</u> S_ε^{2n-1}.

<u>Proof</u> : The proof uses some basic facts from algebraic geometry which we shall not prove.

<u>Fact 1</u> : Let $X \subset \mathbb{R}^k$ be an algebraic set, i.e., the solutions set of a finite number of polynomial equations. Then, there is a proper algebraic subset $\Sigma(X)$ and $r \leq k$ so that $X - \Sigma(X)$ is locally defined by r polynomials, $\{p_1, \ldots, p_r\}$ with $D(p_1, \ldots, p_r)$ of rank r. We call $\Sigma(X)$ the <u>singularity</u> of X.

<u>Fact 2</u> : If $X_0 \supset X_1 \supset X_2 \supset \cdots$ is a decreasing sequence of algebraic sets, then it must stabilize at some point, i.e., $X_N = X_{N+1} = \cdots$

<u>Fact 3</u> : If X is a smooth algebraic set, i.e., if X is an algebraic set with $\Sigma X = \phi$, then X has finitely many components.

<u>Sublemma</u> : <u>If</u> X <u>is an algebraic set and</u> $W \subset X$ <u>is an algebraic subset containing</u> $\Sigma(X)$, <u>then</u> $X - W$ <u>has finitely many components</u>.

<u>Proof</u> : We claim that if $X \subset \mathbb{R}^n$ is an algebraic set then $X - W$ is diffeomorphic to a smooth algebraic set in $\mathbb{R}^{n+1}$. To see this suppose that the ideal of polynomials which vanish on X is generated by $\{f_1, \ldots, f_T\}$ and that the ideal which vanish on W is

generated by $\{f_1, \ldots, f_T, g_1, \ldots, g_S\}$. Consider $\mathbb{R}^n \times \mathbb{R}^1$ with variables $(x_1, \ldots, x_n, t)$. The defining equations for $(X - W) \subset \mathbb{R}^n \times \mathbb{R}^1$ are:

$$f_1(x_1, \ldots, x_n) = \ldots = f_T(x_1, \ldots, x_n) = 0$$

$$t \cdot \left(\sum_{j=1}^{S} g_j^2 (x_1, \ldots, x_n)\right) = 0.$$

Clearly, the projection map onto $\mathbb{R}^n$ induces a diffeomorphism from this algebraic set to $X - W$. One sees easily that the above algebraic set is smooth.

<u>Lemma</u> <u>If</u> X <u>is an algebraic set and</u> $W \subset X$ <u>is an algebraic subset, then</u> $X - W$ <u>has finitely many components</u>.

<u>Proof</u> : Define $\Sigma^1(X) = \Sigma(X)$ and $\Sigma^{n+1}(X) = \Sigma(\Sigma^n(X))$. By Fact 2, we know that $\Sigma^N(X) = \phi$ for some N. Thus, we wrire $X - W$ as

$$\{X - (W \cup \Sigma(X))\} \cup \{\Sigma(X) - (W \cap \Sigma(X)) \cup \Sigma^2(X)\} \cup \ldots \cup \{\Sigma^{N-1}(X) - W \cap \{\Sigma^{N-1}X\}.$$

Each piece in the union is of the form $V - Z$ where $Z \supset \Sigma(V)$. Hence, we have written $X - W$ as a finite union of spaces each of which has finitely many components. It follows that $X - W$ has finitely many components.

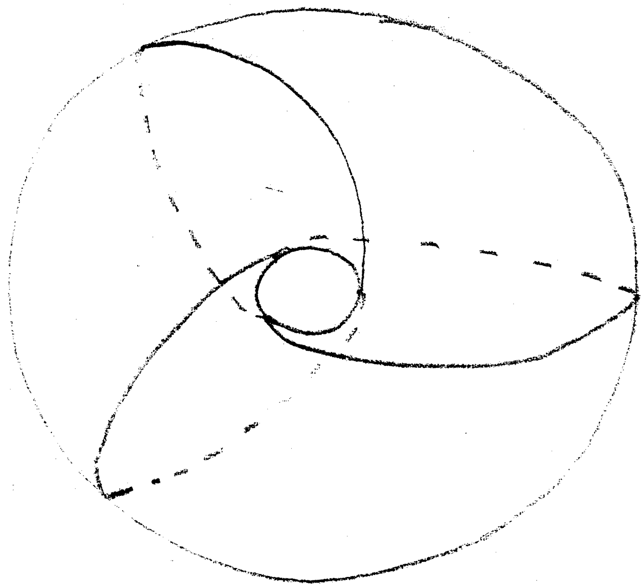
Let us return now to $\{p(z_1, \ldots, z_n) = 0\} \subset \mathbb{C}^n$ a complex hypersurface with an isolated singularity at the origin. Let $X \subset \mathbb{C}^n$ be the solution set. Clearly, $X \subset \mathbb{R}^{2n}$ is a real algebraic set. Consider $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$\psi(x_1, \ldots, x_{2n}) = \sum_{i=1}^{2n} x_j^2$. We wish to show that $\psi \mid (X - 0)$ has only finitely many critical values. Let R and I be the real an imaginary parts of $p(z_1, \ldots, z_n)$. The defining equations for $X \subset \mathbb{R}^{2n}$ are $\{R = 0$ and $I = 0\}$. At all points of X except 0, ∇R and ∇I are linearly independent. Let $Z \subset \mathbb{R}^{2n}$ be defined by :

$$\{(x_1, \ldots, x_{2n}) \mid R = 0,\ I = 0, \text{ and } \nabla R, \nabla I, \text{ and } \nabla\psi \text{ are linearly dependent}\} .$$

Clearly, Z is an algebraic set contained in X. It consists of 0 union the critical points of ψ on $X - \{0\}$. Since $Z - \{0\}$ has only finitely many components, and ψ restricted to any component of $Z - \{0\}$ is constant, ψ has only finitely many critical values on $X - \{0\}$. In particular, ψ has no critical value in some interval of the form $(0, \varepsilon)$.

Examples: 1) Let $p(z_1, z_2) = z_1^3 - z_2^2$. The hypersurface $\{p = 0\}$ has an isolated singularity at the origin. The link is isotopic to a torus knot of type $(2,3)$ in S^3.



2) More generally, if $p(z_1, z_2) = z_1^r + z_2^s$ with r and s relatively prime, then the link $\{p(z_1, z_2) = 0\} \cap S_\varepsilon^3$ is a torus knot of type (r, s).

3) Let $p(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$. Then, the link $\{p = 0\} \cap S_\varepsilon^5$ is diffeomorphic to dodecahedral space : $SO(3)/\{\text{symmetries of the dodecahedron}\}$.

4) Let $p(z_1, z_2, z_3, z_4, z_5) = z_1^2 + z_2^3 + z_3^5 + z_4^2 + z_5^2$. The link $\{p = 0\} \cap S_\varepsilon^9$ is a differentiable 7 manifold which is homeomorphic to S^7 but not diffeomorphic to S^7.

Chapter II: Tangential Structure

The first chapter dealt with manifolds and maps between them. Any serious study of these objects requires the use of infinitesimal versions--tangent planes and differentials. It is the purpose of this chapter to introduce these and to enumerate some of their basic structure.

§1: The Case of Submanifolds of R^n

We begin our study of tangent planes to manifolds with the simplest case--that of $M^n \subset R^N$. The tangent plane to M^n at $p \in M^n$, TM^n_p, is the linear subspace of R^N (through the origin) given in any one of the following three ways:

1) Choose an open set $U \subset R^N$ about $p \in M$ and a differentiable function $F: U \to R^{N-n}$ so that DF_p is of rank $(N-n)$ and $M \cap U = F^{-1}(0)$. We define TM^n_p to be the kernel of $DF_p: R^N \to R^{N-n}$.

2) Choose an open set $U \subset R^N$ about $p \in M$, another open set W, and a diffeomorphism $\Phi: W \to U$ so that $p = \Phi(0)$ and $M \cap U = \Phi((R^n \times \{0\}) \cap W)$. We define TM^n_p to be the image of $D\Phi_0(R^n \times \{0\})$ in R^N.

3) Take the collection of all C^1-curves

$$\gamma: (-\epsilon, \epsilon) \longrightarrow R^N$$

so that

$$\gamma(0) = p \qquad \text{and} \qquad \gamma(-\epsilon, \epsilon) \subset M.$$

Associate to each such curve the vector $\gamma'(0) \in R^N$. All these vectors form a linear subspace of R^N.

<u>Theorem 1.1</u>: <u>The three subspaces defined above all agree.</u>

<u>Proof of 1) = 2)</u>: Let $\Phi: W \to U$ be as in 2) and $F: U \to R^{N-n}$

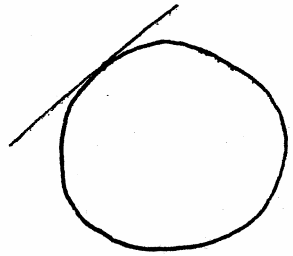
be as in 1). Since $F \circ \Phi(x_1, \ldots, x_n, 0, \ldots, 0) = 0$ for all $(x_1, \ldots, x_n)$ sufficiently close to zero we see that the subspace $D\Phi_0(R^n \times \{0\})$ is contained in the kernel of DF_p. Since Φ is a diffeomorphism, $\dim(D\Phi_0(R^n \times \{0\}) = n$. Since DF_p is of maximal rank, the dimension of the kernel of DF_p is also n. This means that these two linear subspaces have the same dimension and hence are identical.

<u>Proof 2) = 3)</u>: Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a C^1-curve with $\gamma(0) = p$. Then $\Phi^{-1} \circ \gamma\colon (-\epsilon, \epsilon) \to W$ is a C^1-curve whose image is contained in $R^n \times \{0\}$. Hence, $(\Phi^{-1} \circ \gamma)'(0)$ is contained in $R^n \times \{0\}$. Consequently, $D\Phi \circ (\Phi^{-1} \circ \gamma)'(0)$ is contained in $D\Phi_0(R^n \times \{0\})$. But, by the chain rule $D\Phi_p \circ (\Phi^{-1} \circ \gamma)'(0) = (\Phi \circ \Phi^{-1} \circ \gamma)'(0) = \gamma'(0)$. This shows the subspace in 3) is contained in the one defined in 2). To show that it is equal to the one defined in 2) consider the curve $\gamma(t) = \Phi(ta_1, \ldots, ta_n, 0, \ldots, 0)$. Then,
$\gamma'(0) = D\Phi_p(a_1, \ldots, a_n, 0, \ldots, 0)$.

As defined, TM_p^n is an n-dimensional linear subspace of R^N, e.g., it passes through the origin. When we draw pictures we usually translate this plane to pass through the point $p \in M$. When we do this it looks like the "tangent plane".

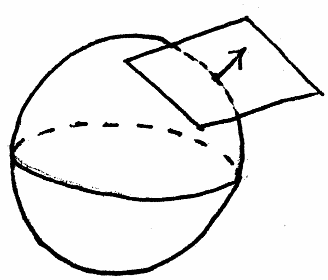
Examples:

Consider the circle $\{x^2 + y^2 = 1\}$ in R^2.

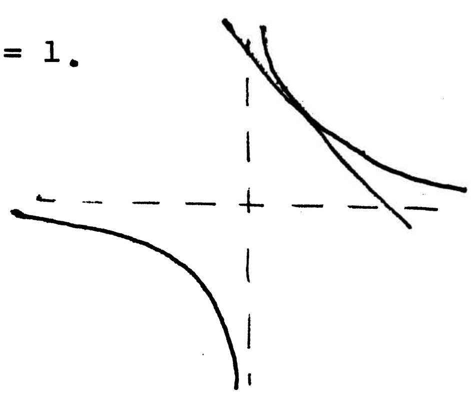


The tangent plane at $(\cos\theta, \sin\theta)$ is the line perpendicular to $(\cos\theta, \sin\theta)$, e.g.,

$TS^1_{(\cos\theta,\ \sin\theta)} = \{(a,b) \mid a\cos\theta + b\sin\theta = 0\}$. The reason is that if $f(x,y) = x^2 + y^2 - 1$, then $Df(x,y) = (2x,2y)$ and kernel $Df(x,y) = \{(a,b) \mid 2xa + 2yb = 0\}$. Likewise, in the case of $S^{n-1} = \{(x_1,\ldots,x_n) \mid \Sigma_{i=1}^n x_i^2 = 1\}$ the tangent plane at $(x_1,\ldots,x_n)$ is the plane perpendicular to $(x_1,\ldots,x_n)$.



Consider $xy = 1$.



The tangent plane at $(x,\frac{1}{x})$ is the plane perpendicular to $(\frac{1}{x},x)$.

Lemma 1.2: *If* $M^{n-1} \subset R^n$ *is defined by one equation* $f(x_1,\ldots,x_n) = 0$ *and if* $\nabla f(x_1,\ldots,x_n)$ *never vanishes along* M, *then* TM_p^{n-1} *is* $\nabla f(p)^{\perp}$.

The proof is left as an exercise.

The vector perpendicular to TM_p^{n-1}, $\nabla f(p)$, points into the side of M on which f is positive.

More generally, if $M^n \subset R^{n+k}$ is the common zeroes of $\Phi = (\Phi_1,\ldots,\Phi_k)$ and if $D\Phi$ has rank k everywhere along M, then $TM_p^n = (\nabla\Phi_1(p),\ldots,\nabla\Phi_k(p))^{\perp}$. If $M^n \subset R^{n+k}$ is a submanifold, then the *normal space to* M *at* p is space perpendicular to TM_p^n. As we have just seen, if M^n is defined by $\{\Phi = 0\}$ ($\Phi = (\Phi_1,\ldots,\Phi_k)$ with $D\Phi$ of rank k), then the normal space to M at p is the linear subspace spanned by $\{\nabla\Phi_1(p),\ldots,\nabla\Phi_k(p)\}$.

Exercise : 1) Prove Lemma 1.2.

§2. Tangent Planes in General

It is not satisfactory just to have tangent planes defined from submanifolds of R^N. We need an abstract definition of the tangent plane to an abstract manifold. The definition is somewhat complicated.

Definition: Let M be a differentiable manifold and let $p \in M$. Consider all pairs (U,f), where U is an open set of M containing p, and $f: U \to R^1$ is a C^∞-function. We define an equivalence relation: $(U,f) \sim (V,g)$ if and only if there is an open set W, $p \in W \subset U \cap V$, so that $f|W = g|W$. The equivalence classes are called germs of C^∞-functions at p. The class of (U,f) is called the germ of f at p.

Exercises: 1) Show that the relation given above is an equivalence relation.

2) Show that if the germ of f and g at $0 \in R^n$ are the same, then $\dfrac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}}(0) = \dfrac{\partial^r g}{\partial x_{i_1} \cdots \partial x_{i}}(0)$.

3) Show that there are two C^∞-functions defined near $0 \in R^1$ with the same Taylor expansions at 0 but which have different germs.

4) Suppose given a germ of a function at $p \in M^n$. Show that given another point $q \neq p$, there is a representative

of the given germ which has any preassigned value at q.

The germs of C^∞-functions at $p \in M$ define an algebra over R. We add two germs by taking representatives with a common domain of definition and adding their values:

$$[U,f] + [V,g] = [U \cap V, (f|(U \cap V)) + (g|(U \cap V))].$$

Likewise,

$$r[U,f] = [U,rf] \qquad \text{for } r \in R,$$

and

$$[U,f]\cdot[V,g] = [U \cap V, (f|(U \cap V))\cdot(g|(U \cap V))].$$

We denote this R-algebra by $\mathcal{G}_p(M)$.

<u>Definition</u>: A <u>derivation</u> on $\mathcal{G}_p(M)$ is a function $D\colon \mathcal{G}_p(M) \to R$ which

1) is R-linear, and

2) satisfies the Leibnitz rule, i.e., satisfies

$$D(\alpha\cdot\beta) = D(\alpha)\cdot\beta(p) + \alpha(p)\cdot D(\beta).$$

<u>Example</u>: Let $\gamma\colon (-\epsilon,\epsilon) \to M$ be a C^1 curve with $\gamma(0) = p$. This curve defines a derivation D_γ by the formula $D_\gamma([U,f]) = f\circ\gamma'(0)$.

<u>Theorem 2.1</u>: <u>The derivations on</u> $\mathcal{G}_p(M^n)$ <u>form a real vector space of dimension</u> n, TM^n_p. <u>If</u> $f\colon M^n \to Q^q$, <u>is a</u> C^∞-<u>map</u>,

<u>then</u> f <u>induces</u> $Df_p\colon TM_p^n \to TQ_{f(p)}^q$. <u>If</u> $f\colon M \to Q$ <u>and</u> $g\colon Q \to S$ <u>are</u> C^∞, <u>then</u> $Dg_{f(p)} \cdot Df_p = D(g \cdot f)_p$.

<u>Proof</u>: Let us begin by defining the real vector space structure on the set of derivations on $\mathcal{G}_p(M)$. If D_1 and D_2 are derivations and $r,s \in R$, then we define $(rD_1 + sD_2)(\alpha) = r \cdot D_1(\alpha) + s \cdot D_2(\alpha)$. This is easily seen to define a real vector space structure. Suppose $f\colon M \to Q$ is a C^∞-map. There is an induced mapping $f^*\colon \mathcal{G}_{f(p)}(Q) \to \mathcal{G}_p(M)$ defined by $[U,\varphi] \to [f^{-1}(U), \varphi \cdot f]$. One checks that f^* is well defined and a map of R-algebras. $Df_p\colon TM_p^n \to TQ_{f(p)}^q$ is defined as follows:

$$Df_p(\Delta)(\alpha) = \Delta(f^*\alpha)$$

for $\alpha \in \mathcal{G}_{f(p)}(Q)$ and $\Delta \in TM_p^n$. Clearly, Df_p is well-defined and R-linear. Also, one sees that $Dg_{f(p)} \cdot Df_p = D(g \cdot f)_p$. As a result, if $f\colon M \to Q$ is a local diffeomorphism at $p \in M$, then $Df_p\colon TM_p \to TQ_{f(p)}$ is a linear isomorphism. Thus, as a vector space, TM_p^n is isomorphic to TR_0^n.

Let us consider TR_0^n. There are n natural derivations $\{\frac{\partial}{\partial x_1}(0), \ldots, \frac{\partial}{\partial x_n}(0)\}$ on $\mathcal{G}_0(R^n)$. We claim that they form a basis for all the derivations. First let us show that the n derivations are linearly independent. If

$\Sigma_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(0)$ is the trivial derivation , then

$$\sum_{i=1}^{n} a_i \frac{\partial x_j}{\partial x_i}(0) = 0, \quad \text{i.e.,} \ a_j = 0.$$

To show that the partial derivatives span the space of all derivations we need a lemma.

<u>Lemma 2.2</u>: <u>Let</u> $\varphi: U \to R^1$ <u>be a</u> C^∞-<u>function defined on an open ball about</u> 0 <u>in</u> R^n. <u>If</u> $f(0) = 0$, <u>then</u> $\varphi = \Sigma_{i=1}^{n} x_i \cdot h_i$ <u>where the</u> h_i <u>are</u> C^∞-<u>functions</u>.

<u>Proof</u>: Define $h_i(x) = \int_0^1 \frac{\partial\varphi}{\partial x_i}(tx)dt$. Clearly each h_i is C^∞. We claim that $\Sigma_{i=1}^{n} x_i h_i(x) = \varphi(x)$. The reason is that

$$\sum_{i=1}^{n} x_i \int_0^1 \frac{\partial\varphi_i}{\partial x_i}(tx)dt = \int_0^1 \nabla\varphi(tx)\cdot\frac{d}{dt}(tx)dt$$

$$= \int_0^1 \frac{d}{dt}(\varphi(tx))dt = \varphi(tx)\Big|_0^1$$

$$= \varphi(x) - \varphi(0)$$

$$= \varphi(x).$$

This completes the proof of Lemma 2.2.

Suppose that φ is a C^∞-function in a ball about 0. Write $\varphi(x) = c + \Sigma_{i=1}^n x_i h_i(x)$. It follows that $h_i(0) = \frac{\partial\varphi}{\partial x_i}(0)$. Let D be any derivation. Then $D(\varphi) = D(c) + \Sigma_{i=1}^n D(x_i \cdot h_i) = c \cdot D(1) + \Sigma_{i=1}^n 0 \cdot D(h_i) + h_i(0) \cdot D(x_i)$ $= c \cdot D(1) + \Sigma_{i=1}^n \frac{\partial\varphi}{\partial x_i}(0) \cdot D(x_i)$. To complete the proof we must show that $D(1) = 0$. But $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1)$ $= 2D(1)$. Hence, $D(1) = 0$. Thus, $D(\varphi) = \Sigma_{i=1}^n D(x_i) \frac{\partial\varphi}{\partial x_i}(0)$. This proves that $\{\frac{\partial}{\partial x_1}(0), \ldots, \frac{\partial}{\partial x_n}(0)\}$ forms a basis for TR_0^n. In particular, TR_0^n is of dimension n, and consequently, so is TM_p^n. A basis for TM_p^n is given by choosing C^∞-local coordinates $(x_1, \ldots, x_n)$ and taking $\{\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)\}$.

<u>Corollary 2.3</u>: <u>Let</u> $f: M^n \to Q^q$ <u>be a</u> C^∞<u>-map. Choose local</u> C^∞<u>-coordinates</u> $(x_1, \ldots, x_n)$ <u>near</u> $p \in M$ <u>and</u> $(y_1, \ldots, y_q)$ <u>near</u> $f(p) \in Q$. <u>These give bases</u>, $\{\frac{\partial}{\partial x_i}(p)\}$ <u>and</u> $\{\frac{\partial}{\partial y_j}(f(p))\}$, <u>for</u> TM_p <u>and</u> $TQ_{f(p)}$. <u>In these bases the matrix for</u> $Df_p: TM_p \to TQ_{f(p)}$ <u>is</u>

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right).$$

<u>In particular</u>, $Df_p(\frac{\partial}{\partial x_i}(p)) = \Sigma_{j=1}^q \frac{\partial f_j}{\partial x_i}(p) \cdot \frac{\partial}{\partial y_j}(f(p))$.

This calculation is actually a statement about how the matrix of partial derivatives transforms under change of coordinates. Of course, it is just the chain rule in disguise.

Definition : Let M and N be C^∞-manifolds and $f\colon M \to N$ a C^∞-mapping. Let P be a submanifold on N We say that f is transverse (or transverse regular) to P provided that for each $x \in f^{-1}(P)$ the subspaces $Df_x(TM_x)$ and $TP_{f(x)}$ span $TN_{f(x)}$. (Of course, $TP_{f(x)}$ is identified with a subspace of $TN_{f(x)}$ since we have the embedding $P \hookrightarrow N$.)

Exercises: 1) Let $f\colon M^n \subset Q^{n+k}$ be an embedding. Suppose that near $f(p)$ there are local C^∞-coordinates in which M^n is given by $\Phi(y_1,\ldots,y_{n+k}) = 0$ for Φ a function of rank k. Show that $Df_p(TM_p^n) \subset TQ_{f(p)}^{n+k}$ equals the kernel of $D\Phi_{f(p)}$.

2) Give a definition of the tangent plane to a C^1-manifold. (Hint: Use C^1-curves.)

3) If $M^n \subset R^N$, then we have two definitions of TM_p^n. Show that the resulting spaces have a natural identification between them.

4) Show that if $f : M^m \to N^n$ is transverse to $P^p \hookrightarrow N^n$, then $f^{-1}(P)$ is either a submanifold of M of dimension $(n - m + p)$ or is empty.

§3. The Tangent Bundle

The collection of all the tangent planes to a manifold $\{TM_p^n\}_{p\in M}$ fit together to make a space and, in fact, a C^∞-manifold. The manifold is called the tangent bundle of M, TM. The underlying set is $\{(p,\tau)\,|\,p \in M \text{ and } \tau \in TM_p\}$. To define the topology and differential structure on TM we begin with an C^∞-atlas $\{U_\alpha,\varphi_\alpha,V_\alpha\}$ for M. Since $V_\alpha \subset R^n$ is an open set, we have a basis for $T(V_\alpha)_q$, namely $\{\frac{\partial}{\partial x_1}(q),\ldots,\frac{\partial}{\partial x_n}(q)\}$. Using this basis gives an identification of $T(V_\alpha)_q$ with R^n. Doing this for every $q \in V_\alpha$ gives us a bijection $V_\alpha \times R^n \longleftrightarrow TV_\alpha$. Since $V_\alpha \times R^n \subset R^n \times R^n = R^{2n}$, this is a bijection between an open set in R^{2n} and TV_α. We use this identification of TV_α with $V_\alpha \times \mathbb{R}^n$ to define a topology and C^∞-manifold structure on TV_α.

