FRAMING AND THE SELF-LINKING INTEGRAL

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Abstract. The Gauss self-linking integral of an unframed knot is not a knot invariant, but it can be turned into an isotopy invariant by adding a correction term which requires adding extra structure to the knot. We collect the different definitions/theorems/proofs concerning this correction term, most of which are well-known (at least as folklore) and put everything together in an accessible format. We then show simply and elegantly how these approaches coincide.

1. Introduction

In 1833 Carl Friedrich Gauss, in his investigation of electromagnetic theory, discovered an integral formula for the linking number of two space curves. If γ_0 and γ_1 are disjoint embeddings of S^1 into S^3— i.e two disjoint space curves, and if $\Phi : S^1 \times S^1 \to S^2$ is the map that assigns to each $(x, y) \in S^1 \times S^1$ the unit vector from $\gamma_0(x)$ to $\gamma_1(y)$, with ω defined as the volume form on S^2, the Gauss integral is

$$\frac{1}{4\pi} \int_{S^1 \times S^1} \Phi^* \omega$$

It is natural to ask what happens if we take γ_0 to equal γ_1, in other words if we want to find a knot invariant analogous to the linking number of a two component link. But in the case of a knot we run into the problem that $\Phi(x, x)$ is not defined (how do we define the direction from a point to itself?). So rather than Φ being a function from $S^1 \times S^1$ to S^2 it is instead a function from $C_2(S^1) := \{(x, y) \in S^1 \times S^1 \mid x \neq y\}$ (the configuration space of two points on S^1) to S^2. The natural way to transport the Gauss integral to the case of a knot is then

$$\eta(\gamma) := \frac{1}{4\pi} \int_{C_2(S^1)} \Phi^* \omega$$

The problem now is that since $C_2(S^1)$ is not a compact space, we are not guaranteed that the integral converges. There are two ways that we might try to solve this problem.

(1) We could compactify the configuration space, and examine by how much $\eta(\gamma)$ 'fails' to be invariant. By Stokes' Theorem, we find that this quantity depends on the boundary of the compactified configuration space. We seek to eliminate this boundary by pasting some extra discs D_0 and D_1 onto our

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space, thus renormalizing the integral. Our invariant will then be $\eta(\gamma)$ plus a correction term which will depend on a "swaddling" map β, the extension of Φ to the extra discs D_0 and D_1. The invariant thus constructed will depend on the initial curve and on a choice of a **homotopy class for the swaddling map** β.

(2) We may look at the linking number of two copies of the knot, when the second copy is "pushed off" to a distance of ε from the first copy, and calculate what happens as $\varepsilon \to 0$. But this linking number will depend on **which direction** we decided to push off the second copy of the knot in relation to the first copy at each point, which implicitly specifies a knot framing. As $\epsilon \to 0$, the limit will not necessarily be an integer, forcing the introduction of a framing dependant correction term which will turn out to be the total holonomy of the curve. The invariant thus constructed will depend on the initial curve, along with a choice of **framing** for it.

It is not at all clear at first what these two constructions should have to do with one another. The aim of this note is to present both approaches in a clear and accessible fashion, and to showing how they relate in basic differential geometric terms. We are not trying to say anything new per se, but rather to present definitions, facts, and proofs most of which are well known, at least as folklore, in a simple and accessible format.

1.1. **Historical remarks.** The importance of the Self-Linking Integral is that it is the most simple and basic example of presentation of a Vassiliev invariant as a configuration space integral. Moreover, as the work of Bott and Taubes [5] (see also [2]) shows, this integral plays a basic role as a correction term for anomalies in the definition of more general finite-type "self-linking invariants". In this regard, this invariant constitutes a basic ingredient in the understanding of the Chern-Simons invariants of knot theory.

The first effective 'renormalization' of the Gauss integral by adding a correction term was carried out by Calugareanu [6], and later by Pohl [11] in the case of a closed space curve with nowhere vanishing torsion.

