The boundary of the Milnor fiber of Hirzebruch surface singularities

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Abstract. We give the first (as far as we know) complete description of the boundary of the Milnor fiber for some non-isolated singular germs of surfaces in \mathbb{C}^3 . We study irreducible (i.e. gcd (m, k, l) = 1) non-isolated (i.e. $1 \le k < l$) Hirzebruch hypersurface singularities in \mathbb{C}^3 given by the equation $z^m - x^k y^l = 0$. We show that the boundary Lof the Milnor fiber is always a Seifert manifold and we give an explicit description of the Seifert structure. From it, we deduce that:

1) L is never diffeomorphic to the boundary of the normalization.

2) L is a lens space iff m = 2 and k = 1.

3) When L is not a lens space, it is never orientation preserving diffeomorphic to the boundary of a normal surface singularity.

1. Introduction.

In [MP] the authors prove, among other facts, that the boundary L of the Milnor fiber of a non-isolated hypersurface singularity in \mathbb{C}^3 is a Waldhausen manifold (non-necessarily "reduziert").

In this paper, we apply the general method of [MP] to the study of Hirzebruch singularities, defined by the equation

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$$z^m - x^k y^l = 0$$

We assume that the germ is irreducible, which amounts to ask that gcd(m, k, l) = 1. We also assume that $1 \le k \le l$ to avoid redundancies and that $m \ge 2$ in order to have a genuine singularity.

Hirzebruch proved in [H] that the boundary \tilde{L} of the normalization is a lens space and he gave an explicit description of the minimal resolution as a bamboo-shaped graph of rational curves. See also [HNK]. We call "bamboo" a connected graph whose vertices have at most two neighbours. We briefly recall this result in section 2.

We prove in theorem 3.1 that L is always a Seifert manifold. Its canonical star-shaped plumbing graph is described in theorem 4.2.

When $m \ge 3$ or when m = 2 and $k \ge 2$ the plumbing graph for L is never a bamboo of rational curves. A little computation shows then (see corollary 4.3) that L is never orientation-preserving diffeomorphic to the boundary of a normal surface singularity.

When m = 2 and k = 1, the plumbing graph is a bamboo of rational curves. But it is different from the Hirzebruch one. Indeed, the corresponding lens spaces do not have the same fundamental group.

In [MP] it is stated that the boundary L_t of the Milnor fiber of a non-isolated hypersurface singularity in \mathbb{C}^3 is never diffeomorphic to the boundary \tilde{L}_0 of the normalization. This result is exemplified here in a very explicit way, because we are able to compare the two corresponding plumbing graphs for any Hirzebruch singularity.

The more general case of germs having equation $z^m - g(x, y) = 0$ is treated in [MPW]. The proofs we present here are self-contained, i.e. independent from [MP] and from [MPW].

The first named author had the idea to study Hirzebruch singularities while reading Egbert Brieskorn beautiful article [B].

We thank Walter Neumann for very pleasant discussions during the meeting and for attraction our attention to the computation of the invariant e_0 for Seifert manifolds.

2. Plumbing graphs.

The 3-dimensional manifolds we consider are compact and oriented. In many cases, they are oriented as the boundary of a complex surface. To describe these manifolds, we use plumbing graphs and we follow [N] as closely as possible. Recall that a vertex of a plumbing graph carries two weights: the genus g of the base space and the Euler number $e \in \mathbb{Z}$. In this paper we always have $g \geq 0$ i.e. the base surfaces are orientable. Particuliarly useful are the bamboos for lens spaces and the star-shaped graphs for "general" Seifert manifolds.

The lens space L(n,q) is defined as the quotient of the sphere $S^3 \subset \mathbb{C}^2$ (oriented as the boundary of the unit 4-ball, equiped with the complex orientation) by the action $C_{n,q}$ of the n-th roots of unity given by $\zeta(z_1, z_2) = (\zeta z_1, \zeta^q z_2)$ with 0 < q < n and gcd(n,q) = 1. The canonical plumbing graph for L(n,q) is the bamboo of rational curves with Euler numbers, from left to right, $(e_1, e_2, ..., e_u)$ defined as $e_i = -b_i$. The integers b_i are defined by $b_i \ge 2$ together with

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \frac{1}{\ddots - \frac{1}{b_u}}}}$$

As in [N], we summarize the continued fraction expansion as $[b_1, b_2, ..., b_u]$.