The map $D\varphi_\alpha : TV_\alpha \to TU_\alpha$ defined by $D\varphi_\alpha(q,\tau) = (\varphi_\alpha(q),\, D(\varphi_\alpha)_q(\tau))$ is a bijection. Push forward the topology and C^∞-manifold structure on TV_α to TU_α via this bijection. This means that we define $X \subset TU_\alpha$ to be open if and only if $(D\varphi_\alpha)^{-1}(X) \subset TV_\alpha$ is open, and we define $\mu : TU_\alpha \to \mathbb{R}^1$ to be C^∞ if and only if $\mu\circ D\varphi_\alpha$ is C^∞.

We let the $\{TU_\alpha\}$ generate a topology on TM. Define $X \subset TM$ to be open if and only if $X \cap TU_\alpha$ is an open subset of TU_α for all α. Clearly, this defines a topology on TM. We claim that if we use this topology on TM to induce one on TU_α, then we get back the topology wich we began with on TU_α. To show this comes down to showing that if $X \subset TU_\alpha$ is an open set (in the topology induced by $D\varphi_\alpha$), then $X \cap TU_\beta$ is an open subset of TU_β (again in the topology induced by $D\varphi_\beta$) for all β. First, note that $TU_\alpha \cap TU_\beta = T(U_\alpha \cap U_\beta)$. Hence, $TU_\alpha \cap TU_\beta$ is an open subset of both TU_α and TU_β. Next, the mapping

$$(D\varphi_\beta)^{-1} \circ (D\varphi_\alpha) \colon (D\varphi_\alpha)^{-1}(T(U_\alpha \cap U_\beta)) \longrightarrow (D\varphi_\alpha)^{-1}(TU_\alpha \cap U_\beta))$$

is a homeomorphism between open sets in $R^n \times R^n$. Thus, if $X \subset TU_\alpha$ is open, i.e., if $(D\varphi_\alpha)^{-1}(X) \subset TV_\alpha$ is open, then so is $(D\varphi_\alpha)^{-1}(X \cap T(U_\beta))$. Hence

$$(D\varphi_\beta)^{-1} \circ (D\varphi_\alpha)\{(D\varphi_\alpha)^{-1}(X \cap T(U_\alpha \cap U_\beta))\} = D\varphi_\beta^{-1}(X \cap T(U_\beta))$$

is open. This means that $X \cap TU_\beta \subset TU_\beta$ is open. This proves that $TU_\alpha \subset TM$ is a homeomorphism onto an open set.

Lastly, we claim that $\{TU_\alpha, D\varphi_\alpha, TV_\alpha\}$ is a C^∞-atlas for TM. The overlap functions for this atlas are $D\varphi_\beta^{-1} \circ D\varphi_\alpha$. If we let $\psi = \varphi_\beta^{-1} \circ \varphi_\alpha$, then ψ is a C^∞-diffeomorphism between open sets in R^n. By the chain rule $(D\varphi_\beta)^{-1} \circ D\varphi_\alpha = D\psi$. Thus, this composition sends

$$(p,(t_1,\ldots,t_n)) \text{ to } (\psi(p),(\sum_{i=1}^{n} \frac{\partial\psi_1}{\partial x_i}(p)\cdot t_i,\ldots,\sum_{i=1}^{n} \frac{\partial\psi_n}{\partial x_i}(p)\cdot t_i)).$$

Clearly, this is a C^∞-mapping between open subsets of $R^n \times R^n$. This defines the C^∞-manifold structure on TM.

There is a C^∞-projection map, π: TM $\to$ M defined by $\pi(p,\tau) = p$. This map is a C^∞-submersion. The "fiber" over x, i.e., $\pi^{-1}(x)$, is the vector space TM^n_x. Thus, we have a family of vector spaces parameterized by the points of M^n. It is a <u>locally trivial family</u> in the sense that for each $x \in M$, there is an open set $U \subset M$ containing x and a C^∞-diffeomorphism ψ which is linear on each fiber and which makes the following diagram commute:

$$\begin{tikzcd} U \times R^n \arrow[rr, "\psi"] \arrow[dr, "p_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

Such a diagram is called a <u>local trivialization</u>.

If $f: M^m \to N^n$ is a C^∞-map, then there is induced a C^∞-map Df: TM $\to$ TN which "covers" f, i.e. so that

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes. Df is linear on each fiber, and in fact, $Df: TM_p \to TN_{f(p)}$ is just Df_p.

The tangent bundle is an example of a more general class of objects--locally trivial vector bundles (or vector bundles for short). Let B be <u>any</u> topological space. A vector bundle over B is a family of vector spaces $\{V_b\}_{b\in B}$ which form a topological space E. The map $E \overset{\pi}{\to} B$ obtained by sending V_b to b is called the projection map. In addition, there are an open covering $\{U_\alpha\}$ of B, vector spaces V_α, and homeomorphisms $\psi_\alpha : U_\alpha \times V_\alpha \to \pi^{-1}(U_\alpha)$, so that

$$\begin{tikzcd} U_\alpha \times V_\alpha \arrow[rr, "\psi_\alpha"] \arrow[dr, "p_1"'] & & \pi^{-1}(U_\alpha) \subset E \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

commutes, and so that ψ_α is a linear isomorphism on each fiber. Such a collection $\{\psi_\alpha\}$ is called collection of local trivializations for the cover $\{U_\alpha\}$.

There is another way to describe a vector bundle. Begin with an open covering $\{U_\alpha\}$ of B, and a finite dimensional vector space V. Take continuous maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$ which satisfy : 1) $g_{\alpha\alpha}(x) = \mathrm{Id}$, 2) $g_{\alpha\beta}(x) \cdot g_{\gamma\alpha}(x) = g_{\gamma\beta}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Given all this data one forms the quotient space $(\coprod_{\alpha\in I} U_\alpha \times V)/\sim$ where $(u \in U_\alpha, v) \sim (u \in U_\beta, g_{\alpha\beta}(u)\cdot v)$

for any $u \in U_\alpha \cap U_\beta$. The quotient space is E, the total space of the vector bundle.

<u>Examples</u>: 1) Let $M^n \subset R^{n+k}$ be a C^∞-submanifold. Define $N(M^n)$, the normal bundle of M^n to be all pairs $\{(p,v) | p \in M^n \text{ and } v \in (TM_p^n)^\perp\}$. The local trivializations come from choosing open sets $W_\alpha \subset R^{n+k}$, which cover M, and C^∞-functions of rank k, $\Phi_\alpha : W_\alpha \to R^k$ so that $\Phi_\alpha^{-1}(0) = M \cap W_\alpha$. At each $p \in M \cap W_\alpha$ we have a basis for $N(M^n)_p$, namely $\{\nabla(\Phi_\alpha)_1(p), \ldots, \nabla(\Phi_\alpha)_k(p)\}$. We use this basis to define a trivialization for $N(M^n)$ over $M \cap W_\alpha$.

2) $B \times R^n$ is a vector bundle, called the <u>trivial vector bundle of dimension</u> n over B.

3) Let $U_0 \subset S^1$ be $S^1 - \{(0,1)\}$ and $U_1 \subset S^1$ be $S^1 - \{(-1,0)\}$. Define $g_{0,1} : U_0 \cap U_1 \to GL(R) = R^*$ by

$$g_{0,1}(\cos(\theta), \sin\theta) = \begin{cases} +1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$$

This defines a vector bundle over S^1 whose total space is the Mobius band.

A map between 2 vector bundles is a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{f} & B' \end{array}$$

such that $\tilde{f}$ is linear on each fiber. (The map $\tilde{f}$ is said to <u>cover</u> f.) A C^∞-<u>bundle map</u> is a bundle map between C^∞-bundles which is a C^∞-map between the total spaces. If $g: M \to N$ is a C^∞-map, then $Dg: TM \to TN$ is a C^∞-bundle map.

A (C^∞-)<u>bundle isomorphism</u> is a (C^∞-)bundle map, covering the identity on the base, which has a (C^∞-)bundle map inverse. It is an easy exercise to show that a (C^∞-)bundle map, covering the identity on the base, is a (C^∞-)bundle isomorphism if and only if it is a linear isomorphism on each fiber.

A (C^∞-) vector bundle is <u>trivial</u> if and only if it is isomorphic to a product bundle $B \times V$.

Let $\pi: E \to B$ be a bundle, and let $A \subset B$ be a subspace. The restriction of E to A, denoted $E|A$ is the family of vector spaces $\bigcup_{a \in A} \pi^{-1}(a)$. It inherits from E a topology and local trivializations. If $\pi: E \to B$ is a C^∞-vector bundle and A is a C^∞-submanifold of B, then $E|A$ is a C^∞-vector bundle.

Let $\pi: E \to B$ be a (C^∞-) vector bundle, and let $\{S_b\}_{b \in B}$ be a collection of linear subspaces, $S_b \subset \pi^{-1}(b)$. The union, $S = \bigcup_{b \in B} S_b$, is a subspace of E. We say that it is a (C^∞-) subbundle if and only if there are local (C^∞-) trivializations, $U_\alpha \times V_\alpha \xrightarrow{\psi} \pi^{-1}(U_\alpha)$, in which $\bigcup_{b \in U_\alpha} S_b$ is given by $(U_\alpha \times V_\alpha')$ for V_α' a linear subspace of V_α. This implies that S has (C^∞-) local trivializations, and, in particular, that S is a (C^∞-) vector bundle in its own right. To study subbundles we need the following lemma.

Lemma 3.1 :a) Let U be an open subset of $\mathbb{R}^n$, and let $\sigma: U \to M(r,s)$ be a C^∞-mapping so that $\sigma(u)$ is of rank k for all $u \in U$. For each $u \in U$ there is an open set V_u, with $u \in V_u$, and C^∞-mappings $\psi_1: V_u \to Gl(s,\mathbb{R})$ and $\psi_2: V_u \longrightarrow GL(r,\mathbb{R})$ so that

$$\psi_1(v)\cdot\sigma(v)\cdot\psi_2(v) = \begin{array}{cc} \begin{array}{cc} k & r-k \end{array} & \\ \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & O \\ \hline O & O \end{array}\right) & \begin{array}{c} k \\ s-k \end{array} \end{array}$$

for all $v \in V_u$.

b) If U is any topological space, then a result similar to the one in part a) holds with the ψ_1 and ψ_2 being continuous.

Proof: We shall prove part a), and leave part b) as an exercise. Since $\sigma(u)$ is of rank k there is a $(k\times k)$-minor of $\sigma(u)$ which is non-singular. We assume for simplicity that it is the minor $(\sigma(u)_{ij})$ $1\le i,j\le k$. There is an open set V_u containing u in which this same minor remains non-singular. We call this minor $M(\sigma(v))$. The map $\alpha: V_u \to GL(s,\mathbb{R})$ given by

$$\alpha(v) = \left(\begin{array}{c|c} M(\sigma(v))^{-1} & O \\ \hline O & Id \end{array}\right)$$

is a C^∞-map. The product $\alpha(v)\cdot\sigma(v)$ has the form

$$\begin{array}{c c} & \begin{array}{cc} \quad k \quad & \quad r-k \quad \end{array} \\ \begin{array}{c} k \\ \\ s-k \end{array} & \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & * \\ \hline * & * \end{array}\right) \end{array}$$

Since each of these matrices is of rank k, further row and column operations will leave it in the form

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right).$$

The row operations are achieved by left multiplying by an element $\beta(v) \in GL(s, \mathbb{R})$, while the column operations are achieved by right multiplying by $\gamma(v) \in GL(r, \mathbb{R})$. There is no choice in the way we perform these row and column operations, and they clearly vary in a C^∞- manner with the matrix $\alpha(v) \cdot \sigma(v)$. Letting $\psi_1(v) = \beta(v) \cdot \alpha(v)$ and $\psi_2(v) = \gamma(v)$ gives the result.

Theorem 3.2: *Let* $\pi : E \to B$ *and* $\pi' : E' \to B$ *be* $(C^\infty$-$)$ *vector bundles, and let* $\tilde{f} : E \to E'$ *be a* $(C^\infty$-$)$ *bundle map covering the identity on* B. *Define* $\operatorname{Ker}(\tilde{f}) = \bigcup_{b \in B} (\text{Kernel } \tilde{f}|\pi^{-1}(b))$ *and* $\operatorname{Im}(\tilde{f}) = \bigcup_{b \in B} \tilde{f}(\pi(b))$. *Both of these are* $(C^\infty$-$)$ *subbundles if and only if the rank of* $\tilde{f}(\pi^{-1}(b))$ *is locally constant.*

Proof : The necessity of the rank being locally constant is clear. We shall prove its sufficiency. It suffices to consider the case when the rank of $\tilde{f}(\pi^{-1}(b))$ is k for all $b \in B$. Suppose given a vector bundle over B which has local trivializations over an open cover $\{U_\alpha\}_{\alpha \in I}$ of B. If $\{V_\beta\}_{\beta \in J}$ is a refinement of $\{U_\alpha\}_{\alpha \in I}$, i.e., if each V_β is contained in some $U_{\alpha(\beta)}$, then the bundle has local trivializations over $\{V_\beta\}_{\beta \in J}$. Applying this, one sees that given two vector bundles over B, there is an open cover $\{U_\alpha\}$ for which both bundles have local trivializations. Choose such a cover for $\pi : E \to B$ and $\pi' : E \to B$, say $\{U_\alpha\}$. If we restrict $\tilde{f}$ to $\pi^{-1}(U_\alpha\}$ and use the local trivializations, then we have

$$\tilde{f} : U_\alpha \times V \to U_\alpha \times W .$$

Equipping V and W with bases, we can view $\tilde{f}$ as a $(C^\infty\text{-})$ mapping $\sigma : U_\alpha \to M(r,s)$. (Here $\dim V = r$ and $\dim W = s$.) By the previous lemma, there is an open cover $\{Z_\beta\}$ of U_α and $(C^\infty\text{-})$ changes of bases

$$\psi_1 : Z_\beta \to GL(s,\mathbb{R}) \quad \text{and} \quad \psi_2 : Z_\beta \to GL(r,\mathbb{R})$$

so that $\psi_1(z) \cdot \sigma(z) \cdot \psi_2(z)$ is the matrix

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right).$$

The maps ψ_1 and ψ_2 define $(C^\infty\text{-})$ bundle isomorphisms

$$\tilde{\psi}_1 : Z_\beta \times \mathbb{R}^s \to Z_\beta \times V \text{ and } \tilde{\psi}_2 : Z_\beta \times \mathbb{R} \to Z_\beta \times W,$$

and hence define new $(C^\infty\text{-})$ trivializations over $\{Z_\beta\}$. In these trivializations $\tilde{f}$ is given by

$$\tilde{f}(z,(t_1, \cdots, t_s)) = (z,(t_1,\cdots , t_k,0, \cdots,0))$$

Thus, in these $(C^\infty\text{-})$ trivializations $(\operatorname{Ker} \tilde{f})|Z_\beta$ is $Z_\beta \times (\{0\}\times \mathbb{R}^{r-k}) \subset Z_\beta \times \mathbb{R}^r$ and $(\operatorname{Im} \tilde{f})|Z_\beta$ is $Z_\beta \times (\mathbb{R}^k \times \{0\}) \subset Z_\beta \times \mathbb{R}^s$. This proves that both $\operatorname{Ker} \tilde{f}$ and $\operatorname{Im} \tilde{f}$ are vector bundles .

<u>Corollary 3.3</u>. <u>Let</u> $\pi : E \to B$ <u>be a</u> $(C^\infty\text{-})$ <u>vector bundle and</u> $E' \subset E$ <u>a</u> $(C^\infty\text{-})$ <u>subbundle</u>. <u>Then there is a quotient</u> $(C^\infty\text{-})$ <u>bundle</u> E/E' <u>and</u> $(C^\infty\text{-})$ <u>mapping</u> $E \to E/E'$.

<u>Proof</u> : There is an open cover $\{U_\alpha\}$ so that $E|U_\alpha \cong U_\alpha \times V$ with $E'|U_\alpha \cong U_\alpha \times V'$. The local trivialization for E/E' over U_α is $U_\alpha \times (V/V')$.

<u>Definition</u> : An <u>exact sequence of vector bundles</u> is diagram of vector bundles and vector bundle maps covering the identity on the base

$$E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$$

where f is a linear injection on each fiber, g is a linear surjection in each fiber, and $\mathrm{Im}(f(\pi_1^{-1}(b)) = \mathrm{Ker}(g|\pi_2^{-1}(b))$ for all b in the base. If $E_1 \overset{f}{\to} E_2 \overset{g}{\to} E_3$ is an exact sequence of vector bundles, then 1) Ker f is a vector bundle whose fibers are the trivial vector space; 2) Ker g = Im f; 3) $f : E_1 \to \mathrm{Im}\, f$ is an isomorphism; and 4) $g : (E_2/\mathrm{Ker}\, g) \to E_3$ is an isomorphism.

If $\pi : E \to B$ is a $(C^\infty-)$ vector bundle and if $f : A \to B$ is a $(C^\infty-)$ map, then there is defined the pullback of E via f, f^*E. The vector space over a A is $\pi^{-1}(f(a))$. The topology (and C^∞- structure) are inherited from $A \times E$. We view f^*E as a subspace of $A \times E$, viz. $\{(a,e) | \ f(a) = \pi(e)\}$. This defines the topology for f^*E. If A and E are C^∞-manifolds and f and π are C^∞- maps, then we view f^*E as the preimage of the diagonal, $\Delta_B \subset B \times B$, under the C^∞-mapping $f \times \pi : A \times E \to B \times B$. One checks easily that $f \times \pi$ is transverse to Δ_B, and hence, that f^*E is a C^∞- manifold with $\pi : f^*E \to A$ a C^∞- mapping. If $\pi : E \to B$ has local triavilizations for the cover $\{U_\alpha\}$, then f^*E has local trivializations for $\{f^{-1} U_\alpha\}$.

The restriction of a bundle to a subspace is a special case of the pull back construction applied to the inclusion map.

Suppose we have two $(C^\infty-)$ vector bundles over B, $\pi : E \to B$ and $\pi' : E' \to B$. Form the product $E \times E' \xrightarrow{\pi \times \pi} B \times B$.

One checks easily that this is a $(C^\infty-)$ vector bundle whose fiber over (b_1,b_2) is $\pi^{-1}(b_1) \times \pi^{-1}(b_2)$. The restriction of this bundle to the diagonal is called the Whitney sum of E and E'. It is denoted $\pi \oplus \pi' : E \oplus E' \to B$. The fiber over b is $\pi^{-1}(b) \times \pi'^{-1}(b)$.

Exercises : 1) Show that if $M^n \subset \mathbb{R}^{n+k}$ is given globally by k-equations $\Phi : \mathbb{R}^{n+k} \to \mathbb{R}^k$ whose differential is of rank k everywhere along M, then $N(M^n)$ is trivial.

2) Show $N(S^{n-1})$ is trivial.

3) Show that if $M^n \subset \mathbb{R}^{n+k}$, then $TM^n \oplus N(M^n)$ is a trivial bundle.

4) Show that $TS^{n-1} \oplus L$ is trivial, where L is a trivial line bundle. (N.B.: This does not imply that TS^{n-1} itself is trivial.)

5) If G is a Lie group, show that TG is trivial. (Hint: Use multiplication by g to identify TG_e with TG_g.)

6) Show that the bundle constructed in Example 3 above is non-trivial.

7) Give the definition of a C^∞- vector bundle in terms of the transition functions.

8) Let $G \times M \to M$ be a C^∞, free, and properly discontinuous action. Show that there is induced a free, properly discontinuous C^∞-action $G \times TM \to TM$, and that $T(M/G) = (TM)/G$.

9) Actually it is possible to define the tangent bundle for any C^r-manifold, $r \geq 1$. It will be a C^{r-1}-manifold.

I. Consider germs of C^1-curves $\gamma: (-\epsilon,\epsilon) \to R^n$ with $\gamma(0) = x$. Define an equivalence relation on these germs :

$$\gamma \sim \mu \quad \text{if and only if} \quad \gamma'(0) = \mu'(0).$$

II. The equivalence classes remain the same if we take any C^1-change of coordinates.

III. The equivalence classes form an n-dimensional vector space called $T\mathbb{R}^n_0$.

IV. Use II and local coordinates to define TM_x for any C^1-manifold.

V. Show that TM is a C^{r-1}-manifold if M is a C^r-manifold.

10) The Whitney sum extends to vector bundles the operation of direct sum on vector spaces. Extend the following operations on vector spaces to operations on vector bundles : tensor product, symmetric product, exterior product, homorphism, and dual.

§4: Orientability

If V is a real vector space of dimension n, then we define an equivalence relation on the set of ordered bases for V. We say that $(e_1,\ldots,e_n) \sim (f_1,\ldots,f_n)$ if and only if when we express the f_i as linear combinations of the e_j

$$f_i = \sum \alpha_{ij} e_j$$

the resulting matrix has determinant greater than zero. There are exactly two equivalence classes and they are represented by $(e_1,e_2,\ldots,e_n)$ and $(-e_1,e_2,\ldots,e_n)$. These equivalence classes are called orientations for V. We equip R^n with its canonical orientation, i.e., the one determined by $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$.

Exercises: 1) The set of all ordered bases for V forms an open subspace of $V \times \ldots \times V$ (n-times). Show that this space has two path components corresponding to the two orientations of V.

2) Show that $GL(n,R)$ has two components as does $O(n)$.

If $L: V \to W$ is a linear isomorphism and if we have an orientation for V, we can push it forward to get an orientation for W. If $W = V$, then the pushed foward orientation may, or may not agree with the original one. If it does we say that $L: V \to V$ is orientation preserving, and otherwise that L is orientation reversing. If we choose a basis $\{e_1, \ldots, e_n\}$ for V and use it to express L as a matrix, then L is orientation preserving if and only if $\det L > 0$.

Let $\pi: E \to M$ be a vector bundle. <u>An orientation for</u> E is an orientation for each vector space $\pi^{-1}(m)$ so that there is an open cover $\{U_\alpha\}$ of M and local trivializations over U_α, $\varphi_\alpha: U_\alpha \times R^n \to E|U_\alpha$ which are orientation preserving on every fiber. An <u>orientation for a manifold</u> is an orientation for $TM \to M$.

<u>Theorem 4.1</u>: <u>Let</u> M <u>be a</u> C^∞-<u>manifold which is connected</u>. <u>Then</u> M <u>either had no orientation or exactly</u> 2 <u>orientations</u>. M <u>has an orientation if and only if it is possible to find</u> <u>a</u> C^∞-<u>atlas</u> $\{U_\alpha, \varphi_\alpha, V_\alpha\}$ <u>for</u> M <u>with</u> $\det(D(\varphi_\beta^{-1} \circ \varphi_\alpha)(p)) > 0$ <u>for all</u> α <u>and</u> β <u>and all</u> $p \in \varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$.

<u>Proof:</u> If M has an orientation, then we can take the opposite orientation obtained by reversing the orientation on every TM_x. Given two orientations for M, the set $x \in M$ for which they agree (or disagree) is an open set. Thus if M is connected, then two orientations either agree or disagree everywhere. Thus the two orientations are either

the same or opposite on every tangent plane. This proves that when M has an orientation it has exactly two.

Suppose that we have an atlas $\{U_\alpha, \varphi_\alpha, V_\alpha)$ for M so that $\det(D(\varphi_\beta^{-1} \circ \varphi_\alpha)(p)) > 0$. Define a orientation for TM_x for each $x \in U_\alpha$ by taking the standard orientation on $TV_\alpha(\varphi_\alpha^{-1}(p))$ and taking its image under $D\varphi_\alpha(\varphi_\alpha^{-1}(p))$. If $p \in U_\alpha \cap U_\beta$ then the orientations defined using φ_α and using φ_β agree since $\varphi_\beta^{-1} \circ \varphi_\alpha$ is orientation preserving. This then defines an orientation for M. Conversely, if M is oriented choose an atlas $\{U_\alpha, \varphi_\alpha, V_\alpha\}$ so that each U_α is connected. Then, for each α, $D\varphi_\alpha\colon TV_\alpha \to TM|U_\alpha$ is either orientation preserving or orientation reversing at all points. If it is orientation reversing, then change the coordinates in V_α by replacing x_1 by $-x_1$. After this change $D\varphi_\alpha(p)$ is orientation preserving for all α and all $p \in V_\alpha$. Hence, $D(\varphi_\alpha^{-1} \circ \varphi_\beta)$ is also always orientation preserving. A manifold is said to be <u>orientable</u> if it admits an orientation and <u>non-orientable</u> if it does not.

