ANNALS OF MATHEMATICS
Vol. 65, No. 3, May, 1957
Printed in U.S.A.

EQUIVARIANT EMBEDDINGS IN EUCLIDEAN SPACE

BY G. D. MOSTOW*

(Received June 4, 1956)

Section 1. Introduction

Let G be a group of transformations on a topological space E. If $p \in E$ we denote by G_p the set of transformations in G which keep p fixed. If H is a subgroup of G, we denote by (H) the totality of subgroups of G of the form xHx^{-1} with x in G. We denote by $L(G, E)$ the totality of (G_p) as p varies over E. The orbits Gp and Gq through points p and q of E are called equivalent if $(G_p) = (G_q)$. Thus G has but a finite number of inequivalent orbits in E if and only if $L(G, E)$ is a finite set. This is the case for example if E is a compact differentiable manifold and G is a compact group of differentiable transformations (cf. Section 7).

The main results are the following.

THEOREM 6.1. *Let G be a compact Lie group operating faithfully on a separable metric finite-dimensional space E. Assume G has only a finite number of inequivalent orbits in E. Then there exist a homeomorphism φ of E into a euclidean space E^n and an isomorphism θ of G into the unitary group on E^n such that φ is equivariant with respect to θ i.e. $\varphi(gp) = \theta(g)\,\varphi(p)$ for all $p \in E$, $g \in G$. Furthermore, if G has no fixed points on E, then θ, φ may be so chosen that $\theta(G)$ has no fixed points on E^n except the origin.*

THEOREM 2.1. *Let G be a compact Lie group of transformations on a completely regular space E. Then at each point p of E there exists a pseudo-section to the orbit through p.* (See Section 3 for definitions).

THEOREM 4.2. *Let G be a compact Lie group of transformations of a separable metric finite dimensional space E. Assume all the orbits are equivalent. Then there exists a finite set of local cross-sections whose orbits cover E.*

Theorem 2.1 on pseudo-sections is a more general version of a theorem first proved by Montgomery and Yang for spaces satisfying suitable connectivity conditions. The proof of Montgomery and Yang is strictly topological; in contrast, our proof hinges essentially on producing a suitable representation of the transformation group.

From the point of view of transformation groups, one can obtain quite directly some information about the conjugacy of subgroups of a compact Lie group. Thus we can obtain the result:

THEOREM 7.1. *In a compact Lie group, any set of (connected) analytic subgroups whose normalizers are mutually non-conjugate (under an inner automorphism) is finite. Any set of semi-simple analytic subgroups which are mutually non-conjugate is finite.*

* This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF 18 (600)-1474.

This result is useful in finding conditions under which $L(G, E)$ is finite. This question will be taken up in a future paper.

It is of interest to note that Theorem 2.1 yields as a consequence the result of Montgomery and Zippin that nearby closed subgroups of a compact Lie group are conjugate (see Corollary 3.2 in Section 3).

Section 2. Faithful representations of orbits

LEMMA 2.1. *Let H be a closed subgroup of the compact Lie group G. Then there is representation α of G by unitary transformations on the finite dimensional complex euclidean space E^n and a point $p \in E^n$ such that $\alpha^{-1}(\alpha(G)_p) = H$. If $H \neq G$, α can be so chosen that $\alpha(G)$ keeps only the origin fixed.*

PROOF. If $G = H$, the lemma is obviously true. We assume therefore $G \neq H$. For any compact Lie group F containing the closed subgroup H, there exists an irreducible representation β_F whose restriction to H contains the trivial unit representation (cf. CHEVALLEY, Theory of Lie Groups, vol. 1, Prop. 5, p. 192, p. 211). Taking F to be a closed subgroup of G properly containing H, the representation β_F is contained in the restriction to F of some representation of G (loc. cit. Prop. 4, p. 191). We denote this representation of G by α_F. Let V_F, E_F denote the representation spaces of β_F, α_F respectively. Select any point q other than the origin in each V_F and set $H_F = \alpha_F^{-1}(\alpha_F(G)_q)$. Set $K = \bigcap_F H_F$ (all $F \supset H$ properly). Then K is a compact subgroup of G containing H. If K contains H properly, then β_K is not the unit representation and thus

$$\beta_K(K)_q \neq \beta_K(K).$$

Now $H_K \cap K = \alpha_K^{-1}(\alpha_K(G)_q) \cap K$ is the totality of elements x of K with $\alpha_K(x)q = q$ and, since $q \in V_K$, coincides with $\beta_K^{-1}(\beta_K(K)_q)$. Thus $K \neq H$ implies $H_K \cap K \neq \beta_K^{-1}(\beta_K(K)) = K$; that is, K is not contained in H_K—a contradiction. Thus $H = K = \bigcap_F H_F$ (all $F \supset H$).

It is next to be observed that any (well-ordered) descending chain of compact subgroups of a compact Lie group is finite; for in a descending chain, only a finite number of subgroups of the same dimension can occur, and only a finite number of dimensions can occur. On the other hand, we can clearly well order a subset of the subgroups F containing H—say $F_1, F_2, \cdots, F_\alpha \cdots$ (α an ordinal less than γ) so as to obtain a strictly descending chain $H_1, H_2, \cdots H_\alpha, \cdots$ (all $\alpha < \gamma$) with the property $H = \bigcap_\alpha H_\alpha$ (all $\alpha < \gamma$). Hence γ is a finite ordinal $n + 1$ and $H = H_{F_1} \cap \cdots \cap H_{F_n}$. Set $\alpha = \alpha_{F_1} + \cdots + \alpha_{F_n}$, $E^n = E_{F_1} + \cdots + E_{F_n}$ (direct sum), and $p = (q_1, \cdots, q_n) \in E^n$ where q_i is a non zero element of V_{F_i} $(i = 1, \cdots, n)$. Then $\alpha^{-1}(\alpha(G)_p) = \bigcap_i \alpha_F^{-1}(\alpha_F(G)_{q_i}) = H_{F_1} \cap \cdots \cap H_{F_n} = H$, as asserted in the lemma.

If $H \neq G$, we could have selected the α_F in the construction above so as to not contain the trivial unit representation of G. For G being compact, α_F is a direct sum of irreducible representations; upon omitting from the sum the trivial representations, we obtain a representation whose restriction to F contains β_F but which does not contain the trivial representation. Selecting for each F such an α_F, we obtain an α which does not contain the trivial representation of G. Hence the only fixed point of $\alpha(G)$ in E^N is the origin.

DEFINITION. Let G be a compact group operating on a topological space E. A G-equivariant map of G into a finite dimensional complex or real euclidean space E^N is a continuous map φ of E into E^N together with a continuous homomorphism θ into the unitary group on E^N such that $\theta(g)\varphi(p) = \varphi(gp)$ for all $p \in E$, $g \in G$. A G-equivariant map is called a G-equivariant homeomorphism if the associated φ is a homeomorphism.

The associated θ of a G-equivariant homeomorphism is an isomorphism if the group G operates *faithfully* on E.

We remark that a complex euclidean space E^N can be identified in a natural way with a real euclidean E^{2N}, and that real euclidean E^N can be extended naturally to a complex euclidean E^N. These natural isomorphisms convert G-equivariant maps into complex euclidean space to G-equivariant maps into real euclidean space and vice-versa.

