

# ON A SECONDARY INVARIANT OF THE HYPERELLIPTIC MAPPING CLASS GROUP

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ABSTRACT. In this paper, we discuss relations among several invariants of 3-manifolds including Meyer's function, the  $\eta$ -invariant, the von Neumann  $\rho$ -invariant and the Casson invariant from the viewpoint of the mapping class group of a surface.

## 1. INTRODUCTION

Let  $\Sigma_g$  be a closed oriented smooth surface of genus  $g$  and  $\text{Diff}_+\Sigma_g$  the group of orientation preserving diffeomorphisms of  $\Sigma_g$  equipped with the  $C^\infty$ -topology. The mapping class group  $\mathcal{M}_g$  is defined to be the group of path components of  $\text{Diff}_+\Sigma_g$ . In this paper, by an automorphism of  $\Sigma_g$ , we mean an element of  $\mathcal{M}_g$ . Next let  $r : \mathcal{M}_g \rightarrow \text{Sp}(2g, \mathbf{Z})$  denote the classical representation defined by the action of  $\mathcal{M}_g$  on the first integral homology group  $H_1(\Sigma_g, \mathbf{Z})$ . It is known that  $r$  is surjective and the Torelli group  $\mathcal{I}_g$  is defined to be the kernel of  $r$ . Namely, it has the trivial action on  $H_1(\Sigma_g, \mathbf{Z})$ .

Our main object here is the signature cocycle  $\tau$ , which is a group 2-cocycle of the Siegel modular group  $\text{Sp}(2g, \mathbf{Z})$ . It was introduced by Meyer (see [21]) to describe a signature formula for surface bundles over a surface. If we pull back the cocycle by the representation  $r$ , then we can think of  $\tau$  as a 2-cocycle of  $\mathcal{M}_g$ . But here, we mainly consider the restriction of  $\tau$  to a subgroup of  $\mathcal{M}_g$ , that is, the hyperelliptic mapping class group  $\Delta_g$ , which is the centralizer with a fixed hyperelliptic involution. As is known,  $\Delta_g = \mathcal{M}_g$  if  $g = 1, 2$  and  $\Delta_g \neq \mathcal{M}_g$  if  $g \geq 3$ . Moreover the fact that  $H^*(\Delta_g, \mathbf{Q}) = 0$  for  $* = 1, 2$  (see [8], [14]) implies the restriction of  $\tau$  to  $\Delta_g$  must be the coboundary of the unique rational 1-cochain  $\phi : \Delta_g \rightarrow \mathbf{Q}$  (i.e.  $\delta\phi = r^*\tau|_{\Delta_g}$  holds). In the following, we call it Meyer's function of the hyperelliptic mapping class group. Namely, it is a secondary invariant of  $\Delta_g$  associated with the signature cocycle.

In the case of genus one, explicit formulas of  $\tau$  and  $\phi$  were given by Meyer himself, Kirby-Melvin [15] and Szczec [34]. On the other hand, geometric aspects of Meyer's function have been studied by Atiyah in [2]. In fact, he related it to various invariants defined for each element of  $\Delta_1 = \mathcal{M}_1 \cong SL(2, \mathbf{Z})$  including Hirzebruch's signature defect, the logarithmic monodromy of Quillen's determinant line bundle, the Atiyah-Patodi-Singer  $\eta$ -invariant and its adiabatic limit, the special value of the Shimizu  $L$ -function and so on.

The purpose of the present paper is to discuss some extensions of Atiyah's results to higher genus case and explain relations among several invariants of 3-manifolds,

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that is, Meyer's function, the  $\eta$ -invariant, the von Neumann  $\rho$ -invariant and the Casson invariant from the viewpoint of the mapping class group of a surface.

Now we describe the contents of this paper. In the next section, we recall the definitions of the signature cocycle and Meyer's function according to [21] and [27]. In Section 3 we give a relation between Meyer's function and the  $\eta$ -invariant under the mapping torus construction. In Section 4 we describe the von Neumann  $\rho$ -invariant for a  $\mathbf{Z}$ -covering by using Meyer's function. In the last section, the Casson invariant of an integral homology 3-sphere is described by Meyer's function through the correspondence between elements of  $\mathcal{M}_g$  and 3-manifolds via Heegaard splittings.