<u>Examples</u>: 1) S^{n-1} is orientable. One way to get an orientation for TS_x on S^{n-1} is to take a basis $\{e_1, \ldots, e_{n-1}\}$ for TS_x^{n-1} so that $\{e_1, \ldots, e_{n-1}, x\}$ forms a basis giving the usual orientation for R^n. It is easy to see that these orientations on TS_x are locally trivial and hence form an

orientation for S^{n-1}.

2) Any hypersurface, $M = \{x \in R^n | f(x) = 0\}$ with $\nabla f(p) \neq 0$ for every $p \in f^{-1}(x)$, is orientable. Again for each $p \in f^{-1}(x)$ take a basis for TM_p, $\{e_1,\ldots,e_{n-1}\}$, so that $(e_1,\ldots,e_{n-1},\nabla f(p))$ is an oriented basis for R^n.

3) More generally if $M^n \subset R^{n+k}$ is defined by k-equations, and some inequalities, on an open set $U \subset R^{n+k}$, then M is orientable.

<u>Exercises</u>: 1) Show that the Möbius band is non-orientable.

2) Suppose that M^n is a connected, oriented manifold and that $\gamma: M \to M$ is a diffeomorphism. Show that $D\gamma_p: TM_p \to TM_{\gamma(p)}$ is orientation preserving either for all $p \in M$ or for no $p \in M$.

3) Let M be a connected, oriented manifold and $\Gamma \times M \to M$ a free, properly discontinuous, differentiable action. Show that M/Γ is orientable if and only if each $\gamma \in \Gamma$ is orientation preserving.

4) Show RP^n is orientable when n is odd and non-orientable when n is even.

5) Suppose M is connected and oriented and Γ is a free, properly discontinuous, differentiable action. Show that if every homomorphism $\Gamma \to \{\pm 1\}$ is trivial, then M/Γ is orientable.

6) Show that if $RP^2 \subset R^n$, then its image cannot be given globally as the zeroes of a function $\Phi: U \to R^{n-2}$ where U is an open set containing RP^2 and $D\Phi_p$ has rank $(n-2)$ for every $p \in RP^2$.

7) Let M^{2n} be a manifold which has the structure of a complex analytic manifold. Show that M^{2n} is orientable.

§5. Vector Fields

Let M be a C^∞-manifold and $A \subset M$ a subspace. A vector field over A is an association to each point $a \in A$ of a vector $\tau_a \in TM^n_a$ so that the resulting map $\mathfrak{X}: A \to TM$ is continuous. If A is an open set of M, then a vector field on A is said to be C^r, or C^∞, if the map $\mathfrak{X}: A \to TM$ is C^r, or C^∞ respectively. If $A \subset M$ is not open, then a vector field on A is said to be C^r, or C^∞, if it admits an extension to a C^r, or C^∞, respectively, vector field on some open set of M containing A.

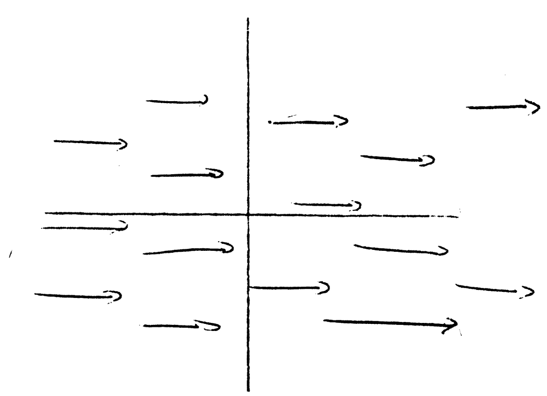
Examples: 1) Associating to $x \in R^n$ the vector $\frac{\partial}{\partial x_i}(x)$ gives a C^∞-vector field which we call $\frac{\partial}{\partial x_i}$.

2) Associating to $(x_1,\ldots,x_n) \in R^n$ the vector $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}(x)$ is a C^∞-vector field on R^n.

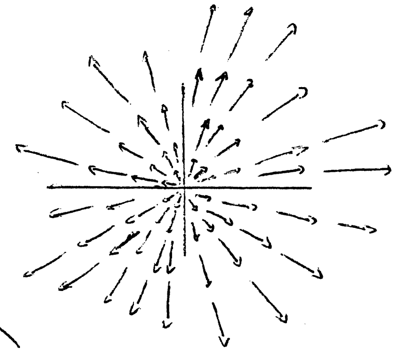
3) Associating to (x,y) the vector $(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y})$ gives a C^∞-vector field on $\mathbb{R}^2$.

Pictures of these vector fields (when we identify $TR^n_x = R^n$) by using the basis $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\}$ are:

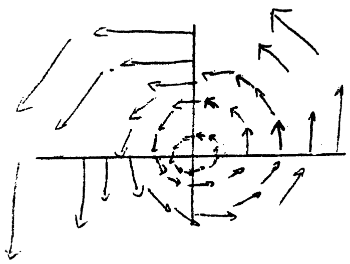
1



2



3



In general, given a vector field on M^n to know if it is continuous, C^r, or C^∞ we restrict to an open set where we have C^∞-coordinates $(x_1,\ldots,x_n)$ and we write out the vector field as:

$$(x_1,\ldots,x_n) = \sum_{i=1}^{n} f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}(x_1,\ldots,x_n)$$

The vector field in U is continuous, C^r or C^∞, if and only if the f_i are continuous, C^r, or C^∞. If $U \subset R^n$ is an open set, then we have C^∞-vector fields $\frac{\partial}{\partial x_i}$ on U. (This is the vector field whose value at $p \in U$ is $\frac{\partial}{\partial x_i}(p) \in TU_p$.)

An integral curve through $p \in M$ for a vector field $\mathcal{X}$ on M is a curve $\gamma\colon (-\epsilon,\epsilon) \to M$, with $\gamma(0) = p$ and $\gamma'(t) = \mathcal{X}(\gamma(t))$ for all $t \in (-\epsilon,\epsilon)$. For example, $\gamma(t) = (t,0,\ldots,0) \in R^n$ is an integral curve through 0 for $\frac{\partial}{\partial x_1}$; $\gamma(t) = \frac{1}{2}(1+t)^2(x_1,\ldots,x_n)$ is an integral curve for $\Sigma_{i=1}^n x_i \frac{\partial}{\partial x_i}$ through the point $\frac{1}{2}(x_1,\ldots,x_n)$; $\gamma(t) = (\cos(t),\sin(t))$ is an integral curve for $(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})$ through $(1,0)$.

An ordinary, first order differential equation in $U \subset R^n$ is a system $\dot{x}_i = \varphi_i(t,x_1,\ldots,x_n)$ for $i = 1,\ldots,n$. A local solution with initial conditions $(t^0,(x_1^0,\ldots,x_n^0))$ is a curve $\gamma: (t^0-\epsilon,t^0+\epsilon) \to U$ so that $\gamma(t^0) = x^0$ and $\gamma'(t) = (\varphi_1(t,\gamma(t)),\ldots,\varphi_n(t,\gamma(t)))$ for all $t \in \{t^0-\epsilon,t^0+\epsilon)$. For example, the system $\dot{x}_i = \delta_{1i}$ has solution with initial conditions $(0,(0,\ldots,0))$ given by $\gamma(t) = (t,0,\ldots,0)$. An ordinary, first order differential equation is <u>time independent</u> if the φ_i do not depend on t, i.e. φ_i is a function of $(x_1,\ldots,x_n)$. A vector field on U corresponds to a time independent differential equation:

$$\mathcal{X}(p) = \sum_{i=1}^{n} \varphi_i(p)\frac{\partial}{\partial x_i} \longleftrightarrow \{\dot{x}_i = \varphi_i(x) \quad \text{for} \quad i = 1,\ldots,n\}.$$

Under this correspondence an integral curve through p for $\mathcal{X}$ corresponds to a local solution of the differential equation with initial condition $(0,p)$.

The standard theory of ordinary differential equations posits the existence, uniqueness and C^∞-variation with initial conditions of the solution. Here we quote a version of this theorem which can be found in Pontrjagin "Ordinary Differential Equations" pp. 150-183 (see especially the proof of theorem 2 beginning on page 159, proof of differentiability beginning on page 170, remark B on page 177, and the discussion on pages 178 and 179).

Theorem 5.1: Let $\dot{x}_i = \varphi_i(t,x_1,\ldots,x_n)$, $i = 1,\ldots,n$ be a C^∞-differential equation in an open subset $U \subset R^n$.

1) Given initial conditions (t^0,x^0) there is $\epsilon > 0$ and a C^∞ solution $\gamma\colon (t^0-\epsilon, t^0+\epsilon) \to U$.

2) Given two solutions defined on intervals containing t^0, γ and ψ, then $\gamma(t) = \psi(t)$ for all t for which both $\gamma(t)$ and $\psi(t)$ are defined.

3) Given initial conditions (t^0,x^0) there is a neighborhood in $R^1 \times R^n$ of (t^0,x^0), N, and $\epsilon > 0$ so that the solution $\gamma_{(\tau,\xi)}(t)$ exists for all $(\tau,\xi) \in N$ and all t within ϵ of τ. Considering $\gamma_{(\tau,\xi)}(t)$ as a function of (t,τ,ξ), it is C^∞ in all variables.

Corollary 5.2: Let $\chi\colon U \to TU$ be a C^∞-vector field on an open set $U \subset R^n$. There is an open set of $U \times \{0\} \subset U \times R$, W, so that if $(p,t) \in W$ then the integral curve for χ through p is defined at time t, $\gamma_p(t)$. This gives a well-defined map $W \to U$ which is a C^∞-mapping.

Corollary 5.3: Let M be a C^∞-manifold and $\chi\colon M \to TM$ a C^∞-vector field. There is an open set containing $M \times \{0\}$ in $M \times R^1$, W, so that if $(p,t) \in W$, then $\gamma_p(t)$ is defined (*where* γ_p *is the integral curve for* $\mathcal{X}$ *through* p). *The resulting map* $W \to M$ *is* C^∞.

Proof: The existence, uniqueness, and C^∞-variation is a purely local question. Hence, we can always work in a coordinate system and apply 5.2.

Theorem 5.4: *If* M *is compact and if* $\mathcal{X}: M \to TM$ *is a* C^∞ *vector field, then the integral curve for* $\mathcal{X}$ *through* $p \in M$ *can be defined for all* $t \in R^1$. *The resulting map* $M \times R^1 \to M$ *is* C^∞.

Proof: The only part of 5.4 that isn't contained in 5.3 is the existence of $\gamma_p(t)$ for all $t \in R^1$. First, we prove that there is $\epsilon > 0$ (independent of p) so that $\gamma_p(t)$ is defined for all $p \in M$ and all t with $|t| < \epsilon$. This is a consequence of the fact that if $W \subset M \times R^1$ is an open set containing $M \times \{0\}$ and if M is compact, then $M \times (-\epsilon, \epsilon) \subset W$ for some $\epsilon > 0$. (Exercise: Prove this statement.)

If $\gamma_p(s) = q$, then by the uniqueness of the solution

$$\gamma_p(t+s) = \gamma_q(t) \text{ whenever both are defined.}$$

Thus, if $\gamma_p(t)$ is defined for $t \in (-\tau, \tau)$ we can extend it to be defined in $(-\tau-\epsilon/2, \tau+\epsilon/2)$ by setting $q = \gamma_p(\tau-\epsilon/2)$ and $r = \gamma_p(-\tau+\epsilon/2)$ and defining

$$\gamma_p(t) = \gamma_q(t-\tau+\varepsilon/2) \qquad \text{for } t \in (\tau-\varepsilon/2, \tau+\varepsilon/2)$$

and

$$\gamma_p(t) = \gamma_r(t+\tau-\varepsilon/2) \qquad \text{for } t \in (-\tau-\varepsilon/2, -\tau+\varepsilon/2).$$

Continuing in this manner we can eventually define $\gamma_p(t)$ for all $t \in R^1$.

As we have seen a vector field on a manifold becomes a (time independent) ordinary first order system of differential equations in local coordinates. Ofter we think of the manifold as the possible states of some physical system (called the configuration space) and the vector field as a dynamic or motion law describing how states evolve with time. In this case an integral curve will describe the state of the system at time t if it begins in state $\gamma(0)$ at time 0.

Example: Consider n point masses with masses $m_1, \ldots, m_n$ in R^3 which move according to the gravitational force law and Newton's equations. The state space for this system is an open subset of.

$R^{6n} = \{\{(v_1,p_1), \ldots, (v_n,p_n) \mid v_i \neq v_j \text{ for } i \neq j\}$. Here v_i represents the position vector in R^3 of the i^{th}-mass and p_i represents its momentum vector. The vector field describing the motion is:

$$\mathcal{X}(v_1,\ldots,v_n,p_1,\ldots,p_n)$$

$$= (p_1/m_1,\ldots,p_n/m_n,\sum_{i\neq 1} Gm_i m_1 (v_i-v_1)/\|v_i-v_1\|^2,\ldots,\sum_{i\neq n} Gm_i m_n (v_i-v_n)/\|v_i-v_n\|^2)$$

where G is the universal gravitational constant. Written as a differential equation it becomes:

$$\begin{cases} \dot{v}_i = p_i/m_i & \text{(definition of momentum)} \\ \dot{p}_i = \sum_{j\neq i} Gm_j m_i (v_j-v_i)/\|v_j-v_i\|^2 & \text{(Newton's law).} \end{cases}$$

An integral curve describes how the positions and momenta change with time. For this reason vector fields are some times called flows.

<u>Example</u>: Let $(x,y) = (-y,x)$ be a flow in R^2. The integral curves are:

1) circles of any radius $r > 0$, and
2) the constant path $\gamma(t) = (0,0)$. (A so-called <u>fixed point</u> for the flow.)

<u>Example</u>: Let $\mathcal{X}(x) = e^x$ be a vector field on R^1. The integral curve with $\gamma(0) = 0$ is $\gamma(t) = \ln(\frac{1}{1-t})$. Notice that this curve is only defined for $t < 1$. What happens is that starting at 0 one flows all the way to $+\infty$ by the time $t = 1$.

In terms of the tangent bundle, a vector field on M is a function $\mathcal{X}: M \to TM$ so that $\pi \circ \mathcal{X}(m) = m$. It is C^∞ if $\mathcal{X}$ is a C^∞-mapping between C^∞-manifolds. A zero of a vector field is an $m \in M$ such that $\mathcal{X}(m)$ is the zero vector in TM_x.

Theorem 5.5: Let $\mathcal{X} : M \to TM$ be a C^∞-vector field with $\mathcal{X}(p)$ non-zero. There is an open set $U \subset M$, containing p, and C^∞ coordinates on U, $(x_1,\ldots,x_n)$, in which $\mathcal{X}$ becomes $\frac{\partial}{\partial x_n}$.

Proof: Choose local C^∞-coordinates near p, $(x_1',\ldots,x_n')$. Suppose $\mathcal{X}(p) = (p, \Sigma_{i=1}^n a_i \frac{\partial}{\partial x_i'}(p))$. There is an invertible $(n\times n)$-matrix (λ_{ij}) so that

$$\sum_{j=1}^{n} \lambda_{ij} a_j = \delta_{nj}.$$

Consider (λ_{ij}) as a linear automorphism of R^n and use it to change coordinates from $(x_1',\ldots,x_n')$ to $(y_1,\ldots,y_n)$. In the new coordinates $\mathcal{X}(p) = (p, \frac{\partial}{\partial y_n})$. We can, in addition, suppose that p is the origin in $(y_1,\ldots,y_n)$-space. Let V be $\{(y_1,\ldots,y_n) | y_n = 0\}$. Inside $V \times R^1$ there is an open set W containing $V \times \{0\}$, so that the integral curves define a C^∞-mapping $\Psi: W \to U$. Let the coordinates in W be $(y_1,\ldots,y_{n-1},t)$. One checks easily that $D\Psi(0)$ is the identity matrix and hence invertible. Thus, the image under Ψ of $(y_1,\ldots,y_{n-1},t)$ form local coordinates near p. Call these $(x_1,\ldots,x_n)$. Clearly, the vector field is $\frac{\partial}{\partial x_n}$ in these coordinates.

<u>Corollary 5.6:</u> <u>Let</u> M <u>be a</u> C^∞-<u>manifold and</u> $\mathcal{X}$ <u>a</u> C^∞-<u>vector field on</u> M <u>which is never zero. The integral curves of</u> $\mathcal{X}$ <u>define a one-dimensional foliation on</u> M.

As we have seen, non-zero vector fields locally all look the same. This is not true at the zero of a vector field. For example in R^1 the vector field $t \mapsto t\frac{\partial}{\partial t}$ can not be changed into $t \mapsto -t\frac{\partial}{\partial t}$ by any C^∞ (or even any C^0) change of coordinates.

§6: Algebraic Structure of Vector Fields

Let $\mathfrak{X}(M)$ be the set of all C^∞-vector fields on M. Our first construction is to give $\mathfrak{X}(M)$ a topology--the compact open topology. The sub-basic open sets are denoted $\langle K,U\rangle$ where $K \subset M$ is compact and $U \subset TM$ is open. Such a set $\langle K,U\rangle$ consists of all the vector fields whose values for points in K lie in U; i.e., $\langle K,U\rangle = \{\mathcal{X} \in \mathfrak{X}(M) \mid \mathcal{X}(K) \subset U\}$. A set $X \subset \mathfrak{X}(M)$ is open if and only if $X = \bigcup_{\alpha\in I} \left(\bigcap_{i=1}^{n(\alpha)} \langle K_{\alpha_i}, U_{\alpha_i}\rangle \right)$, i.e., if and only if X is an arbitrary union of finite intersections of the $\langle K,U\rangle$'s.

Next, comes the real vector space structure of $\mathfrak{X}(M)$:

$$(r\mathcal{X}_1 + s\mathcal{X}_2)(p) = r\mathcal{X}_1(p) + s\mathcal{X}_2(p).$$

Lastly along these lines, we have the module structure of $\mathfrak{X}(M)$ over the ring of real valued C^∞-functions on M, $\mathcal{C}^\infty(M)$. It is given by $(f\cdot\mathcal{X})(p) = f(p)\cdot\mathcal{X}(p)$.

There is another type of structure: The vector fields act as derivations on the C^∞-functions. The formula is $\mathcal{X}(f)(p) = \mathcal{X}_p$ (germ of f at p). Since $\mathcal{X}_p(\alpha\cdot\beta) = \alpha(p)\mathcal{X}_p(\beta) + \beta(p)\mathcal{X}_p(\alpha)$, it follows that $\mathcal{X}(f\cdot g) = f\cdot\mathcal{X}(g) + g\cdot\mathcal{X}(f)$. Thus, the vector fields are a module (over the C^∞-functions) of derivations. It is a straightforward exercise to show that the C^∞-vector fields are the module of <u>all</u> derivations on the C^∞-functions. Notice also that the module structure is compatible with the derivation structure in that :

<u>6.1:</u> $$(f\cdot\mathcal{X})(g) = f\cdot(\mathcal{X}(g)).$$

There is one more very important piece of algebraic structure: the Lie bracket. This is a bilinear pairing which produces from two vector fields $\mathcal{X}$, $\mathcal{Y}$ and third $[\mathcal{X},\mathcal{Y}]$. To define $[\mathcal{X},\mathcal{Y}]$ we give its value at $p \in M$ as a derivation: $[\mathcal{X},\mathcal{Y}]_p(f) = \mathcal{X}_p(\mathcal{Y}(f)) - \mathcal{Y}_p(\mathcal{X}(f))$. One checks that this is a continuous, bilinear map

$$[\ ,\ ]: \mathcal{F}(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M).$$

In local coordinates $(x_1,\ldots,x_n)$ if $\mathcal{X} = \Sigma_{i=1}^n f_i \frac{\partial}{\partial x_i}$ and $\mathcal{Y} = \Sigma_{i=1}^n g_i \frac{\partial}{\partial x_i}$, then

$$[\mathcal{X},\mathcal{Y}] = \sum_{i,j=1}^{n} \left(f_i \frac{\partial g_j}{\partial x_i}\cdot \frac{\partial}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}\right).$$

In addition to the above properties $[\ ,\ ]$ satisfies:

<u>6.2:</u> $$\begin{cases} \text{a)} & [\mathcal{X},\mathcal{Y}] = -[\mathcal{Y},\mathcal{X}] \quad \text{and} \\ \text{b)} & [[\mathcal{X},\mathcal{Y}],\mathcal{Z}] + [[\mathcal{Z},\mathcal{X}],\mathcal{Y}] + [[\mathcal{Y},\mathcal{Z}],\mathcal{X}] = 0 \end{cases}$$

The second relation is called the Jacobe identity. In general, any real vector space V together with a bilinear operation satisfying 6.2 a) and b) is called a Lie algebra.

Suppose that we are given a manifold M^n. A C^∞-k-plane field, $\mathfrak{F}^k$, is a collection of k-dimensional linear subspaces $\mathfrak{F}^k_p \subset TM_p$, for each $p \in M$. These subspaces are required to vary in a C^∞-manner with p, i.e., to be a subbundle of TM. An integral submanifold for a k-plane field on M^n is a C^∞-submanifold $N^k \subset M$ so that $TN_p = \mathfrak{F}^k_p$ for each $p \in N$. (Thus, N is tangent to the k-plane field.) One might be tempted to think, in analogy with the case of flows, that such integral submanifolds always exist at least locally. This is not true, however. We say that $\mathfrak{F}^k$ is integrable if it has integral submanifolds through every point. If $\mathfrak{F}^k$ is integrable, then the Lie bracket of any two vector fields in $\mathfrak{F}^k$ must be in $\mathfrak{F}^k$. (Exercise : Prove this.) Thus, we have an obstruction to integrability -- the vector fields in $\mathfrak{F}^k$ must form a Lie subalgebra of the Lie algebra of all vector fields. The analogue of the 1-dimensional theorem is the following:

Theorem (Frobenius) : Let $\mathfrak{F}^k$ be a C^∞-k-plane field in M. $\mathfrak{F}^k$ is integrable if and only if the vector space of vector fields whose values lie in $\mathfrak{F}^k$ is closed under the Lie bracket.

We shall not prove this theorem is this course.

<u>Exercises</u>: 1) Verify that the Jacobi identity holds for vector fields and the Lie bracket.

2) Let G be a Lie group. Consider the vector space of left invariant vector fields inside all vector fields on G. (A vector field $\mathcal{X}$ is left invariant if $(Dg)(\mathcal{X}) = \mathcal{X}$ where Dg denotes the differential of the diffeomorphism given by left multiplying by g.) Show that this vector space can be identified with TG_e and hence is finite dimensional. Show that it is closed under the bracket operation and hence becomes a finite dimensional sub-Lie algebra of all the vector fields. It is called the Lie algebra of G.

3) Show that if $f\colon M \to P$ is a C^∞-diffeomorphism then $[Df(\mathcal{X}), Df(\mathcal{Y})] = Df([\mathcal{X}, \mathcal{Y}])$.

4) Let $\mathcal{F}^2$ be the plane field in $\mathbb{R}^3$ given by $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z})$. Show that $\mathcal{F}^2$ is not integrable. Indeed, show that if we define four arcs in $\mathbb{R}^3$ tangent to $\mathcal{F}^2$ and lying above the unit square in (x,y)-space, then these four arcs cannot make a closed path in $\mathbb{R}^3$. Thus, the interior of the unit square cannot be lifted to $\mathbb{R}^3$ to be everywhere tangent to the foliation.

5) Show that if A is any associative algebra, then there is a Lie algebra structure on A defined by

$$[a,b] = ab - ba .$$

6) Show that the Lie algebra of $GL(n,\mathbb{R})$ can be identified with vector space of all $(n\times n)$-matrices so that the Lie bracket becomes $[X,Y] = X\cdot Y - Y\cdot X$. Show that the Lie algebra of $SL(n,\mathbb{R})$ is the subalgebra of matrices of trace zero. Show that the Lie algebra of $O(n)$ is the subalgebra of all skew symmetric matrices.