The following is a fundamental result about extensions of G-equivariant maps due to A. GLEASON (Proc. Amer. Math. Soc., v. 1 (1950), pp. 35–43).

GLEASON'S LEMMA. *Let G be a compact group operating on a completely regular* (resp. *normal*) *space E and let F be a compact* (resp. *closed*) *subset invariant under G. Then any G-equivariant map of F into E^N can be extended to a G-equivariant map of E into E^N.*

For the sake of completeness, we repeat the proof of this lemma. Let φ be a continuous map of F into E^N, and let θ be a homomorphism of G into the unitary group of E^N with $\varphi(gp) = \theta(g)\,\varphi(p)$ for all $p \in F$ and $g \in G$. Extend φ to a continuous map ψ of E into E^N (Tietze Extension Lemma). Set

$$\Phi(p) = \int_G \theta(g)^{-1}\psi(gp)\,dg,$$

for all $p \in E$. Then Φ gives the desired extension, since

$$\int_G \theta(g)^{-1}\psi(gp)^{-1}\,dg = \int_G \varphi(p)\,dg = \varphi(p) \quad \text{for} \quad p \in F$$

and

$$\Phi(g_1 p) = \int_G \theta(g)^{-1}\psi(gg_1 p)\,dg = \int \theta(gg_1^{-1})^{-1}\psi(gg_1^{-1}g_1 p)\,dg$$

$$= \int_G \theta(g_1)\theta(g)^{-1}\psi(gp)\,dg = \theta(g_1)\,\Phi(p)$$

for all $p \in E$.

THEOREM 2.1. *Let G be a compact Lie group of transformations on a completely regular space E. Let $p_1, \cdots, p_n$ be any finite set of points of E. Then there is a G-equivariant map (φ, θ) of E with φ a homeomorphism on the orbits through $p_1, \cdots$ p_n and $\theta(G)$ keeping only the origin fixed if G has no fixed point on E.*

PROOF. Set $H_i = G_{p_i}$. By Lemma 2.1, there is a representation α_i of G by unitary transformations on euclidean space E^{N_i} and a point $q_i \in E^{N_i}$ such that $\alpha_i^{-1}(\alpha_i(G)_{q_i}) = H_i$ and $\alpha_i(G)$ keeps only the origin fixed if $H_i \neq G$, $i = 1, \cdots, n$. Let $\varphi_i(gp)_i = \alpha_i(g)p_i$, $i = 1, \cdots, n$. Then φ_i is a homeomorphism of the orbit through p_i. Set $E^N = E^{N_1} + \cdots + E^{N_n}$ (direct), and identify E^{N_i} with a subspace of E^N in the natural way. Let $\varphi(p) = \varphi_i(p)$ for p in the orbit Gp_i through p_i $(i = 1, \cdots, n)$, set $\theta = \alpha_1 + \cdots + \alpha_n$ and set $F = Gp_1 \cup \cdots \cup Gp_n$. Then (φ, θ) is clearly a G-equivariant homeomorphism of F into E^N. By Gleason's lemma, (φ, θ) can be extended to a G-equivariant map of E into E^N, which we denote by (φ, θ) also. If G has no fixed points on E, then $H_i \neq G$ for all i and hence $\theta(G)$ keeps only the origin of E^N fixed.

Section 3. Existence of pseudo-sections

DEFINITION. Let G be a compact Lie group of transformations on a topological space E, and let $p \in E$. A pseudo-section to the orbit Gp at p is a closed subset K containing p satisfying: (1) K is invariant under the isotropy group G_p; (2) there exists a continuous cross-section map f into G of a neighborhood U of the coset G_p in G/G_p such that the mapping $(u, q) \to f(u)q$ is a homeomorphism of the product space $U \times K$ onto a neighborhood of p; (3) $gK \cap K$ is empty if $g \in G - G_p$.

A pseudo-section is closely related to a notion employed by Koszul (Colloques Intern. de CNRS, Strassbourg, 1953 pp. 137–41) and its existence has been proved by him in the case of compact groups of differentiable transformations on a differentiable manifold.

In case all the orbits in E are equivalent, we call a pseudo-section at p a "local cross-section at p".

If K is a local cross-section at p to the orbit through p, then gK is disjoint from K for any element g of G which is not in the isotropy subgroup G_p. Thus $G_q \subset G_p$ for all q in K. Since no compact Lie group is conjugate to a proper subgroup of itself, $G_q = G_p$ for all q in K. As a result distinct points of K belong to distinct orbits. A local cross-section at p can be characterized as a closed subset K such that (1) distinct points of K lie in distinct orbits (2) $G_p = G_q$ for each q in K (3) GK is a neighborhood of p. Thus a local cross-section K at p is a local cross-section at all of its points which are interior to the set GK. We define a "local cross-section" to be a local cross-section at *some* point.

If K is a pseudo-section at p and $g \in G$, then gK is a pseudo-section at gp.

LEMMA 3.1. *Let G, G' be compact Lie groups of transformations on the space E, E' respectively. Let φ be a continuous map of E into E', let θ be a homomorphism of G into G', and assume $\varphi(gq) = \theta(g)\varphi(q)$ for all $g \in G$, $q \in E$. Let $p \in E$, let K' be a pseudo-section to the orbit $G'\varphi(p)$ at $\varphi(p)$ and assume φ is* 1–1 *on the orbit Gp. Then $K = \varphi^{-1}(K')$ is a pseudo-section to the orbit Gp at p.*

PROOF. By hypothesis, $G'_{\varphi(p)}K = K$ and there exists a continuous cross-section map f of an open neighborhood U of $G'_{\varphi(p)}$ in $G'/G'_{\varphi(p)}$ such that $F'\colon (u, q) \to f(u)q$ is a homeomorphism of $U \times K'$ onto a neighborhood N' of $\varphi(p)$. Inasmuch as

φ is one-to-one on the orbit G_p, $\theta^{-1}(G'_{\varphi(p)}) = G_p$ and θ induces a homeomorphism of G/G_p onto $G'/G'_{\varphi(p)}$; we identify G/G_p with $G'/G'_{\varphi(p)}$. To verify that K is a pseudo-section, we observe first that $G_pK = \theta^{-1}(G'_{\varphi(p)})\varphi^{-1}(K') = \varphi^{-1}(G'_{\varphi(p)}K') = \varphi^{-1}(K') = K'$. Next, $F:(u, q) \to f(u)q$ is clearly a continuous map of $U \times K$ into E and it is one-to-one also; for if u_1, u_2 and q_1, q_2 are distinct elements in U and K respectively, then $\varphi(f(u_i)q_i) = \theta(f(u_i))\varphi(q_i)$ are distinct elements of N since $\theta(f(u_i))$ and $\varphi(q_i)$ $(i = 1, 2)$ are distinct elements of G' and K' respectively. Finally, F is an open mapping of $U \times K$ onto the neighborhood $\varphi^{-1}(N')$ of p since $F = \varphi^{-1}F'$, and $gK \cap K$ is empty if $g \in G - G_p$. Thus K is a pseudo-section.