Who the first person was to extend the invariant to curves that may have a non-vanishing torsion I do not know. The 'holonomy' construction (the second method above) appears the more common, and is used for instance by Polyakov [13] (see also Tze [15]), by Bott and Taubes [5] and by Bar-Natan [3]. Meanwhile, the swaddling construction (the first method mentioned above) is preferred by Dylan Thurston [14] and appears more recently in papers by Poirier ([12] and by Lescop [10]. The Poirier paper also gives a brief explanation for the equivalence between the two constructions ([12], remark 6.17).

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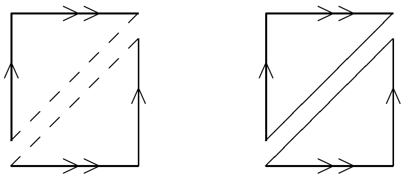
2. The swaddling map construction

Our problem when attempting to transport the Gauss self-linking integral to knots is what to do about points of the form (x, x), for which the Gauss map is not defined. We cannot simply ignore them, since this would force us to integrate over the space $C_2(S^1)$ which is not compact, so that the Gauss integral would not be guaranteed to converge. If we want to find an invariant based on the concept of self-linking, we have no choice but to extend the Gauss function to points of the form (x, x) in some way. The problem is that as points in $C_2(S^1)$ approach points on the diagonal, the Gauss map has two limits— the forward and the backward sweeping tangents.

2.1. **Compactifying the configuration space.** Let us define $\overline{C_2(S^1)}$, a compactification of $C_2(S^1)$, by pasting two copies of the diagonal, Δ_0 and Δ_1 to its missing diagonal $\{(x, x)|x \in S^1\}$, as shown in the diagram below.



Points on Δ_0, which are limits of the form $\lim_{y\to x^+}(x, y)$, shall be denoted (x, x^+), with points on Δ_1 correspondingly denoted (x, x^-). At these boundaries of $C_2(S^1)$, the Gauss map converges to the tangent vector to the curve, sweeping either forwards or backwards depending on whether its input converges to a point in Δ_0 or to a point in Δ_1. This allows us to solve the problem of how to extend the Gauss map to the diagonal. Φ can be extended smoothly to a function $\overline{\Phi_\gamma} : \overline{C_2(S^1)} \to S^2$, defined as $\overline{\Phi_\gamma}(x, x^\pm) := \pm\dot\gamma$ on the boundary.

2.2. **Checking invariance.** Let $H : S^1 \times I \to S^3$ be a one parameter family of curves. For $t \in I$, let us define

$$\eta_t(\gamma) := \frac{1}{4\pi}\int_{\overline{C_2(S^1)}\times\{t\}} \overline{\Phi^*}\omega \tag{2.1}$$

Invariance of η means that $\eta_0(\gamma) = \eta_1(\gamma)$ for all γ. But:

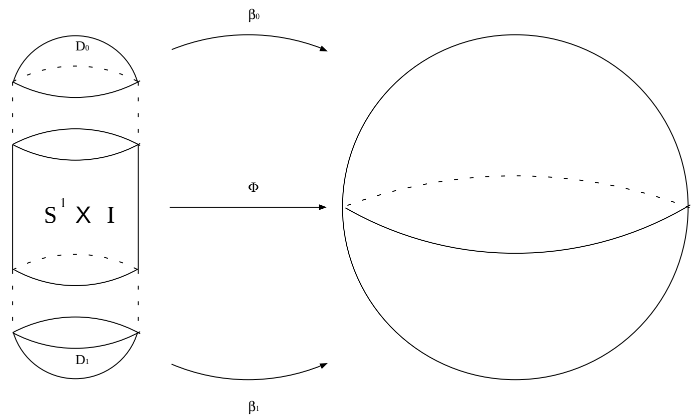
$$0 = \int_{\overline{C_2(S^1)}\times I} d\overline{\Phi^*_H}\omega = \int_{\partial(\overline{C_2(S^1)}\times I)} \overline{\Phi^*_H}\omega = \eta_1 - \eta_0 + 2\int_{S^1\times I} \overline{\Phi^*_H}\omega \tag{2.2}$$

The first equality holds because d and $\overline{\Phi^*_H}$ commute, and since ω is a 2-form defined on a 2-manifold, $d\omega = 0$. The second equality is the Stokes theorem. The third equality is simply the fact that $\partial\overline{C_2(S^1)} = 2S^1$.