Chapter III: Differential Forms

In the first chapter we dealt with the basic properties of the spaces on which we shall do calculus--manifolds. In the second chapter we dealt with the infinitesimal structure of manifolds--the tangential structure. This chapter develops the differential and integral calculus on manifolds. What, in modern terminology are called differential forms, were originally called integrals. There are two parts to an integral--the thing being integrated, e.g., the integrand, and the region over which it is being integrated, e.g., the interval of integration. The integrand could not stand alone. As first year calculus students are wont to ask, "What is the 'dx' anyway?" Originally, in higher dimensions, one considered integrals of integrands over regions and studied the change of the result as the region was deformed. This separated somewhat the two ingredients. In the modern point of view the two are completely separated.

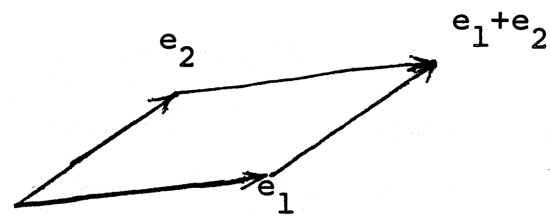
The integrands are differential forms, and they have an existence independent of any operation of integrating. They live in an appropriate infinite dimensional algebra. What they are actually is templates or models for integrands.

When we supply the region of integration we can make an integral using the model or "form". The region of integration is usually a submanifold or a union of pieces of submanifolds. Of course, the models or forms carry with them the dimension of the region over which they can be integrated. It runs from 0 to the dimension of the ambient manifold in which we are working.

Before taking up the study of forms, we must prepare their habitat . This requires the introduction of the multi-linear algebra of finite dimensional real vector spaces.

§1. Multilinear Algebra

Let us begin by studying order pairs of vectors in R^2, $\{e_1,e_2\}$. If the vectors are linearly independent, then they give an orientation. Compare this with the standard orientation and assign a $+$ if they agree and a $-$ if they disagree. Next consider the parallelogram that they span:



Let its area be A. Multiply the two quantities together. The result, $\pm A$, is called the signed area of the parallelogram generated by $\{e_1,e_2\}$. If e_1 and e_2 are linearly dependent, then assign 0. Call the resulting association $\alpha\colon R^2 \times R^2 \to R$. A map $\varphi\colon R^2 \times R^2 \to R$ is <u>bilinear</u> if it is linear in each factor, i.e. if $\varphi(re_1 + se_1', e_2) = r\varphi(e_1,e_2) + s\varphi(e_1',e_2)$ and $\varphi(e_1, re_2 + se_2') = r\varphi(e_1,e_2) + s\varphi(e_1,e_2')$. It is alternating if $\varphi(e_1,e_2) = -\varphi(e_2,e_1)$.

<u>Theorem 1.1</u>: <u>The map</u> $\alpha\colon R^2 \times R^2 \to R$ <u>defined above is bilinear and alternating. If</u> $\varphi\colon R^2 \times R^2 \to R$ <u>is any bilinear</u>, <u>alternating map, then there</u> $r \in R$ <u>such that</u> $\varphi(e_1,e_2) = r\alpha(e_1,e_2)$

for all $(e_1,e_2) \in R^2 \times R^2$.

Proof: Suppose $e_1 = (a,b)$ and $e_2 = (c,d)$. By definition $\alpha(e_1,e_2) = ad - bc$. From this formula it is immediate that α is bilinear and alternating. Suppose φ is any bilinear alternating map. Then $\varphi((a,b),(c,d)) = ad\varphi((1,0),(0,1)) + bc\varphi((0,1),(1,0)) + ac\varphi((1,0),(1,0)) + bd\varphi((0,1),(0,1))$. By the fact that φ is alternating, we have $\varphi(e,e) = 0$ and $\varphi((0,1),(1,0)) = -\varphi((1,0),(0,1))$. Thus, $\varphi((a,b),(c,d)) = (ad - bc)\varphi((1,0,(0,1))$. Letting $r = \varphi((1,0),(0,1))$ gives the result

$$\varphi(e_1,e_2) = r\alpha(e_1,e_2).$$

Corollary 1.2: *If* $L: R^2 \to R^2$ *is a linear map, then* $\alpha(L(1,0),L(0,1)) = \det L$.

Let V be a finite dimensional real vector space. A map $\varphi: V \times \ldots \times V \to R$ is *multilinear* if it is linear in each variable, i.e., if for each i:

$$\varphi(v_1,\ldots,v_{i-1},rv_i+sv_i',v_{i+1},\ldots,v_n)$$

$$= r\varphi(v_1,\ldots,v_n) + s\varphi(v_1,\ldots,v_i',\ldots,v_n).$$

It is alternating if for all $i < j$:

$$\varphi(v_1,\ldots,v_i,\ldots,v_j,\ldots,v_n) = -\varphi(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_n).$$

Consider all multilinear alternating maps

$$\underbrace{V \times \ldots \times V}_{k \text{ factors}} \longrightarrow R$$

This set is a real vector space under the obvious operations:

$$(r\varphi+s\varphi')(e_1,\ldots,e_n) = r\varphi(e_1,\ldots,e_n)+s\varphi'(e_1,\ldots,e_n).$$

This real vector space is denoted $\Lambda^k(V^*)$. By convention $\Lambda^0(V^*) = R$.

<u>Examples</u>: 1) Let V be n-dimensional with $\{b_1,\ldots,b_n\}$ a basis. One element $\varphi \in \Lambda^n(V^*)$ is described as follows. Given $\{e_1,\ldots,e_n\}$, then express $e_1 = \Sigma_{j=1}^n \alpha_{ij} b_j$. Define $\varphi(e_1,\ldots,e_n)$ to be $\det(\alpha_{ij})$. This is clearly a multilinear alternating map since "det" is. In more geometric terms, we use $\{b_1,\ldots,b_n\}$ to identify V with R^n. The $\{e_1,\ldots,e_n\}$ span an n-dimensional parallelepiped in R^n. Its signed n-dimensional volume is $\det(\alpha_{ij})$. As we shall see below all other elements of $\Lambda^n(V^*)$ are multiples of this one.

2) $\Lambda^1(V^*) = \mathcal{L}(V,R)$, the space of real valued, linear maps on V (also called <u>the dual space to</u> V).

Lemma 1.3: *If $\{e_1,\ldots,e_k\} \in V$ are linearly dependent and if $\varphi \in \Lambda^k(V^*)$, then $\varphi(e_1,\ldots,e_k) = 0$.*

Proof: Suppose $\alpha_1 e_1 +\ldots+ \alpha_k e_k = 0$ with not all the $\alpha_i = 0$. We can suppose that $\alpha_1 \neq 0$. Then
$e_1 = (-\alpha_2/\alpha_1)e_2 +\ldots+ (-\alpha_k/\alpha_1)e_k$. Hence,
$\varphi(e_1,\ldots,e_k) = \Sigma_{i=2}^{k}(-\alpha_i/\alpha_1)\varphi(e_i,e_2,\ldots,e_k)$. By the alternating property $\varphi(v_1,\ldots,v_k)$ vanishes if two of the v_i's are equal. Consequently, $\varphi(e_1,\ldots,e_k) = 0$.

Theorem 1.4: 1) $\Lambda^k(V^*) = 0$ *for* $k > \dim V$.

2) *If the dimension of* V *is* n, *then the dimension of* $\Lambda^k(V^*)$ *is* $\binom{n}{k}$. *In particular,* $\Lambda^n(V^*)$ *is one dimensional.*

Proof: Part 1) follows immediately from the previous lemma since any $\{e_1,\ldots,e_k\}$ must be linearly dependent if $k > \dim V$.

Let $\{b_1,\ldots,b_n\}$ be a basis for V. We claim that $\varphi \in \Lambda^k(V^*)$ is determined by the $\binom{n}{k}$ numbers $\varphi(b_{i_1},\ldots,b_{i_k})$, $1 \leq i_1 <\ldots< i_k \leq n$ and that any collection of $\binom{n}{k}$ numbers occurs for some φ. First, suppose that φ and ψ yield the same collection of numbers. Then if $\{b_{i_1},\ldots,b_{i_k}\}$ is any k-tuple of the basis vectors, then
$\varphi(b_{i_1},\ldots,b_{i_k}) = \psi(b_{i_1},\ldots,b_{i_k})$. The reason is that either

some basis element appears twice in the collection, in which case φ and ψ both vanish since they are alternating, or by a finite number of interchanges we can put the elements in ascending order. Each such interchange switches the sign of both φ and ψ evaluated on the collection, and, after we have achieved ascending order, φ and ψ take the same value. Now consider $\varphi(e_1,\ldots,e_k)$ and $\psi(e_1,\ldots,e_k)$. Express $e_i = \Sigma_{j=1}^{n} \alpha_{ij} b_j$ and use the multilinearity to show that:

$$\varphi(e_1,\ldots,e_k) = \sum_{(j_1,\ldots,j_k)} \alpha_{1j_1} \cdot \alpha_{2j_2} \cdots \alpha_{kj_k} \varphi(b_{j_1},\ldots,b_{j_k})$$

and

$$\psi(e_1,\ldots,e_k) = \sum_{(j_1,\ldots,j_k)} \alpha_{ij_1} \cdot \alpha_{2j_2} \cdots \alpha_{kj_k} \psi(b_{j_1},\ldots,b_{j_k}).$$

From this, and the fact that φ and ψ evaluate the same on k-tuples of basis vectors, it follows that $\varphi(e_1,\ldots,e_k) = \psi(e_1,\ldots,e_k)$.

Conversely suppose given $1 \le i_1 < i_2 < \ldots < i_k \le n$. We shall construct $\varphi \in \Lambda^k(V^*)$ so that $\varphi(b_{i_1},\ldots,b_{i_k}) = 1$, and $\varphi(b_{j_1},\ldots,b_{j_k}) = 0$ for all other sequences $j_1 < \ldots < j_k$. Given $\{e_1,\ldots,e_k\}$ express each e_i as a column vector in the basis $\{b_1,\ldots,b_n\}$. They produce an $(n\times k)$-matrix

$$\begin{array}{c} e_1 \quad \cdot \quad \cdot \quad \cdot \quad e_k \\ \left(\begin{array}{ccc} & & \\ & \alpha_{ij} & \\ & & \end{array} \right) \end{array}$$

Pick out the rows $\{i_1, i_2, \ldots, i_k\}$ to form a $(k \times k)$-matrix and take its determinant. One checks easily that this is the required multilinear map.

Elements in $\Lambda^k(V^*)$ are called k-covectors. There is a pairing

$$\Lambda^k(V^*) \times \Lambda^\ell(V^*) \xrightarrow{\wedge} \Lambda^{k+\ell}(V^*)$$

given by

$$(\varphi, \omega) \longmapsto \varphi \wedge \omega$$

where

$$\varphi \wedge \omega(e_1, \ldots, e_{k+\ell})$$
$$= \frac{1}{k!\,\ell!}\Big(\sum_\sigma (-1)^\sigma \varphi(e_{\sigma(1)}, \ldots, e_{\sigma(k)}) \cdot \omega(e_{\sigma(k+1)}, \ldots, e_{\sigma(k+\ell)})\Big).$$

Here, σ runs through the permutations of the set $\{1, \ldots, k+\ell\}$, and $(-1)^\sigma = \pm 1$ means the sign of the permutation, i.e., $(-1)^\sigma$ is $+1$ if and only if σ is a product of an even number of interchanges.

This operation is easily seen to be bilinear, associative and to be <u>graded commutative</u>, i.e., to satisfy

(1.5) $$\varphi \wedge \omega = (-1)^{k\cdot\ell}\omega \wedge \varphi.$$

Consequently, this multiplication makes $\bigoplus_{k=0}^{n} \Lambda^k(V^*)$ a graded algebra with an associative, graded-commutative multiplication. The unit of this algebra is the $1 \in R = \Lambda^0(V^*)$. The algebra is called the <u>exterior algebra on</u> V^*.

<u>Proposition 1.6</u>: <u>Let</u> V <u>be an</u> n-<u>dimensional real vector space. The pairing</u>

$$\Lambda^k(V^*) \times \Lambda^{n-k}(V^*) \xrightarrow{\wedge} \Lambda^n(V^*) \cong R$$

<u>is non-singular in the sense that if</u> $\varphi \wedge \omega = 0$ <u>for all</u> $\omega \in \Lambda^{n-k}(V^*)$, <u>then</u> $\varphi = 0$.

<u>Proof</u>: Let $\{b_1,\ldots,b_n\}$ be a basis for V and fix $1 \leq i_1 <\ldots< i_{n-k} \leq n$. Choose $\omega \in \Lambda^{n-k}(V^*)$ so that $\omega(b_{i_1},\ldots,b_{i_{n-k}}) = 1$ and $\omega(b_{j_1},\ldots,b_{j_{n-k}}) = 0$ for all other sequences $1 \leq j_1 < j_2 <\ldots< j_{n-k} \leq n$. Then $\varphi \wedge \omega(b_1,\ldots,b_n) = \pm\varphi(b_{r_1},\ldots,b_{r_k})$ where $\{b_{r_1},\ldots,b_{r_k},b_{i_1},\ldots,b_{i_{n-k}}\} = \{b_1,\ldots,b_n\}$. Thus, if $\varphi \wedge \omega = 0$, then $\varphi(b_{r_1},\ldots,b_{r_k}) = 0$. Consequently, if $\varphi \wedge \omega = 0$ for all

$\omega \in \Lambda^{n-k}(V^*)$, then $\varphi(b_{r_1},\ldots,b_{r_k}) = 0$ for all $1 \leq r_1,\ldots,r_k \leq n$, and hence $\varphi = 0$.

Example: Let $V = T\mathbb{R}^n_0$. A basis for V is $\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\}$. Denote the dual basis for $V^* = \Lambda^1(V^*)$ by $\{dx_1,\ldots,dx_n\}$. This means that $dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. Using the multiplication, $\wedge$, in $\bigoplus_{\ell=0}^{n} \Lambda^{\ell}(V^*)$ we define elements $dx_{i_1} \wedge\ldots\wedge dx_{i_k} \in \Lambda^k(V^*)$. The collection $\{dx_{i_1} \wedge\ldots\wedge dx_{i_k}\}_{1\leq i_1<\ldots<i_k\leq n}$ is a basis for $\Lambda^k(V^*)$.

One can define multi-vector as well as multi-covectors. Let V be a finite dimensional real vector space. Define $\Lambda^k V$ to be a quotient of the free abelian group generated by $\underbrace{V \times\ldots\times V}_{k\text{-times}}$. The relations that define the quotient are:

1) $e_1\wedge\ldots\wedge e_i\wedge\ldots\wedge e_j\wedge\ldots\wedge e_n = -e_1\wedge\ldots\wedge e_j\wedge\ldots\wedge e_i\wedge\ldots\wedge e_n$

and

2) $e_1\wedge\ldots\wedge(re_i+se_i')\wedge\ldots\wedge e_n = r(e_1\wedge\ldots\wedge e_n)+s(e_1\wedge\ldots\wedge e_i'\wedge\ldots\wedge e_n)$.

Exercises: 1) Verify the claim that $\bigoplus_{\ell=0}^{n} \Lambda^{\ell}(V^*)$ is an associative, graded commutative algebra under $\wedge$.

2) Show that $\Lambda^k V$ is naturally identified with $\Lambda^k((V^*)^*)$. (Here $V^* = \mathcal{L}(V,\mathbb{R})$ is the dual space to V.)

3) Show that if V is of dimension n, then $\Lambda^k V$ is of dimension $\binom{n}{k}$.

4) Let V be a vector space with basis $\{b_1,\ldots,b_n\}$. Show that if $\varphi(\Lambda^k(V^*))$ satisfies

a) $\varphi(b_1,\ldots,b_k) \neq 0$, and

b) $\varphi(b_{i_1},\ldots,b_{i_k}) = 0$ if some $i_j > k$, then

$\varphi(e_1,\ldots,e_k)$ is the volume of the dimensional parallelepiped spanned by the projections of $\{e_1,\ldots,e_k\}$ into the subspace spanned by $\{b_1,\ldots,b_k\}$.

§2. Differential Forms--The Definition

We have seen an intimate connection between volumes and the exterior algebra. It is this connection that makes the exterior algebra the correct place for forms to live. Before giving the definition, we must first study the parameterized version of the exterior algebra. This is part of a general pattern--any natural, algebraic operation on vector spaces has an extension to vector bundles. Let $E \xrightarrow{\pi} B$ be a vector bundle. We shall define another vector bundle $\Lambda^k(E^*) \xrightarrow{\pi'} B$. The fiber $\pi'^{-1}(b)$ is $\Lambda^k(\pi^{-1}(b)^*)$ as a vector space. To give the local trivializations for $\Lambda^k(E^*)$ we have need for a lemma.

<u>Lemma 2.1</u>: <u>Let</u> $\alpha\colon V \to W$ <u>be a linear map between finite dimensional real vector spaces. Then</u> α <u>induces a linear</u> $\Lambda^k(\alpha^*)\colon \Lambda^k(W^*) \to \Lambda^k(V^*)$. <u>This defines</u> $\Lambda(\alpha^*)\colon \bigoplus_\ell \Lambda^\ell(W^*) \to \bigoplus_\ell \Lambda^\ell(V^*)$ <u>an algebra homomorphism</u>.

<u>Proof</u>: We give the definition of $\Lambda^k(\alpha^*)$ and leave the checking of the properties as an exercise:

$$\Lambda^k(\alpha^*)(\varphi)(e_1,\ldots,e_k) = \varphi(\alpha(e_1),\ldots,\alpha(e_k)).$$

Let $E \xrightarrow{\pi} B$ be a vector bundle with local trivializations

$$\begin{tikzcd} U_\alpha \times V \arrow[rr, "\varphi_\alpha"] \arrow[dr] & & E|U_\alpha \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

over the elements of an open cover $\{U_\alpha\}$ of B. Define

$$U_\alpha \times \Lambda^k(V^*) \xrightarrow{\{\Lambda^k(\varphi_\alpha^*)\}^{-1}} \bigcup_{b \in U_\alpha} \Lambda^k(\pi^{-1}(b)^*).$$

This map is a bijection and a linear isomorphism on each fiber. We use it to define a topology on $\bigcup_{b\in U_\alpha} \Lambda^k(\pi^{-1}(b)^*)$. To show that, in the resulting topology on $\Lambda^k(E^*)$, the maps given above define a local trivialization we must know that the overlap functions are homeomorphisms

$$(U_\alpha \cap U_\beta) \times \Lambda^k(V^*) \xrightarrow{\Lambda^k(g_{\beta\alpha}^*)} (U_\alpha \cap U_\beta) \times \Lambda^k(V^*).$$

This follows from the general fact that the map defined in Lemma 2.1, $\mathrm{Auto}(V) \to \mathrm{Auto}(\Lambda^k(V^*))$, is continuous. (Actually, it is C^∞.)

Thus we have defined the structure of a locally trivial vector bundle for $\Lambda^k(E^*) \xrightarrow{\pi'} B$. If $E \xrightarrow{\pi} M$ is a C^∞-vector bundle over a C^∞-manifold, then $\Lambda^k(E^*) \xrightarrow{\pi'} M$ is also a C^∞-vector bundle. This is a consequence of the fact that the mapping in Lemma 2.1 is C^∞.

As examples, $\Lambda^1(E^*)$ (also written E^*) is called the dual bundle to E. Its fibers are the dual vector spaces to the fibers of E. $\Lambda^0(E^*)$ is the trivial one dimensional bundle $B \times R$. If $E \overset{\pi}{\to} M$ is the tangent bundle TM, then E^* is denoted T^*M and is called the cotangent bundle of M.

<u>Definition</u>: <u>A differential</u> k-<u>form on</u> M is a C^∞ section of $\Lambda^k(T^*M) \to M$.

In local C^∞-coordinates $(x_1,\ldots,x_n)$, $\Lambda^k(T^*M_p)$ has a basis $\{dx_{i_1} \wedge \ldots \wedge dx_{i_k}\}_{1 \le i_1 < \ldots < i_k \le n}$. Thus in these local coordinates, a k-form is

$\Sigma_{1 \le i_1 < \ldots < i_k \le n} f_{i_1 \ldots i_k}(x_1,\ldots,x_n) dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, where the $f_{i_1 \ldots i_k}$ are C^∞-functions.

It is important to know not only the local expressions which are k-forms but also the transformation law. To understand this we study the behavior of k-forms under C^∞-maps. Let $f\colon M \to N$ be a C^∞-map and $\omega\colon N \to \Lambda^k(T^*N)$ a differential k-form on N. We define $f^*\omega$, a differential k-form on M. Its value at $p \in M$, $f^*\omega_p \in \Lambda^k(T^*M_p)$, is determined by giving $f^*\omega_p(e_1,\ldots,e_k)$ for $\{e_1,\ldots,e_k\} \in TM_p$. We define

$$f^*\omega_p(e_1,\ldots,e_k) = \omega_p(Df_p(e_1),\ldots,Df_p(e_k)).$$

We claim that f^* preserves both the real vector space structure and wedge product, $\wedge$, in the sense that

$$(2.2) \quad \begin{cases} f^*(\omega + \mu) = f^*\omega + f^*\mu & \text{and} \\ f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu & . \end{cases}$$

Verifying this is left as an exercise.

<u>Lemma 2.3</u>: <u>If</u> ω <u>is a</u> C^∞-k-<u>form on</u> N, <u>then</u> $f^*\omega$ <u>is a</u> C^∞-k-<u>form on</u> M.

<u>Proof</u>: To prove this statement it suffices to prove it near each point $p \in M$. Let $(x_1,\ldots,x_n)$ be local C^∞-coordinates near p and $(y_1,\ldots,y_m)$ be local C^∞-coordinates near $f(p)$. Then $f = (f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$ with the f_i C^∞-functions. We claim that $f^*(dy_i) = \Sigma_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j$. The reason is that

$$(dy_i)_p(Df_p(\frac{\partial}{\partial x_j})) = (dy_i)_p(\sum_{i=1}^n \frac{\partial f_\ell}{\partial x_j}(p)\cdot\frac{\partial}{\partial y_\ell}$$

$$= \frac{\partial f_i}{\partial x_j}(p).$$

From this, the claim follows directly.

This shows that $f^*(dy_i)$ is a C^∞-form. A general C^∞ k-form locally has an expression

$$\omega = \sum \varphi(y_1,\ldots,y_n)dy_{i_1} \wedge\ldots\wedge dy_{i_k}.$$

By 2.2 and the above calculation

$$f^*\omega = \sum \varphi(f_1(x),\ldots,f_n(x))\left(\sum_{j=1}^{n} \frac{\partial f_{i_1}}{\partial x_j}dx_j\right)\wedge\ldots\wedge\left(\sum_{j=1}^{n} \frac{\partial f_{i_k}}{\partial x_j}dx_j\right).$$

This is clearly a C^∞-form as well.

<u>Corollary 2.4</u>: <u>Let</u> $(x_1,\ldots,x_n)$ <u>and</u> $(y_1,\ldots,y_n)$ <u>be two sets of local</u> C^∞-<u>coordinates on an open set</u> U <u>of</u> M. <u>Suppose that</u> ω <u>is a differential</u> k-<u>form in</u> U <u>which is given by</u>

$$\varphi(y_1,\ldots,y_n)dy_{i_1}\wedge\ldots\wedge dy_{i_k}$$

<u>in the</u> $(y_1,\ldots,y_n)$ <u>system. Then, it is given by</u>

$$\sum \varphi(y_1(x_1,\ldots,x_n),\ldots,y_n(x_1,\ldots,x_n))\left(\sum_{j=1}^{n}\frac{\partial y_{i_1}}{\partial x_j}dx_j\right)\wedge\ldots\wedge\left(\sum_{f=1}^{n}\frac{\partial y_{i_k}}{\partial x_j}dx_j\right)$$

<u>in the</u> $(x_1,\ldots,x_n)$ <u>system</u>.

<u>Corollary 2.5</u>: <u>Let</u> $(x_1,\ldots,x_n)$ <u>and</u> $(y_1,\ldots,y_n)$ <u>be two</u> C^∞-<u>coordinate systems on an open set</u> U <u>in</u> M^n. <u>Let</u> ω <u>be a differential</u> n-<u>form</u>, $\omega = f(y_1,\ldots,y_n)dy_1 \wedge\ldots\wedge dy_n$. <u>Then, expressed in the</u> $(x_1,\ldots,x_n)$ <u>system</u>

$$\omega = f(y_1(x_1,\ldots,x_n),\ldots,y_n(x_1,\ldots,x_n))\det\left(\frac{\partial y_i}{\partial x_j}\right)dx_1\wedge\ldots\wedge dx_n.$$

<u>Proof</u>: The form $dy_1 \wedge\ldots\wedge dy_n$ evaluates on $(\Sigma_{j=1}^n \alpha_{1j} \frac{\partial}{\partial y_j},\ldots,\Sigma_{j=1}^n \alpha_{nj} \frac{\partial}{\partial y_j})$ to give $\det(\alpha_{ij})$. Applying this and 2.4 yields the result.