Lemma 3.2. *Let G be a compact group of linear transformations of the real or complex finite dimensional linear space V. For any $v \in V$, there exists a pseudo-section at v to the orbit through v.*

Proof. By the well-known unitary trick, an inner product may be introduced on V which is preserved by the elements of G. For any $v \in V$, the orbit Gv is a submanifold of V. Let L denote the affine subspace perpendicular to the tangent plane to Gv at v. Clearly L is invariant under G_v. Let $\mathfrak{G}$ and $\mathfrak{G}_v$ denote the Lie algebra of G and the Lie subalgebra of G_v respectively. Let $X_1, \cdots, X_n$ be a base for the Lie algebra $\mathfrak{G}$ with $X_{s+1}, \cdots, X_n$ a base for $\mathfrak{G}_v$. Let $\mathfrak{C}$ denote the linear subspace spanned by $X_1, \cdots, X_s$, let γ denote the map $t_1X_1 + \cdots + t_nX_n \to \exp t_1X_1 \exp t_2X_2 \cdots \exp t_nX_n$ of $\mathfrak{G}$ into G, and let $\mathfrak{W}$ be a neighborhood of zero in $\mathfrak{G}$ on which the map γ is one-to-one and regular. Since G_v is a closed subgroup of G, $\gamma(\mathfrak{W}) \cap G_v = \gamma(\mathfrak{W} \cap \mathfrak{G}_v)$ for $\mathfrak{W}$ suitably small. Selecting such a small $\mathfrak{W}$, we deduce that the projection π of G onto G/G_v maps $\gamma(\mathfrak{W} \cap \mathfrak{C})$ homeomorphically and bi-differentiably onto a neighborhood U of the coset G_v in G/G_v. Set $f(\pi(g)) = g$ for $g = \gamma(\mathfrak{W} \cap \mathfrak{C})$ and set $F(u, q) = f(u)q$ for $u \in U$, $q \in L$. The map F is differentiable and regular at the point (G_v, v) of $(G/G_v) \times L$ and hence by the implicit function theorem F is a homeomorphism of a neighborhood $U \times K_1$ of (G_v, v) onto a neighborhood of v in V. Since $G_vv = v$ and G_v preserves distance, there is a neighborhood K_2 of v in K_1 which is invariant under G_v. Since F is a homeomorphism, $gK_2 \cap K_2$ is empty for $g \in \pi^{-1}(U) - G_v$. Let e denote the minimum distance between v and $g\,v$ for $g \in G - \pi^{-1}(U)$, and let K be closed ball in K_2 with center v and radius $e/4$. Then $gK \cap K$ is empty for all $g \in G - G_v$. Hence K is a pseudo-section at v.

Theorem 3.1. *Let G be a compact Lie group of transformations on a completely regular space E. Then at each point p of E, there exists a pseudo-section to the orbit through p.*

Proof. Let $p \in E$. By Theorem 2.1, there is a G-equivariant map (φ, θ) of E into some E^N with φ one-to-one on the orbit through p. By Lemma 3.2, there exists a pseudo-section K' at $\varphi(p)$ to the orbit through $\varphi(p)$. Set $K = \varphi^{-1}(K')$. By Lemma 3.1, K is a pseudo-section at p to the orbit through p.

Note. The hypothesis that G be compact is not superfluous. It is easy to find examples of groups of linear transformations which do not admit pseudo-sections.

Corollary 3.1. *Let G be a compact Lie group of transformations on a com-*

pletely regular space E, let U be a neighborhood of the identity in G, and let $p \in E$. Then there is a neighborhood N of p such that for each $q \in N$, $G_q \subset gG_pg^{-1}$ with $g \in U$.

PROOF. Let K be a pseudo-section to the orbit Gp at p. Then gK is disjoint from K for g not in G_p. Hence $G_q \subset G_p$ for $q \in K$. Since $G_{gq} = gG_qg^{-1}$, the neighborhood UK has the desired property.

NOTE. We could have taken for N the neighborhood UL where L is the set of all $q \in E$ with $G_q \subset G_p$.

From the foregoing we deduce the following result of MONTGOMERY and ZIPPIN, *A Theorem on Lie Groups*, Bull. Amer. Math. Soc., v. 48 (1942), pp. 448–452.

COROLLARY 3.2. *Let G be a compact Lie group, let U be a neighborhood of the identity in G, and let H be a closed subgroup of G. There exists a neighborhood V of the identity such that any subgroup in the subset VH is conjugate to a subgroup of H by an element in U.*

PROOF. Let E be the set of all closed subsets of G topologized by the metric $d(A, B) = \sup c(p, B) + \sup c(A, q)$, where $A \in E$, $B \in E$, and $c(p, q)$ is a right invariant metric on the compact group G. The group G operates on E by left translation and the map $(g, A) \to gA$ of $G \times E$ into E is continuous. Clearly the isotropy subgroup of the point $H \in E$ is the subgroup H, i.e. $G_H = H$. Let L be the set of all points A in E with $G_A \subset G_H$ and set $N = UL$. Then as remarked above, N is a neighborhood of H in E and therefore contains a ball with center H and radius d_0.

Let V be a closed ball with center at the identity of G and radius d_0. If F is a closed subset of VH which meets H, then

$$d(FH, H) = \sup_{\substack{f \in F \\ h \in H}} c(fh, H) + \sup_{h \in H} c(FH, h) = \sup_{f \in F} c(f, H)$$

$$\leqq \sup_{g \in VH} c(g, H) \leqq \sup_{g \in V} c(g, H) \leqq d_0$$

and consequently $FH \in N$.

Suppose now that F is a subgroup in VH. Since $\bar{F}$ is a closed subgroup in VH, no generality is lost when we assume that F is closed. Then $FH \in UL$. Obviously $F \subset G_{FH}$. It follows immediately that $gFg^{-1} \subset G_H \subset H$ with $g \in U$. Proof of the Corollary is now complete.

NOTE. In their result, Montgomery and Zippin do not impose the hypothesis that G is compact, i.e. they assume that G is a Lie group and H a compact subgroup.

It can be proved that Corollary 3.2 implies Corollary 3.1 and hence the two are equivalent.

Section 4. Finite spanning set of cross-sections

A covering of a topological space is called *star-finite* if each set of the covering meets at most a finite number of others; the covering is called *star-bounded* if there is a finite number b such that each set of the covering meets at most b others. Such a number b is called a *bound* of the covering.

We require the following fact:

THEOREM 4.1. *Any open covering of a finite dimensional separable regular space admits a star-bounded open refinement.*

Inasmuch as an n-dimensional separable regular (and hence metric) space can be embedded in a bounded portion of euclidean $2n + 1$ space, Theorem 4.1 will follow immediately from

THEOREM 4.1′. *Let O be a bounded open set in euclidean r-space E^r. Let S be an open covering of O. Then there exists a star-bounded open refinement of O.*

PROOF. Inasmuch as any open set in E^r is a union of disjoint *connected open* sets, there is no generality lost if we add the hypothesis that O is connected. Assume therefore that O is connected as well as bounded and open.