## 2. SIGNATURE COCYCLE AND MEYER'S FUNCTION

In this section, we quickly review the definitions of the signature cocycle and Meyer's function. First we take two elements  $A, B \in \mathrm{Sp}(2g, \mathbf{Z})$  and define the vector space  $V_{A,B}$  by

$$V_{A,B} = \{(x, y) \in \mathbf{R}^{2g} \times \mathbf{R}^{2g} \mid (A^{-1} - I)x + (B - I)y = 0\},$$

where  $I$  denotes the identity matrix. We also define the pairing map on  $\mathbf{R}^{2g} \times \mathbf{R}^{2g}$  by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} = (x_1 + y_1) \cdot J(I - B)y_2,$$

where  $\cdot$  denotes the standard inner product in  $\mathbf{R}^{2g}$  and  $J$  is the  $2g \times 2g$  real matrix corresponding to the multiplication by  $\sqrt{-1}$  on  $\mathbf{C}^g = \mathbf{R}^{2g}$ . We can easily check that this pairing map is a symmetric bilinear form on  $V_{A,B}$  (possibly degenerate), and so we define  $\tau(A, B)$  to be the signature of the pairing map  $\langle \cdot, \cdot \rangle_{A,B}$  on the vector space  $V_{A,B}$ .

From Novikov additivity,  $\tau(A, B)$  satisfies so called the cocycle condition, namely

$$\tau(A, B) + \tau(AB, C) = \tau(A, BC) + \tau(B, C),$$

so that  $\tau$  is a group 2-cocycle of the Siegel modular group and represents the cohomology class  $-4c_1 \in H^2(\mathrm{Sp}(2g, \mathbf{Z}), \mathbf{Z})$ . We call this 2-cocycle  $\tau$  the signature cocycle [21].

**Remark 2.1.** (i) The signature cocycle  $\tau$  is defined for  $\mathrm{Sp}(2g, \mathbf{Z})$ , but we can regard it as a 2-cocycle of  $\mathcal{M}_g$  in terms of the homology representation  $r$ . We use the same letter  $\tau$  for simplicity. (ii)  $\tau(A, B)$  just presents  $\sigma(W)$ , the signature of the 4-manifold  $W$ , which is a surface bundle over the pair of pants with three boundary components  $M_A$ ,  $M_B$  and  $-M_{AB}$ , where  $M_A$  denotes a mapping torus with monodromy  $A$ . (iii) By definition,  $\tau$  is a bounded 2-cocycle. In fact, the range of  $|\tau|$  is clearly bounded by  $2g$ .

Now in the following, we consider the hyperelliptic mapping class group of a surface  $\Sigma_g$ . We fix an involution  $\iota \in \mathcal{M}_g$  with  $2g + 2$  fixed points. The centralizer

$$\Delta_g = \{f \in \mathcal{M}_g \mid f\iota = \iota f\}$$

is called the hyperelliptic mapping class group of  $\Sigma_g$  with respect to  $\iota$ . The conjugacy class of  $\Delta_g$  in  $\mathcal{M}_g$  does not depend on a choice of  $\iota$ .

As was mentioned in the introduction, the rational cohomology groups of  $\Delta_g$  are trivial for dimensions 1 and 2. Hence the cohomology class represented by  $\tau$  is a

torsion element in  $H^2(\Delta_g, \mathbf{Z})$ . More precisely, we can show that the order of  $[\tau]$  in  $H^2(\Delta_g, \mathbf{Z})$  is  $2g + 1$  (see [9]). Thus there exists the uniquely defined mapping

$$\phi : \Delta_g \rightarrow \frac{1}{2g+1}\mathbf{Z}$$

so that the coboundary of  $\phi$  coincides with the restriction of  $\tau$  to  $\Delta_g$ , where  $(1/(2g+1))\mathbf{Z}$  denotes an additive group  $\{n/(2g+1) \in \mathbf{Q} \mid n \in \mathbf{Z}\}$ . Namely, we have

$$\delta\phi(f_1, f_2) = \phi(f_2) - \phi(f_1 f_2) + \phi(f_1) = \tau(f_1, f_2)$$

for  $f_1, f_2 \in \Delta_g$ . We call it Meyer's function of the hyperelliptic mapping class group.

