2

LINEAR SYMPLECTIC GEOMETRY

The geometry of a bilinear skew form is very different from that of a symmetric form and its main features, for example compatible complex structures and Lagrangian subspaces, appear later as significant elements of the global theory. Moreover, it is increasingly apparent that the linear theory is a profound paradigm for the nonlinear theory. For example, the fact that all symplectic vector spaces of the same dimension are isomorphic translates into the statement that all symplectic manifolds are locally diffeomorphic (Darboux's theorem in Chapter 3). A more startling example is Gromov's nonsqueezing theorem which implies that a ball can be symplectically embedded into a cylinder if and only if it has a smaller radius. We shall discuss the linear version of this result in Section 2.4.

The theory is enriched by the interplay between symmetric and skew-symmetric forms. This appears in many guises: either directly, as in the theorem that a nondegenerate symmetric bilinear form and a nondegenerate skew-symmetric form have a common diagonalization, or indirectly, for example in the theorem that the eigenvalues of a symplectic matrix occur in pairs of the form $\lambda, 1/\lambda$. In the presence of a skew-symmetric form, a symmetric form gives rise to other related geometric objects such as Lagrangian subspaces and almost complex structures, and we shall also explore the elementary theory of these objects.

We begin this chapter by discussing symplectic vector spaces (Section 2.1), linear symplectomorphisms (Section 2.2), and Lagrangian subspaces (Section 2.3). These can be thought of as the three fundamental notions in linear symplectic geometry. In Section 2.4 we prove the linear nonsqueezing theorem and discuss its consequences for the action of the symplectic linear group on ellipsoids. We prove that a linear transformation is symplectic (or anti-symplectic) if and only if it preserves the *linear symplectic area* of ellipsoids. Section 2.5 is about almost complex structures. Finally, in Section 2.6 on symplectic vector bundles, we develop the theory of the first Chern class from scratch.

The material of the first three sections in this chapter is essential for the study of symplectic manifolds in Chapter 3. The linear nonsqueezing theorem will play an important role in the proof of symplectic rigidity in Chapter 12. However, the various discussions of the Maslov index and also the subsections on trivializations and the first Chern class may be omitted at first reading.

2.1 Symplectic vector spaces

The archetypal example of a symplectic vector space is the Euclidean space $\mathbb{R}^{2n}$ with the skew-symmetric form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j .$$

More generally, a **symplectic vector space** is a pair (V, ω) consisting of a finite dimensional real vector space V and a nondegenerate skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$. This means that the following conditions are satisfied:

(skew-symmetry) For all $v, w \in V$

$$\omega(v, w) = -\omega(w, v).$$

(nondegeneracy) For every $v \in V$

$$\omega(v, w) = 0 \quad \forall\, w \in V \qquad \Longrightarrow \qquad v = 0.$$

The vector space V is necessarily of even dimension since a real skew-symmetric matrix of odd dimension must have a kernel. (See also Exercise 2.13 below.)

A **linear symplectomorphism** of the symplectic vector space (V, ω) is a vector space isomorphism $\Psi : V \to V$ which preserves the symplectic structure in the sense that

$$\Psi^* \omega = \omega,$$

where $\Psi^* \omega(v, w) = \omega(\Psi v, \Psi w)$ for $v, w \in V$. The linear symplectomorphisms of (V, ω) form a group which we denote by $\mathrm{Sp}(V, \omega)$. In the case of the standard symplectic structure on Euclidean space we use the notation $\mathrm{Sp}(2n) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

The **symplectic complement** of a linear subspace $W \subset V$ is defined as the subspace

$$W^{\omega} = \{v \in V \,|\, \omega(v, w) = 0 \quad \forall\, w \in W\}.$$

The symplectic complement need not be transversal to W. A subspace W is called

isotropic if $W \subset W^{\omega}$,
coisotropic if $W^{\omega} \subset W$,
symplectic if $W \cap W^{\omega} = \{0\}$,
Lagrangian if $W = W^{\omega}$.

Note that W is isotropic if and only if ω vanishes on W, and W is symplectic if and only if $\omega|_W$ is nondegenerate. The following exercise shows that Lagrangian subspaces are closely related to linear symplectomorphisms.