Let $B = \bar{O} - O$, and let $c(p) = \frac{1}{2}d(p, B)$ where $d(p, q)$ is the euclidean metric in E^r. The function $c(p)$ is continuous on the compact set O and therefore attains its maximum at a point $p_0 \in O$. Set $a = c(p_0)$, and we denote the set consisting of p_0 by H_0. Inductively, we define $H_{n+1} = \sum_p S(p, c(p))$ $(p \in H_n)$ where $S(p, c)$ is the closed ball with center p and radius c. We next define the family of sets $H(t)$, $0 \leqq t < \infty$ as follows: $H(t + na) = \sum_p S(p, tc(p)/a)$ $(p \in H_n)$ for $0 \leqq t \leqq a$. Clearly $H(na) = H_n$ $(n = 0, 1, \cdots)$. The proof of Theorem 4.1′ is arranged in a series of remarks.

1. $H(t)$ is compact. We prove this for t between na and $(n + 1)a$ by induction on n. The assertion is true for $n = 0$. Assuming by induction that $H_n = H(na)$ is compact, let $q \in \overline{H(t)}$, $na < t \leqq (n + 1)a$. Then $q = \lim q_k$ with

$$q_k \in S(p_k, (t - na)c(p_k)/a)$$

where each p_k is in H_n. H_n being compact, we can assume without loss of generality that $\lim p_k = p$ where $p \in H_n$. Hence $c(p) = \lim c(p_k)$ and therefore $d(q, p) = \lim d(q_k, p_k) \leqq (t - na)\, c(p)/a$. Consequently $q \in H(t)$, $H(t)$ is closed and therefore compact for $na \leqq t \leqq (n + 1)a$. Hence $H(t)$ is compact for all t.

2. If $t < t'$, then $H(t) \subset \operatorname{int} H(t')$. This follows at once from the observation that if $c < c'$ then $S(p, c)$ is in the interior of $S(p, c')$.

3. $\sum_t H(t) = O$ $(0 \leqq t < \infty)$. By the preceding remark, $\sum_t H(t)$ is open. We now prove that it is closed in O. Clearly it equals $\sum_{n=1}^{\infty} H_n$. Suppose therefore that q is in the closure of $\sum_n H_n$. Then there is a point $p \in \sum_n H_n$ with $d(q, p) < c(q)$. Say for definiteness $p \in H_n$. Then $q \in S(p, c(p)) \subset H_{n+1}$, and therefore $\sum_t H(t)$ is closed in O. But O being connected, we infer $\sum_t H(t) = O$.

4. If $s < t$ and $q \in H(t)$, then $d(q, H^{(s)}) \leqq t - s$. Suppose first that $na \leqq s < t \leqq (n + 1)a$ for some n. Then there is a point $p \in H_n$ with $d(p, q) \leqq (t - na)c(p)/a$. Let q_1 be the point on the line segment pq at the distance $(s - na)c(p)/a$ from p. Then $q_1 \in H(s)$ and $d(q_1, q) = (t - s)c(p)/a \leqq t - s$. Now let s and t be arbitrary with $0 \leqq s < t$. Then there are integers k and h such that $ka \leqq s \leqq (k + 1)a \leqq \cdots \leqq ha \leqq t \leqq (h + 1)a$. By the foregoing result, there is a point q_1 in $H(ha)$ with $d(q, q_1) \leqq t - ha$. Inductively we get a point q_n in $H(na)$ such that $d(q_n, q_{n-1}) \leqq a$ $(n = 1, \cdots h - k)$. We then have $d(q, H_s) \leqq d(q, q_1) + \cdots +$

$d(q_{h-k}, H_s) \leqq (t - ha) + (ha - (h-1)a + \cdots + ((k+1)a - s) = t - s$. Proof is now complete.

Let $\mathfrak{S}$ be an open covering of O. For each integer n, we select a finite covering $\mathfrak{R}_n$ of the compact set $H_n - \text{int } H_{n-1}$ by open sets in O each of which lies in some set of $\mathfrak{S}$ and in the open set int $H_{n+1} - H_{n-2}$. Let $\mathfrak{R}$ denote the union of $\mathfrak{R}_n$ for all n. Any set of $\mathfrak{R}_n$ meets at most the sets of $\mathfrak{R}_{n+k}$, $k = -2, -1, 0, 1, 2$. Hence $\mathfrak{R}$ is a star-finite open refinement of $\mathfrak{S}$.

5. Given a positive number t, there exists a positive number L satisfying the condition: if A is a set in O of diameter less than L and A meets $H(t)$, then A lies in a set of $\mathfrak{R}$. For let L_1 be the Lebesgue number of the finite open covering of $H(t + 2a)$ by $\mathfrak{R}$. Let $L_2 = d(H(t), O - H(t + 2a))$ and set $L = \min (L_1, L_2)$. Clearly L satisfies the required condition. We *define* the number $L(t)$ to be the maximum of the numbers satisfying the condition. Clearly $L(t)$ decreases to zero as t increases to infinity.

6. $L(t + s) \geqq L(t) - s$. For let A be a set in O of diameter less than $L(t) - s$ and meeting $H(t + s)$. Then there is a point q in A with $d(q, H(t)) \leqq s$ by Remark 4 above. Let A_1 be a ball of diameter s meeting both $H(t)$ and A, and set $A_2 = A + A_1$. A_2 has a diameter less than $L(t)$ and meets $H(t)$; therefore it lies in some set of $\mathfrak{R}$. Hence A lies in a set of $\mathfrak{R}$, and thus $L(t + s) \geqq L(t) - s$.

It follows directly from Remark 6 that $| L(t + s) - L(t) | < | s |$ and hence $L(t)$ is a continuous positive function of t, $0 \leqq t < \infty$. Moreover $L(t + s)/L(t) \geqq \frac{1}{2}$ if $s \leqq L(t)/2$.

7. We denote by D_u, $u > 0$, the decomposition of E^r formed by planes $x_i = nu/(r)^{\frac{1}{2}}$ $(n = 0, \pm 1, \cdots)$, where $x_1, \cdots x_r$ form an orthonormal base of linear functions on E^r. Each cube of the decomposition has diameter u. We define the sequences of numbers t_n and u_n as follows: $t_0 = 0$, $t_{n+1} = t_n + \frac{1}{2}L(t_n)$; $u_n = L(t_n)/2^{[t_n]+3}$ where $[t_n]$ is the largest integer less than or equal to t_n. In proving Theorem 4.1′ no generality is lost in assuming $a = 1$ and we *henceforth* assume $a = 1$. Then $L(t) \leqq 1$ and $\frac{1}{4} \leqq u_{n+1}/u_n \leqq 1$.

8. $d(H_{n+1}, B) \geqq \frac{1}{2}d(H_n, B)$. For given $q \in H_{n+1}$, there is a point $p(q)$ in H_n with $d(p(q), q) \leqq \frac{1}{2}d(p(q), B)$. Therefore $d(q, B) \geqq \frac{1}{2}d(p(q), B) \geqq \frac{1}{2}d(H_n, B)$, so that $d(H_{n+1}, B) \geqq \frac{1}{2}d(H_n, B)$. Since $d(H_0, B) = 1$, we conclude $d(H_n, B) \geqq \frac{1}{2}^n$.