**Remark 2.2.** (i) Meyer's function  $\phi$  is a class function of  $\Delta_g$ . That is, for two elements  $f_1, f_2 \in \Delta_g$ ,  $\phi(f_2 f_1 f_2^{-1}) = \phi(f_1)$  holds. Consequently, we can regard  $\phi$  as an invariant of surface bundles over the circle. (ii) The relation  $\delta\phi = \tau|_{\Delta_g}$  ( $g \geq 2$ ) implies that  $\phi$  is a homomorphism on the Torelli group  $\mathcal{I}_g \cap \Delta_g$ , because  $\tau$  is originally defined on the Siegel modular group  $\mathrm{Sp}(2g, \mathbf{Z})$ .

The group presentation of  $\Delta_g$  was given by Birman and Hilden (see [5]). By using its defining relations, we can determine the values of  $\phi$  on generators of  $\Delta_g$ .

**Example 2.3.** For the generators  $\zeta_1, \zeta_2, \dots, \zeta_{2g+1}$  of  $\Delta_g$  ( $g \geq 2$ ), the values of Meyer's function  $\phi$  are equal to  $(g+1)/(2g+1)$ . This is shown as follows. First each generator is mutually conjugate (in fact,  $\zeta_{i+1} = \xi \zeta_i \xi^{-1}$  holds for  $\xi = \zeta_1 \cdots \zeta_{2g+1}$ ), so that  $\phi$  has the same value on generators. Moreover substituting a defining relation of  $\Delta_g$  for  $\delta\phi = \tau$ , we can evaluate  $\phi(\zeta_i)$  explicitly.

**Example 2.4.** Let  $\psi_h \in \Delta_g$  ( $1 \leq h \leq g-1$ ) be a BSCC-map of genus  $h$ . Namely, it is a Dehn twist along a bounding simple closed curve on  $\Sigma_g$  which is invariant under the action of the hyperelliptic involution  $\iota$  and separates  $\Sigma_g$  into two subsurfaces of genus  $h$  and  $g-h$ . Then  $\psi_h$  is presented by  $\psi_h = f(\zeta_1 \cdots \zeta_{2h})^{4h+2} f^{-1}$  for some automorphism  $f \in \Delta_g$ . Thus we have  $\phi(\psi_h) = -4h(g-h)/(2g+1)$ .

**Remark 2.5.** (i) There exist other constructions of Meyer's function of genus 2, which are due to Kasagawa [12] and Iida [11]. (ii) Meyer's function plays an important role in the study of certain singular fibrations. In fact, Matsumoto [20] and Endo [9] defined the local signature of hyperelliptic Lefschetz fibrations by using Meyer's function. See also [18] for Meyer's function of plane curves.

### 3. ETA-INVARIANT

In this section, we discuss a relation between Meyer's function and the  $\eta$ -invariant of a surface bundle over the circle. Let  $M$  be an oriented closed Riemannian 3-manifold. By using the spectrum of the signature operator, the  $\eta$ -invariant of  $M$  is defined to be the value at zero of the  $\eta$ -function associated with the signature operator. Instead of giving the precise definition of the  $\eta$ -invariant, we recall the following index theorem due to Atiyah, Patodi and Singer (see [4]). Let  $W$  be a compact oriented Riemannian 4-manifold with product metric near the boundary  $\partial W = M$ . Then the  $\eta$ -invariant of  $M$  is given by

$$\eta(M) = \frac{1}{3} \int_W p_1 - \sigma(W),$$

where  $p_1$  is the first Pontrjagin form for the Riemannian connection on the tangent bundle  $TW$  of  $W$ . As is well-known, if  $W$  is a closed 4-manifold, then the signature  $\sigma(W)$  is described by the integral of  $p_1$ . That is, the above formula gives the extension of the Hirzebruch signature formula to 4-manifold with boundary. Here we have to remark that the  $\eta$ -invariant is not a topological invariant, but a spectral invariant.

Now for an automorphism  $f \in \mathcal{M}_g$ , let  $M_f$  be the mapping torus corresponding to  $f$ . Namely, it is the identification space  $\Sigma_g \times \mathbf{R}/(x, t) \sim (f(x), t + 1)$ . When  $f \in \mathcal{M}_g$  is of finite order (or a periodic automorphism), we endow  $M_f$  with the metric which is induced from the product of the standard metric on the circle and an  $f$ -invariant metric of  $\Sigma_g$ . If we restrict ourselves to the hyperelliptic mapping class group  $\Delta_g$ , we obtain the following theorem.