Exercise 2.1 Let (V, ω) be a symplectic vector space and $\Psi : V \to V$ be a linear map. Prove that Ψ is a linear symplectomorphism if and only if its graph

$$\Gamma_\Psi = \{(v, \Psi v) \mid v \in V\}$$

is a Lagrangian subspace of $V \times V$ with symplectic form $(-\omega) \times \omega$. (The symplectic form is standard in the target and reversed in the source.) □

The next lemma shows that W is symplectic if and only if W^ω is symplectic. It also shows that every Lagrangian subspace has half the dimension of V and that W is isotropic if and only if W^ω is coisotropic.

Lemma 2.2 *For any subspace $W \subset V$,*

$$\dim W + \dim W^\omega = \dim V, \qquad W^{\omega\omega} = W.$$

Proof: Define a map ι_ω from V to the dual space V^* by setting

$$\iota_\omega(v)(w) = \omega(v, w).$$

Since ω is nondegenerate ι_ω is an isomorphism. It identifies W^ω with the annihilator $W^\perp$ of W in V^*. But for any subspace W of any vector space V we have $\dim W + \dim W^\perp = \dim V$. □

The following result is the main theorem of this section. It asserts that all symplectic vector spaces of the same dimension are linearly symplectomorphic.

Theorem 2.3 *Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that*

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \qquad \omega(u_j, v_k) = \delta_{jk}.$$

Such a basis is called a **symplectic basis**. *Moreover, there exists a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$ such that*

$$\Psi^*\omega = \omega_0.$$

Proof: The proof is by induction over n. Since ω is nondegenerate there exist vectors $u_1, v_1 \in V$ such that

$$\omega(u_1, v_1) = 1.$$

Hence the subspace spanned by u_1 and v_1 is symplectic. Let W denote its symplectic complement. Then (W, ω) is a symplectic vector space of dimension $2n-2$. By the induction hypothesis, there exists a symplectic basis $u_2, \ldots, u_n, v_2, \ldots, v_n$ of W. Hence the vectors $u_1, \ldots, u_n, v_1, \ldots, v_n$ form a symplectic basis of V. The linear map $\Psi : \mathbb{R}^{2n} \to V$ defined by

$$\Psi z = \sum_{j=1}^{n} x_j u_j + \sum_{j=1}^{n} y_j v_j$$

satisfies $\Psi^* \omega = \omega_0$ as required. □

A symplectic basis is sometimes called an **ω-standard basis.**

Corollary 2.4 *Suppose that ω_t is a smooth family of nondegenerate skew-symmetric bilinear forms on $\mathbb{R}^{2n}$ depending on a parameter t. Then there exists a smooth family of matrices $\Psi_t \in \mathbb{R}^{2n \times 2n}$ such that $\Psi_t^* \omega_t = \omega_0$ for every t.*

Proof: Theorem 2.3 and Gram–Schmidt. □

Corollary 2.5 *If V is a $2n$-dimensional real vector space then a skew-symmetric bilinear form ω on V is nondegenerate if and only if the n-fold exterior power is nonzero:*

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0.$$

Proof: Assume first that ω is degenerate. Let $v \neq 0$ such that $\omega(v, w) = 0$ for all $w \in V$. Now choose a basis $v_1, \ldots, v_{2n}$ of V such that $v_1 = v$. Then $\omega^n(v_1, \ldots, v_{2n}) = 0$. Conversely, suppose that ω is nondegenerate. Then, since ω_0^n is a volume form, it follows from Theorem 2.3 that $\omega^n \neq 0$. □

Lemma 2.6 *Any isotropic subspace is contained in a Lagrangian subspace. Moreover, any basis $u_1, \ldots, u_n$ of a Lagrangian subspace Λ can be extended to a symplectic basis of (V, ω).*

Proof: Let W be an isotropic subspace. If the subspace W_1 is obtained by adjoining some vector $v \in W^\omega - W$ to W, then ω vanishes on W_1. Hence a maximal isotropic subspace must be Lagrangian. Because $\Lambda \subset W$ implies $W^\omega \subset \Lambda^\omega$, it follows that the Lagrangian subspaces are precisely the maximal isotropic subspaces. This proves the first statement.