9. $d(H(t), O - H(s)) \geqq (s - t)/2^{n+2}$ if $n \leqq t \leqq s < n + 2$. For any t, $H(t)$ contains all points within the distance $\frac{1}{2}(t - [t])\ d(H_{[t]}, B)$ of $H_{[t]}$. If $[s] = [t]$ then $H(s)$ contains all points within $\frac{1}{2}(s - t)\ d(H_{[t]}, B)$ of $H(t)$. If $[s] = [t] + 1$, then $H_{[s]}$ contains all points within $\frac{1}{2}([s] - t)\ d(H_{[t]}, B)$ of $H(t)$ and $H(s)$ contains all points within $\frac{1}{2}(s - [s])\ d(H_{[s]}, B)$ of $H_{[s]}$. Since $\sum_q S(q, a)$ (all $q \in S(p, b)$) $= S(p, a + b)$ for balls in E^r, we infer that $H(s)$ contains all points within $\frac{1}{2}(s - [s])\ d(H_{[s]}, B) + \frac{1}{2}([s] - t)\ (d(H_{[t]}, B)$ of $H(t)$. Hence $d(H(t), O - H(s)) \geqq \frac{1}{2}(s - t)\ d(H_{[s]}, B) \geqq (s - t)/2^{n+2}$.

10. Let $\mathfrak{G}_n$ denote the collection of closed cubes from the decomposition D_{u_n} which meet $H(t_n) - H(t_{n-1})$. $\mathfrak{G}_n$ is a finite collection and the set $G_n = \sum Q$

(all $Q \in \mathfrak{G}_n$) is compact. $G_n \subset H(t_n + \frac{1}{2}L(t_n))$ by Remark 9, and hence $G_n \subset H(t_{n+1})$. On the other hand, $d(H(t_{n-2}), O - H(t_{n-1})) \geqq (t_{n-1} - t_{n-2})/2^{[t_{n-2}]+2} \geqq L(t_n)/2^{[t_n]+3}$ and therefore G_n does not meet $H(t_{n-2})$, that is $G_n \subset O - H(t_{n-2})$. As a result $G_n \cap G_{n+k}$ is empty if $|k| \geqq 3$.

11. Let $\mathfrak{F}_n$ be the collection of open cubes obtained by enlarging each cube of $\mathfrak{G}_n$ to an open cube with same center and side $(1 + e_n)u_n$ where e_n is positive and satisfies

$$e_n < \min (1, e_{n-1}, \tfrac{1}{2}d(G_{n-3}, G_n), \tfrac{1}{2}d(G_n, G_{n+3})).$$

Then $(1 + e_n)u_n/(1 + e_{n+k})\, u_{n+k} \leqq 1$ if $k = 0, -1, -2$ and $\leqq 8$, if $k = 1, 2$ respectively. Hence each set of $\mathfrak{F}_n$ meets sets from only $\mathfrak{F}_{n+k}$ ($k = -2, -1, 0, 1, 2$). If $Q \in \mathfrak{F}_n$, then Q meets no more than 3^r sets from each of $\mathfrak{F}_n$, $\mathfrak{F}_{n-1}$, $\mathfrak{F}_{n-2}$ and no more than 10^r sets from $\mathfrak{F}_{n+1}$ and 34^r sets from $\mathfrak{F}_{n+2}$. Set $b = 3.3^r + 10^r + 34^r$, and set $\mathfrak{F} = \sum_n \mathfrak{F}_n$ (all $n = 0, 1, \cdots$). Then $\mathfrak{F}$ is a star bounded open refinement of $\mathfrak{S}$ with bound b. Proof of Theorem 4.1′ is now complete.

THEOREM 4.2. *Let G be a compact Lie group of transformations on a separable metric finite-dimensional space E. Assume all the orbits are equivalent. Then there exists a finite set of local cross-sections whose orbits cover E.*

PROOF. Let X denote the space of orbits of G in E, and let π denote the continuous map of E onto X which sends each point of E into its orbit under G. Clearly π is a homeomorphism on local cross-sections and therefore X is a finite dimensional separable regular space. A subset of X is called "liftable" if it is the image under π of a subset of a local cross-section in E. Let $\mathfrak{S}$ denote the collection of open liftable subsets of X. Clearly $\mathfrak{S}$ is an open covering of X. Let $\mathfrak{F}_1$ be a star-bounded open refinement of $\mathfrak{S}$ with bound b. The space X is normal and therefore the covering $\mathfrak{F}_1$ is shrinkable to a covering $\mathfrak{F}$ by closed sets whose interiors cover X; $\mathfrak{F}$ is a fortiori star-bounded with bound b.

Now select from $\mathfrak{F}$ a maximal subcollection $\mathfrak{M}_1$ of disjoint sets. Inductively, select in $\mathfrak{F} - (\mathfrak{M}_1 + \cdots + \mathfrak{M}_n)$ a maximal subcollection of disjoint closed sets and denote it by $\mathfrak{M}_{n+1}$. Then $\mathfrak{F} = \mathfrak{M}_1 + \cdots + \mathfrak{M}_k$ with $k \leqq b + 1$. For otherwise, there is a set $V \in \mathfrak{M}_{b+2}$ which meets some set of $\mathfrak{M}_i$ ($i = 1, \cdots, b + 1$). Since no set of $\mathfrak{M}_i$ is in $\mathfrak{M}_j$ for $i \neq j$, V meets more than b sets—a contradiction.

Now set $L_i = \sum V$ (all $V \in \mathfrak{M}_i$) $i = 1, \cdots, k$. Each point in L_i has a neighborhood meeting only a finite number of sets of $\mathfrak{M}_i$ and thus L_i is closed, $i = 1, \cdots, k$.

We assert that each L_i is liftable $i = 1, \cdots, k$. In proving this, assume for definiteness that $i = 1$. For each $V \in \mathfrak{M}_1$, there corresponds a homeomorphism φ_V of V into E such that $\pi \cdot \varphi_V =$ identity and $\varphi_V(V)$ is a local cross-section in E. Let H denote the isotropy subgroup G_p for some definite point p in E. For each $V \in \mathfrak{M}_1$ select an element p_V in $\varphi_V(V)$ and an element g_V in G such that $G_{g_V p_V} = g_V G_{p_V} g_V^{-1} = H$. Then set $K_1 = \sum_V g_V \varphi_V(V)$ (all $V \in \mathfrak{M}_1$). It is easily verified that K_1 is closed, that $G_q = H$ for all $q \in K_1$, and that distinct points of K_1 lie on distinct orbits. It follows at once that K_1 is a local cross-section, and hence L_i is liftable, $i = 1, \cdots, k$. Let $K_1, \cdots, K_k$ denote local cross-sections mapping onto $L_1, \cdots, L_k$ by π. Then $GK_1 \cdots + GK_k = E$.

Section 5. Union of homeomorphisms

Let G be a compact Lie group of transformations of a space E having no fixed points, and let φ be a G-equivariant map of E into euclidean space with associated homomorphism θ. The map φ is called an *n.t. map* if the representation θ does not contain the trivial representation, i.e. if the origin is the only point fixed under $\theta(G)$.