**Theorem 3.1.**  *$\eta(M_f) = \phi(f)$  holds for any periodic automorphism  $f \in \Delta_g$ .*

This theorem is shown by using an explicit formula of the  $\eta$ -invariant for a periodic automorphism of the mapping class group  $\mathcal{M}_g$  (see [23]). More precisely, it holds that

$$\eta(M_f) = \frac{1}{n} \sum_{k=1}^{n-1} \tau(f, f^k)$$

for  $f \in \mathcal{M}_g$  of the order  $n$ . The formula is based on the index theorem, due to Atiyah, Patodi and Singer. The point is to use the notion of 2-framing of 3-manifolds, which is a trivialization of twice the tangent bundle  $2TM = TM \oplus TM$  as a  $\text{Spin}(6)$ -bundle (see [3] for details).

**Example 3.2.** We can explicitly calculate all the values of Meyer's function  $\phi$  on periodic automorphisms for the genus 1 and 2 cases by using the homological action of each periodic automorphism (see [24]). In particular, we notice a relation to the Nielsen-Thurston theory of surface automorphisms [6]. As is known, the theory classifies the automorphisms of  $\Sigma_g$  into the following three types: (i) periodic, (ii) reducible and (iii) pseudo-Anosov. We say an automorphism of  $\Sigma_g$  is reducible, if it has a representative which leaves some essential 1-submanifold of the surface invariant. We then easily see that (i) and (ii) have some overlap, although (iii) cannot have any intersection with others. Hence, a given periodic automorphism is either reducible or irreducible. In fact, there are several works concerning the characterization of the reducibility of periodic automorphisms (see [13], [25] and their references).

From direct computations of Meyer's function, if the genus of  $\Sigma_g$  is 1 or 2, then we obtain the following. That is, a periodic automorphism  $f$  is reducible if and only if  $\phi(f) = 0$  (i.e.  $\eta(M_f) = 0$ ). Thus we can say that the reducibility of periodic automorphisms of a surface with low genus is characterized by the vanishing of Meyer's function or the  $\eta$ -invariant. In general, the same statement as above does not hold for higher genus case, but we can get the similar result for automorphisms of prime orders of hyperelliptic Riemann surfaces (see [26]).

Now as an immediate corollary of Theorem 3.1, we have the next assertion.

**Corollary 3.3.** *Let  $f \in \mathcal{M}_g$  be a periodic automorphism. If  $f \in \Delta_g$ , that is,  $f$  commutes with the fixed hyperelliptic involution  $\iota$ , then  $\eta(M_f) \in (1/(2g+1))\mathbf{Z}$  holds.*

Thus in some sense, we may regard the  $\eta$ -invariant as an obstruction for distinguishing a given periodic automorphism to be hyperelliptic or not.

**Example 3.4.** Let  $f \in \mathcal{M}_3$  be a periodic automorphism of order 3 so that the quotient orbifold of  $\Sigma_3$  by its cyclic action is homeomorphic to  $S^2(3, 3, 3, 3, 3)$  (2-sphere with five cone points of index 3). Then an easy computation shows that the  $\eta$ -invariant of corresponding mapping torus is given by

$$\eta(M_f) = -\frac{2}{3} \notin \frac{1}{7}\mathbf{Z}.$$

Hence, Corollary 3.3 implies that  $f$  cannot commute with the hyperelliptic involution  $\iota$ .

**Remark 3.5.** Let  $G$  be the finite cyclic group generated by a periodic automorphism  $f \in \mathcal{M}_g$ . Since  $H^*(G, \mathbf{Q}) = 0$  for  $* = 1, 2$ , we can define Meyer's function  $\phi_G : G \rightarrow \mathbf{Q}$  of  $G$ . Thereby as an analogue of Theorem 3.1, we see that  $\eta(M_f) = \phi_G(f)$  holds for any periodic automorphism  $f \in \mathcal{M}_g$  (see [1]).