The Seifert manifolds (with unique Seifert foliation) are described by a star-shaped graph. See [N] corollary 5.7. All vertices, except possibly the central one, have genus zero and Euler number $e \leq -2$.

We now consider Hirzebruch singularity $z^m - x^k y^l = 0$. The boundary \tilde{L} of its normalization is the lens space L(n,q) where n and q are computed as follows. Let $d_k = gcd(m,k)$ and $d_l = gcd(m,l)$. Then

$$n = \frac{m}{d_k d_l}$$

To get q let λ_0 be the smallest integral positive solution of the equation

$$\lambda l \equiv -kd_l \pmod{m}$$

in the unknown λ . This solution λ_0 is divisible by d_k and we have

1

$$q = \frac{\lambda_0}{d_k}$$

The special case $d_k = 1 = d_l$ is more pleasant. Then

$$n = m$$
 and $q = \lambda_0$

where λ_0 is the smallest positive solution of the equation $\lambda l \equiv -k \pmod{m}$. See [BPV].

The description we give below in theorem 4.1 for the boundary L of the Milnor fiber is in sharp contrast with the classical result (essentially Hirzebruch thesis) about the boundary \tilde{L} of the normalisation. For instance, if m is fixed, \tilde{L} depends only on the residue classes $(\mod m)$ of k and l. This is not the case for L. See section 5 below for an example.

3. Vertical monodromies.

Let $f(x, y, z) = z^m - x^k y^l$ be an irreducible germ (i.e. gcd(m, k, l) = 1) of hypersurface in \mathbb{C}^2 with a singular point (i.e. $2 \le m$) at the origin. Recall that we assume that $1 \le k \le l$ to avoid redundancies. In this paper, we use for technical reasons a polydisc $B(\alpha) = B_{\alpha}^2 \times B_{\alpha}^2 \times B_{\epsilon}^2$ with $0 < \alpha \leq \epsilon$ and $\alpha^{k+l} < \epsilon^m$ in place of the standard Milnor ball $B_{\epsilon}^6 = \{P \in \mathbb{C}^3 \text{ with } |P| \leq \epsilon\}$. The equation of f being quasi-homogeneous, for any $B(\alpha)$ there exists η with $0 < \eta \ll \alpha$ such that the restriction of f on $B(\alpha) \cap f^{-1}(B_{\eta}^2 \setminus \{0\})$ is a locally trivial fibration on $(B_{\eta}^2 \setminus \{0\})$ and such that this fibration does not depend on α up to isomorphism. Let S be the boundary of $B(\alpha)$. The condition $\alpha^{k+l} < \epsilon^m$ implies that we may choose η with $0 < \eta \ll \alpha$ such that $L_t = f^{-1}(t) \cap S$ is contained in $\{(x, y, z) \in C^3 \text{ such that } |x| = \alpha \text{ or } |y| = \alpha$ for all t with $0 \leq |t| \leq \eta$. For such a η , if $t \in B_{\eta}^2 \setminus 0$ we say that $F_t = B(\alpha) \cap f^{-1}(t)$ is "the" Milnor fiber of f and that $L_t = F_t \cap S$ is "the" boundary of the Milnor fiber of f. From now on, we write $L = L_t$ for a chosen t such that $0 < |t| \leq \eta$.

We will now describe L as the union of $M' = L \cap \{|x| = \alpha\}$ and $M'' = L \cap \{|y| = \alpha\}$.

Theorem 3.1. The boundary L of the Milnor fiber of $z^m - x^k y^l$ is a Seifert manifold. Moreover, the projection on the z-axis is constant on each Seifert leaf.

Proof of theorem 3.1. Let $\varphi : M' \to \mathbb{C}^3$ be defined by $\varphi(x, y, z) = (x, z, f(x, y, z))$. Hence we have $\varphi(M') \subset S^1_{\alpha} \times B^2_{\epsilon} \times \{t\}$. The singular locus $\Sigma(f)$ of f satisfies the equation $\frac{\partial f}{\partial y} = 0$ i.e. $lx^k y^{l-1} = 0$. But we have $M' \subset \{|x| = \alpha\}$. Hence we have $\Sigma(\varphi) = \bigcup_{i=1}^m (S^1_{\alpha} \times \{0\} \times \{z_i\})$ where $z_i^m = t$.