LEMMA 5.1. *Let G be a compact Lie group of transformations of a space E, and let φ be a G-equivariant homeomorphism of E into E^N. Then there is a G-equivariant homeomorphism φ_1 of E into E^{2N} with $|\varphi_1(p)| = 1$ for all $p \in E$. If φ is an* n.t. *map, then φ_1 can be chosen so as to be an* n.t. *map.*

PROOF. We introduce the functions

$$\alpha(r) = ((1 + r^2)/(4 + r^2))^{\frac{1}{2}} \text{ and } \beta(r) = (1 - \alpha^2(r))^{\frac{1}{2}} \text{ on } 0 \leqq r < \infty;$$

we define maps A and B of euclidean space minus the origin into the ball of radius 1 as follows: $A(v) = \alpha(|v|)|v|^{-1}v$ and $B(v) = \beta(|v|)|v|^{-1}v$ for $v \in E^N$.

We form $E^N \times E^1$, and set $\psi(p) = (\varphi(p), w)$ where w is a fixed non-zero vector in E^1. Set $\varphi_1(p) = \psi(p)/|\psi(p)|$ and set $\theta_1 = \theta + \theta_0$ (direct) where $\theta_0(G)$ consists only of the identity transformation of E^1. Then φ_1 is G-equivariant.

If G has no fixed points on E, then $\varphi(E)$ does not contain the origin of E^N. The map $\varphi_1(v) = (A(v), B(v))$ of E into the unit sphere of $E^N \times E^N$ is equivariant with respect to $\theta + \theta$ (direct) and is a homeomorphism. Clearly it is an n.t. map if φ is.

LEMMA 5.2. *Let G be a compact Lie group of transformations on a metric space E, and let T_1, T_2 be invariant subsets with $E = T_1 + T_2$ and T_2 closed. Assume there exists a G-equivariant homeomorphism φ_i of T_i into E^{n_i} $(i = 1, 2)$. Then there exists a G-equivariant homeomorphism φ of E into euclidean space E^N, which is an* n.t. *map if each φ_i is an* n.t. *map.*

PROOF. By Lemma 5.1 we may assume that $|\varphi_1(p)| = 1$ for all $p \in T_1$. By Gleason's lemma, φ_2 can be extended to a G-equivariant map of E into E^{n_2}, which we denote by φ_2 also. Let $d_1(x, y)$ denote the metric on E. Then $d_1(gx, gy)$ regarded as a function on $G \times E \times E$ is continuous. Consequently $d(x, y) = \sup_g d_1(gx, gy)$ (all $g \in G$) is continuous on $E \times E$. Moreover $d(x, y)$ is a metric on E; it is equivalent to $d_1(x, y)$ since every d_1 ball contains a concentric d ball by definition of d, and every d ball contains a concentric d_1 ball by the continuity of the function d. It is clear too that $d(gx, gy) = d(x, y)$.

Set $d(x) = \inf_t (d(x, t) + |\varphi_2(x) - \varphi_2(t)|)$ (all $t \in T_2$). The function $d(x)$ is continuous on E, zero on T_2, and non-zero on $T_1 - T_2$. In addition $d(gx) = d(x)$ for all $g \in G$. Define φ as the map of E into $E^{n_1} \times E^{n_2} = E^{n_1+n_2}$ given by:

$$\begin{aligned} \varphi(x) &= (d(x)\varphi_1(x), \quad (1 + d(x))\varphi_2(x)) && \text{for } x \in T_1 \\ &= (0, \varphi_2(x)) && \text{for } x \in T_2. \end{aligned}$$

The map φ is clearly continuous, G-equivariant, and is n.t. if φ_1 and φ_2 are n.t. It is clear too that φ is one-to-one, that it is a homeomorphism on T_2 and on T_1 also. To complete the proof that φ is a homeomorphism, it suffices to demonstrate that if $x_n \in T_1 - T_2$, $\bar{x} \in T_2$, and $\varphi(x_n) \to \varphi(\bar{x})$, then $x_n \to \bar{x}$. To this end, we observe that $d(x_n) \to d(\bar{x})$ and $\varphi_2(x_n) \to \varphi_2(\bar{x})$. By definition of the function $d(x)$, there exists a point t_n of T_2 with $d(x_n, t_n) < 2d(x_n)$ and $|\varphi_2(x_n) - \varphi_2(t_n)| < 2d(x_n)$. Since $d(\bar{x}) = 0$, $\lim \varphi_2(t_n) = \lim \varphi_2(x_n) = \varphi_2(\bar{x})$. Since φ_2 is a homeomorphism on T_2, $\lim t_n = \bar{x}$ and hence $\lim x_n = \bar{x}$. Proof of the Lemma is now complete.

Section 6. The embedding theorem

Throughout this section E denotes a finite dimensional separable metric space and G a compact Lie group of transformations on E with $L(G, E)$ finite, i.e. with at most a finite number of inequivalent orbits. By "euclidean space" we understand finite dimensional real or complex euclidean space with a distinguished origin. If H_1 and H_2 are closed subgroups of G, we mean by $H_1(\leqq)H_2$ that H_1 is conjugate in G to a subgroup of H_2, and by $H_1(<)H_2$ that H_1 is conjugate to a proper subgroup of H_2. If $H_1(\leqq)H_2$ then $H_1'(\leqq)H_2'$ for any H_1' in (H_1) and H_2' in (H_2). The relation $(\leqq)$ is clearly transitive. Furthermore if $H_1(\leqq)H_2$ and $H_2(\leqq)H_1$ then H_1 is conjugate to H_2; for H_1 and H_2 must have the same dimension and the same number of connected components. Upon carrying H_1 into a subgroup H_1' of H_2 by an inner automorphism, we find that H_1' and H_2 have the same Lie algebra, and therefore the same connected component of the identity. Since they have the same number of connected components, $H_1' = H_2$ and therefore H_1 and H_2 are conjugate.

In the set $L(G, E)$ we define $(H_1) < (H_2)$ if $H_1(<)H_2$. This relation is well defined and is a partial ordering. We set $E_p =$ the set of all $q \in E$ with $(G_p) = (G_q)$, $T_p =$ the set of all $q \in E$ with $(G_p) \leqq (G_q)$ and $S_p =$ the set of all q with $(G_p) < (G_q)$; that is $T_p = E_p + S_p$. According to a theorem of Montgomery and Zippin (Bull. Amer. Math. Soc., v. 48 (1942), pp. 448–452) (cf. also Corollary 3.1 above), $G_{q_1}(\leqq)G_q$ for all points q_1 is some neighborhood of q. It follows immediately that S_p and T_p are closed sets of E. It is to be noticed that E_p, S_p, and T_p are invariant under G for any $p \in E$. Also, all orbits in E_p are equivalent.