From (2.2), we learn that $\eta_1 = \eta_0$ if and only if $\int_{S^1\times I} \overline{\Phi^*_H}\omega = 0$. But we have no reason to assume that this would generally be the case.

2.3. **Introducing a correction term.** Let us then "cap off" the cylinder $S^1 \times I$ by pasting two D^2's to it, making it isomorphic to S^2, as shown in the illustration below.

Let us define a "swaddling" map β as a continuous map which wraps S^2 in D^2 such that on the boundary $\beta|_{\partial D^2} = \dot\gamma$. In our case we have two such maps, β_0 and



β_1. $\overline{\Phi_H}$ maps the boundaries of $S^1 \times I$ in "antipodally"— i.e. if $x \times \{0\} \in S^1 \times \{0\}$ maps to $y \in S^2$, $x \times \{1\} \in S^1 \times \{1\}$ maps to $-y$. Let us now define β_0 as a map that maps the border of a disc to $\pm\dot{\gamma}$, with the sign corresponding to that of $\overline{\Phi_H}(S^1 \times \{0\})$, and β_1 to be $-\beta_0$. By abuse of notation, let us now define our total map β as the difference between these two maps. Our correction term will then be defined by the equation

$$\tau_\beta(\gamma) := \frac{1}{2\pi}\int_{D^2} \beta^*\omega$$

The motivation for this is that just like with γ, invariance of τ_β means that $\tau_{\beta,0}(\gamma) = \tau_{\beta,1}(\gamma)$ for all γ. But:

$$0 = \int\limits_{D^2\times I} d\beta^*\omega = \int\limits_{\partial(D^2\times I)} \beta^*\omega = \tau_{\beta,1} - \tau_{\beta,0} + 2\int\limits_{S^1\times I} \beta^*\omega$$

So τ_β is "at a distance" of $2\int\limits_{S^1\times I} \beta^*\omega$ from being invariant.

We have found that $\partial\tau_\beta = \partial\eta$, proving the following:

Theorem 2.1. *$\eta - \tau_\beta$ is an invariant of ordered pairs of a knot and an integer specifying the homotopy class for the map β.*

Indeed, a simple Stokes' theorem argument shows that for two homotopic β's give us the same τ_β (we can push the difference to the boundary, where the two β's will coincide). Moreover, our new invariant assumes integer values, because β wraps the disc around S^2 a whole number of times (it has to, since $\pi_2(S^2) = \mathbb{Z}$, and so $\int_{D^2} \beta^*\omega$ assumes values in $4\pi\mathbb{Z}$).

Thus we find that $sl(\gamma) := \eta(\gamma) - \tau_\beta(\gamma)$ (the "self-linking number" of γ) is an integer-valued invariant of closed space curves along with a choice of homotopy class of swaddling maps β. But just how much information are we adding about the knot when we specify a homotopy class for a swaddling map?

2.4. **Relating τ_β to the total torsion of a space curve.** In passing, we may note that for a curve with nowhere vanishing curvature, τ_β corresponds to the notion of the correction term for the self-linking number as it was first defined by Calugareanu and later by Pohl, as the total torsion of a space curve. In section 2.4, we discovered that the correction term τ_β is equal, modulo $\mathbb{Z}$, to the area on

S_2 covered by the map β. By the Gauss-Bonnet Theorem, this is equal to the total curvature of $\dot{\gamma}$. But if γ has nowhere vanishing curvature, then this equals simply the total torsion of the space curve γ.

3. The holonomy construction

In the previous section, we transported the concept of a linking number from links to knots by compactifying the configuration space and pasting pieces onto it in order to "force" the Gauss integral to converge and to give us an invariant. There is another way to approach the problem however. We already know that the linking number is a link invariant— well then, let's pretend that our knot is a link! If we take two copies of our knot that are only an ε apart, and then see what happens when ε goes to zero, we may utilize the known invariance of the linking number for links in order to directly conjure up an invariant for knots.