The set of singular values $\Delta(\varphi) = \varphi(\Sigma(\varphi))$ of the map φ is the union of the *m* circles $S^1_{\alpha} \times \{z_i\} \times \{t\}$ where $z_i^m = t$.

We fill $\varphi(M')$ with the circles $S^1_{\alpha} \times \{c\} \times \{t\}$ where $c \in B^2_{\epsilon}$ and $|c^m - t| \leq \alpha^{k+l}$. As $\Delta(\varphi)$ is the union of m of these circles, we pull-back this (trivial) fibration of $\varphi(M')$ in circles to obtain a Seifert foliation on M'. The Seifert leaves are defined as the intersection $M' \cap \{z = c\}$.

Replacing φ by the restriction to M'' of the morphism $(x, y, z) \mapsto (y, z, f(x, y, z))$ we see that, in a symmetric way, the intersections $M'' \cap \{z = c\}$ fill M'' with a Seifert foliation in circles. The Seifert leaves of M' and of M'' are defined by the same equation $L \cap \{z = c\}$, so they coincide on $T = M' \cap M''$. End of proof of theorem 3.1.

Let $\pi_x : M' \to S^1_{\alpha}$ (resp $\pi_y : M'' \to S^1_{\alpha}$) be the restriction to M' (resp M'') of the projection on the x-axis (resp the y-axis). Let $a \in S^1_{\alpha}$. Now let $G' = \pi_x^{-1}(a)$ and $G'' = \pi_y^{-1}(a)$.

Theorem 3.2. π_x and π_y are locally trivial differentiable fibrations over S^1_{α} . Moreover: 1) The fibers of π_x (resp π_y) are diffeomorphic to the Milnor fiber of the plane curve germ $z^m - y^l$ (resp $z^m - x^k$).

2) The fibers of π_x (resp π_y) meet transversaly the Seifert leaves of M' (resp M'') constructed in the proof of theorem 3.1.

Proof of theorem 3.2. The singular locus of π_x is defined by $lx^ky^{l-1} = 0$ and $mz^{m-1} = 0$. But, if $(x, y, z) \in M'$ we have $|x| = \alpha$ and $z^m - x^ky^l = t$ with 0 < |t|. So π_x has no singular point. It is easy to see that the restriction of π_x to $\partial M'$ is a submersion onto S^1_{α} . As M' is a compact differentiable manifold, π_x is a differentiable fibration. The situation is symmetric for π_y . Now, we have chosen $a \in S^1_{\alpha}$ and t such that $0 < |t| \leq \eta$ where η is very small. By definition we have $G' = \{(a, y, z) \text{ with } z^m - a^k y^l = t \text{ and } (y, z) \in S^1_{\alpha} \times B^2_{\epsilon}\}$ and also $G'' = \{(x, a, z) \text{ with } z^m - x^k a^l = t \text{ and } (x, z) \in S^1_{\alpha} \times B^2_{\epsilon}\}$. Hence, the assertion 1) is obvious.

To prove 2) let b be any l^{th} root of $(a^{-k}(c^m - t))$ and let $P = (a, b, c) \in G'$. The Seifert leaf containing P is parametized by $(e^{i\theta}a, e^{-i\theta\frac{k}{t}}b, c)$ with, say, $\theta \in \mathbf{R}$. Hence, the Seifert leaves are oriented and transverse to the hyperplane $H_a = \{x = a\}$ for all $a \in S^1_{\alpha}$. The situation is symmetric for M''. End of proof of theorem 3.2.

Remarks. 1. If k = l = 1 the germ f has an isolated singular point at the origin. In this case, theorem 3.2 shows that G' and G'' are discs and that M' and M'' are solid torii. Hence L is a lens space, diffeomorphic to $L_0 = \tilde{L}$.

2. If we assume that $\dim \Sigma(f) = 1$ then we have $l \ge 2$ and the x-axis $D' = \{(x,0,0) \text{ with } x \in \mathbb{C}\}$ is a component of $\Sigma(f)$. Then, theorem 3.2 implies that G' is never diffeomorphic to a disc and that M' is not a solid torus. When $D' \subset \Sigma(f)$ we say in [MP] that M' is the **vanishing zone** around D'. When $k \ge 2$ then $D'' = \{(0, y, 0) \text{ with } y \in \mathbb{C}\}$ is the second component of $\Sigma(f)$ and M'' is the vanishing zone around D'.