To prove the second statement it suffices to consider the case $V = \mathbb{R}^{2n}$ with the standard symplectic structure. Given a Lagrangian subspace Λ the subspace $\Lambda' = J_0 \Lambda$ is also Lagrangian and can be identified with the dual space Λ^* via the isomorphism $\iota_{\omega_0} : \mathbb{R}^{2n} \to \mathbb{R}^{2n*}$ of Lemma 2.2. Hence we may choose $\{v_1, \ldots, v_n\} \subset \Lambda'$ to be the basis dual to $u_1, \ldots, u_n$. □

Linear symplectic reduction

Every coisotropic subspace $W \subset V$ gives rise to a new symplectic vector space obtained by dividing W by its symplectic complement. This construction of a subquotient is called symplectic reduction.

Lemma 2.7 *Let (V, ω) be a symplectic vector space and $W \subset V$ be a coisotropic subspace. Then the following hold:*
(i) *The quotient $V' = W/W^\omega$ carries a natural symplectic structure ω' induced by ω.*
(ii) *If $\Lambda \subset V$ is a Lagrangian subspace then $\Lambda' = ((\Lambda \cap W) + W^\omega)/W^\omega$ is a Lagrangian subspace of V'.*

Proof: Denote $[w] = w + W^\omega \in V'$ for $w \in W$. By the definition of coisotropic, W^ω is an isotropic subspace of W and $\omega(v, w) = 0$ whenever $v \in W^\omega$ and $w \in W$. Hence $\omega(w_1, w_2)$ depends only on the equivalence classes $[w_1]$ and $[w_2]$ in $V' = W/W^\omega$. Hence ω induces a 2-form ω' on V'. Moreover, if $w \in W$ and $\omega(v, w) = 0$ for all $v \in W$, then $w \in W^\omega$ and hence ω' is nondegenerate. This proves (i).

To prove (ii) we first show that $\widetilde{\Lambda} = (\Lambda \cap W) + W^\omega$ is a Lagrangian subspace of V:

$$\begin{aligned} \widetilde{\Lambda}^\omega &= (\Lambda \cap W)^\omega \cap W \\ &= (\Lambda + W^\omega) \cap W \\ &= (\Lambda \cap W) + W^\omega \\ &= \widetilde{\Lambda}. \end{aligned}$$

Now let $w \in W$ such that $\omega'([w], [v]) = 0$ for all $[v] \in \Lambda' = \widetilde{\Lambda}/W^\omega$. Then $\omega(w, v) = 0$ for all $v \in \widetilde{\Lambda}$ and hence $w \in \widetilde{\Lambda}^\omega = \widetilde{\Lambda}$. This implies $\Lambda' \in \mathcal{L}(V', \omega')$. Thus we have proved (ii). □

Example 2.8 Let (V_0, ω_0) and (V_1, ω_1) be symplectic vector spaces of the same dimension, $\Lambda_0 \subset V_0$ be a Lagrangian subspace, and $\Psi : V_0 \to V_1$ be a linear symplectomorphism. Consider the symplectic vector space $V = V_0 \times V_0 \times V_1$ with symplectic form $\omega = \omega_0 \times (-\omega_0) \times \omega_1$. Then

$$W = \Delta \times V_1 \subset V$$

is a coisotropic subspace with isotropic complement $W^\omega \simeq \Delta \times \{0\}$ and quotient $V' = W/W^\omega \simeq V_1$. The subspace

$$\Lambda = \Lambda_0 \times \Gamma_\Psi \subset V$$

is Lagrangian and intersects W transversally. The reduced Lagrangian subspace is isomorphic to $\Lambda_1 = \Psi\Lambda_0$. □

Exercises

Exercise 2.9 Identify a matrix with its graph as in Exercise 2.1 and use a construction similar to that in Example 2.8 to interpret the composition of symplectic matrices in terms of symplectic reduction. □

Exercise 2.10 Let (V, ω) be a symplectic vector space and $W \subset V$ be any subspace. Prove that the quotient $V' = W/W \cap W^\omega$ carries a natural symplectic structure. □