In the approach we are now taking, we have one 'stationary' curve, which we shall label γ_0, for which we choose a smooth framing $n(t)$. We then take the curve obtained made up of points $t + \varepsilon \cdot n(t)$ for $t \in \gamma_0$, with $0 < \varepsilon \in \mathbb{R}$, which we shall denote γ_ε. Now we let the 'mobile' γ_1 descent towards the 'stationary' γ_0. What we want to know is what happens the self-linking integral when they touch.

The classical approach here is to take the limit as $\varepsilon \to 0$ of the Gauss integral, which involves writing and partially calculating an explicit integral. This leads us to what physicists call the 'point-splitting regularization integral'. Tze [15] quotes a 'simplified approach' which he credits to Anshelevich, as quoted in an article about the twisting of strands of DNA [9] (!), which leads to the conclusion that the correction term must $\frac{1}{2\pi}$ times the total holonomy of the curve. But the same result may be obtained in a more elementary way by making use of a technique we have already used— that of a 'swaddling map'. This will also help us to visualize why and when the two constructions for the error term of the self-linking integral will coincide.

3.1. **Two Ways of Looking at the Same Thing.**

3.1.1. *Don't take limits- compactify!* Rather than thinking of limits of integrals, let's compactify the space of pairs

$$\mathbb{L}^{\overrightarrow{n}}(S^1) := \{(t, t + \varepsilon \cdot n(t)) \mid t \in \{\text{a closed space curve}\}, 0 < \varepsilon \in \mathbb{R}\}$$

by pasting something onto the boundary $\mathbb{L}_0^{\overrightarrow{n}}(S^1) := \overline{\mathbb{L}}^{\overrightarrow{n}}(S^1)\mid_{\varepsilon=0}$, where the overline denotes topological closure. Our problem, as usual, is that $\Phi(x, x)$ is not defined, and what this second approach gives is a way of defining it via a limit which keeps track of the information which is relevant to the Gauss integral— direction— and thus tells us what it wants $\Phi(x, x)$ to be.

Thus, the space we must paste on should consist of pairs (x, θ) in which we store the "address" x of the point, as well as the direction from which x is coming in on γ_ε. We may depict the newly created boundary of our compactified space as a continuum of pieces which can be schematically depicted like this:

The leftmost point of the semicircle corresponds to a point $\gamma_\varepsilon(t)$ coming in to γ_t on a backward sweeping tangent, the rightmost one corresponds to the point coming in on a forward sweeping tangent, and the apex corresponds to the point coming in straight off the normal. The tangent and the normal at each point define the framing for the knot, so that we see that we have not lost any information. Let us define

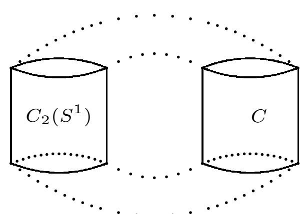
$$\tau_\phi := \frac{1}{2\pi}\int_{\mathbb{L}^{\vec{n}}(S^1)_0} \overline{\Phi^*_\gamma}\omega \tag{3.1}$$

Our main claim in this section is:

Theorem 3.1. *$\eta + \tau_\phi$ is an invariant of framed knots.*

To see that $\eta+\tau_\phi$ is in fact the limit of the integral of the pullback of the volume form, note that the domain at every $\varepsilon > 0$ where ε is fixed is cobordant to the integral at 'the bottom', where $\varepsilon = 0$, and hence via a standard Stokes' theorem argument, the "point-splitting regularization integral" used by [15] is the same as the integral along $\mathbb{L}_0^{\vec{n}}(S^1)$.

3.1.2. *"C-swaddling".* Let us eliminate the boundary of our configuration space this time in a different way— rather than pasting two discs onto the $S^1 \times I$ boundary, let us instead paste another $S^1 \times I$ to it (which we shall call C for cylinder), completing the cylinder $C_2(S^1)$ to a torus $C_2(S^1) \cup_\partial C$ as shown in the illustration below:



Then we define a new swaddling map $\phi: C \to S^2$ which maps the boundaries of C to the curves $\pm\dot{\gamma}$ in such a way that $\overline{\Phi_H}$ and ϕ combine to give us a continuous map $\overline{C_2(S^1)} \cup_\partial C \to S^2$.