We now proceed to the definition of the vertical monodromy. Let $h': G' \to G'$ be the diffeomorphism defined by the first return along the (oriented) leaves of M'. Theorem 3.2 implies that h' is a monodromy for the fibration π_x .

Definition. We call h' the **vertical monodromy** for D'.

Likewise, the first return along the (oriented) Seifert leaves of M'' is a diffeomorphism $h'': G'' \to G''$. We call it the vertical monodromy for D''.

In conclusion, we know that M' is the mapping torus of h' acting on G' and that M'' is the mapping torus of h'' acting on G''. We wish now to describe in details the vertical monodromies.

Notations. Let d = gcd(k, l); $\overline{l} = \frac{l}{d}$; $\overline{k} = \frac{k}{d}$; $d_l = gcd(m, l)$; $d_k = gcd(m, k)$.

Remark. As f is assumed to be irreducible, we have gcd(m, k, l) = 1 and \bar{k} is prime to d_l (resp \bar{l} is prime to d_k). Moreover, G' has d_l boundary components and G'' has d_k boundary components.

Theorem 3.3. The vertical monodromy h' (resp h'') has finite order \bar{l} (resp \bar{k}). Moreover: 1. If $\bar{l} \ge 2$ (resp $\bar{k} \ge 2$) then h' (resp h'') has exactly m fixed points and any non-fixed point has order \bar{l} (resp \bar{k}).

2. At each fixed point h' (resp h'') acts locally as a rotation of angle $-(\bar{k}/\bar{l})2\pi$ (resp $-(\bar{l}/\bar{k})2\pi$).

Proof of theorem 3.3. As in the proof of theorem 3.2, we consider $P = (a, b, c) \in G'$. We have seen that $(e^{i\theta}a, e^{-i\theta \frac{k}{l}}b, c)$ for, say, $\theta \in \mathbf{R}$ is a parametrization of the Seifert leaf which contains P. Hence

$$(\star) \qquad h'(P) = (a, e^{-2i\pi\frac{\kappa}{l}}b, c)$$

As $k/l = \bar{k}/\bar{l}$ with \bar{k} prime to \bar{l} , we see that h' has order \bar{l} on each P = (a, b, c) with $b \neq 0$.

Then, if $\bar{l} \geq 2$, it is clear that h'(P) = P iff b = 0. Then $c^m = t$ and h' has exactly m fixed points, i.e. the points $\{(a, 0, z_i)\}$ where $z_i^m = t$.

The formula (\star) implies directly the last statement of theorem 3.3. End of proof of theorem 3.3.

Corollary 3.4. The intersection $T = M' \cap M''$ is a torus.

Proof of corollary 3.4. Indeed, T is the mapping torus of h' acting on the d_l boundary components of G'. As \bar{k} is prime to \bar{l} the formula (*) in the proof of theorem 3.3 implies that h' permutes transitively the boundary components of G'. End of proof of corollary **3.4**.

Remark. G', G'' and $F_t = f^{-1}(t) \cap B$ are oriented by the complex structure. L is oriented as the boundary of F_t and this orientation induces one on M' and M''.

Theorem 3.5. Orient $T = M' \cap M''$ as the boundary of M''. Orient $\partial G'$ (resp $\partial G''$) as the boundary of G' (resp G''). Then the intersection number on T of $\partial G'$ with $\partial G''$ is equal to -m.

Proof of theorem 3.5. Let $\pi: L \to B^2_{\alpha} \times B^2_{\alpha}$ be the restriction on L of the projection $(x, y, z) \mapsto (x, y)$. The restriction of π to $T = M' \cap M''$ is a regular covering of order m. Moreover, we have $\pi(G') = \{a\} \times S^1_{\alpha}$ and $\pi(G'') = S^1_{\alpha} \times \{a\}$. The complex structure of \mathbf{C}^2 induces an orientation on $B^2_{\alpha} \times B^2_{\alpha}$. Let $S^1_{\alpha} \times S^1_{\alpha} = \pi(T)$ be oriented as the boundary of $B^2_{\alpha} \times S^1_{\alpha}$. The intersection number of $\{a\} \times S^1_{\alpha}$ with $S^1_{\alpha} \times \{a\}$ in $S^1_{\alpha} \times S^1_{\alpha}$ is equal to (-1). The covering projection π being compatible with orientations, this proves that the intersection number we are looking for is equal to (-m). End of proof of theorem 3.5.