Lemma 6.1. *Let $p \in E$. Then there is a G-equivariant homeomorphism of E_p into euclidean space, which is* n.t. *if $G_p \neq G$.*

Proof. By Theorem 4.2 there exists in E_p a finite set of local cross-sections to the orbits $K_1, \cdots, K_k$ such that $E_p = GK_1 + \cdots + GK_k$. By Lemma 2.1, there exists a representation α of G into the unitary group on the euclidean space E^n and a point v other than the origin of E^n such that (1) $\alpha^{-1}(\alpha(G)_v) = G_p$ and (2) α does not contain the trivial representation of G if $G_p \neq G$ Let V denote the one-dimensional subspace spanned by v and the origin. Let r_i be an integer such that K_i can be embedded homeomorphically in E^{r_i} $(i = 1, \cdots, k)$. We identify E^{r_i} with the subspace $V + \cdots + V$ of $E^n + \cdots + E^n = E^{nr_i}$, and obtain thereby a homeomorphism ψ_i of K_i into E^{nr_i} with the property that $\beta_i(G_p)\varphi_i(q) = \varphi_i(q)$ for all $q \in K_i$ where $\beta_i = \alpha + \cdots + \alpha$ (r_i times) $(i = 1, \cdots, k)$. As a result the map $\bar{\varphi}_i: (gG_p, q) \to (\beta_i(g)\psi_i(q), \alpha(g)v)$, where $g \in G$, $q \in K_i$ is a well-defined continuous one-to-one map of $(G/G_p) \times K_i$ into $E^{n(r_i+1)}$ $(i = 1, \cdots, k)$. It is clear too that the inverse mapping is continuous.

Let π_i denote the map $(gG_p, q) \to gq$ of $(G/G_p) \times K_i$ onto GK_i. Each π_i is well-defined since $G_q = G_p$ for all $q \in K_i$. π_i is a homeomorphism in a set $U \times K_i$ where U is a neighborhood in G/G_p by definition of a pseudo-section and hence π_i is a homeomorphism throughout $(G/G_p) \times K_i$ $(i = 1, \cdots, k)$. Set $\varphi_i = \bar{\varphi}_i \cdot \pi^{-1}$. Then φ_i is a G-equivariant homeomorphism of GK_i which is n.t. if $G_p \neq G$. Since each GK_i is closed in E, we can construct a G-equivariant homeomorphism φ of E_p in euclidean space by repeated applications of Lemma 5.2. The map φ is n.t. if $G_p \neq G$.

THEOREM 6.1′. *Let G be a compact Lie group operating on a separable metric finite dimensional space E. Assume $L(G, E)$ is finite. Then there exists a G-equivariant homeomorphism of E into eulidean space E_n which is* n.t. *if G has no fixed points in E.*

PROOF. The set of conjugacy classes $L(G, E)$ is partially ordered by the relation $\leqq$ introduced above. We define the *length* of $L(G, E)$ as the maximum number of elements appearing in a linearly ordered subset. The theorem is proved by induction on the length of $L(G, E)$.

If the length of $L(G, E)$ is 1, then $E_p = T_p$ for any $p \in E$, and therefore E_p is closed in E. Now there exists a finite set of points $p_1, \cdots, p_r$ in E such that $E = E_{p_1} + \cdots + E_{p_r}$. By Lemma 6.1 there is a G-equivariant homeomorphism of E_{p_i} into euclidean space which is n.t. if $G_{p_i} \neq G$, $i = 1, \cdots, r$. By repeated applications of Lemma 5.2, there exists a G-equivariant homeomorphism of E into euclidean space which is n.t. if $G_p \neq G$ for all $p \in G$, that is, if G has no fixed points in E.

Assume inductively that the theorem is true whenever the length is less than $L(G, E)$. There obviously exists in E a finite set of points $p_1, \cdots, p_r$ such that $E = T_{p_1} + \cdots + T_{p_r}$. Each $T_{p_i} = E_{p_i} + S_{p_i}$ and hence length $L(G, S_{p_i}) \leqq$ length $L(G, E) - 1$, $i = 1, \cdots, r$. By the induction hypothesis there is a G-equivariant homeomorphism of S_{p_i} which is n.t. if G has no fixed point on S_{p_i} and a similar assertion holds for E_{p_i}, $i = 1, \cdots, r$. By Lemma 5.2, a similar assertion holds for each T_{p_i} and also for $T_{p_i} + \cdots + T_{p_r} = E$. Proof of the theorem is now complete.

Theorem 6.1 mentioned in the introduction is simply a restatement of Theorem 6.1′ coupled with the observation that the unitary representation which is associated with a G-equivariant map is faithful if G operates faithfully on E.

If G is a compact group operating faithfully on a space E and there is a G-equivariant homeomorphism of E into euclidean space, then E is separable, metric, and finite dimensional; also G is a Lie group. We show in Section 7 that $L(G, E)$ is finite. Thus the hypotheses on E of Theorem 6.1 are necessary and sufficient for the existence of a G-equivariant homeomorphism into euclidean space.

Section 7. Groups acting differentiably. Applications

We collect first several remarks about compact Lie groups of differentiable transformations. Numbers 1, 2, and 3 below were noted independently by Montgomery and Yang. We include them here for the sake of completeness.

Throughout this section G denotes a compact Lie group of differentiable transformations, M denotes a *differentiable* manifold, and E_n denotes a real euclidean n-space with distinguished origin and n finite.

1. Let G operate on M, and let $p \epsilon M$. There is a pseudo-section to the orbit through p which is a closed ball submanifold (of lower dimension in general).

PROOF. The isotropy subgroup G_p is a compact group of differentiable transformations keeping the point p fixed. Hence by a result of Bochner admissible coordinates may be introduced in a neighborhood of p with respect to which G_p is a group of orthogonal transformations. Since G_p keeps invariant the tangent space at p to the orbit Gp, it keeps invariant a complementary subspace K in the new coordinates. With the help of the implicit function theorem one can see that the mapping $(g, q) \to gq$ is a homeomorphism of $U \times K_1$ onto a neighborhood of p, where U is a differentiable local cross-section to the coset G_p in G and K_1 is a ball neighborhood of p in K. Select a ball K_2 in K_1 with center p so that $gK_2 \cap K_2$ is empty for $g \epsilon G - G_p$. It follows that the ball submanifold K_2 is a pseudo-section.

2. If M is compact, then $L(G, M)$ is finite.

PROOF. We use induction on dim M. Let $P(n)$ denote the assertion that $L(G, M)$ is finite if dim $M \leqq n$. Let $Q(n)$ denote the assertion that $L(G, E^n)$ is finite if G is a compact group of linear transformation of E^n. The well-known "unitary trick" tells us that a compact group of linear transformations of E^n is equivalent to a compact group of orthogonal transformations. Since the latter keeps the unit sphere S^{n-1} invariant and sends rays into rays, we see that $Q(n)$ is equivalent to $P(n - 1)$ if $M = S^{n-1}$. Also, no generality is lost in assuming G operates faithfully for the subgroup of G operating trivially is in every isotropy subgroup.

The assertion $P(0)$ is true, for then G is simply a finite group of permutations of a finite set.