As the title to this section suggests, we would like to show that in 3.1.1 and in 3.1.2 we have done one and the same thing (modulo 4π)— that in point of fact, there is no difference between compactifying the space as in section 3.1.1. by storing framing information on the boundary, and between eliminating the boundary of $\overline{C_2(S^1)}$ by the "C-swaddling" method as we have done in this section.

3.1.3. *Every framing gives a C-swaddling.* The integral on the bottom splits into two parts— the "normal" pieces in which we have just the standard Gauss self-linking integral, and the "bumps". On the bumps, θ goes from the tangent to the normal to minus the tangent— i.e. it traces out the image of a line between two points of the form $\{x\} \times \{1\}$, $\{x\} \times \{0\}$ on $S^1 \times I$, thus defining a C-swaddling, as the path can be assumed to be a great circle for all point on the knot.

We thus see that the "C-swaddling" construction gives us τ_ϕ, in just the same way as the construction in the first section gave us τ_β.

3.1.4. *Every C-swaddling gives a framing.* The fact that every C-swaddling gives a framing follows from the following lemma.

Lemma 3.2. *Every C-swaddling map is homotopic via C-swaddling maps to a map in which the path on S^2 from $\dot{\gamma}(s)$ to $-\dot{\gamma}(s)$ for each s in the knot is a great circle between the two points.*

Proof. Step 1 Every C-swaddling map is homotopic to a boundary-fixing diffeomorphisms from the cylinder to itself composed with a mapping from the cylinder to S^2 taking the boundaries of the cylinder to $\dot{\gamma}$ and to $-\dot{\gamma}$ and mapping the line $s \times \{i\}$, $i \in \mathbf{I}$ to a great circle from $\dot{\gamma}(s)$ to $-\dot{\gamma}(s)$ for each $s \in S^1$. Let us fix the second map in this composition and call it g. By [8] there is a $\mathbb{Z}$-worth of boundary-preserving diffeomorphisms of the cylinder $S^1 \times \mathbf{I}$ to itself up to homotopy by such diffeomorphisms. The generator of the this homotopy group is a Dehn twist about a boundary-parallel curve.

Step 2 Let p_s be the image of $s \times I$ in the S^2 under a power of the generator which we found in step 1. Let D_ε be a disc in S^2 of radius ε centred at the image of $s \times \{1\}$. The homotopy which fixes $s \times \{0\}$ and $s \times \{1\}$ and revolves $p_s \cap \partial D_\varepsilon$ in a full circle around ∂D_ε for all $s \in S^1$ undoes the Dehn twist.

Concretely, let's look at φ_t $t \in \mathbf{I}$ which takes p_s to a smoothing of the curve obtained by taking a radius of D_ε at angle $2\pi t$, then travelling around the circumference of D_ε clockwise until we hit p_s and continuing with p_s. $\varphi_1(p_s)$ can be taken by homotopy to a curve in the image of the cylinder on S^2, which is p_s with a Dehn twist added.

Step 3 Thus for each mapping g considered in step 1, the map from the cylinder has a single representative g composed with the identity. There are a $\mathbb{Z}$-worth of choices of g, corresponding to taking the path on the cylinder from $s \times i$, $i \in \mathbf{I}$, to the great circle which wraps around the circle n times for all $n \in \mathbb{Z}$.

□

Choosing the midpoint for each such great circle gives us a normal to the knot at s. In other words, given a family of forward-sweeping tangents to the knot, a C-swaddling map gives us a smooth family of normals to the knot, thus giving the knot a framing.

3.2. **Holonomy through C-swaddling.** To show that the correction term τ_ϕ that we get is the total holonomy, we must first represent $\phi^*\omega$ as the pullback of an element of $SO(3)$. For this purpose, as the ϕ swaddling map is a smooth extension of the Gauss map, let us redefine $\overline{\Phi_\gamma}(x, y)$ to be $\Phi_\gamma(x, y)$ when $x \neq y$ and ϕ on the boundary.