4. The Seifert structure on the boundary of the Milnor fiber.

Theorem 4.1. The Seifert invariants (associated to the Seifert structure described in section 3) for the boundary L of the Milnor fiber of a Hirzebruch singularity are as follows:

1. The genus q of the base space is equal to (m-1)(d-1) where d = qcd(k, l).

2. The integral Euler number e is equal to m.

3. Let $\bar{l} = \frac{l}{d}$ and $\bar{k} = \frac{k}{d}$. Then *L* has 2m (possibly) exceptional leaves. There are *m* of them with Seifert invariants (α', β') defined by $\alpha' = \bar{l}$ and β' given by $(-\bar{k})\beta' \equiv 1 \mod \bar{l}$ and $0 < \beta' < \bar{l}$ in normalized form.

There are m of them with Seifert invariants (α'', β'') defined by $\alpha'' = \bar{k}$ and β'' given by $(-\bar{l})\beta'' \equiv 1 \mod \bar{k}$ and $0 < \beta'' < \bar{k}$.

Comments. 1. The singularity is isolated iff k = l = 1. Of course in this case we have L = L. The theorem above says that L has no exceptional leaf, that q = 0 and that e = m. Hence L is the lens space L(m, m-1). We are happy to see that this agrees with Hirzebruch result.

Assume from now on that $1 \leq k$ and that $2 \leq l$.

2. Under this hypothesis L is a lens space iff m = 2 and k = 1. (Quick proof: To get a lens space we need g = 0 and the theorem says that this is equivalent to d = 1. Then we can admit at most two exceptional leaves. Hence k = 1 and m = 2). The lens space is L(2l, 1). On the other hand $\tilde{L} = L(1, 1) = S^3$ when l is even and $\tilde{L} = L(2, 1) = P^3(\mathbf{R})$ when l is odd.

3. If $3 \le m$ or if m = 2 and $2 \le k$ then at least one of the two following statements is true:

i) g is strictly positive

ii) L has strictly more than two exceptional leaves.

We describe the canonical plumbing graph in the next theorem. Its proof follows immediately from theorem 4.1 and from the recipes in [N].

Theorem 4.2. 1. If k = l = 1 the canonical plumbing graph is a bamboo of rational curves, having (m-1) vertices with Euler number equal to (-2). This is the singularity A_{m-1} .

Assume from now on that $1 \leq k$ and that $2 \leq l$.

2. If k = 1 and m = 2 the plumbing graph has just one vertex with g = 0 and e = -2l.

3. Assume either that $3 \le m$ or that m = 2 and $2 \le k$. Then the canonical plumbing graph is never a bamboo of rational curves. More precisely:

3a. If k = l the graph has just one vertex with g = (m-1)(d-1) and e = m. Notice that g is strictly positive because d = k = l > 1.

3b. If k divides l but $k \neq l$ the graph is star-shaped with m branches. The central vertex has g = (m-1)(d-1) and e = 0. Each branch has just one vertex (tied to the central vertex by an edge). Its weights are g = 0 and $e = -\frac{l}{k}$.

3c. If k does not divide l then the graph is star-shaped with 2m branches. The central vertex has g = (m-1)(d-1) and e = -m.

There are m branches which are a bamboo of rational curves with $e'_i = -b'_i$ and b'_i defined by $b'_i \ge 2$ and

$$\frac{\alpha'}{\alpha'-\beta'} = [b_1', ..., b_u']$$

The vertex carrying the number 1 is joined to the central vertex by an edge.

There are also m branches which are a bamboo of rational curves with $e''_i = -b''_i$ and b''_i defined by $b''_i \ge 2$ and

$$\frac{\alpha^{\prime\prime}}{\alpha^{\prime\prime}-\beta^{\prime\prime}}=[b_1^{\prime\prime},...,b_v^{\prime\prime}]$$

Again, the vertex carrying the number 1 is joined to the central vertex by an edge.

Corollary 4.3. If L is not a lens space, it is never orientation preserving diffeomorphic to the boundary of a normal surface singularity.