Exercise 2.11 Let $A = -A^T \in \mathbb{R}^{2n \times 2n}$ be a nondegenerate skew-symmetric matrix and define $\omega(z, w) = \langle Az, w \rangle$. Prove that a symplectic basis for $(\mathbb{R}^{2n}, \omega)$ can be constructed from the eigenvectors $u_j + iv_j$ of A. **Hint:** Use the fact that the matrix $iA \in \mathbb{C}^{2n \times 2n}$ is self-adjoint and therefore can be diagonalized. □

Exercise 2.12 Consider a smooth family of symplectic forms $\omega_t(z, w) = \langle A_t z, w \rangle$ on $\mathbb{R}^{2n}$. Prove Corollary 2.4 by considering the family of subspaces $E_t \subset \mathbb{C}^{2n}$ generated by the eigenvectors of A_t corresponding to eigenvalues with a positive imaginary part. □

Exercise 2.13 Show that if β is any skew-symmetric bilinear form on the vector space W, there is a basis $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_p$ of W such that $\beta(u_j, v_k) = \delta_{jk}$ and all other pairings $\beta(b_1, b_2)$ vanish. A basis with this property is called a **standard basis** for (W, β), and the integer $2n$ is the **rank** of β. □

Exercise 2.14 Show that if W is an isotropic, coisotropic or symplectic subspace of (V, ω) then any standard basis for (W, ω) extends to a symplectic basis for (V, ω). □

Exercise 2.15 Show that any hyperplane W in a $2n$-dimensional symplectic vector space (V, ω) is coisotropic. Thus $W^\omega \subset W$ and $\omega|_W$ has rank $2(n-1)$. **Hint:** By Exercise 2.13 the 2-form $\omega|_W$ has even rank. Hence there is some nonzero vector $w \in W$ such that $\omega(w, x) = 0$ for all $x \in W$. Show that this vector w spans W^ω. □

Exercise 2.16 Let $\Omega(V)$ denote the space of all symplectic forms on the vector space V. By considering the action of $\mathrm{GL}(2n, \mathbb{R})$ on $\Omega(V)$ given by

$$\omega \mapsto \Psi^* \omega$$

show that $\Omega(V) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$. □

Exercise 2.17 (The Gelfand–Robbin quotient) It has been noted by physicists for a long time that symplectic structures often arise from boundary value problems. The underlying abstract principle can be formulated as follows. Let H be a Hilbert space and $D : \mathrm{dom}(D) \to H$ be a symmetric linear operator with a closed graph and a dense domain $\mathrm{dom}(D) \subset H$. Prove that the quotient

$$V = \mathrm{dom}(D^*)/\mathrm{dom}(D)$$

is a symplectic vector space with symplectic structure

$$\omega([x],[y]) = \langle x, D^*y\rangle - \langle D^*x, y\rangle$$

for $x, y \in \mathrm{dom}(D^*)$. Here $[x] \in \mathrm{dom}(D^*)/\mathrm{dom}(D)$ denotes the equivalence class of $x \in \mathrm{dom}(D^*)$. Show that self-adjoint extensions of D are in one-to-one correspondence to Lagrangian subspaces $\Lambda \subset V$. If D has a closed range, show that the kernel of D^* determines a Lagrangian subspace

$$\Lambda_0 = \{[x] \mid x \in \mathrm{dom}(D^*),\ D^*x = 0\}.$$

In applications D is a differential operator on a manifold with boundary, V is a suitable space of boundary data, and the symplectic form can, via Stokes' theorem, be expressed as an integral over the boundary. **Hints:** The symmetry condition asserts that $\langle x, Dy\rangle = \langle Dx, y\rangle$ for all $x, y \in \mathrm{dom}(D)$. Recall that the domain of the operator $D^* : \mathrm{dom}(D^*) \to H$ is defined as the set of all vectors $y \in H$ such that the linear functional $\mathrm{dom}(D) \to \mathbb{R} : x \mapsto \langle y, Dx\rangle$ extends to a bounded linear functional on H. Thus for $y \in \mathrm{dom}(D^*)$ there exists a unique vector $z \in H$ such that $\langle z, x\rangle = \langle y, Dx\rangle$ for all $x \in \mathrm{dom}(D)$ and one defines $D^*y = z$. The symmetry condition is equivalent to $\mathrm{dom}(D) \subset \mathrm{dom}(D^*)$ and $D^*y = Dy$ for $y \in \mathrm{dom}(D)$. A symmetric operator is called **self-adjoint** if $D^* = D$. Show that a linear operator $D : \mathrm{dom}(D) \to H$ is self-adjoint iff $\mathrm{graph}(D) \subset H \times H$ is a Lagrangian subspace with respect to the standard symplectic structure. Interpret the exercise as linear symplectic reduction. □