There are two points of view which amount to the same thing. One can view a differential form as a section of $\Lambda^k(T^*M)$ (the abstract point of view) and any time it is necessary to make computations take local coordinates and express the form in them. Or one can view the form as a collection of expressions in terms of local coordinate systems in a C^∞-atlas which satisfy the transformation law (2.4) as we pass from one system to another. These two points of view are, of course, mathematically equivalent.

The structure of a graded ring (under $\wedge$) for $\bigoplus_{\ell=0}^{n} \Lambda^\ell(T^*M_p)$ produces the structure of a graded commutative, associative algebra on the differential forms of all degrees. Thus, if ω is a k-form and μ is a ℓ-form, we define $\omega \wedge \mu$, a $(k+\ell)$-form by $(\omega\wedge\mu)_p = \omega_p \wedge \mu_p$. Clearly, from (1.5), it follows that $\omega \wedge \mu = (-1)^{k\cdot\ell} \mu \wedge \omega$ and $(\omega \wedge \mu) \wedge \nu = \omega \wedge (\mu \wedge \nu)$. In addition, the differential k-forms are a module over the C^∞-functions. The structure is given by

$$(\varphi \cdot \omega)_p = \varphi(p) \cdot \omega_p$$

for φ a C^∞-function and ω a differential k-form.

These products are preserved under the pull back operation, i.e., $f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu$ and $f^*(\varphi \cdot \omega) = f^*\varphi \cdot f^*\omega$. Actually, the C^∞-functions are C^∞-sections of $M \times R$ and hence 0-forms. The multiplication of a k-form by a function is just a special case of the wedge product ($\wedge$) of forms.

All of this structure can be summarized by saying that the differential forms (of all degrees) on M are an associative, graded commutative algebra under wedge product. The subalgebra in degree 0 is the algebra of C^∞-functions on M. If $f: M \to N$ is a C^∞-map, then f induces f^* a graded algebra homomorphism from the graded algebra of differential forms on N to that of differential forms on M. (In short, the graded algebra of differential forms is a <u>contravariant</u> functor for C^∞-manifolds and C^∞-maps between them.)

The space of C^∞-vector fields on M was also given the structure as a module over the ring of C^∞-functions. It turns out that the modules of 1-forms and vector fields are dual modules over the C^{∞}-functions. The duality is given by $\langle\omega,\chi\rangle(p) = \langle\omega(p),\chi(p)\rangle$. This is easily seen to be an injection and to be linear over the functions. It requires a little work to show that it is onto as well. We leave this to the reader.

If $(\tau_1,\ldots,\tau_k)$ are vectors in R^n we define the k-dimensional volume spanned by $(\tau_1,\ldots,\tau_k)$ as follows.

a) If $(\tau_1,\ldots,\tau_k)$ lie in $R^k \subset R^n$, then $\mathrm{vol}_k(\tau_1,\ldots,\tau_k)$ is the usual volume of the parallelepiped that they span, i.e. let $\tau_i = \Sigma\, a_{ij}e_j$ and take $|\det(a_{ij})|$.

b) If $g \in O(n)$, then $\mathrm{vol}_k(\tau_1,\ldots,\tau_k) = \mathrm{vol}_k(g\tau_1,\ldots,g\tau_k)$.

Since any k-tuple of vectors can be rotated into $R^k \subset R^n$, there can be at most one such volume function. To show that this indeed exists we must show that if $g\colon R^k \to R^k$ is an element of $O(n)$ then $|\det(\tau_1,\ldots,\tau_k)| = |\det(g\tau_1,\ldots,g\tau_k)|$. This follows from the fact that determinants multiply and the fact that the linear map $g|R^k$ is in $O(k)$ and hence has determinant ± 1.

Let $M^k \subset R^n$ be an oriented C^{∞}-submanfiold. Associated to M is a k-form called the volume form, ω_{vol}. It is defined by $\langle\omega_{vol},(\tau_1,\ldots,\tau_k)\rangle = \pm\mathrm{vol}(P(\tau_1,\ldots,\tau_k))$ where

$P(\tau_1,\ldots,\tau_k)$ is the parallelepiped spanned by $\{\tau_1,\ldots,\tau_k\}$, i.e. $\mathrm{vol}(P(\tau_1,\ldots,\tau_k))$ is just the k-dimensional volume defined in the previous paragraph. The ambiguous $\pm$ measures the difference of the orientation of TM_p given by $\{\tau_1,\ldots,\tau_k\}$ and the one coming from the orientation of M. If $(x_1,\ldots,x_k)$ are local C^∞-coordinates for M and if $\varphi\colon M^k \to R^n$ is the inclusion map, then

$$\omega_{vol} = \mathrm{vol}_k\left(\frac{\partial\varphi}{\partial x_1},\ldots,\frac{\partial\varphi}{\partial x_k}\right)dx_1 \wedge\ldots\wedge dx_k.$$

Example: If $M^2 \subset R^3$ is an oriented hypersurface, then we associate to it a Gauss map $G\colon M^2 \to S^2$. This map assigns to each $p \in M^2$ the outward unit normal to M^2 at p. (The outward normal, ν, is the one so that if (e_1,e_2) is an oriented basis for TM_p, then (e_1,e_2,ν) is an oriented basis for R^3.) One sees that G is a C^∞-mapping. In fact, if $\varphi\colon M \to R^3$ is the embedding, and (x,y) is an oriented local C^∞-coordinate system for M, then

$$G(p) = \frac{\frac{\partial\varphi}{\partial x} \times \frac{\partial\varphi}{\partial y}}{\left\|\frac{\partial\varphi}{\partial x} \times \frac{\partial\varphi}{\partial y}\right\|}.$$

The pullback to M^2 of the volume form on S^2 via the Gauss map is called the curvature form for M. In local coordinates it is

$$\{G(p)\cdot(\frac{\partial G}{\partial x}(p)\times\frac{\partial G}{\partial y}(p))\}dx \wedge dy.$$

Exercises: 1) Let $U \subset R^n$ be an open set. Show that the module of 1-forms on U is free over the functions on U. A basis is $\{dx_1,\ldots,dx_n\}$.

2) Let M^n be a C^∞-manifold. Show that the 1-forms on M are a free module over the C^∞-functions if and only if there are 1-forms $\omega_1,\ldots,\omega_n$ so that $\{\omega_1(p),\ldots,\omega_n(p)\}$ is a basis for T^*M_p for every $p \in M$. (Hint: Show that there is a C^∞-function μ: $R^n \to R^+$ so that $\mu(0) = 1$ and $\mu(x) = 0$ if $\|x\| \geq 1$.) Dually, show that the C^∞-vector fields are free over the C^∞-functions if and only if there are vector fields $\chi_1,\ldots,\chi_n$ so that for every $p \in M$, $\{\chi_1(p),\ldots,\chi_n(p)\}$ is a basis for TM_p.

3) Show that a C^∞-vector bundle $E \overset{\pi}{\to} M$ is trivial if and only if its C^∞-sections, as a module over the C^∞-functions is free. It turns out that not all vector bundles are trivial, and hence not all the modules of sections are free. They are all projective modules, however.

*4) Show that TS^2 is not trivial. In fact show that if χ: $S^2 \to TS^2$ is any C^∞-vector field, then χ must have a zero.

5) Show that M^n is orientable if and only if

$\Lambda^n TM^* \xrightarrow{\pi} M^n$ is a trivial bundle, if and only if the module of n-forms on M^n is free over the functions, if and only if there is an n-form ω so that $\omega(p) \neq 0$ for all $p \in M^n$.

6) Show that the differential 1-forms and the vector fields are dual modules over the C^∞-functions.

7) Define C^r-forms on a C^r-manifold, and show that they form an associative, graded commutative algebra over the ring of C^r-functions.

§3: Integration

In this section we shall define the integral of a k-form on M^n over oriented C^∞-submanifolds of dimension k in M^n. In the end of course, such integration comes down to ordinary k-dimensional Lebesque integration on subsets of R^k. Let us begin with the simplest case. We take the case where the ambient manifold, M, is an open subset of R^k, U (and hence n = k). As region of integration we take a compact subset of U whose boundary is finitely many pieces of C^∞-hypersurfaces $N_i^{k-1} \subset U$. More precisely, let $\{\varphi_1,\ldots,\varphi_\ell\}$ be C^∞-functions defined on U, so that $\{\varphi_i = 0\}$ defines a C^∞-submanifold of U (i.e., $D\varphi_i(x) \neq 0$ for any x such that $\varphi_i(x) = 0$). We call $N = \{p \in U | \varphi_1(p) \geq 0,\ldots,\varphi_\ell(p) \geq 0\}$ <u>a manifold with piecewise smooth boundary</u>. Actually, N is a union of two pieces $\text{int } N = \{p \in U | \varphi_1(p) > 0,\ldots,\varphi_\ell(p) > 0\}$ and $\partial N = N - \text{int } N$. The set (int N) is an open subset of U and hence is a manifold of dimension k (possibly empty). Without further assumptions ∂N can be quite nasty.

<u>Examples</u>: 1) $\overline{B_R(p)} \cap U$.

2) The cube $\displaystyle\mathop{\times}_{i=1}^{k} [0,1]$.

3) $\{(x,y) | a \leq x \leq b \text{ and } \varphi(x) \leq y \leq \psi(x)\}$.

Let $\omega = f(x_1,\ldots,x_k)dx_1 \wedge \ldots \wedge dx_k$ be a differential k-form in $U \subset R^k$. Suppose $N \subset U$ is a compact manifold with piecewise smooth boundary. We define

$$\int_N \omega$$

as follows:

$$(3.1) \int_N \omega = \int_N f(x_1,\ldots,x_k)dx_1 \wedge \ldots \wedge dx_k = \int_U f(x_1,\ldots,x_k)\chi_N dx_1 \ldots dx_k.$$

Here, $dx_1 \ldots dx_k$ denotes the usual Lebesque measure on R^k and χ_N is the characteristic function of N, i.e. $\chi_N(p) = 0$ if $p \notin N$ and $\chi_N(p) = 1$ if $p \in N$. Actually, this definition is valid for any measurable set N contained in U. (As we shall see the measure of ∂N is 0, and hence N is measurable, and $\int_N \omega = \int_{\text{int } N} \omega$.) A slightly more intrinsic definition of this integral is the following:

$$\int_N \omega = \int_N \langle \omega, (\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}) \rangle dx_1 \ldots dx_k$$

$$= \int_U \langle \omega, (\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_k}) \rangle \cdot \chi_N \, dx_1 \ldots dx_k.$$

In this formulation we do not need the expression for ω in local coordinates though we still need the local coordinates to do the integration.

There is a subtle point here about orientations and orderings of the variables. The Lebesque measure on R^k $dx_1 \ldots dx_k$ is independent of orientation and thus of ordering of the variables, i.e., $dx_1 dx_2 \ldots dx_k = dx_2 dx_1 \ldots dx_k$. This is because the volume or measure of a set is independent of any orientation. However, this is not true for forms, $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k = -dx_2 \wedge dx_1 \wedge \ldots \wedge dx_k$. The reason is that these forms measure oriented volumes. In our equation relating the integral of forms to that of measures then is slightly peculiar:

$$\int_N dx_1 \wedge \ldots \wedge dx_k = \int_U \chi_N \, dx_1 \ldots dx_k = \text{volume } (N),$$

$$\int_N dx_2 \wedge dx_1 \wedge \ldots \wedge dx_k = -\int_U \chi_N \, dx_2 dx_1 \ldots dx_n = - \text{ volume } (N).$$

How do we account systematically for the minus sign in the second equation? The point is that int N, being an open subset of R^k, has a natural orientation $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$ at every point and $\int_N dx_{i_1} \wedge \ldots \wedge dx_{i_k} = \pm \int_U \chi_N \, dx_{i_1} \ldots dx_{i_k}$ where the $\pm$ sign is $\langle dx_{i_1} \wedge \ldots \wedge dx_{i_k}, (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}) \rangle$.

The first basic theorem about this integration is that it is independent of the C^∞-coordinates. This is a consequence of the fact that $dx_1 \wedge \ldots \wedge dx_k$ is a form which measures volume, as well as of the chain rule.

<u>Theorem 3.2</u>: <u>Let</u> $U \subset R^k$ <u>be an open set, and let</u> ω <u>be a differential</u> k-<u>form. Suppose</u> $(x_1,\ldots,x_k)$ <u>and</u> $(y_1,\ldots,y_k)$ <u>are two sets of</u> C^∞-<u>coordinates in</u> U <u>which give the standard orientation. Let</u> $N \subset U$ <u>be a compact manifold with piecewise smooth boundary. If we use</u> (3.1) <u>to define</u> $\int_N \omega$ <u>using either set of coordinates, then the two results are equal</u>.

<u>Proof</u>: Suppose that in the $(x_1,\ldots,x_k)$-coordinates $\omega = f(x_1,\ldots,x_k)dx_1 \wedge\ldots\wedge dx_k$. Consider the change of coordinates as a diffeomorphism

$$(x_1,\ldots,x_k) = (\Phi_1(y_1,\ldots,y_k),\ldots,\Phi_k(y_1,\ldots,y_k)).$$

Calculating in the $(y_1,\ldots,y_k)$ system ω becomes

$$f(\Phi_,(y),\ldots,\Phi_k(y))\det(\frac{\partial\Phi_i}{\partial y_j})dy_1 \wedge\ldots\wedge dy_k.$$

Since $\Phi = (\Phi_1,\ldots,\Phi_k)$ is orientation preserving, $\det(\frac{\partial\Phi_i}{\partial y_j}) > 0$. The two Lebesque integrals which we must compare are

$$\int_N f(x_1,\ldots,x_k)dx_1\cdot\ldots\cdot dx_k$$

and

$$\int_{\Phi^{-1}(N)} f(\Phi_1(y_1,\ldots,y_k),\ldots,\Phi_k(y_1,\ldots,y_k))\det(\frac{\partial\Phi_i}{\partial y_j})dy_1 \ \ldots\ dy_k.$$

We claim that in general if $\Phi: V \to U$ is a diffeomorphism, then

$$\int_N f(x_1,\ldots,x_k)dx_1 \ldots dx_k$$
$$= \int_{\Phi^{-1}(N)} f(\Phi_1(y),\ldots,\Phi_k(y))\left|\det\left(\frac{\partial\Phi_i}{\partial y_j}\right)\right| dy_1\cdot\ldots\cdot dy_k,$$

or letting $P = \Phi^{-1}(N)$ and dropping the notation $dx_1 \ldots dx_k$ or $dy_1\cdot\ldots\cdot dy_n$ from the Lebesque integral, we claim

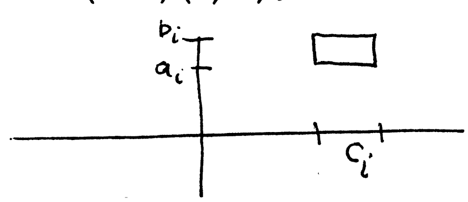
$$(3.3) \qquad \int_P f(\Phi(y_1,\ldots,y_k))\left|\det\left(\frac{\partial\Phi_i}{\partial y_j}\right)\right| = \int_{\Phi(P)} f(x_1,\ldots,x_k).$$

It is in this form that we shall establish the result. First notice that in dimension 1 it is the usual change of variable formula, viz. if $a < b$ then
$\int_{[a,b]} f\circ g\cdot|g'| = \int_a^b (f\circ g)|g'| = \int_{g([a,b])} f$. (Here, $\int_{g([a,b])} f$ is $\int_{g(a)}^{g(b)}(f)$ if $g' > 0$ and is $\int_{g(b)}^{g(a)}(f)$ if $g' < 0$.) Consequently, the result holds in dimension 1. We shall prove the result by induction on k. There are several steps in the proof, and only the final one uses the inductive hypothesis.

<u>Step I</u>: Let $A \subset R^{k-1}$ be a bounded set and suppose that $\rho: W \to R^1$ is C^∞ (W is an open set in R^{k-1} containing $\bar{A}$.) The k-dimensional volume of
$\Gamma(\rho) = \{(x_1,\ldots,x_{k-1},\rho(x_1,\ldots,x_{k-1})) \mid (x_1,\ldots,x_{k-1}) \in A\}$
is zero, i.e.,

$\int_{\Gamma(\rho)} f = 0$ for any measurable function f.

<u>Proof</u>: We show that for every $\epsilon > 0$, $\Gamma(\rho)$ is contained in a set of measure less than ϵ. Since $\bar{A} \subset U$, there is a finite union of cubes $C_1, \ldots, C_T \subset U$ so that $A \subset \bigcup_{i=1}^{T} C_T$. Thus it suffices to consider the case when $A \subset C \subset U$ with C a cube. Let α be the side length of C. Subdivide C into smaller cubes of side length α/ℓ. There will be ℓ^{k-1} of these smaller cubes. Since A is compact $|\frac{\partial f}{\partial x_i}(x)| \leq M$ for all $i = 1, \ldots, k-1$ and all $x \in A$. Thus, the variation of f on any one of the smaller cubes will be at most $M \cdot (k-1) \cdot (\alpha/\ell)$. Thus, above each of the smaller cubes C_i is an k-dimensional cube $C_i \times [a_i, b_i]$ containing $\Gamma(\rho | C_i)$ where $b_i - a_i = M(k-1)(\alpha/\ell)$.



Thus, $\Gamma(\rho) \subset \bigcup_{i=1}^{\ell^{k-1}} C_i \times [a_i, b_i]$. The measure of the cubes is $\Sigma_{\ell^{k-1}} (\frac{\alpha}{\ell})^{k-1} \cdot M(k-1) \cdot \frac{\alpha}{\ell} = \dfrac{\alpha^k (M \cdot (k-1))}{\ell}$.

As we take ℓ larger and larger this volume approaches 0.

Step II: Let X^{k-1} be a C^∞-hypersurface in $U^k \subset R^k$ and let $K \subset X$ be a compact set. Then $\mathrm{vol}_k(K) = 0$.

Proof: Near any $x \in X^{k-1}$, X^{k-1} defines one of the coordinates implicitly as a C^∞-function of the other $(k-1)$. Thus, there is an open set of X containing x in which X has the form $F(\rho)$. Applying step I, we see that any bounded set inside such an open set has measure zero. We cover K by finitely many such open sets and find that the measure on k-dimensional volume of K is zero.

Step III: Let $N \subset U^k$ be given by $\{(p|\varphi_1(p) \geq 0,\ldots,\varphi_\ell(p) \geq 0\}$, where each φ_i defines a C^∞-hypersurface in U. Suppose that N is compact. Then

$$\int_N f = \int_{\mathrm{int}\, N} f.$$

Proof: Since $N - \mathrm{int}\, N = \bigcup_{i=1}^{\ell} \{\varphi_i = 0\} \bigcap_{j\neq i} \{\varphi_j \geq 0\}$ and since each of the sets in the union is a compact subset of a C^∞-hypersurface, it follows from step II that $\mathrm{vol}_k(N - \mathrm{int}\, N) = 0$. Hence,

$$\int_{N-\mathrm{int}\, N} f = 0$$

or

$$\int_N f = \int_{\mathrm{int}\, N} f.$$

Since $\Phi: V \to U$ is a diffeomorphism $P = \Phi^{-1}(N)$ is a manifold with piecewise smooth boundary and $\Phi^{-1}(\text{int } N) = \text{int } P$. Hence, the upshot of steps I, II, and III is that to prove the theorem for compact manifolds with piecewise smooth boundary it suffices to prove the theorem for open sets with compact closures, i.e. for bounded open sets.

<u>Step IV</u>: Given P and Φ as in (3.3), then if the result holds for any rectangle $\displaystyle\bigtimes_{i=1}^{n} [a_i, b_i]$ contained in P, then it holds for P.

<u>Proof:</u> As we remarked above it suffices to prove (3.3) for the open set (int P). But being an open set, (int P) is a union of a countable number of closed rectangles with disjoint interiors int $P = \bigcup_{i=1}^{\infty} R_i$ (see exercise 1). Since the Lebesque integral is countably additive

$$\int_P f \circ \Phi \cdot \left|\det \frac{\partial \Phi_i}{\partial y_j}\right| = \sum_{i=1}^{\infty} \int_{R_i} f \circ \Phi \cdot \left|\det \frac{\partial \Phi_i}{\partial y_j}\right|.$$

Likewise,

$$\int_{\Phi(P)} f = \sum_{i=1}^{\infty} \int_{\Phi(R_i)} f.$$

From these equalities Step IV follows immediately.

<u>Step V</u>: Given P and Φ suppose that for each $p \in P$ there

is an open set U_p containing p so that the result holds for any rectangle contained in $U_p \cap P$. Then the result holds for P.

Proof: The open sets $\{U_p\}_{p\in P}$ give an open covering of P. Let $\{U_1,\ldots,U_T\}$ be a finite subcover. There is $\epsilon > 0$ so that any rectangle of diameter less than ϵ in P is contained in some U_i. Take a decomposition as in Step IV: int $P = \bigcup_{i=1}^{\infty} R_i$. By subdividing the R_i we can assume that each has diameter less than ϵ. (Only finitely many have diameter $> \epsilon$ to begin with.) Since the result holds for each of the new, smaller rectangles, it holds for P.

Step VI: Suppose given $\{P,\Phi\}$ and $\{\Phi(P),\Psi\}$ as in (3.3) for which the result holds. Then it holds for $\{P,\Psi\circ\Phi\}$ as well.

Proof: If $\rho\colon V \to U$ is a diffeomorphism denote by $J(\rho)$ its Jacobian determinant, viz. $\det(\frac{\partial\rho_i}{\partial y_j})$. Then if $(x_1,\ldots,x_k) = \Phi(y_1,\ldots,y_k)$, we have

$$\begin{aligned}\int_P f(\Psi\circ\Phi(y))\cdot|J(\Psi\circ\Phi(y))| &= \int_P f\circ\Psi(\Phi(y))\cdot|J(\Psi(x))|\cdot|J(\Phi(y))| \\ &= \int_{\Phi(P)} f\circ\Psi(x)\,|J\Psi(x)| \\ &= \int_{\Psi(\Phi(P))} f\end{aligned}$$

<u>Step VII</u>: <u>The result holds for sufficiently small rectangles</u>. By this we mean that given $\Phi: V \to U$ and $p \in V$ we shall show that there is $\epsilon > 0$ so that (3,3) holds for any rectangle contained in the ϵ-ball about p. Given $p \in V$, there is an $(k-1)$-tuple, $1 \le i_1 < i_2 < \dots < i_{k-1} \le k$, so that

$$\left(\frac{\partial \Phi_{i_j}}{\partial y_r} \right) \quad ; \text{ for } j, r = 1, \dots, k-1,$$

is invertible near p. For simplicity let us assume that the missing index is k. Define $\Gamma: V \to W$ by
$\Gamma(y_1, \dots, y_k) = (\Phi_1(y_1, \dots, y_k), \dots, \Phi_{k-1}(y_1, \dots, y_k), y_k)$.
Then Γ is a diffeomorphism on some open set V_p containing p. Let $\Psi = \Phi \circ \Gamma^{-1}$. Clearly, $\Psi(z_1, \dots, z_k) = (z_1, \dots, z_{k-1}, \rho(z_k))$. We claim that if R is any rectangle in V_p, then the result holds for $\{R, \Gamma\}$ and $\{\Gamma(R), \Psi\}$. By Step VI the result holds also for $\{R, \Psi \circ \Gamma\} = \{R, \Phi\}$.