#### 4. VON NEUMANN RHO-INVARIANT

Next we discuss a relation to the von Neumann  $\rho$ -invariant. Let  $\Gamma$  be a discrete group and  $M$  an oriented closed Riemannian 3-manifold. Moreover, we assume that we are given a surjective homomorphism from the fundamental group  $\pi_1 M$  to  $\Gamma$ . Then we can take a  $\Gamma$ -covering  $\Gamma \rightarrow \hat{M} \rightarrow M$ . Lifting the metric and the signature operator to  $\hat{M}$ , the  $\eta$ -invariant of  $\hat{M}$ , denoted by  $\eta^{(2)}(\hat{M})$ , is defined. We call it the von Neumann  $\eta$ -invariant. It is also known that  $\eta^{(2)}(\hat{M})$  fits an index theorem as before.

The  $\eta$ -invariant and the von Neumann  $\eta$ -invariant depend on the Riemannian metric, but Cheeger-Gromov showed in [7] that the difference of them is independent of the Riemannian metric. We denote the difference  $\eta^{(2)}(\hat{M}) - \eta(M)$  by  $\rho^{(2)}(\hat{M})$  and call it the von Neumann  $\rho$ -invariant.

In a certain sense, this invariant  $\rho^{(2)}$  is an extension of the classical  $\rho$ -invariant due to Atiyah-Patodi-Singer. That is, the difference between the  $\eta$ -invariant  $\eta_\gamma$  twisted by a unitary representation  $\gamma : \pi_1 M \rightarrow U(n)$  and the original  $\eta$ -invariant does not depend on the choice of the Riemannian metric.

For an automorphism  $f \in \Delta_g$ , we consider the  $\mathbf{Z}$ -covering  $\mathbf{Z} \rightarrow \hat{M}_f \rightarrow M_f$  associated with a surjective homomorphism  $\pi_1 M_f \rightarrow \pi_1 S^1 \cong \mathbf{Z}$ . A simple observation tells us that Meyer's function is not multiplicative for coverings. But if we take a limit, we see that it is related to the von Neumann  $\rho$ -invariant. Namely, we have

$$\textbf{Theorem 4.1.} \quad \rho^{(2)}(\hat{M}_f) = \lim_{k \rightarrow \infty} \frac{\phi(f^k) - k\phi(f)}{k}.$$

To be more precise, the von Neumann  $\rho$ -invariant  $\rho^{(2)}(\hat{M}_f)$  is given by the limit of the deviation from the multiplicativity of  $\phi$  for a finite covering  $M_{f^k} \rightarrow M_f$ . This is shown by using Propositions 2.2 and 3.1 in [23] and the approximation theorem of the  $\eta$ -invariant, due to Vaillant [35] and Lück-Schick [19], which states that

$$\eta^{(2)}(\hat{M}) = \lim_{k \rightarrow \infty} \frac{\eta(M_{(k)})}{[\Gamma : \Gamma_k]}$$

holds for a descending sequence of normal subgroups  $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$  such that  $[\Gamma : \Gamma_k] < \infty$  and  $\cap_k \Gamma_k = \{1\}$ , and  $\Gamma/\Gamma_k$ -covering  $M_{(k)} = \hat{M}/\Gamma_k \rightarrow M$ .

**Example 4.2.** As an example, we consider the genus one case. In this case, an element  $A \in SL(2, \mathbf{Z})$  is classified into the following three cases:

(i) Elliptic case (namely,  $|\text{tr } A| < 2$ ). Let  $A_n \in SL(2, \mathbf{Z})$  have the order  $n$  ( $n = 3, 4, 6$ ). We can take

$$A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

An easy calculation shows that

$$\begin{aligned} \phi(A_3) &= -\phi(A_3^2) = -2/3, \quad \phi(A_4) = -\phi(A_4^3) = -1, \quad \phi(A_4^2) = 0, \\ \phi(A_6) &= -\phi(A_6^5) = -4/3, \quad \phi(A_6^2) = -\phi(A_6^4) = -2/3, \quad \phi(A_6^3) = 0. \end{aligned}$$

Needless to say,  $\phi(A_n^n) = \phi(I) = 0$  holds. Hence we have

$$\rho^{(2)}(\hat{M}_{A_n}) = \begin{cases} 2/3, & n = 3, \\ 1, & n = 4, \\ 4/3, & n = 6 \end{cases}$$

in terms of Theorem 4.1. It should be noted that  $\rho^{(2)}(\hat{M}_f) = 0$  for any involution  $f \in \mathcal{M}_g$  (see [23], [28]).