3.2.1. *Transporting the pullback of the volume form to SO(3).* Since we are now moving into SO(3), we shall convert the discussion into the language of framings. Let us break $\overline{\Phi_\gamma}(x, y)$ into the mapping ϕ from S^1 to $SO(3)$ composed on a mapping $e_1(x, y)$ from $SO(3)$ to S^2 . Following [11], let e_2 be the unit vector normal to e_1 on the plane spanned by e_1 and the tangent, extending smoothly to the normal defined by ϕ on the boundary. We shall then define e_3 to be $e_1 \times e_2$ at every point. The following lemma is due to William Pohl [11].

Lemma 3.3.

$$e_1^*\omega = d(de_3 \wedge e_2)$$

Proof. For every $1 \leq i_3$ in $\mathbb{N}$, e_i defines a function x_i by means of the relation $e_i^0 \cdot v = x_i(v)$ for any vector $v \in \mathbb{R}^3$, when e_i^0 denoted the restriction of e_i to the point $\overrightarrow{0}$. The volume form in $\mathbb{R}^3$ is then given by the expression

$$x_1(dx_2 \wedge dx_3) + \text{ cyclic permutations.}$$

Let us pull back the volume form via e_1. $e_i^* x_j = e_i \cdot e_j^0 = \delta_{i,j}$, therefore

$$e_1^*\omega = d(e_1^* x_2) \wedge d(e_1^* x_3) = de_1 \cdot e_2^0 \wedge de_1 \cdot e_3^0$$

by Leibnitz's rule. There was nothing special about our choice of 0 as the point by which to define the functions x_i, therefore we have $de_1 \cdot e_2 \wedge de_1 \cdot e_3$.

But $de_i \cdot e_i = 0$, and by differentiating this equality we find that $de_i \cdot e_j = -de_j \cdot e_i$ and so this equals

$$(de_3 \cdot e_1) \wedge (de_2 \cdot e_1) \tag{3.2}$$

Now we remember that the e^i's are an orthonormal to one another, and therefore they satisfy the equality

$$de_3 \cdot de_2 = \sum_{i=1}^{3} (de_3 \cdot e_i) \wedge (de_2 \cdot e_i)$$

But according to (3.1) this is exactly $e_1^*\omega$, and so we have found that

$$e_1^*\omega = de_3 \cdot de_2 = de_3 \cdot de_2 + d^2 e_3 \cdot e_2 = d(de_3 \wedge e_2)$$

□

3.2.2. *Relating τ_ϕ to the total torsion of a space curve.* In γ has nowhere vanishing curvature, we can use Lemma (3.2) to show that this correction term as well is equal to the total torsion of the curve γ. Here e_1 is the tangent, e_2 the normal, and e_3 the binormal, so

$$de_3 \cdot e_2 = db \cdot n$$

$-\tau \cdot n \cdot n = -\tau$, so by the Frenet equations, $b' \cdot n = -\tau$ so $db \cdot n = -\tau \cdot ds$.

Thus, we see that for a curve with a nowhere-vanishing curvature,

$$\tau_\phi = \int_{S^1} \tau ds$$

which is again the total torsion of the space curve γ.

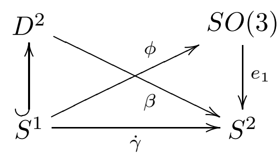
3.2.3. *Making sense of it all.* The last step of our argument is just the Stokes' theorem. The domain of $\overline{\Phi_\gamma}$ is our "cylinder compactification" of $C_2(S^1)$. Pulling back the volume form via this map, when restricted to C, will then by Stokes' Theorem be equivalent to pulling back $de_3 \wedge e_2$ via ϕ along C's boundary. But here ϕ gives us the tangent, e_2 the normal, and e_3 the binormal, $\phi^* de_3 \wedge e_2$ is the triple product $(\dot\gamma, n, \dot n)$. Here though we have $\int_{S^1} (\dot\gamma, n, \dot n) ds = \int_{S^1} (\dot\gamma, n, (n(s) + \dot n(s) ds)) = \int_{S^1} (\dot\gamma, n, \dot n(s + ds))$. The last integral is measuring "by how far" the normal has strayed from its initial position at $t = 0$ by the time we get to $t = L$. In other words, τ_ϕ is measuring the **total holonomy** of the curve γ, with respect to the Reimannian connection on the normal bundle to the curve.