Let us study $\{R, \Gamma\}$ first. We write $\nu = (y_1, \dots, y_{k-1})$ and $\zeta = (z_1, \dots, z_{k-1})$. Then $\Gamma(\nu, y_k) = (\Gamma_{y_k}(\nu), y_k)$ where $\Gamma_{y_k}(\nu)$ is the ζ^{th}-coordinate of $\Gamma(\nu, y_k)$. Note that $J(\Gamma(\nu, y_k)) = J(\Gamma_{y_k}(\nu))$. Let R be a rectangle contained in V_p (the domain where Γ is a diffeomorphism). We write $R = R_\zeta \times [a, b]$. Then

$$\int_R f(\Gamma(\nu,y_k))\cdot|J(\Gamma(\nu,y_k))| = \int_a^b\int_{R_\zeta\times\{y_k\}} f(\Gamma_{y_k}(\nu))\cdot|J(\Gamma_{y_k}(\nu))|.$$

By induction

$$\int_{R_\zeta\times\{y_k\}} f(\Gamma_{y_k}(\nu))\cdot|J(\Gamma_{y_k}(\nu))| = \int_{\Gamma_{y_k}(R_\zeta\times\{y_k\})} f.$$

Thus,

$$\int_R f(\Gamma(\nu,y_k))\cdot|J(\Gamma(\nu,y_k))| = \int_a^b\int_{\Gamma_{y_k}(R_\zeta\times\{y_k\}} f = \int_{\Gamma(R)} f.$$

Let us consider $\{\Gamma(R),\Psi\}$. By Step IV it suffices to prove the result for $\{S,\Psi\}$ where S is an arbitrary rectangle in $\Gamma(R)$. We have $\Psi(z_1,\ldots,z_k) = (z_1,\ldots,z_{k-1},\rho(z_k))$. Thus $J(\Psi(z_1,\ldots,z_k)) = \frac{d\rho}{dz_k}(z_k)$. Let $\zeta = (z_1,\ldots,z_{k-1})$ and write S as $S_\zeta \times [c,d]$.

$$\int_S g(\Psi(\zeta,z_k))\cdot|J(\Psi(\zeta,z_k))| = \int_{S_\zeta}\left(\int_c^d g(\zeta,\rho(z_k))\cdot|\rho'(z_k)|\right).$$

By the fundamental theorem of calculus

$$\int_c^d g(\zeta,\rho(z_k))\cdot|\rho'(z_k)| = \int_{\rho(\{\zeta\}\times[c,d])} g.$$

Thus, we have

$$\int_S g(\Psi(\zeta,z_k))\cdot|J(\Psi(\zeta,z_k))| = \int_{S_\zeta}\left(\int_{\rho(\{\zeta\}\times[c,d])} g\right).$$

Since $\Psi(S) = \bigcup_{\zeta\in S_\zeta} \rho(\{\zeta\}\times[c,d])$, Fubini's theorem tells us that

$$\int_{S_\zeta}\left(\int_{\rho(\{\zeta\}\times[c,d])} g\right) = \int_{\Psi(S)} g.$$

This completes the proof of Step VII and of theorem 3.2.

<u>Exercises</u>: 1) Show that if $A \subset R^k$ is a bounded, open set, then $A = \bigcup_{i=1}^{\infty} R_i$ where the R_i are rectangles with sides parallel to the coordinate hyperplanes and where the $\{R_i\}$ have disjoint interiors.

2) Show that if $A \subset R^k$ is bounded, $\bar{A} \subset U$, with U open, then $A \subset \bigcup_{i=1}^{T} R_i \subset U$ where the R_i are rectangles with disjoint interiors.

At this stage we have only begun. We have defined $\int_N \omega$ if N is a compact k-dimensional manifold with piecewise smooth boundary contained in an open set U in R^k and ω in a k-form defined on U. The next step is to define $\int_N \omega$ where $N^k \subset M^k$ is a compact manifold with piecewise smooth boundary and ω is a k-form on M^k (which is oriented). By exercise 1 below, one can write $N = \bigcup_{i=1}^{T} N_i$ with

$(\text{int } N_i) \cap (\text{int } N_j) = \emptyset$ for $i \neq j$, where each N_i is a manifold with piecewise smooth boundary contained in the image of a coordinate path. We let

$$(3.4) \qquad \int_N \omega = \sum_{i=1} \int_{N_i} \omega$$

where $\int_{N_i} \omega$ is defined by (3.1) using the local coordinates in an open set containing N_i. The result is independent of the decomposition; for, if we have two such, $N = \bigcup_{i=1}^{T} N_i$ and $N = \bigcup_{j=1}^{S} N_j'$, then their intersection gives a third such decomposition: $N = \bigcup_{i=1}^{T} \bigcup_{j=1}^{S} (N_i \cap N_j')$, which is finer than either of the initial two (finer in the sense that each element in the third decomposition is contained in some element of each of the first two). Clearly,

$$\int_{N_i} \omega = \sum_{j=1}^{S} \int_{N_i \cap N_j'} \omega$$

if we use the same C^∞-coordinates, $(x_1, \ldots, x_k)$, to calculate all the integrals. On the other hand Theorem 3.2 tells us that the integrals are independent of the coordinates. Hence,

$$\sum_{i=1}^{T} \int_{N_i} \omega = \sum_{i=1}^{T} \sum_{j=1}^{S} \int_{N_i \cap N_j'} \omega = \sum_{j=1}^{S} \int_{N_j'} \omega.$$

Now let M^n be a C^∞-manifold of any dimension, n, and let ω be a k-form on M^n. Suppose $N^k \subset P^k$ is a compact submanifold with piecewise smooth boundary, and P^k is oriented. Let $\varphi: P^k \to M^k$ be a C^∞ map. We define

$$\int_{\varphi(N^k)} \omega = \int_{N^k} \varphi^* \omega.$$

Examples: 1) Let $\omega = \Sigma_{i=1}^n f_i dx_i$, and let $\gamma: [a,b] \to R^n$ be a C^∞-curve. Then $\int_\gamma \omega = \int_a^b \Sigma_{i=1}^n f_i(\gamma(t)) \cdot \gamma_i'(t) dt$.

2) Let $\rho: M^2 \longrightarrow \mathbb{R}^n$ be a C^∞-mapping. Suppose that M^2 is oriented. Define a 2-form ω_{vol} on M^2 as follows. Let $\tau_1, \tau_2 \in TM_p$. Then, $\omega_{vol}(\tau_1, \tau_2)$ is the area of the parallelogram $(D\rho(\tau_1), D\rho(\tau_2))$ in $\mathbb{R}^n$. The sign is + if and only if (τ_1, τ_2) give the orientation for TM_p. The area of $\rho(M^2)$ is $\int_{M^2} \omega_{vo}$

3) There is a differential 1-form on the circle called $d\theta$. If we consider $i: S^1 \hookrightarrow R^2$, the unit circle, then $d\theta = i^*(-y\, dx + x\, dy)$. We claim that $\int_{S^1} d\theta = 2\pi$. To see this define $\mu: [0,2\pi] \to S^1$ by $\mu(t) = (\cos(t), \sin(t))$. Since $S^1 = \text{im}(\mu)$ and $\mu | [0,2\pi)$ is injective $\int_{S^1} d\theta = \int_0^{2\pi} \mu^* \, d\theta$. But $\mu^*(d\theta) = -\sin t \cdot \cos'(t) dt + \cos(t) \sin'(t) dt = dt$. Thus $\int_0^{2\pi} \mu^* d\theta = \int_0^{2\pi} dt = 2\pi$.

More generally, if $\gamma: S^1 \to R^2 - \{0\}$ is any C^∞-mapping, then the winding number of $\gamma(S^1)$ about origin is defined to be

$$w(\gamma) = \frac{1}{2\pi} \int_{\gamma(S^1)} \left(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy \right).$$

We shall show later in this chapter that $w(\gamma)$ is always an integer.

Exercises: 1) If M^k is a C^∞-manifold and $N^k \subset M^k$ is a subset, then N^k is a C^∞ submanifold with piecewise smooth boundary if for each $p \in M$ there is an open set U containing p and C^∞-functions $\{\varphi_1,\ldots,\varphi_\ell\}$: $U \to R$ so that $N = \{p \in U | \varphi_1(p) \geq 0,\ldots,\varphi_\ell(p) \geq 0\{$ and so that $\{\varphi_i = 0\}$ defines a C^∞-hypersurface in U. Show that if $N^k \subset M^k$ is a compact C^∞-submanifold with piecewise smooth boundary, then there are:

1) a finite collection of open sets $U_\alpha \subset M^k$ so that $N \subset \bigcup_{\alpha=1}^{T} U_\alpha$,

2) a decomposition $N = \bigcup_{\alpha=1}^{T} N_\alpha$ with $N_\alpha \subset U_\alpha$ such that:

   a) each U_α has C^∞-coordinates,

   b) each N_α is a compact submanifold of U_α with piecewise smooth boundary, and

   c) the interiors of the $\{N_\alpha\}_{\alpha=1}^{T}$ are disjoint.

2) Show that if we have two such decompositions of N, $N = \bigcup_{\alpha=1}^{T} N_\alpha$ and $N = \bigcup_{\beta=1}^{S} N'_\beta$, then their intersection $N = \bigcup_{\alpha=1}^{T} \bigcup_{\beta=1}^{S} N_\alpha \cap N'_\beta$, is also such a decomposition.

§4: Exterior Differentiation

The previous section dealt with setting up the many-dimensional analogue of integral calculus. In this section we establish the analogue of differentiation. It is called exterior differentiation.

We denote the module of k-forms on a C^∞-manifold M by $A^k(M)$. The exterior differentiation is a linear map $d: A^k(M) \to A^{k+1}(M)$. It satisfies the following properties:

1) d is natural for C^∞-mappings, i.e.

   $d(f^*\omega) = f^*d\omega$.

2) $d(d\omega) = 0$,

3) $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^{\deg \omega}\, \omega \wedge d\mu$, and

4) if $U \subset R^n$ is an open set and $x_i: U \to R$ is one of the coordinate functions then $d(x_i) = dx_i$.

Theorem 4.1: There is a unique exterior differentiation satisfying properties 1) - 4). If, in local coordinates, $\omega = \Sigma\, \varphi\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$, then $d\omega = \Sigma\, \Sigma_{j=1}^{n} \frac{\partial \varphi}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

Proof: Let us show first of all that there is at most one exterior differentiation satisfying 1) - 4).

Suppose d and d' are exterior differentiations satisfying 1) - 4). Let f be a function defined in a neighborhood

U of 0 in R^n. Then, $f = \Sigma_{i=1}^n x_i h_i + f(0)$ by Lemma 2.2 of Chapter II. Thus

$$d(f) = \sum_{i=1}^{n} dx_i \cdot h_i + \sum_{i=1}^{n} x_i dh_i + d(f(0)).$$

Since $f(0)$ is a constant function $d(f(0)) = f(0) \cdot d(1)$. But $d(1) = d(1 \cdot 1) = 1d(1) + d(1) \cdot 1 = 2d(1)$. It follows that $d(1) = 0$, and hence, that $d(f(0)) = 0$. Thus, $d(f)_0 = \Sigma_{i=1}^n dx_i \cdot h_i(0)$. Exactly the same argument shows that $d'(f) = \Sigma_{i=1}^n d'(x_i) \cdot h_i(0)$. But property 4) ensures that $d'(x_i) = dx_i$. Thus $d(f)_0 = d'(f)_0$.

This argument is valid at any point $p \in U$. Hence, $df = d'f$ for all functions defined in open sets in R^n. Clearly $d(dx_i) = 0$ and $d'(dx_i) = 0$. Thus, if $\omega = \Sigma f_I dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ then both $d\omega$ and $d'\omega$ are given by

$$\sum \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

This proves that $d = d'$ on $A^*(U)$ for U an open set in R^n. But if $\omega \in A^k(M^n)$ and $d\omega \neq d'\omega$, then there is a coordinate patch $U \subset M^n$ so that $d(\omega|U) \neq d'(\omega|U)$. Since we have just shown that this is impossible, it follows that $d\omega = d'\omega$ for all $\omega \in A^*(M)$.

In showing there is at most one such d, we gave a formula for $d\omega$ if ω is expressed in local coordinates. If we knew that this formula gave the same value for $d\omega$ what ever coordinates we used, then it would define $d\omega$ for all forms $\omega \in A^*(M)$. The fact that $d\omega$ is independent of the coordinates follows immediately from the uniqueness argument. The only axiom that needs to be checked is $d^2 = 0$. This will follow if we can show that $d^2(f) = 0$ for all C^∞-functions. In local coordinates

$$d^2(f) = d\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j .$$

By the skew symmetry of the wedge product of 1-forms, this is equal to

$$\sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j .$$

Since f is C^∞, its cross partials are equal. This proves that $d^2(f) = 0$.

<u>Definition</u>: A form ω is said to be <u>closed</u> if and only if $d\omega = 0$. A form ω is said to be <u>exact</u> if and only if there is a form μ so that $d\mu = \omega$.

Since $d^2 = 0$ every exact form is closed. The converse does not hold. The amount by which it fails is a strong invariant of a manifold.

<u>Exercises:</u> 1) Show that $d\theta$ on S^1 is closed but not exact. Hint: If it were exact show that $\int_{S^1} d\theta$ would be 0.

2) Let $f(z) = h(x,y) + ik(x,y)$ be a holomorphic function in some region $U \subset \mathbf{C}^1$. Show that $h(x,y)dx - k(x,y)dy$ is a closed 1-form in U.

3) Let M^n be a connected manifold. Show that the only closed functions are the constants.

4) Show that every n-form on M^n is closed.

5) Consider $T^n = R^n/\mathbf{Z}^n$. Show that the 1-forms $dx_1,\ldots,dx_n$ on R^n define 1-forms $dx_1,\ldots,dx_n$ on T^n. Show that on T^n these 1-forms are not exact. (Of course, on R^n, dx_i is exact since it is $d(x_i)$.)

6) In R^3 we identify 1-forms and 2-forms with vector fields and 3-forms with functions according to the following scheme:

$$f\, dx \wedge dy \wedge dz \longleftrightarrow f$$

$$f_1\, dx + f_2\, dy + f_3\, dz \longleftrightarrow f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{2y} + f_3 \frac{\partial}{2z}$$

$$f_1\, dy \wedge dz + f_2\, dx \wedge dz + f_3\, dx \wedge dy \longleftrightarrow f_1 \frac{\partial}{\partial x} - f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}.$$

Show that when we make these identifications we have a commutative diagram:

$$\begin{array}{ccccccc}
\text{Functions} & \xrightarrow{d} & \text{1-forms} & \xrightarrow{d} & \text{2-forms} & \xrightarrow{d} & \text{3-forms} \\
\| & & \updownarrow & & \updownarrow & & \updownarrow \\
\text{Functions} & \xrightarrow{\nabla} & \text{Vector fields} & \xrightarrow{\nabla\times} & \text{Vector fields} & \xrightarrow{\nabla\cdot} & \text{Functions}
\end{array}$$

Thus $d^2 = 0$ contains the results, curl(grad) = 0 and div(curl) = 0.

7) Show that if (F_1,F_2,F_3) is a vector field in R^3 with ω the corresponding 2-form and if $S \subset R^3$ is an oriented C^∞- surface, then

$$\int_S \omega = \int_S (F_1,F_2,F_3)\cdot\vec{n}\, d\,\sigma$$

where $\vec{n}(p)$ is the unit normal to S at p which completes an oriented basis for TS_p to an oriented basis for TR^3_p. The formula on the right hand side is the usual formula for integrating a vector field over a surface in 3-space.

8) Suppose ω is a closed 1-form on M^n (which is connected). Show that $\omega = df$ if and only if

$$\int_\Gamma \omega = 0 \text{ for every closed curve } \Gamma \text{ in } M.$$

(Note: A closed curve is a map γ: $[a,b] \to M$ so that $\gamma(a) = \gamma(b)$.)

<u>Hint:</u> If $\omega = df$, and γ: $[a,b] \to M$ is a curve then $\gamma^*\omega = df\circ\gamma$. Hence, by the Fundamental Theorem of Calculus

$\int_a^b \gamma^*\omega = f(\gamma(b)) - f(\gamma(a))$. Conversely, if $\int_\Gamma \omega = 0$ for all closed curves show that if γ and μ are C^∞-paths from a to b, then $\int_\gamma \omega = \int_\mu \omega$. Define $f: M \to R$ by $f(x) = \int_{\gamma_x} \omega$, where γ_x is any C^∞-path from a fixed point $p_0 \in M$ to x. Show $df = \omega$.

§5. Complex Valued Differential Forms

Heretofore, we have been studying real valued functions and real valued differential forms. There is, however, an obvious extension to complex valued forms. If we work globally and abstractly, we consider complex valued multilinear, skew-symmetric forms on V (a real vector space)

$$\alpha : \underbrace{V \times V \times \ldots \times V}_{k} \longrightarrow \mathbb{C}$$

These form a vector space $\Lambda^k_{\mathbb{C}}(V^*)$. It is naturally isomorphic to $\Lambda^k(V) \otimes_R \mathbb{C}$. If $E \overset{\pi}{\to} B$ is a real vector bundle, then we can form $\Lambda^k_{\mathbb{C}}(E^*)$ with fibers $\Lambda^k_{\mathbb{C}}(\pi^{-1}(b)^*)$. Again, $\Lambda^k_{\mathbb{C}}(E^*) = \Lambda^k(E^*) \otimes_R \mathbb{C}$. A C^∞-complex valued k-form on M is a C^∞-section of $\Lambda^k_{\mathbb{C}}(T^*M)$. Such sections form a complex vector space which we denote by $A^k(M;\mathbb{C})$. One sees that $A^k(M;\mathbb{C}) = A^k(M;R) \otimes_R \mathbb{C}$ as a module over the C^∞-functions.

In local coordinates a complex valued k-form is

$$\sum_I \varphi_I \, dx_{i_1} \wedge \ldots \wedge dx_{i_\ell}$$

where the φ_I are complex valued, C^∞ functions on M.

The operations of integration and exterior differentiation have complex analogues. The map

$$d\colon A^k(M;\mathbb{C}) \longrightarrow A^{k+1}(M;\mathbb{C})$$

is just the linear extension of the real exterior differentiation:

$$d(\varphi_I\, dx_{i_1}\wedge\ldots\wedge dx_{i_\ell}) = \sum_{j=1}^{n} \frac{\partial\varphi_I}{\partial x_j}\, dx_j \wedge dx_{i_1}\wedge\ldots\wedge dx_{i_\ell}.$$

The integral $\int_{\rho(N)}\omega$ can be evaluated by writing $\omega = \mu + i\nu$ with μ and ν real and setting

$$\int_{\rho(N)}\omega = \int_{\rho(N)}\mu + i\int_{\rho(N)}\nu.$$

All this is very straightforward. It becomes more interesting when we consider complex valued forms on a <u>complex</u>-manifold. Suppose that $(z_1,\ldots,z_n)$ are holomorphic coordinates on U, an open subset of $\mathbb{C}^n$, with $z_j = x_j + iy_j$. Then, we have complex valued functions $\{z_j\colon U \to \mathbb{C}\}$ and $\{\bar{z}_j\colon U \to \mathbb{C}\}$. The first set are holomorphic and the second are anti-holomorphic. Their differentials $\{dz_j, d\bar{z}_j\}_{j=1}^{n}$ form a basis (over the complex-valued, C^∞-functions) for the complex-valued 1-forms in U. In fact, we see that $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$. Hence, $dx_j = \frac{1}{2}(dz_j + d\bar{z}_j)$ and $dy_j = \frac{1}{2i}(dz_j - d\bar{z}_j)$.

Let us denote $dz_{i_1}\wedge\ldots\wedge dz_{i_\ell}$ by dz_I and

$d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_k}$ by $d\bar{z}_J$ where I is the multi-index $(i_1, \ldots, i_\ell)$ and J is the multi-index $(j_1, \ldots, j_k)$. The exterior derivative, d, decomposes as $\partial + \bar{\partial}$ where:

$$\partial(\varphi \, dz_I \wedge d\bar{z}_J) = \sum_{r=1}^{n} \frac{\partial \varphi}{\partial z_r} dz_r \wedge dz_I \wedge d\bar{z}_J$$

and

$$\bar{\partial}(\varphi \, dz_I \wedge d\bar{z}_J) = \sum_{r=1}^{n} \frac{\partial \varphi}{\partial \bar{z}_r} d\bar{z}_r \wedge dz_I \wedge d\bar{z}_J.$$

If we make a holomorphic change of coordinates $\mu(\zeta_1, \ldots, \zeta_n) = (z_1, \ldots, z_n)$, then $\frac{\partial \mu_i}{\partial \bar{\zeta}_j} = 0$. (These are just the Cauchy-Riemann equations in several variables.) Thus,

$$\mu^* \, dz_j = \sum_{i=1}^{n} \frac{\partial \mu_j}{\partial \zeta_i} d\zeta_i$$

and

$$\mu^* \, d\bar{z}_j = \sum_{i=1}^{n} \frac{\partial \mu_j}{\partial \bar{\zeta}_i} d\bar{\zeta}_i.$$

Consequently, a holomorphic change of coordinates leaves invariant the module (over the complex valued, C^∞-functions) spanned by the differentials of local holomorphic functions. This submodule is denoted $A^{1,0}(M) \subset A^1(M;\mathbb{C})$. Likewise, we have the module spanned by the anti-holomorphic differentials $A^{0,1}(M) \subset A^1(M;\mathbb{C})$. Clearly, this gives a direct sum decomposition

$$A^{1,0}(M) \oplus A^{0,1}(M) = A^1(M;\mathbb{C}).$$

Proposition 5.1:

a) Let $\omega \in A^{1,0}(M)$ be a closed 1-form. Then, in local holomorphic coordinates $\omega = \Sigma f_i dz_i$ where the f_i are holomorphic functions.

b) If $\omega \in A^{1,0}(M)$ is exact and if M is compact, then $\omega = 0$.

Proof: We have $\omega = \Sigma_{i=1}^n f_i(z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n)dz_i$. Thus $d\omega = \partial\omega + \bar{\partial}\omega = \Sigma_{j=1}^n \Sigma_{i=1}^n \frac{\partial f_i}{\partial z_j} dz_j \wedge dz_i + \frac{\partial f_i}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_i$. Since $\{dz_i \wedge d\bar{z}_j\}$, $i,j = 1,\ldots,n$ are linearly independent over the C^∞-functions, it follows that if $d\omega = 0$, then $\frac{\partial f_i}{\partial \bar{z}_j} = 0$ for all i,j. This is the definition of the function f_i being holomorphic.

If ω is in $A^{1,0}(M)$ and $\omega = d\varphi$, then φ is a function on M and $\bar{\partial}\varphi = 0$. This means that φ is holomorphic. But a compact complex manifold has only locally constant holomorphic functions. Hence $d\varphi = 0$.

Example: Let $U \subset \mathbb{C}^1$ be an open set and let z be the holomorphic variable. If $f(z)$ is a holomorphic function, then we claim that $f(z)dz$ is a closed 1-form. As we have seen $\bar{\partial}(f(z)dz) = 0$ since f is holomorphic. On the other hand, $\partial(f(z)dz) = \frac{\partial f}{\partial z} dz \wedge dz = 0$. Thus $d(f(z)dz) = 0$. This shows that on a complex curve any holomorphic 1-form is closed. If the curve is compact, then none (save 0) are exact.

<u>Exercises</u>: 1) Let $C = \mathbb{C}/L$ where $L \subset \mathbb{C}$ is a lattice i.e., $L \otimes_{\mathbb{Z}} R \cong_R \mathbb{C}$. Show that dz induces a holomorphic 1-form on C. Show that dz is not exact on C.

2) a) Let $f(z)$ be holomorphic in $U = \{z \mid 0 < |z| < 1\}$ and suppose that $f(z)$ has a pole with zero residue at 0, i.e., near 0, $f(z) = \Sigma_{n=-k}^{+\infty} a_n z^n$ with $a_{-1} = 0$. Show that $f(z)dz$ is exact on U.

b) Show that $\frac{dz}{z}$ is not exact in U.

§ 6. Manifolds with Corners

Let M^k be a C^∞-manifold and $N^k \subset M^k$ a subset. Suppose given any point $p \in M$ that there are C^∞-functions defined in an open set U of M containing p, $\{\varphi_1,\ldots,\varphi_\ell\}$, so that

1) $N \cap U = \{q \in U | \varphi_1(q) \geq 0,\ldots,\varphi_\ell(q) \geq 0\}$, and

2) if $q \in N \cap U$ and $\varphi_{i_1}(q) = 0,\ldots,\varphi_{i_r}(q) = 0$, then $\{D\varphi_{i_1}(q),\ldots,D\varphi_{i_r}(q)\}$ are linearly independent elements in $\mathcal{L}(TM_q, R) = T^*M_q$.

The subset N^k is called a <u>manifold with corners</u>.