Proof of corollary 4.3. L is not a lens space iff we are in case 3. We claim that the intersection form associated to the canonical plumbing graph is never negative definite. In cases 3a and 3b this is obvious since the self-intersection of the central vertex is ≥ 0 .

Let us suppose that we are in case 3c. We compute the rational Euler number e_0 of the Seifert structure on L. By definition

$$e_0 = e - \sum \frac{\beta_i}{\alpha_i}$$

¿From theorem 4.1 we deduce that

$$e_0 = m - m\frac{\beta'}{\bar{l}} - m\frac{\beta''}{\bar{k}}$$

Hence:

$$\bar{k}\bar{l}e_0 = m(\bar{k}\bar{l} - \beta'\bar{k} - \beta''\bar{l})$$

We shall prove later in this section that $(\bar{k}\bar{l} - \beta'\bar{k} - \beta''\bar{l}) = 1$. See lemma 4.6. Hence

$$e_0 = \frac{m}{\bar{k}\bar{l}} > 0$$

The conclusion follows from [N] Corallary 6 p.300. End of proof of corollary 4.3.

Proof of theorem 4.1. We shall compute the Seifert invariants from the data provided by the theorems proved in section 3.

We first determine the genus g. The Euler characteristic $\chi(G')$ is equal to (-ml+m+l). The classical formula for ramified coverings implies that the Euler characteristic χ' of the quotient of G' by the action generated by h' is equal to (-md + d + m). An analogous computation shows that $\chi'' = \chi'$. Hence the Euler characteristic χ of the base space of the Seifert foliation is equal to 2(-md + d + m) and we get g = (m-1)(d-1).

The computation of the Seifert invariants (α, β) is routine if we use the dictionary which translates Nielsen invariants into Seifert's.

It is sufficient for us to consider the following special case. Suppose that the angle of rotation at a fixed point of a monodromy h of finite order acting on an oriented surface is equal to $\frac{\omega}{\lambda}2\pi$ with $gcd(\omega, \lambda) = 1$. Define σ as the integer which satisfies $0 < \sigma < \lambda$ and $\omega\sigma \equiv 1 \pmod{\lambda}$. In the mapping torus of h, the Seifert invariant (α, β) for the exceptional leaf which corresponds to the fixed point is given by $\alpha = \lambda$ and $\beta = \sigma$ in normalized form. See [M]. The result follows now immediately from theorem 3.3.

The delicate part of the proof is to determine the Euler number e. As we feel that this invariant is rather elusive, we prefer to deal with closed objects.

Let \hat{G}' be the closed surface obtained from G' by attaching a disc on each of its $d_l = gcd(m, l)$ boundary components. We have seen (in the proof of Corollary 3.4) that the monodromy h' permutes them transitively. Let \hat{h}' be "the" finite order extension of h' on \hat{G}' . There is exactly one orbit of \hat{h}' which corresponds to the center of these discs. Its Nielsen invariant σ/\bar{l} is given by

$$\frac{\sigma}{\overline{l}} \equiv -m\frac{\beta'}{\overline{l}} \quad in \ \mathbf{Q} \bmod \mathbf{Z}$$

because the sum of all Nielsen quotients is equal to zero in $\mathbf{Q} \mod \mathbf{Z}$ for a closed surface.

Let \hat{M}' be the mapping torus of \hat{h}' acting on \hat{G}' . It is a closed Seifert manifold. It has m exceptional leaves with Seifert invariant (α', β') and one with Seifert invariant $(\hat{\alpha}', \hat{\beta}')$ which we choose to be defined as

$$\frac{\hat{\beta}'}{\hat{\alpha}'} = -m\frac{\beta'}{\alpha'}$$

where $\hat{\beta}'$ and $\hat{\alpha}'$ are by necessity chosen to be relatively prime. This choice has the advantage that the Euler number \hat{e}' for \hat{M}' is equal to zero, because the rational Euler number for \hat{M}' is equal to zero, as \hat{M}' is the mapping torus of a finite order monodromy acting on a closed surface. See [P].

We proceed along the same path with G'' and h'' to get a closed Seifert manifold \hat{M}'' with analogously defined Seifert invariants.

We now state a lemma about glueings of Seifert manifolds. The statement is painful (sorry!).

Lemma 4.4. Let V' and V'' be two closed oriented Seifert manifolds. Let H'_0 be a leaf in V' and let H''_0 be one in V''. Let N' be a foliated closed tubular neighborhood of H'_0 in V' and let N'' be one for H''_0 in V''.