(ii) Parabolic case (namely,  $|\text{tr } A| = 2$ ). We can take  $A_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  ( $b \in \mathbf{Z}$ ).

Then we obtain  $\rho^{(2)}(\hat{M}_{A_b}) = -\text{sgn}(b)$ , where  $\text{sgn}(b) = b/|b|$  if  $b \neq 0$  and 0 if  $b = 0$ . This result follows from Atiyah's calculation in [2].

(iii) Hyperbolic case (namely,  $|\text{tr } A| > 2$ ). Since Meyer's function of genus one satisfies  $\phi(A^k) = k\phi(A)$  for a hyperbolic element  $A \in SL(2, \mathbf{Z})$  (see [21]), we have  $\rho^{(2)}(\hat{M}_A) = 0$  by virtue of Theorem 4.1.

As an immediate corollary of Theorem 4.1 and Remark 2.2 (ii), we obtain the following for higher genus case.

**Corollary 4.3.** *If  $f$  is an automorphism in  $\mathcal{I}_g \cap \Delta_g$ , then  $\rho^{(2)}(\hat{M}_f) = 0$ .*

Here we have a remark. If we restrict the above theorem to the level 2 subgroup of  $\mathcal{M}_g$ , we can describe a relation among the von Neumann  $\rho$ -invariant, the first Morita-Mumford class  $e_1 \in H^2(\mathcal{M}_g, \mathbf{Z})$  (see [29], [33] and Section 5 below) and the Rochlin invariant of a spin 3-manifold in a framework of the bounded cohomology [10]. Roughly speaking, the pull-back of  $e_1$  into  $H_b^2(S^1, \mathbf{Z}) \cong \mathbf{R}/\mathbf{Z}$  via a holonomy homomorphism is given by the linear combination of the Rochlin invariant and the von Neumann  $\rho$ -invariant (see [17], [28] for details). We describe it more explicitly in the appendix of the present paper.

## 5. CASSON INVARIANT

In this last section, we explain a relation between Meyer's function and the Casson invariant. The Casson invariant  $\lambda(M)$  is an integer valued invariant defined for an oriented integral homology 3-sphere  $M$ . Roughly speaking, it counts the number of conjugacy classes of irreducible representations of the fundamental group  $\pi_1 M$  into the Lie group  $SU(2)$ .

On the other hand, from the theory of characteristic classes of surface bundles, due to Morita (see [30], [31]), the Casson invariant  $\lambda$  can be interpreted as a secondary invariant associated with the first Morita-Mumford class  $e_1$ , through the

correspondence between elements of the mapping class group and 3-manifolds via Heegaard splittings.

Let us state more precisely. We define  $\mathcal{K}_g$  to be the subgroup of  $\mathcal{M}_g$  generated by Dehn twists along bounding simple closed curves on  $\Sigma_g$ . This is also a subgroup of the Torelli group  $\mathcal{I}_g$ , because clearly, the action of a generator of  $\mathcal{K}_g$  on the homology of  $\Sigma_g$  is trivial. Next we fix a Heegaard splitting of the 3-sphere  $S^3 = \mathcal{H}_g \cup_{\iota_g} -\mathcal{H}_g$ , where  $\mathcal{H}_g$  denotes the handle body of genus  $g$  and  $\iota_g$  is the gluing map of this splitting.

For an automorphism  $f \in \mathcal{K}_g$ , we construct a 3-manifold  $M^f$  by regluing two handle bodies via the composition  $\iota_g f$ . Then it is easy to see that the resulting manifold  $M^f$  is again a homology 3-sphere, because  $f$  acts on  $H_1(\Sigma_g, \mathbf{Z})$  trivially. Thereby we can evaluate its Casson invariant  $\lambda(M^f)$ . In short, Morita's results claim that there exists a homomorphism  $\lambda^* : \mathcal{K}_g \rightarrow \mathbf{Z}$  such that  $\lambda^*(f) = \lambda(M^f)$ . More precisely,  $\lambda^*$  consists of the sum of two homomorphisms. One is Morita's homomorphism  $d_0 : \mathcal{K}_g \rightarrow \mathbf{Q}$ , which is the core of the Casson invariant from the viewpoint of the mapping class group. The other is the Johnson homomorphism, which is interpreted as Massey higher products of mapping tori (see [16]).