<u>Theorem 6.1: Let M^k be a C^∞-manifold and $N^k \subset M^k$. Then N^k is a manifold with corners if and only if for every $p \in M^k$ there is a local C^∞-coordinate system $(x_1,\ldots,x_k)$, valid in an open set U containing p, and $s \geq 0$ so that</u>

$$N \cap U = \{(x_1,\ldots,x_k) \in U | x_1 \geq 0,\ldots,x_s \geq 0\}.$$

<u>Proof</u>: If such coordinates exist near p, then we define φ_i to be x_i for $i \leq s$. Conversely, suppose $N^k \subset M^k$ is a manifold with corners and let $p \in M$. Let $\varphi_1,\ldots,\varphi_\ell$ be the functions defined in an open set U containing p so that $N \cap U = \{q | \varphi_1(q) \geq 0,\ldots,\varphi_\ell(q) \geq 0\}$. After renumbering we can assume that $\varphi_1(p) = 0,\ldots,\varphi_s(p) = 0, \varphi_{s+1}(p) \neq 0,\ldots,\varphi_\ell(p) \neq 0$.

There are two cases to consider--$p \in N$ and $p \notin N$. If $p \notin N$, then, since N is closed in M, there is an open set U containing p which misses N with coordinates $(x_1,\ldots,x_n)$. We can assume $x_1 < 0$ throughout U. Thus $N \cap U = \{(x_1,\ldots,x_n) \in U | x_1 \geq 0\}$. If $p \in N$, then $\varphi_{s+1}(p) > 0,\ldots,\varphi_\ell(p) > 0$. By restricting to a smaller open set V containing p, we can assume $\varphi_{s+1} > 0,\ldots,\varphi_\ell > 0$ throughout V. Thus, $N \cap V = \{q \in V | \varphi_1(q) \geq 0,\ldots,\varphi_s(q) \geq 0\}$. Since $\{D\varphi_1(p),\ldots,D\varphi_s(p)\}$ are linearly independent, there are s of the variables, which, after renumbering, we can assume are $(x_1,\ldots,x_s)$ so that

$$\left(\frac{\partial\varphi_i}{\partial x_j}(p)\right) \quad , \quad i,j = 1,\ldots,s,$$

is invertible. Consider the map $\Psi: V \to R^n$ given $\Psi(x_1,\ldots,x_n) = (\varphi_1(x_1,\ldots,x_n),\ldots,\varphi_s(x_1,\ldots,x_n),x_{s+1},\ldots,x_n)$. The map is a local diffeomorphism near p, say in W. Define new coordinates $(y_1,\ldots,y_n)$ valid in W by $y_i = \Psi_i(x_1,\ldots,x_n)$. In these coordinates $N \cap W = \{(y_1,\ldots,y_n \in W | y_1 \geq 0,\ldots,y_s \geq 0\}$.

<u>Definition</u>: Let $N^k \subset M^k$ be a manifold with corners. Define $C_i(N)$, $i \geq 0$, to be the union of those $p \in N$ so that near p there are C^∞-coordinates so that p is the origin and $N = \{(x_1,\ldots,x_n) | x_1 \geq 0,\ldots,x_i \geq 0\}$.

Lemma 6.2:

a) $N = \bigcup_{i=0}^{k} C_i(N)$.

b) $C_i(N) \cap C_j(N) = \emptyset$ for $i \neq j$.

c) $\overline{C_i(N)} = \bigcup_{j \leq i} C_j(N)$.

d) $C_i(N) \subset M - \bigcup_{j>i} C_j(N)$ is a codimension i submanifold.

Proof: Part a) is immediate from 6.1. Part b) follows from the fact that $\{(x_1,\ldots,x_n) | x_1 \geq 0,\ldots,x_i \geq 0\}$ and $\{(x_1,\ldots,x_n) | x_1 \geq 0,\ldots,x_j \geq 0\}$ are not locally diffeomorphic at the origin if $i \neq j$.

Parts c) and d) follow easily.

Definition: $\partial N = \bigcup_{i\geq 1} C_i(N)$ and $\partial^s N = C_1(N)$.

Corollary 6.3: $\partial^s N \subset M - (\partial N - \partial^s N)$ is a C^∞-submanifold of codimension 1.

Let $N^k \subset M^k$ be a manifold with corners and suppose that M^k is oriented. The manifold, int N, being an open subset of M receives an orientation from that of M^k. We give $\partial^s N^k$, which is a (k-1) dimensional manifold, an orientation. At $p \in \partial^s N^k$ choose a tangent vector ν which points into the region $M - N$, i.e. if locally N is given by $\varphi \leq 0$ choose a curve γ so that $\gamma(0) = p$, $\varphi(\gamma(t))>0$ for $t > 0$ and $\gamma'(0) \neq 0$. The vector $\gamma'(0)$ is a tangent vector pointing into $M - N$. If $(\tau_1,\ldots,\tau_{k-1})$ is a basis for $T(\partial^s N)_p$, it gives the orientation for $T(\partial^s N)_p$ if and only if $(\nu,\tau_1,\ldots,\tau_{k-1})$ gives the orientation for TM_p.

If $N^k \subset M^k$ is a manifold with corners, then M can be covered by open sets $\{U_\alpha\}$ with C^∞-coordinates $(x_1^\alpha,\ldots,x_k^\alpha)$ such that:

$$N \cap U_\alpha = \{(x_1^\alpha,\ldots,x_n^\alpha) \mid x_1^\alpha \geq 0,\ldots,x_s^\alpha \geq 0\}.$$

Consider $X_i^{k-1} = \{(x_1^\alpha,\ldots,x_n^\alpha) \mid x_1^\alpha \geq 0,\ldots,x_i^\alpha = 0,\ldots,x_s^\alpha \geq 0\}$ in U_α. Clearly, $\partial N \cap U_\alpha = \bigcup_{i=1}^{s} X_i^{k-1}$. Also, each X_i^{k-1} is a C^∞-submanifold with corners inside $Y_i^{k-1} = \{(x_1^\alpha,\ldots,x_n^\alpha) \mid x_i^\alpha = 0\}$. Thus, if $N^k \subset M^k$ is a compact manifold with corners, then ∂N is measurable and $\partial N - \partial^s N$ is a set of measure 0 (by Lemma 6.2). Thus, if ω is any $(k-1)$ form on M^k and if $\partial^s N$ is oriented, then

$$(6.3) \qquad \int_{\partial N} \omega = \int_{\partial^s N} \omega.$$

<u>Exercises</u>: 1) Give an abstract definition of a manifold with corners.

2) Show that a manifold with boundary is the same thing as a manifold with corners with the condition that $C_2 = \emptyset$

§7. Statement of Stokes' Theorem

We have established the multi-variable version of differential and integral calculus. A natural question to ask at this point is "What is the generalization of the Fundamental Theorem of Calculus?" The answer is Stokes' Theorem.

Theorem 7.1 (Stokes' Theorem): *Let $N^k \subset M^k$ be a compact submanifold with corners inside an oriented manifold, M^k. Let ω be a C^∞-$(k-1)$-form on P^n and $\varphi: M^k \to P^n$ a C^∞-mapping. Then,*

$$\int_{\varphi(N^k)} d\omega = \int_{\varphi(\partial N)} \omega \quad .$$

Example: $k = 1$: Let $\gamma: [a,b] \to P^n$ be a C^∞-curve and let $\gamma: P \to R^1$ be a C^∞-function. Stokes' theorem says

$$\int_\gamma df = \int_{\partial\gamma} f = f(\gamma(b)) - f(\gamma(a))$$

If we pull these forms back to forms on $[a,b]$, then this equation becomes

$$\int_a^b d(f\circ\gamma) = f\circ\gamma(b) - f\circ\gamma(a).$$

Of course, $d(f\circ\gamma)$ is just $(f\circ\gamma)'(t)dt$. Hence, in this case

Stokes' Theorem becomes the Fundamental Theorem of Calculus. The proof in higher dimensions succeeds by reducing the problem to the Fundamental Theorem of Calculus.

§8. Partitions of Unity

The main technical result that we need to prove Stokes' theorem is an ability to reduce the problem to a sum of ones which are non-zero only in one coordinate patch. This is achieved by using a partition of unity. As might be expected, this technique is very important to many different problems in the theory of manifolds. In fact, the existence of a partition of unity is the main difference between the C^∞-category and the analytic category.

Let $\{U_\alpha\}_{\alpha\in I}$ be open subsets of M^n so that $A \subset \bigcup_{\alpha\in I} U_\alpha$; this is called an <u>open cover</u> of A. <u>A C^∞-partition of unity for A subordinate to the cover</u> $\{U_\alpha\}_{\alpha\in I}$ is a collection of C^∞-functions $\varphi_\alpha: U \to R$ where U is some open set containing A such that:

1) $\varphi_\alpha(p) \geq 0$ for all $p \in U$.
2) Given $p \in U$ there is an open set V containing p so that all but finitely many of the φ_α vanish on V.
3) $\Sigma_{\alpha\in I}\, \varphi_\alpha(p) = 1$ for all $p \in A$.
4) The support of φ_α, i.e. $\overline{\{x \in U | \varphi_\alpha(x) \neq 0\}}$, is contained in U_α for all $\alpha \in I$.

<u>Theorem 8.1</u>: <u>Given</u> $A \subset M^k$ <u>and any open cover</u> $\{U_\alpha\}_{\alpha\in I}$ <u>of</u> A,

<u>there is a</u> C^∞-<u>partition of unity for</u> A <u>subordinate to</u> $\{U_\alpha\}_{\alpha\in I}$.

<u>Proof</u>: <u>Step I</u>: Let

$$f_a(x) = \begin{cases} e^{-1/(x-a)^2} & x > a \\ 0 & x \leq a. \end{cases}$$

Then f_a is a C^∞-function on R^1.

<u>Step II</u>: Given $a < b$ in R, there is a C^∞-function $f\colon R^1 \to [0,1]$ which is positive exactly on (a,b).

<u>Proof</u>: $f_a(x)\cdot f_{-b}(-x)$ is such a function.

<u>Step III</u>: Given a rectangle $R = \bigtimes_{i=1}^{n} [a_i,b_i]$ there is a C^∞ function $\varphi_R\colon R^n \to [0,1]$ which is positive exactly on (int R).

<u>Proof</u>: Let $\varphi_{(a_i,b_i)}(t)$ be a function positive exactly on (a_i,b_i). Then define $\varphi_R(t_1,\ldots,t_n) = \prod_{i=1}^{n} \varphi_{(a_i,b_i)}(t_i)$.

<u>Step IV</u>: Let $C \subset U \subset R^n$ with C compact and U open. There is a C^∞-function $f\colon R^n \to [0,1]$ so that $f|C = 1$ and $f|(R^n - U) = 0$.

<u>Proof</u>: For each point, p, of C there is a rectangle

$R = \times_{i=1}^{n} [a_i, b_i]$ so that $p \in (\text{int } R)$ and $R \subset U$. Cover C by the interiors of finitely many such rectangles $\{R_1, \ldots, R_T\}$. Consder $g: R^n \to [0,\infty)$ defined by $g = \Sigma_{i=1}^{T} \varphi_{R_i}$ where φ_{R_i} is positive exactly on $(\text{int } R_i)$. Then $g|(R^n - U) = 0$ and $g|C > 0$. Let $\varepsilon > 0$ be the minimum value of g on C. Take a C^∞-function $\varphi_{(0,\varepsilon)}$ which is positive exactly on $(0,\varepsilon)$ and 0 elsewhere. Define

$$\psi(t) = \left(\int_0^t \varphi_{(0,\varepsilon)}(\tau)d\tau\right) / \left(\int_0^\varepsilon \varphi_{(0,\varepsilon)}(\tau)d\tau\right).$$

Clearly, $0 < \psi(t) < 1$ for $0 < t < \varepsilon$, and

$$\psi(t) = \begin{cases} 0 & ; \text{ for } t \leq 0 \\ 1 & ; \text{ for } t \geq \varepsilon. \end{cases}$$

Define $f: R^n \to [0,1]$ by $f = \psi \circ g$. This is the required function.

<u>Step V</u>: Suppose given $A \subset M$ and an open cover $\{U_\alpha\}_{\alpha\in I}$ of A. Suppose $\{V_\beta\}_{\beta\in J}$ is a <u>refinement</u> of $\{U_\alpha\}_{\alpha\in I}$. This means that there is a function $\mu: J \to I$ so that $V_\beta \subset U_{\mu(\beta)}$ for all $\beta \in J$. Then, if A has a C^∞-partition of unity subordinate to $\{V_\beta\}_{\beta\in J}$, $\{\varphi_\beta\}$ then it has one subordinate to $\{U_\alpha\}_{\alpha\in I}$.

<u>Proof</u>: For each $\alpha \in I$ consider $\mu^{-1}(\alpha) \subset J$. Form $\psi_\alpha = \Sigma_{\beta\in\mu^{-1}(\alpha)} \varphi_\beta$. (Since for each $x \in U_\alpha$ all but finitely many of the φ_β are zero in a neighborhood of x, this sum makes sense and is a C^∞-function.) Clearly ψ_α is supported in U_α and $\{\psi_\alpha\}_{\alpha\in I}$ is a C^∞-partition of unity subordinate to $\{U_\alpha\}_{\alpha\in I}$.

<u>Step VI:</u> Let $A \subset M^n$ be compact and $\{U_\alpha\}_{\alpha\in I}$ be an open cover of A. Then there is a C^∞-partition of unity of A subordinate to $\{U_\alpha\}_{\alpha\in I}$.

<u>Proof</u>: By Step V we are allowed to assume that each U_α is a coordinate patch. Since A is compact, $A \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_T}$. We claim that there are compact sets $C_i \subset U_{\alpha_i}$ so that $A \subset (\bigcup_{i=1}^{T} \text{int } C_{\alpha_i})$. If this is so, then there are larger compact sets V_i, with $C_{\alpha_i} \subset \text{int } V_i \subset U_{\alpha_i}$. (See exercise 2.) Choose a C^∞-function $f_i\colon U_{\alpha_i} \to R^+$ which is 1 on C_{α_i} and 0 on $U_{\alpha_i} - (\text{int } V_i)$. Extend f_i to a C^∞ function on M^n by deferring it to be zero outside of U_{α_i}. Since $A \subset (\cup C_{\alpha_i})$, it follows that $\Sigma_{i=1}^{T} f_i(a) > 0$ for all $a \in A$. Hence, there is an open set U, $A \subset U \subset M^n$, so that $\Sigma_{i=1}^{T} f(p) > 0$ for any $p \in U$. Define $\varphi_i\colon U \to R^+$ by

$$\varphi_i(p) = \frac{f_i(p)}{f_1(p) + \ldots + f_T(p)}.$$

Clearly, $\mathrm{supp}(\varphi_i) \subset (U_{\alpha_i} \cap U)$ and

$$\sum_{i=1}^{T} \varphi_i(p) = 1 \quad \text{for all} \quad p \in U.$$

It remains to construct the compact sets $C_{\alpha_i} \subset U_{\alpha_i}$ as required. For each $a \in A$, $a \in U_{\alpha_i}$, there is a compact ball $\overline{B_\varepsilon(a)} \subset U_{\alpha_i}$. Choose finitely many such balls $B_1, \ldots, B_K$ whose interiors cover A. Associate to each B_i some U_{α_j} which contains it and let C_{α_j} be the union of the balls associated to U_{α_j}. This is a finite collection of compact sets as required.

<u>Step VII</u>: Any open subset U of M^n is an increasing union of compact sets:

$$A_1 \subset \mathrm{int}\, A_2 \subset \ldots; \quad \bigcup_{i=1}^{\infty} A_i = U.$$

<u>Proof</u>: Suppose that this is true for any open subset of R^n. Let $\{U_i, \varphi_i, V_i\}_{i=1}^{\infty}$ be a <u>countable</u> atlas of M. (Here is the first and only time that we use the fact that M is paracompact.) For each $U_i \subset M$, let $A_{i,1} \subset \mathrm{int}\, A_{i,2} \subset \ldots$ be compact sets whose union is $U \cap U_i$. Let $B_i = \bigcup_{j \leq i} A_{j,i}$.

Clearly, $B_i \subset \text{int } B_{i+1}$; each B_i is compact; and $\bigcup_{i=1}^{\infty} B_i = U$. It remains to show this result for any $U \subset \mathbb{R}^n$. Let

$$A_i = \{x \in U \mid \|x\| \le i \text{ and } d(x, R^n - U) \ge \frac{1}{i}\}.$$

This is the required sequence of compact sets filling out U.

<u>Step VIII</u>: Let $A \subset M^n$ be an open set and $\{U_\alpha\}_{\alpha \in I}$ and open cover of A. Then A has a C^∞-partition of unity subordinate to $\{U_\alpha\}_{\alpha \in I}$.

<u>Proof</u>: Let $A = \bigcup_{i=1}^{\infty} C_i$ where each C_i is compact and $C_i \subset \text{int } C_{i+1}$. For each i, there is a C^∞-partition of unity for $C_i - (\text{int } C_{i-1})$ subordinate to the cover $\{U_\alpha \cap (\text{int } C_{i+1} - C_{i-2})\}_{\alpha \in I}$, $\{\varphi_{i,\alpha}\}_{\alpha \in I}$. The full collection $\{\varphi_{i,\alpha}\}$ for $\alpha \in I$ and $i = 1, \ldots$ is locally finite in the sense that all but finitely many vanish near any point. Thus $\sigma(x) = \Sigma_{i,\alpha}\, \varphi_{i,\alpha}(x)$ is a C^∞ function on A. It is everywhere positive. Thus, define

$$\psi_{i,\alpha}(x) = \varphi_{i,\alpha}(x)/\sigma(x).$$

This collection of C^∞-functions is a C^∞-partition of unity for the refinement $\{U_\alpha \cap (\text{int } C_{i+1} - C_{i-2})\}$ of $\{U_\alpha\}$. By Step V, these $\psi_{i,\alpha}$ can be amalgamated into a C^∞-partition of unity for $\{U_\alpha\}_{\alpha \in I}$.

<u>Step IX</u>: Let $A \subset M^n$ be an arbitrary set and $\{U_\alpha\}_{\alpha\in I}$ be an open covering for A. Then there is a C^∞-partition of unity for A subordinate to $\{U_\alpha\}_{\alpha\in I}$.

<u>Proof:</u> Let $U = \bigcup_{\alpha\in I} U_\alpha$. By Step VIII there is a C^∞-partition of unity for U subordinate to $\{U_\alpha\}$. This gives a C^∞-partition for A as well.

<u>Exercises</u>: 1) Let M^n be a compact manifold. Show that M^n is a submanifold of some R^N.

a) Find a finite atlas $\{U_\alpha, h_\alpha\}_{\alpha=1}^k$ for M^n with $h_\alpha\colon U_\alpha \to V_\alpha \subset R^n$.

b) Choose a partition of unity $\{\varphi_\alpha\}_{\alpha=1}^k$ subordinate to this cover.

c) Let $\psi\colon R \to R$ be a C^∞-function so that $\psi(t) = 0$ for $t \leq 0$ and $\psi(t) = 1$ for $t \geq \frac{1}{2k}$. Let $\rho_\alpha = \psi\cdot\varphi_\alpha$. Show that given $x \in M$ there is an α, $1 \leq \alpha \leq$, such that $\rho_\alpha = 1$ in a neighborhood of x.

d) Define $i\colon M^n \to R^{k\cdot n}$ by $i(x) = (\rho_1(x)h_1(x),\ldots,\rho_k(x)h_k(x))$ where $\rho_\alpha\cdot h_\alpha$ is defined to be $0 \in R^n$ outside of U_α.

e) Show that i is a one-to-one immersion and hence an embedding.

2) Show that if $C \subset U \subset R^n$ with C compact and U open in R^n, then there is a compact set V with $C \subset \operatorname{int} V \subset U$.

3) Show that the function f_a of Step I is C^∞.

§9: Proof of Stokes' Theorem

Let $N^k \subset M^k$ be a compact C^∞-submanifold with corners. Suppose that M is oriented. For each $p \in N^k$ there is a natural number $s \geq 0$, and a C^∞-coordinate system centered at p and valid in an open set U containing p, so that $N \cap U = \{(x_1,\ldots,x_k) \in U | x_1 \geq 0,\ldots,x_s \geq 0\}$. There is a rectangle

$$R = \{(x_1,\ldots,x_k) | -\epsilon \leq x_i \leq \epsilon\}$$

contained in U. We see that $N \cap R$ is a possibly smaller rectangle. Since N is compact we can cover it by finitely many interiors of such rectangles: $N \subset \bigcup_{i=1}^{T} \operatorname{int} R_i$. Let $\{\varphi_1,\ldots,\varphi_T\}$ be a C^∞-partition of unity subordinate to this open cover. Let ω be a $(k-1)$-form on M^k. Clearly, $\int_N d\omega = \int_N d(\Sigma_{i=1}^T \varphi_i \cdot \omega))$. But $d(\varphi_i \cdot \omega)$ is supported in R_i. Thus

$$\int_N d\omega = \sum_{i=1}^{T} \int_{R_i \cap N} d(\varphi_i \cdot \omega) .$$

On the other hand,

$$\int_{\partial N} \omega = \int_{\partial N} \sum_{i=1}^{T} \varphi_i \cdot \omega = \sum_{i=1}^{T} \int_{\partial N} \varphi_i \cdot \omega = \sum_{i=1}^{T} \int_{R_i \cap \partial N} \varphi_i \cdot \omega.$$

But $\partial(R_i \cap N) = (\partial R_i \cap N) \cup (R_i \cap \partial N)$ and $\int_{\partial R_i \cap N} \varphi_i \cdot \omega = 0$ since φ_i vanishes identically on ∂R_i. Hence,

$$\sum_{i=1}^{T} \int_{R_i \cap \partial N} \varphi_i \cdot \omega = \sum_{i=1}^{T} \int_{\partial(R_i \cap N)} \varphi_i \cdot \omega.$$

Thus, if we can show that

$$\int_{R_i \cap N} d(\varphi_i \cdot \omega) = \int_{\partial(R_i \cap N)} \varphi_i \cdot \omega,$$

then it will follow that $\int_N d\omega = \int_{\partial N} \omega$. That is to say, it suffices to prove Stokes' theorem for a rectangle in R^k.

Let $R \subset U \subset \mathbb{R}^n$ be a rectangle:

$$R = \mathop{\times}_{i=1}^{n} [a_i, b_i].$$

Let ω be a (k-1)-form on U of the following form:

$$\omega = \varphi(x_1, \dots, x_k) dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_k.$$

Since any form is a sum of forms of this type, it suffices to restrict attention to them. We have

$$d\omega = (-1)^{(i-1)} \frac{\partial \varphi}{\partial x_i} dx_1 \wedge \dots \wedge dx_k.$$

Let $R' = \mathop{\times}_{j \neq i} [a_j, b_j]$. Then $R = R' \times [a_i, b_i]$. Let ζ denote the (k-1) variables of R'.

$$\int_R d\omega = \int_{R'} \left(\int_{a_i}^{b_i} (-1)^{(i-1)} \frac{\partial\varphi}{\partial x_i}\right)$$

$$= \int_{R'} (-1)^{(i-1)} (\varphi(\zeta, b_i) - \varphi(\zeta, a_i)).$$

On the other hand, since ω has no dx_i-term, its integral along all faces except $R' \times \{b_i\}$ and $R' \times \{a\}$ must vanish.

By the way we orient the faces the orientation for $R' \times \{b_i\}$ is $(-1)^{(i-1)}$-times the usual orientation on R'. The one for $R' \times \{a_i\}$ is $(-1)^i$-times the usual orientation for R'. Thus

$$\int_{\partial R} \omega = (-1)^{(i-1)} \left[\int_{R' \times \{b_i\}} \omega - \int_{R' \times \{a_i\}} \omega\right]$$

$$= (-1)^{i-1} \left[\int_{R'} \varphi(\zeta, b_i) - \int_{R'} \varphi(\zeta, a_i)\right]$$

$$= (-1)^{(i-1)} \int_{R'} (\varphi(\zeta, b_i) - \varphi(\zeta, a_i)).$$

This completes the proof of Stokes' Theorem for $N^k \subset M^k$ and C^∞-forms on M^k.

If $\rho\colon M^k \to P^n$ is a C^∞-mapping, then $\int_{\rho(N)} \omega = \int_N \rho^*\omega$. Likewise, $\int_{\rho(\partial N)} \omega = \int_{\partial N} \rho^*\omega$. Thus, Stokes' theorem for k-forms on M^k implies the general version of the theorem.