Assume now dim $M = n$ and $P(n - 1)$ is true. Hence $Q(n)$ is true. Now since M is compact, there is a finite number of ball-submanifold pseudo-sections $K_1, \cdots, K_s$ through points $P_1, \cdots, P_s$ respectively such that $M = GK_1 + \cdots + GK_s$ and G_{p_i} is equivalent to a linear group on K_i. If g is not in G_{p_i}, then gK_i does not meet K_i so that $G_q \subset G_{p_i}$ for all $q \epsilon K_i$. Hence $(G_q) \leqq (G_{p_i})$ for all $q \epsilon GK_i$, and therefore the number of elements in $L(G, GK_i)$ is no greater than the number of elements in $L(G_p, K_i)$, the latter being finite by $Q(n)$. Hence $L(G, E)$, which has no more elements than $\sum_i L(G, GK_i)$ is a finite set.

In view of the equivalence between $Q(n)$ and $P(n - 1)$ when $M = S^{n-1}$, we conclude

3. $L(G, E^n)$ is finite if G is a compact group of linear transformations on E^n.

4. If $L(G, M)$ is finite, one can follow through our construction of the G-equivariant embedding of M in euclidean space and obtain after slight modifications a differentiable G-equivariant embedding. If M is a compact differentiable manifold, a short proof can be given based on the following method.

Let B denote the set of differentiable functions on M. Let $\{U_\alpha\}$ be a finite covering of M by coordinate neighborhoods and let $\{V_\alpha\}$ be an open covering with each $V_\alpha \subset U_\alpha$. For each $f \in B$ define $\| f \| = \sup_p \left(| f(p) | + \sum_\alpha | \partial f / \partial x_\alpha^i(P) | \right)$ (all α with $p \in V_\alpha$, all $p \in M$). B is a Banach space with $\| f \|$ as norm. If $g \in G$ and f is a function on M (resp. on G) we define gf to be the function $f \cdot g^{-1}$. We say a function f on M (resp on G) is a *representation function* if the linear span of the set of functions Gf is finite dimensional. The representation functions on G are continuous and by the Peter-Weyl theorem approximate uniformly any continuous function on G.

We assert now that the representation functions in B form a dense subset of B. For given any $f \in B$ and any positive number e, there is a neighborhood U of the identity in G such that $\| gf - f \| < e/2$ for all $g \in U$. Let $s = \sup_g \| gf \|$ (all $g \in G$). Let v be a non-negative continuous function on G vanishing outside U with $\int_G v(g)\, dg = 1$, the Haar measure of G being one. For any continuous function w on G, we set $f_w = \int_G w(g) gf\, dg$; the function f_w is in B. Now $\| f_v - f \| = \| \int_G v(g) gf\, dg - f \| = \| \int_G v(g)(gf - f)\, dg \| \leqq \int_G v(g) e/2\, dg \leqq e/2$. Next select a representation function u on G such that $| v(g) - u(g) | < e/2s$ for all $g \in G$. Then $\| f - f_u \| \leqq \| f - f_v \| + \| f_v - f_u \| \leqq e$. Moreover f_u is a representation function on M for

$$g_1(f_u) = g_1 \int_G u(g) gf\, dg = \int_G u(g) g_1 gf\, dg = \int_G u(g_1^{-1} g_1 g) g_1 gf\, dg$$

$$= \int_G u(g_1^{-1} g) gf\, dg = f_{g_1 u}.$$

Since f_u depends linearly on u, it follows that Gf_u lies in a finite dimensional subspace of B. Thus f_u is a representation function on M lying in an e-neighborhood of f, and therefore the representation functions in B are dense in B.

Let $f_1, \cdots, f_n$ be the component functions of a differentiable embedding φ of M into E^n. We can assume that M is a metric space. Then select approximating representation functions $h_1, \cdots, h_n$ whose functional matrix has the same rank as the functional matrix of $f_1, \cdots, f_n$, i.e. dim M. Each point lies in a neighborhood on which the mapping φ_1: $p \to (h_1(p) \cdots, h_n(p))$ is one-to-one and regular. Take a finite covering by such neighborhoods and let b denote the Lebesgue number of this covering. Then we select representation functions $k_1, \cdots, k_n$ which are so close to $f_1, \cdots, f_n$ respectively, that if $k_i(p) = k_i(q)$, $i = 1, \cdots, n$ then $d(p, q) < b$. Select from the linear span in B of each Gh_i and Gk_j a base with first base vector h_i and k_j respectively, and with respect to which the operations of G are orthogonal. Let $h_{i,1}, \cdots h_{is_i}$ and $k_{j,1}, \cdots, k_{j,t_j}$ denote the bases for the linear spans of Gh_i and Gk_j respectively. Then $p \to (h_{1,1}(p) \cdots, k_{n,t_n}(p))$ is a differentiable, regular G-equivariant homeomorphism of M into a euclidean space.

The foregoing proof of the existence of a G-equivariant embedding in euclidean space applies with a slight modification to compact subsets of a differentiable manifold. However it cannot be generalized to arbitrary differentiable manifolds for a compact Lie group of differentiable transformations can have an infinite number of inequivalent orbits on an open manifold.

5. If the transformation group G is not compact, then $L(G, E)$ can be infinite even if E is euclidean space and G is an algebraic Lie group of linear transformations. For let G be the algebraic linear group in E^3 whose Lie algebra is the set A of matrices $M(a, b)$ of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Lie algebra A is abelian. Let $B(u)$ be the set of all $M(a, b)$ with $a + bu = 0$, and let $H(u)$ be the analytic subgroup corresponding to $B(u)$. Then $H(u)$ is the isotropy subgroup of the vector $(0, 1, u)$. Thus G has an infinite number of distinct isotropy subgroups and being abelian, $L(G, E^3)$ is infinite.

THEOREM 7.1. *Let G be a compact Lie group. Then there exist at most a finite number of mutually non-conjugate subgroups which are normalizers of analytic subgroups. Moreover, there exist at most a finite number of mutually non-conjugate semi-simple analytic subgroups.*

PROOF. Let A denote the Lie algebra of G, let E denote the exterior algebra of A, and let P denote the projective space of one dimensional linear subspaces of E. Each linear subspace B of A determines a point in P by the Grassman correspondence; this point we denote by B^*. The adjoint representation of G on A induces a representation π of G by projective transformations of P and clearly a subgroup N of G keeps a linear subspace B invariant if and only if $\pi(N)$ keeps the point B^* fixed. If H is an analytic subgroup of G and B is its Lie algebra, then $xHx^{-1} = H$ if and only if $Ad\,(x)(B) = B$, and therefore if and only if $\pi(x)(B^*) = B^*$. Consequently a subgroup N is a normalizer of some analytic subgroup of G if and only if $N = \pi^{-1}(\pi(G)_{B^*})$ with B a Lie subalgebra of A. Since $L(\pi(G), P)$ is finite, G has at most a finite number of mutually non-conjugate normalizers of analytic subgroups.

In order to prove the second part of the theorem, it suffices to prove that there are only a finite number of distinct semi-simple analytic subgroups which have the same normalizer. Upon considering the corresponding Lie algebra, it suffices to prove that a Lie algebra contains only a finite number of distinct semi-simple ideals. This follows in turn from the fact that (1) the linear span of the semi-simple ideals in a Lie algebra is semi-simple and (2) a semi-simple Lie algebra is the direct sum of *all* its minimal ideals and therefore has but a finite number of ideals.

THE JOHNS HOPKINS UNIVERSITY