Let s' be a section in V' (as usual possibly outside some discs in the base space) giving rise to an Euler number e' for V' and a Seifert invariant (a', b') for H'_0 . In a similar manner, let s'' be a section in V'' giving rise to the Euler number e'' for V'' and to the Seifert invariant (a'', b'') for H''_0 .

Seifert invariant (a'', b'') for H_0'' . Let $\check{V}' = V' \setminus Int(N')$ and $\check{V}'' = V'' \setminus Int(N'')$. Let V be such that $V = \check{V}' \cup \check{V}''$ and $\check{V}' \cap \check{V}'' = \partial\check{V}' \cap \partial\check{V}''$. This intersection is a torus and we write T for it. Suppose that the leaves H' from V' and H'' from V'' coincide on T (hence V is Seifert foliated).

Let m' be a meridian for N' on T and let m'' be one for N''. Let IN(m', m'') be the intersection number of m' and m'' on T, where T is oriented as the boundary of \check{V}'' .

Then the Euler number e for V (corresponding to a section s essentially built from s' and s'') is given by the equality $e = e' + e'' + \bar{e}$ where \bar{e} is computed from the equation

$$IN(m', m'') = a'b'' + a''b' + a'a''\bar{e}$$

Proof of lemma 4.4. As the section s is built from s' and s'' it follows from the definition of the Euler number as an obstruction (evaluated on a fundamental cycle) that e is the sum of e' and e'' plus a contribution coming from the fact that s' and s'' do not necessarily match along the torus T. The formula of theorem 3.5 will determine that contribution.

Following Seifert conventions we have

m' = a's' + b'H' with a' > 0 and m'' = a''s'' + b''H'' with a'' > 0

By hypothesis, we have H' = H'' = H. Let us choose an orientation (arbitrarily) for H. From Seifert conventions, this choice orients s' and s'' via IN(s', H) = +1 on T oriented as $\partial N'$ and IN(s'', H) = +1 on T oriented as $\partial N''$. This orients m' on $T = \partial N'$ via a' > 0 and m'' on $T = \partial N''$ via a'' > 0.

Notice that a change of orientation of H induces a change of orientation on both m' and m'' and hence the intersection number IN(m', m'') does not change. Let us compute that intersection number.

$$IN(m',m'') = IN((a's' + b'H), (a''s'' + b''H))$$

= $a'a''IN(s',s'') + a'b''IN(s',H) + a''b'IN(H,s'') + b'b''IN(H,H)$

We have:

1) IN(H, H) = 0 because the intersection form is alternating.

2) IN(s', H) = +1 from Seifert conventions, because T is oriented as the boundary of \check{V}'' which is the same as being oriented as the boundary of N'.

3) IN(H, s'') = +1 because IN(s'', H) = +1 if T is oriented as the boundary of N'' and two sign changes occur from the last equality to get the first one.

4) $IN(s', s'') = \bar{e}$. To see that the sign is correct, one way to argue is to go back to the definition of Euler numbers. Another way is to remark that this is the good sign in order to be sure that the sum $e' + e'' + \bar{e}$ remains constant under changes of s' (or s'') near the fiber H'_0 (or H''_0).

End of proof of lemma 4.4.

We now use lemma 4.4 to complete the determination of e. To make the argument simpler let us assume that

$$(d_k = gcd(m,k) = 1 \ ; \ d_l = gcd(m,l) = 1 \ ; \ d = gcd(k,l) = 1)$$

Recall that in this case \hat{M}' has m exceptional leaves with Seifert invariant $\alpha' = l$ and β' defined by $0 < \beta' < l$ and $(-k)\beta' \equiv 1 \pmod{l}$. \hat{M}' has one more exceptional leaf with Seifert invariant $(\hat{\alpha}', \hat{\beta}')$ defined by

$$\frac{\hat{\beta}'}{\hat{\alpha}'} = -m\frac{\beta'}{l}$$

As gcd(m, l) = 1 we have that $\hat{\alpha}' = l$. We have already seen that e' = 0.