Now assume  $g \geq 2$ . We then have

**Theorem 5.1.** *Meyer's function essentially coincides with Morita's homomorphism on  $\mathcal{K}_g \cap \Delta_g$ . To be more precise,*

$$\phi(f) = \frac{1}{3}d_0(f)$$

*holds for any automorphism  $f \in \mathcal{K}_g \cap \Delta_g$ .*

Therefore, in principle, we can say that the Casson invariant of an integral homology 3-sphere is determined by Meyer's function. Here, as an example, we evaluate Morita's homomorphism on a typical element in  $\mathcal{K}_g \cap \Delta_g$ .

**Example 5.2.** Let  $\psi_h \in \mathcal{K}_g \cap \Delta_g$  be a BSCC-map of genus  $h$ . Then the value of Morita's homomorphism on  $\psi_h$  is given by

$$d_0(\psi_h) = -\frac{12}{2g+1}h(g-h)$$

in terms of Theorem 5.1 and Example 2.4.

Finally let us review the definition of Morita's homomorphism very briefly. It is a secondary invariant associated with the first Morita-Mumford class  $e_1 \in H^2(\mathcal{M}_g, \mathbf{Z})$ .

Let  $e \in H^2(\mathcal{M}_{g,*}, \mathbf{Z})$  be the Euler class of the central  $\mathbf{Z}$ -extension  $\mathbf{Z} \rightarrow \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g,*}$ , where  $\mathcal{M}_{g,1}$  and  $\mathcal{M}_{g,*}$  denote the mapping class groups of  $\Sigma_g$  relative to an embedded disc  $D \subset \Sigma_g$  and a base point  $* \in D$  respectively. The center  $\mathbf{Z}$  is generated by the Dehn twist parallel to  $\partial D$ . We define  $e_1$  to be the Gysin image (integration along the fiber) of  $e^2$ . This is an element of  $H^2(\mathcal{M}_g, \mathbf{Z})$  and called the first Morita-Mumford class.

Now it is known that there exist two canonical 2-cocycles over  $\mathbf{Q}$  representing  $e_1$ . One is the signature cocycle  $\tau$  ( $e_1 = [-3\tau]$  holds) and the other is the intersection cocycle  $c$  (the latter one is defined once we fix a certain crossed homomorphism of  $\mathcal{M}_g$ ). Therefore, we have the uniquely defined mapping  $d : \mathcal{M}_g \rightarrow \mathbf{Q}$  so that  $\delta d = c + 3\tau$ . The uniqueness follows from the fact that  $\mathcal{M}_g$  is perfect for  $g \geq 3$ . We denote the restriction of  $d$  to the subgroup  $\mathcal{K}_g$  by  $d_0$ . Then Morita showed

that  $d_0$  does not depend on the choice of the crossed homomorphism and it serves a generator of  $H^1(\mathcal{K}_g, \mathbf{Z})^{\mathcal{M}_g}$ . This  $d_0$  is the map, which we have called Morita's homomorphism.

The point of the proof of Theorem 5.1 is to construct a crossed homomorphism of  $\mathcal{M}_g$  so that the restriction of it to the hyperelliptic mapping class group  $\Delta_g$  is zero map. Using it to define the intersection cocycle, we can obtain a relation between Meyer's function and Morita's homomorphism.

**Remark 5.3.** In [32], Morita gave an interpretation of  $d_0$  in terms of Hirzebruch's signature defect of certain framed 3-manifolds. Combining it and Theorem 5.1, we have a generalization of Atiyah's result mentioned in the introduction.

## 6. APPENDIX

We give here a description of the first Morita-Mumford class  $e_1$  via the Rochlin invariant and the von Neumann  $\rho$ -invariant in the bounded cohomology  $H_b^*$ .