<u>Exercises</u>: 1) Show that if $\omega = f(x,y)\,dx + g(x,y)\,dy$ is a closed form defined everywhere on the square $0 < x, y < 1$ then $\omega = dh$ for some C^∞-function defined on the square.

<u>Hint</u>: Define $h(s,t) = \int_{\frac{1}{2}}^{s} f(x,\tfrac{1}{2})\,dx + \int_{\frac{1}{2}}^{t} g(s,y)\,dy$.

2) State and prove the analogous theorem for the n-dimensional rectangle and the n-dimensional open ball.

3) Suppose M^n is compact, without boundary, and oriented. Show that there is an n-form ω so that $\int_{M^n} \omega > 0$.

§10: Applications of Stokes' Theorem and Examples

The first remark to be made is that Stokes' Theorem is valid not only for C^∞-forms but also for C^1-forms. Note that if ω is a C^1-form, then $d\omega$ is a continuous form. We proved Stokes' Theorem by using a C^∞-partition of unity to reduce to the case of a rectangle in $\mathbb{R}^k$. The case of the rectangle is then proved by invoking the Fundamental Theorem of Calculus. Since the Fundamental Theorem of Calculus holds for C^1-functions, the proof given in Section 9, is valid mutatis-mutandis for C^1-forms.

Our first application of this is to prove the Cauchy Integral Formula for C^1, complex-valued functions satisfying the Cauchy-Riemann equations. Suppose $f : U \to \mathbb{C}^1$, $f(z) = R(z) + iI(z)$ with R and I real-valued functions, is a C^1-function with:

$$\text{(C-R)} \quad \begin{cases} \dfrac{\partial R}{\partial x} = \dfrac{\partial I}{\partial y} & \text{and} \\[2ex] \dfrac{\partial R}{\partial y} = -\dfrac{\partial I}{\partial x} & . \end{cases}$$

There is a short hand notation for this. If we let $\bar{z} : \mathbb{C} \to \mathbb{C}$ be given by $\bar{z} = x - iy$, then $\dfrac{\partial}{\partial \bar{z}} = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}$. Equations (C-R) above become simply $\dfrac{\partial f}{\partial \bar{z}} = 0$. Suppose that $B(z)$ is a closed ball centered about $z \in \mathbb{C}$, and suppose $B(z) \in U$.

The Cauchy Integral Formula states

$$2\pi i \cdot f(z) = \int_{\partial B(z)} \frac{f(\zeta)}{(\zeta - z)} \, d\zeta .$$

We shall deduce this as a consequence of Stokes' Theorem. First, we note that the integrand, $\frac{f(\zeta)}{(\zeta - z)}$, is a holomorphic function of ζ for $\zeta \in (U - \{z\})$. Thus, a result in Section 5 implies that $\frac{f(\zeta)}{(\zeta - z)} d\zeta$ is a closed 1-form in $U - \{z\}$. Let $B' \subset B(z)$ be any ball containing z in its interior. Stokes' Theorem, applied to $B(z) - B'$, tells us that

$$\int_{\partial B(z)} \frac{f(\zeta)}{(\zeta - z)} \, d\zeta \quad \int_{\partial B'(z)} \frac{f(\zeta)}{(\zeta - z)} \, d\zeta .$$

Lemma 10.1: *If* $f : U \to \mathbb{C}^1$ *is a* C^1*-function satisfying the* (C-R) *equations and if* $f(0) = 0$, *then*

$$f(z) = z \cdot h(z)$$

where $h(z)$ *is a continuous function*.

Proof: Consider $f(z)$ as a complex valued function of two real variables (x,y). Let $A(x,y) = \int_0^1 \frac{\partial f}{\partial x}(tx,ty)\, dt$ and $B(x,y) = \int_0^1 \frac{\partial f}{\partial y}(tx,ty)\, dt$. By Lemma 2.2 of Chapter II $f(x,y) = x\,A(x,y) + y\,B(x,y)$. Since $\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}$, we have $A = -iB$ or $B = iA$. Thus,

$$f(x,y) = (x + iy)\,A(x,y) = z \cdot A(x,y), \text{ or}$$

$$f(z) = z \cdot A(z) .$$

Now let us return to the proof of the Cauchy Integral Formula. Applying the above lemma to $f(\zeta)$ we see that $f(\zeta) = (\zeta-z)\, A(\zeta) + f(z)$. Thus,

$$\int_{\partial B(z)} \frac{f(\zeta)}{(\zeta-z)}\, d\zeta = \int_{\partial B(z)} \frac{f(z)}{(\zeta-z)}\, d\zeta + \int_{\partial B(z)} A(\zeta)\, d\zeta$$

As we have seen the integral on the left hand side is independent of the radius of $B(z)$. Likewise, the first integral on the right hand side is independent of the radius of $B(z)$. Consequently, so is $\int_{\partial B(z)} A(\zeta)\, d\zeta$. Since $A(\zeta)$ is continuous at $\zeta = z$, this integral goes to zero as the radius of $B(z)$ goes to zero. Thus,

$$\int_{\partial B(z)} \frac{f(\zeta)}{(\zeta-z)}\, d\zeta = \int_{\partial B(z)} \frac{f(z)}{(\zeta-z)}\, d\zeta .$$

We can evaluate the right hand integral directly.

Let $\gamma(\theta) = z + re^{i\theta}$ for $0 \leq 0 \leq 2\pi$.

Then, $\int_{\partial B(z)} \frac{f(z)}{(\zeta-z)}\, d\zeta = f(z) \int_0^{2\pi} \frac{1}{re^{i\theta}}\, d(re^{i\theta})$

$$= f(z) \int_0^{2\pi} \frac{re^{i\theta}}{re^{i\theta}}\, i d\theta$$

$$= f(z) \int_0^{2\pi} i d\theta = 2\pi i \cdot f(z).$$

This completes the proof of the Cauchy Integral Formula.

From this formula we can easily prove that f has an absolutely convergent power series in $(w - z)$ which represents it on $B(z) \subset U$. If $w \in B(z)$, then $\int_{\partial B(z)} \frac{f(\zeta)}{(\zeta - w)} d\zeta = 2\pi i \cdot f(w)$. The reason is that $\frac{f(\zeta)}{\zeta - w} d\zeta$ is a closed form in $U - \{w\}$. Hence, by Stokes' Theorem $\int_{\partial B(z)} \frac{f(\zeta)}{\zeta - w} d\zeta = \int_{\partial B(w)} \frac{f(\zeta)}{(\zeta - w)} d\zeta$ provided that $B(w) \subset \mathrm{int}(B(z))$. (To see this apply Stokes' Theorem to $\overline{B(z) - B(w)}$.) On the other hand, if $w \in \mathrm{int}(B(z))$ and ζ is in $\partial B(z)$, we have

$$\frac{1}{\zeta - w} = \frac{1}{(\zeta - z) - (w - z)} = \frac{1}{\zeta - z} \left(\frac{1}{1 - \frac{w-z}{\zeta - z}}\right) = \frac{1}{\zeta - z} \left(\sum_{n=0}^{\infty} \left(\frac{w-z}{\zeta - z}\right)^n\right),$$

where the power series on the right converges uniformly (since $\left|\frac{w-z}{\zeta - z}\right| < 1$).

Thus,

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(z)} \frac{f(\zeta)}{(\zeta - z)} d = \frac{1}{2\pi i} \int_{\partial B(z)} \sum_{n=0}^{\infty} \frac{(w-z)^n}{(\zeta - z)^{n+1}} f(\zeta)\, d\zeta$$

$$= \sum_{n=0}^{\infty} \left\{\frac{1}{2\pi i} \int_{\partial B(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta\right\} \cdot (w - z)^n .$$

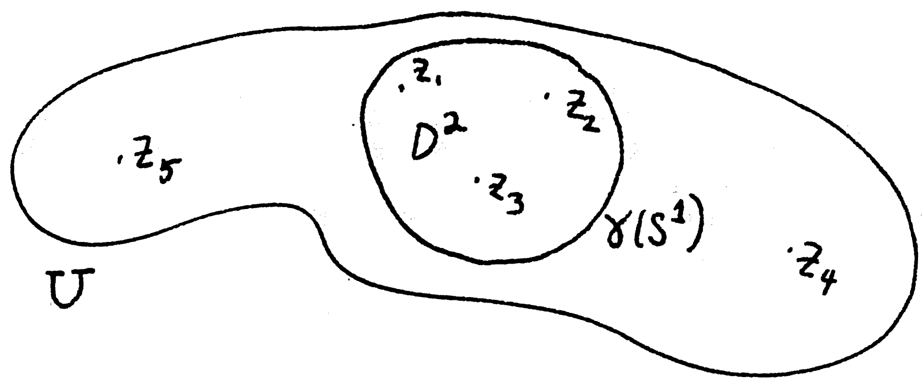
This gives an absolutely convergent power series expansion for $f(w)$ in $\mathrm{int}(B(z))$.

<u>Example I</u>: Suppose f is a meromorphic function on $U \subset \mathbb{C}^1$ with poles at $\{z_1, \ldots, z_N\}$ only. In a neighborhood of z_j, f has a Laurent expansion

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_j)^n$$

we define a_{-1} to be the residue of f at z_j.

Let $\gamma : S^1 \to U - \{z_1, \ldots, z_N\}$ be a C^∞-embedding with $\gamma(S^1)$ bounding $D^2 \subset U$ which contains $\{z_1, \ldots, z_T\}$ in its interior but misses the other poles .



Then, $\int_{\gamma(S^1)} f(z)\,dz = \sum_{j=1}^{T} (2\pi i)\cdot(\text{Residue of } f \text{ at } z_j)$.

To see this, form the 2-manifold with boundary $D^2 - \coprod_{j=1}^{T} B_\delta(z_j)$.

Applying Stokes' Theorem to this manifold (and using the fact that $f(z)\,dz$ is closed in $U - \{z_1, \ldots, z_N\}$) we see that

$$\int_{\gamma(S^1)} f(z)\,dz = \sum_{j=1}^{T} \int_{\partial B_\delta(z_j)} f(z)\,dz .$$

In $B_\delta(z_j)$ f has a Laurent series

$$f(z) = \sum_{n=-k}^{\infty} a_n (z - z_j)^n .$$

Clearly, if $a_{-1} = 0$, then $f(z)\,dz = d\left(\sum_{n=k}^{\infty} \frac{a_n}{n+1} (z - z_j)^{n+1}\right)$

and hence $\int_{\partial B_\delta(z_j)} f(z)\,dz = 0$. This proves that in general (when a_{-1} is non-zero)

$$\int_{\partial B_\delta(z_j)} f(z)\, dz = \int_{\partial B_\delta(z_j)} \frac{a_{-1}}{(z - z_j)}\, dz = 2\pi i\, a_{-1}\,.$$

Adding up the results at each pole gives the formula that we claimed.

Example II: S^2 is not diffeomorphic to $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

There is a closed C^∞- 1-form, dx, on T^2 and a C^∞-embedded $S^1 \subset T^2$ so that $\int_{S^1} dx = 1$. If $\gamma : S^1 \to S^2$ is any C^∞-embedding, then the image of $\gamma(S^1)$ misses a point of S^2. Hence $\gamma(S^1)$ lies in a single coordinate patch, and in fact in a rectangle $R \subset \mathbb{R}^2$. Since any closed form on R is exact (See Exercise 2, Section 9), it follows that $\int_{\gamma(S^1)} \omega = 0$ for all closed 2-forms ω on S^2. This distinguishes T^2 and S^2 .

Example III: The Winding Number .

We gave a description of the winding number for $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$, $w(\gamma)$, as $\frac{1}{2\pi} \int_{\gamma(S^1)} \left(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy\right)$.

The form $\frac{x\,dx + y\,dy}{x^2 + y^2}$ is exact on $\mathbb{R}^2 - \{0\}$ since it is $d(\log\sqrt{x^2 + y^2})$. The sum of this form with i times the integral of the winding number integral is, when expressed in complex coordinates, equal to $\frac{dz}{z}$. Since $\int_{\gamma(S^1)} d(\log(\sqrt{x^2+y^2})) = 0$, the winding number is also given by

$$w(\gamma) = \frac{1}{2\pi i} \int_{\gamma(S^1)} \frac{dz}{z}\,.$$

Near any point $p \in \mathbb{C}^1 - \{0\}$ there is an analytic function, $\log(z)$, whose differential is $\frac{dz}{z}$. The real part of $\log(z)$ is $\log(|z|)$ and the imaginary part is the amplitude of z : $\log(Re^{i\theta}) = \log(R) + i\theta$. Of course, this is a description (locally) of infinitely many functions whose imaginary parts all differ by multiples of 2π. Given a C^∞-path $\gamma : I \to \mathbb{C}^1 - \{0\}$ and a choice of $\log(z)$ near $\gamma(0)$, we can analytically continue this to a well-defined C^∞-function of t, $\log(\gamma(t))$. Of course, it is not necessarily true that, if $\gamma(t_1) = \gamma(t_2)$, then $\log(\gamma(t_1)) = \log(\gamma(t_2))$. For example, if $\gamma(t) = e^{it}$, and we start with $\log(1) = 0$, then $\log(\gamma(t)) = i \cdot t$. Hence, $\gamma(2\pi) = \gamma(0)$, but $\log(\gamma(2\pi)) = 2\pi i$ and $\log(\gamma(0)) = 0$. It is true, however, that if $\gamma(t_1) = \gamma(t_0)$, then $\log(\gamma(t_1)) = \log(\gamma(t_0)) + k2\pi i$ for some $k \in \mathbb{Z}$. The reason is that for each t, $\log(\gamma(t))$ is one of the infinite possible values, $\log(\gamma(t)) + i\,\mathrm{Amp}(\gamma(t))$, and these all differ by multiples of $2\pi i$. If $\gamma : I \to \mathbb{C}^1 - \{0\}$ is a C^∞-path and if we analytically continue $\log(z)$ along γ to get a function $\log(\gamma(t))$, then $\int_{\gamma(I)} \frac{dz}{z} = \log(\gamma(1)) - \log(\gamma(0))$. In case γ is actually a map of S^1 into $\mathbb{C}^1 - \{0\}$, i.e., in case $\gamma(1) = \gamma(0)$, we see that $\int_{\gamma(I)} \frac{dz}{z} = \log(\gamma(1)) - \log(\gamma(0)) = k \cdot 2\pi i$. Thus the winding number, $\frac{1}{2\pi i} \int_{\gamma(S^1)} \frac{dz}{z}$, is an integer.

Example IV: The closed forms of degree 0 on a connected manifold are the constant functions.

If f is a closed function, then $df = 0$. This means that, in local coordinates, $(x_1, \ldots, x_n)$, $\frac{\partial f}{\partial x_i} = 0$. Thus, f must be locally constant. If the space on which f is defined is connected, then f is constant.

Example V: Closed 1-forms on S^1.

Consider S^1 as $\mathbb{R}^1/\mathbb{Z}$. Any function on S^1, f, can be lifted to a periodic function, $\tilde{f}$, on $\mathbb{R}^1$;

$$\tilde{f}(x+1) = \tilde{f}(x) \quad \text{such that} \quad \tilde{f}(x) = \tilde{f}([x]) \ .$$

If f is C^∞ on S^1, then $\tilde{f}$ will be C^∞ on $\mathbb{R}^1$. Likewise, any C^∞-1-form on S^1 lifts to a periodic C^∞-1-form on $\mathbb{R}^1$. This gives identifications of the C^∞-functions and C^∞ 1-forms on S^1 with periodic C^∞-functions and 1- forms on $\mathbb{R}^1$. Clearly, this identification is compatible with exterior differentiation.

The question that we address is when is a C^∞ 1-form on S^1 exact. In light of the above discussion, the question is equivalent to the question of which C^∞- periodic 1-forms are differentials of periodic functions on $\mathbb{R}^1$. Let $f(t)\,dt$ be a C^∞-periodic 1-form. Then, $f(t)\,dt = d(\int_0^t f(s)\,ds + C)$. The functions on the right hand side are the only solutions to the equation

$$f(t)\,dt = dh \ .$$

Thus, we must decide when $\int_0^t f(s)\,ds$ is a periodic function of t. Clearly,

$$\int_0^{t+1} f(s)\,ds - \int_0^t f(s)\,ds = \int_t^{t+1} f(s)\,ds$$

Since f is periodic $\int_t^{t+1} f(s)\,ds = \int_0^1 f(s)\,ds$.

Thus, $f(t)\,dt$ is the differential of periodic function if and only if $\int_0^1 f(s)\,ds = 0$. If we let $f(t)\,dt$ represent the form ω on S^1, then

$$\int_{S^1} \omega = \int_0^1 f(s)\,ds \,.$$

Thus, a 1-form ω on S^1 is exact if and only if $\int_{S^1} \omega = 0$. As a consequence, we see that if ω is any 1-form on S^1, then

$$\omega = d\left(\int_0^t \omega\right) + \left(\int_{S^1} \omega\right) dt$$

In particular, every 2-form on S^1 is an exact 1-form plus a multiple of dt. The multiple is given by integrating the original form over S^1.

<u>Example VI</u>: <u>Closed forms on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$</u>.

The forms on this torus are identified with doubly periodic forms on $\mathbb{R}^2$:

$$\omega(x,y) = \omega(x+1,y) = \omega(x,y+1) \,.$$

Once again, this identification is compatible with exterior differentiation.

<u>1-forms</u>: A 1-form $= f(x,y)\,dx + g(x,y)\,dy$ is closed if and only if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. If this is so, then there is a function $h(x,y)$ such that $\frac{\partial h}{\partial x} = f$ and $\frac{\partial h}{\partial y} = g$. Consequently, $dh = \omega$. The question is when can this h be chosen to be doubly periodic. The function h is determined up to a constant

and is given by the formula :

$$h(s,t) = \left(\int_0^s f(x,0)\, dx + \int_0^t g(s,y)\, dy \right) + C .$$

Thus, h is doubly periodic if and only if

$$\int_0^1 f(x,0)\, dx = 0 \quad \text{and} \quad \int_0^1 g(s,y)\, dy = 0 .$$

Since $f\,dx + g\,dy$ is closed and periodic, these two equations are equivalent to :

$$\int_0^1 f(x,0)\, dx = 0 \quad \text{and} \quad \int_0^1 g(0,y)\, dy = 0 .$$

As a result any closed 1-form on T^2 is an exact 1-form plus a multiply of dx plus a multiply of dy :

$$f\,dx + g\,dy = d\,h + \left(\int_0^1 f(t,0)\, dt \right) dx + \left(\int_0^1 g(0,t)\, dt \right) dy .$$

This expression is unique except for the fact that h can be altered by a constant.

<u>2-forms</u>: Let ω be a 2-form on T^2 . We represent it as $\phi(x,y)\, dx \wedge dy$ where ϕ is a doubly periodic C^∞-function on $\mathbb{R}^2$. We claim that ω is exact if and only if

$$\int_0^1 \int_0^1 \phi(x,y)\, dx \wedge dy = 0.$$

Certainly, if ω is exact $\int_{T^2} \omega = 0$. We have $\gamma : I^2 \to T^2$ a C^∞-map which is onto and $1 - 1$ on interior (I^2) . Thus, $\int_{T^2} \omega = \int_{I^2} \gamma^*\omega$. But $\gamma^*\omega = \phi(x,y)\, dx \wedge dy$. Thus,

$$\int_{T^2} \omega = \int_{I^2} \phi(x,y)\, dx \wedge dy .$$

Hence, if ω is exact, $\iint \phi(x,y)\, dx \wedge dy = 0$.

Conversely, suppose that $\int_{I^2} \phi(x,y)\, dx \wedge dy = 0$ with $\phi(x,y)$ a doubly periodic function. Let

$$g(s,t) = \int_0^s \phi(x,t)\, dx - s \cdot \left(\int_0^1 \phi(x,t)\, dx \right)$$

$$f(t) = \int_0^{2\pi} \phi(x,t)\, dx .$$

Clearly, f is a periodic function of t and $\int_0^1 f(y)\, dy = 0$. Also, $g(s,t)$ is doubly periodic. It is clearly periodic in t since ϕ is. As for the periodicity in s, we have:

$$g(s+1,t) - g(s,t) = \int_s^{s+1} \phi(x,t)\, dx - (s+1-s) \int_0^1 \phi(x,t)\, dx$$

$$= \int_0^1 \phi(x,t)\, dx - \int_0^1 \phi(x,t)\, dx = 0 .$$

Consider the 1-form

$$\omega = -\left(\int_0^y f(t)\, dt \right) dx + g(x,y)\, dy .$$

We claim that $d\omega = \phi(x,y)\, dx \wedge dy$.

We see that $d\left(\int_0^y f(t)\, dt \right) dx = - \frac{\partial}{\partial y} \left(\int_0^y f(t)\, dt \right) \cdot dx \wedge dy$

$$= - f(y)\, dx \wedge dy .$$

On the other hand

$$d(g\,dy) = \frac{\partial g}{\partial x}\, dx \wedge dy$$

$$= \left(\phi(x,y) - \int_0^1 \phi(x,y)d\right) dx \wedge dy$$

$$= \phi(x,y)\, dx \wedge dy - f(y)\, dx \wedge dy\ .$$

Thus, $d\omega = \phi(x,y)\, dx \wedge dy$.

As a consequence of this calculation, we see that every closed 2-form on T^2, ω, is an exact form plus a multiple of $dx \wedge dy$. The multiple of $dx \wedge dy$ is obtained by integrating ω over T^2 .

The general result for T^n $(= \mathbb{R}^n/\mathbb{Z}^n)$ is that every closed k-form is an exact form plus multiples of $dx_{i_1} \ldots dx_{i_k}$, $i_1 < \ldots < i_k$. The multiples are determined by integrating the closed k-form over the various sub-tori of dimension k .

Example VII: If M^n is a C^∞- manifold with the property that every C^∞-map $\gamma : S^1 \to M$ has its image contained in an open ball in some coordinate chart, then all closed 1-forms on M are exact .

Proof: To show that a 1-form ω is exact it suffices, by Problem 8 of Section 4, to show that

$$\int_{\gamma(S^1)} \omega = 0 \quad \text{for all } C^\infty\text{-maps} \quad \gamma : S^1 \to M^n\ .$$

If $\gamma(S^1) \subset U \subset M^n$ where U has C^∞-coordinates which make it an open ball in $\mathbb{R}^n$, then

$$\int_{\gamma(S^1)} \omega = \int_{\gamma(S^1)} \omega|U .$$

Since $\omega|U$ is exact (see Problem 2, Section 9), it follows that $\int_{\gamma(S^1)} \omega|U = 0$ and hence that $\int_{\gamma(S^1)} \omega = 0$.

Exercises 1): Show that every closed 1-form on T^n is of the form $dh + \sum_{i=1}^{n} a_i\, dx_i$. Show that this expression is unique, except for the fact that a constant can be added to h.

2) Show that T^n and S^n are not diffeomorphic .

3) Let $p(z)$ be a (complex) polynomial without repeated roots. Let $X \subset \mathbb{C}^2$ be the complex curve $\{y^2 = p(z)\}$

a) Show that X is non-singular (i.e., a complex 1-manifold).

b) Show that dy and dz, when restricted to X, define holomorphic 1-forms which satisfy

$$2ydy = p'(z)$$

c) Let $t(z)$ be an entire function (that is - an everywhere defined, holomorphic function). Show that $\frac{f(z)dz}{y}$ is an everywhere defined, closed 1-form on X .

d) Let $\gamma : I \to \mathbb{C}^1$ be a C^∞-path which misses the roots of $p(z)$. By a lifting of γ we mean a C^∞-path $\tilde{\gamma} : I \to X$ which sits above γ in that $\tilde{\gamma}(t) = (\gamma(t), \mu(t))$. Show that there are exactly 2 liftings of γ . Such a lifting is determined by closing a point $(\gamma(0), \mu(0))$

where $\mu(0)$ is one of the 2 square roots of $p(\gamma(0))$.

e) Consider $\int_{\gamma(I)} \frac{f(z)dz}{\sqrt{p(z)}}$ There are two values depending on the choice of $\sqrt{p(z)}$ along γ . Show that these two values are equal to

$$\int_{\tilde{\gamma}_0(I)} \frac{f(z)dz}{y} \quad \text{and} \quad \int_{\tilde{\gamma}_1(I)} \frac{f(z)dz}{y}$$

where $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are the two liftings of γ .