Similarly, \hat{M}'' has *m* exceptional leaves with invariant $\alpha'' = k$ and β'' defined by $0 < \beta'' < k$ and $(-l)\beta'' \equiv 1 \pmod{k}$. \hat{M}'' has one more exceptional leaf with invariant $(\hat{\alpha}'', \hat{\beta}'')$ defined by

$$\frac{\hat{\beta}''}{\hat{\alpha}''} = -m\frac{\beta''}{k}$$

We have $\hat{\alpha}'' = k$ because gcd(m, k) = 1 and e'' = 0.

As gcd(m, l) = 1 the boundary $\partial G'$ is connected and $\partial G''$ is connected because gcd(m, k) = 1. As a consequence, the intersection number $IN(\partial G', \partial G'')$ is equal to IN(m', m'') UP TO SIGN.

Lemma 4.5. We have the equality $IN(m', m'') = -IN(\partial G', \partial G'')$.

Proof of lemma 4.5. The result comes from a comparison between the orientation of meridians coming from Seifert conventions and the orientation coming from $\partial G'$ (or $\partial G''$). What happens is that for one meridian both orientations agree and that for the other one they disagree. Which one it is depends on the orientation selected for H. End of proof of lemma 4.5.

We go on with the determination of the Euler number. The formula

$$IN(m', m'') = a'b'' + a''b' + a'a''\bar{e}$$

of lemma 4.4 translates into

$$m = l(-m\beta'') + k(-m\beta') + kl\bar{e}$$

Hence we have

(†)
$$m(1+l\beta''+k\beta') = kl\bar{e}$$

Lemma 4.6. We have the equality: $(\star\star)$ $1 + l\beta'' + k\beta' = kl$.

; From lemma 4.6 and formula (†) we deduce that $\bar{e} = m$ and hence that e = m because e' = 0 = e''. This completes the computation of e.

Proof of lemma 4.6. By definition we have

$$l\beta'' \equiv -1 \pmod{k}$$
 and $k\beta' \equiv -1 \pmod{k}$

Because gcd(k, l) = 1 we deduce that

$$l\beta'' + k\beta' \equiv -1 \pmod{kl}$$

In other words there exists an integer q such that

$$1 + l\beta'' + k\beta' = qkl$$

As $0 < \beta' < l$ and $0 < \beta'' < k$ the only possibility is q = 1. End of proof of lemma 4.6.

By carefully dividing by adequate gcd's an analogous argument works without assuming that $(d_k = gcd(m, k) = 1$; $d_l = gcd(m, l) = 1$; d = gcd(k, l) = 1). End of proof of theorem 4.1.

5. Examples.

Example 1. Let us consider the Hirzebruch singularity $z^{12} - x^5 y^{11} = 0$

The boundary \hat{L} of the normalization is the lens space L(12, 5). Its plumbing graph is a bamboo of three rational curves with Euler numbers successively $\{-3, -2, -3\}$.

The Seifert structure of the boundary L of the Milnor fiber is as follows: (g = 0 and e = 12). L has 24 exceptional leaves. There are 12 of them with Seifert invariant ($\alpha = 11, \beta = 2$) and 12 of them with Seifert invariant ($\alpha = 5, \beta = 4$).

The plumbing graph of L is star-shaped. The central vertex has weights g = 0 and e = -12. There are 24 bamboos of rational curves attached to the central vertex. Among them, 12 have Euler numbers equal successively to $\{-2, -2, -2, -2, -3\}$ and 12 of them have just one vertex with Euler number equal to $\{-5\}$.

Example 2. Let us consider the Hirzebruch singularity $z^{12} - x^{17}y^{11} = 0$. In order to make the comparison between examples 1 and 2 easier, we drop the restriction $k \leq l$.

The boundary \hat{L} of the normalization is the same as in example 1, because 5 is congruent to 17 (mod 12).

But the boundaries L of the Milnor fibers are different. In fact, the Seifert invariants for the exceptional leaves differ. L has 12 leaves with Seifert invariant ($\alpha = 11$, $\beta = 9$) and 12 leaves with Seifert invariant ($\alpha = 17$, $\beta = 3$).

The plumbing graph of L is again star-shaped, as it should be. The central vertex has again weights g = 0 and e = -12. There are 24 bamboos of rational curves attached to the central vertex. Among them, 12 have Euler numbers equal successively to $\{-6, -2\}$ and 12 of them have Euler numbers successively equal to $\{-2, -2, -2, -2, -2, -2, -2\}$.

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