3.3. **Equivalence to total torsion (again).** Here again we have an easy proof that the correction term of the self-linking integral equals the total torsion. When our curve has a Frenet frame (t, n, b) with curvature κ and torsion τ, the Frenet equations give us

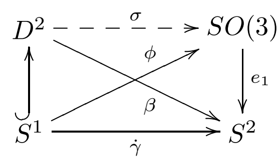
$$\tau_\phi = \frac{1}{2\pi} \int_{S^1} (\dot\gamma, n, \dot n) ds = \frac{1}{2\pi} \int_{S^1} (\dot\gamma, n, (\tau b - \kappa t)) ds = \frac{1}{2\pi} \int_{S^1} \tau ds$$

4. Equivalence of the two constructions

In the previous sections, we have presented two alternative ways of introducing a correction term to the Gauss self-linking integral for a knot, making it an invariant. We do not know yet whether these two methods are equivalent, and there is no reason a priori to assume that this should be the case. Why should choosing a homotopy class for a swaddling map have anything to do with choosing a framing for a knot? In both of these approaches, we reach the image on S^2 via the Gauss map of the tangent bundle to an embedding into S^3 of S^1, but in the first approach we come to this image by first embedding S^1 into D^2 and then getting to S^2 via the swaddling map β, while in the second approach we first map to $SO(3)$ by choosing a framing (we shall call this map ϕ), and then map down from there onto S^2. The situation is schematically depicted in the commutative diagram below:



In this notation, $\tau_\beta(\gamma) = \frac{1}{2\pi}\int_{D^2}\beta^*\omega$, while $\tau_\phi(\gamma) = \frac{1}{2\pi}\int_{S^1}\phi^*\tau$ for τ a pre-image of omega via the map e_1. Equality of these terms would follow from the existence of a map σ such that the following diagram commutes:



For in that case

$$\tau_\beta(\gamma) = \frac{1}{2\pi}\int_{D^2}\beta^*\omega = \frac{1}{2\pi}\int_{D^2}\sigma^* d\tau = \frac{1}{2\pi}\int_{S^1}\phi^*\tau = \tau_\phi(\gamma)$$

The second equality stems from the fact that the diagram is commutative, and the fourth we have already shown. So all the action takes place around the middle equality. We have shown that when $\beta^*\omega$ is transported to SO(3), it becomes the d of something. So we may use Stokes' theorem to go from left to right.

We can also see this easily from the swaddling map construction- let us choose a β mapping, pasting two discs onto the boundaries of $C_2(S^1)$, making it a compact space. Let us choose our discs such that corresponding points on D_1 and on D_0 map to antipodal points on S^2 via β. Cutting out a small neighbourhood of the centres of the discs, we may glue a cylinder between them, connecting them into a shape isomorphic to the cylinder on which our ϕ map was defined. Now every β map can be smoothly extended to a σ map, because the two discs with the narrow tube connecting them is homotopically a cylinder.

But as Tahl Novik observed, going from right to left in this set of equalities we have to watch out, because $\pi_1(SO(3)) = \mathbb{Z}/2$, and for a path belonging to the non-trivial homotopy class of $SO(3)$, there can exist no pre-image via a σ mapping.

Let us note that the cylinder of the C-swaddling construction can be 'cut' into 2 discs if and only if it is homotopic to a cylinder of which the 'middle circle' is constant- in other words as a framing it is homotopic to the constant framing.

Then and only then can we 'pinch closed' that sphere, turning the cylinder into two discs tangent at a point without loss of information. For elements of the non-trivial homotopy class, this is by definition going to be impossible. Notice that by 'pinching' the cylinder into discs, we are separating the backward sweeping tangents and the forward sweeping tangents, which is impossible in the non-trivial homotopy class in which these two families of tangent vectors are one and the same.