First, we note that  $e_1$  is a bounded cohomology class (see Remark 2.1 (iii)) and consider  $e_1$  on  $\mathcal{M}_{g,*}$  rather than on  $\mathcal{M}_g$  for a technical reason. If we pull back  $e_1$  by a holonomy homomorphism  $f : \pi_1 S^1 \rightarrow \mathcal{M}_{g,*}$  of a surface bundle over the circle, it is clearly vanishing because  $H^2(S^1, \mathbf{Z}) = 0$ . However Kitano showed in [17] that  $e_1/48$ , which depends on the spin structure of  $\Sigma_g$ , makes sense as a bounded cohomology class in  $H_b^2(S^1, \mathbf{Z}) \cong \mathbf{R}/\mathbf{Z}$  and it is given by the Rochlin invariant  $\mu$  if the image of  $f$  is contained in the Torelli group  $\mathcal{I}_{g,*}$ . Combining Theorem 4.1 with the result of Kitano, we have the following on the level 2 subgroup  $\mathcal{M}_{g,*}(2) = \text{Ker}\{\mathcal{M}_{g,*} \rightarrow \text{Sp}(2g, \mathbf{Z}/2)\} \supset \mathcal{I}_{g,*}$ . That is, for  $f : \mathbf{Z} \rightarrow \mathcal{M}_{g,*}(2)$ , the pull-back  $f^*e_1/48 \in H_b^2(\mathbf{Z}, \mathbf{Z})$  is represented by  $\mu(M_f, \tilde{\alpha}) - \rho^{(2)}(\tilde{M}_f)/16 \in H^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z})$ . The correspondence between them is given by the isomorphism  $H_b^2(\mathbf{Z}, \mathbf{Z}) \cong H_b^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z}) \cong H^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z})$ .

Here let us review the definition of the Rochlin invariant briefly. Let  $(M, \alpha)$  be an oriented spin 3-manifold with a spin structure  $\alpha$ . Then there exists a compact oriented spin 4-manifold  $(W, \beta)$  such that  $\partial W = M$  and  $\beta|_M = \alpha$ . The Rochlin invariant  $\mu(M, \alpha) \in \mathbf{Q}/\mathbf{Z}$  is defined by

$$\mu(M, \alpha) = \frac{\sigma(W)}{16} \pmod{\mathbf{Z}}$$

and it does not depend on the choice of the 4-manifold  $W$  by virtue of Rochlin's theorem.

Now Miller and Lee show in [22] that the Rochlin invariant of a spin 3-manifold is a spectral invariant. To be more precise, it is described by the  $\eta$ -invariants as follows. Let  $W, M$  be as in Section 3 and assume further that  $W$  (hence also  $M$ ) is a spin manifold. Let  $\mathcal{D}$  denote the Dirac operator of  $M$  acting on the spinor fields. This is a self-adjoint elliptic operator, so that the  $\eta$ -invariant  $\eta_{\mathcal{D}}(M)$  is defined. We then have

$$\text{ind}(\mathcal{D}) = -\frac{1}{24} \int_W p_1 - \frac{1}{2} \{ \hbar + \eta_{\mathcal{D}}(M) \},$$

where  $\hbar$  is the dimension of the space of harmonic spinors on  $M$ . Combining the above formula and the index theorem mentioned in Section 3, we get

$$\sigma(W) + 8 \text{ind}(\mathcal{D}) = -\eta(M) - 4 \{ \hbar + \eta_{\mathcal{D}}(M) \}.$$



The basic spin representations  $\mathcal{S}^\pm$  of  $\text{Spin}(4)$  are quaternionic and hence the index of the Dirac operator is always even. We therefore obtain

$$\mu(M, \alpha) = -\frac{1}{16}\eta(M) - \frac{1}{4}\{\hbar + \eta_{\mathcal{D}}(M)\} \mod \mathbf{Z},$$

and it shows that the Rochlin invariant is a spectral invariant.

Let us fix a spin structure  $\alpha$  of  $\Sigma_g$  ( $g \geq 2$ ). For each automorphism  $f \in \mathcal{M}_{g,*}(2)$ , there exists the uniquely defined spin structure  $\tilde{\alpha}$  on  $M_f$  such that the restriction to each fiber is  $\alpha$  and to the  $S^1$ -orbit of  $*$   $\in \Sigma_g$  is the bounding spin structure. Applying the above formula to  $(M_f, \tilde{\alpha})$  and substituting it for our description of the first Morita-Mumford class, we can conclude that

$$f^*e_1/48 = -\frac{1}{16}\eta^{(2)}(\hat{M}_f) - \frac{1}{4}\{\hbar + \eta_{\mathcal{D}}(M_f)\} \mod \mathbf{Z}$$

holds for an automorphism  $f \in \mathcal{M}_{g,*}(2)$ .

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