But for elements of the trivial homotopy class of $SO(3)$, no such difficulty arises. Stokes' theorem takes us from $\tau_\phi(\gamma)$ to $\tau_\beta(\gamma)$. We have proved then the following theorem:

Theorem 4.1. $\tau_\beta + 4\pi = \tau_\phi$

The "$+4\pi$" correction is an idea of Tomotada Ohtsuki's, to remind us that for 'minimal' representatives of ϕ and β the area ϕ covers on S^2 with the cylinder C (in this case 'minimal' would be taken to mean that each "vertical" line between the boundaries of the cylinder is mapped to the minimal length line between the tangent and minus the tangent on S^2, with appropriate sign) is the entire ball, plus the area β covers with the two discs (and here 'minimal' means simply the minimal such positive area). In any event, modulo 4π the correction terms are equal. The isomorphism between the two correction terms means that in a very real sense choosing a swaddling map β along with the homotopy class in which it sits is exactly the same thing as choosing a framing that is null-homotopic as an element of $SO(3)$. We have shown then that β can be lifted to σ, but only for 'half' our possible choices of ϕ.

4.1. **So which "half" is it?** We have shown then that for framings which give us an element of the trivial homotopy class of $SO(3)$, $\tau_\phi(\gamma) = \tau_\beta(\gamma)$. We have yet to show what framings those are.

Let us recall the mappings defined in section 2.3. $\overline{\Phi_H}$ mapped $S^1 \times I$ to S^2, sending the two components of the boundary to the tangent bundle of the knot γ in antipodal ways, while β_0 and β_1 took the boundary of a disc, and mapped it to $\dot\gamma$ and to $-\dot\gamma$ correspondingly. Thus, we have a $\mathbb{Z}_2$ action on $\overline{C_2(S^1)}$, whose action is to flip: $(x, y) \to (y, x)$. $\overline{\Phi_H}$ then descends to the quotient

$$\Phi_H : \overline{C_2(S^1)}/\mathbb{Z}_2 \to S^2/\mathbb{Z}_2 \simeq \mathbb{R}P^2$$

But $\overline{C_2(S^1)}/\mathbb{Z}_2$ is also just $\mathbb{R}P^2$, so Φ_H is in fact a map from $\mathbb{R}P^2$ to itself.

$\mathbb{R}P^2$ is a non-orientable space, therefore only the degree of $\overline{\Phi_\gamma}$ is only defined mod 2. But the flipping action $(x, y) \to (y, x)$ is precisely the non-trivial path in $\mathbb{R}P^2$, hence the degree of $\overline{\Phi_\gamma}$ must be 1 (this follows from the topological assertion that the degree of a map is π_1 of that map).

This gives us a complete characterization of the framings for which ϕ lifts to σ- they are exactly those framings for which the mod 2 degree of the 'extended Gauss mapping' $\overline{\Phi_\gamma}$ is 1. This leads us to the rather startling conclusion that, given the blackboard framing, our 'trivial' knot turns out not to be the circle at all, but rather the boundary of the Moebius band.

5. A combinatorial description of the invariant

By adding a correction term, we have shown that the Gauss self-linking integral can be made to be an invariant of framed knots. It so happens [4] that this invariant coincides with the so-called 'writhing number' of the curve, obtained by taking the number of positive crossings and subtracting the number of negative crossings. Thus, we have constructed an invariant analogous to the *linking number* of two disjoint space curves.

There is also another combinatorial description of our invariant, which is to my mind more appealing [11], [1]. Let us imagine the knot as a roller-coaster, with us sitting in a car facing forwards. At every point, the tracks face away from the knot in the direction of the normal vector, and as the car travels along the rails, our head is always pointing "up". Let us also assume that our head is locked in place, such that we can only look straight ahead (in the direction of the tangent).

The roller coaster starts up, and we start moving along the track. The car rises and falls, twists and loops, swooshing along. Every now and again, we may see another portion of track coming up directly into our field of vision— Pohl calls such points 'cross-tangents'. We count these with appropriate sign, depending on the orientation of the tracks (which way the car has gone down them or will go down them, and the direction in which we are currently travelling). The roller coaster stops when we return to our initial point, and we sum up all the cross-tangents, with appropriate signs. And we get what we have calculated in this paper— the Gauss self-linking integral with the appropriate correction term, determined by the direction the tracks faced away from the knot at each point.

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