Exercise 2.18 Consider the linear operator

$$D = J_0 \frac{\partial}{\partial t}, \qquad J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

on the Hilbert space $H = L^2([0,1], \mathbb{R}^{2n})$ with $\mathrm{dom}(D) = W_0^{1,2}([0,1], \mathbb{R}^{2n})$ (the Sobolev space of absolutely continuous functions which vanish on the boundary and whose first derivative is square integrable). Show that in this case the Gelfand–Robbin quotient is given by $V = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with symplectic form $(-\omega_0) \times \omega_0$. □

2.2 The symplectic linear group

In this section we shall examine the group $\mathrm{Sp}(V, \omega)$ of linear symplectomorphisms in more detail. Since, in view of Theorem 2.3, all symplectic vector spaces of the same dimension are isomorphic it suffices to consider the case $V = \mathbb{R}^{2n}$ with the standard symplectic form ω_0. In this case we can think of the elements of $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$ as real $2n \times 2n$ matrices Ψ which satisfy

$$\Psi^{\mathrm{T}} J_0 \Psi = J_0.$$

Recall from Chapter 1 that this is equivalent to the condition

$$\Psi^*\omega_0 = \omega_0.$$

Recall also that matrices which satisfy this condition are called **symplectic** and that they have determinant 1 (Lemma 1.14). In the complex case we denote the group of symplectic matrices by $\mathrm{Sp}(2n, \mathbb{C})$.

We may identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way with $z = (x, y)$ corresponding to $x + iy$ for $x, y \in \mathbb{R}^n$. Then multiplication by J_0 in $\mathbb{R}^{2n}$ corresponds to multiplication by i in $\mathbb{C}^n$. With this identification the complex linear group $\mathrm{GL}(n, \mathbb{C})$ is a subgroup of $\mathrm{GL}(2n, \mathbb{R})$ and $\mathrm{U}(n)$ is a subgroup of $\mathrm{Sp}(2n)$.

The following lemma demonstrates the close connection between symplectic and complex linear maps. It is the first step on the way to proving the important Proposition 2.22 which states that the unitary group $\mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{Sp}(2n)$. As usual, we denote the orthogonal group by $\mathrm{O}(2n)$.*

Lemma 2.19

$$\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n).$$

Proof: A real $2n \times 2n$ matrix Ψ satisfies the following three identities:

$$\begin{aligned} \Psi \in \mathrm{GL}(n, \mathbb{C}) \quad &\Longleftrightarrow \quad \Psi J_0 = J_0 \Psi, \\ \Psi \in \mathrm{Sp}(2n) \quad &\Longleftrightarrow \quad \Psi^{\mathrm{T}} J_0 \Psi = J_0, \\ \Psi \in \mathrm{O}(2n) \quad &\Longleftrightarrow \quad \Psi^{\mathrm{T}} \Psi = \mathbb{1}. \end{aligned}$$

Any two of these conditions imply the third. Now the subgroup $\mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ consists of those matrices

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$$

which satisfy

$$X^{\mathrm{T}} Y = Y^{\mathrm{T}} X, \qquad X^{\mathrm{T}} X + Y^{\mathrm{T}} Y = \mathbb{1}.$$

(See Exercise 1.13.) This is precisely the condition on $U = X + iY$ to be unitary. □

Lemma 2.20 *Let $\Psi \in \mathrm{Sp}(2n)$. Then*

$$\lambda \in \sigma(\Psi) \quad \Longleftrightarrow \quad \lambda^{-1} \in \sigma(\Psi)$$

*For notational convenience we identify $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{U}(n)$ here with subgroups of $\mathrm{GL}(2n, \mathbb{R})$. However, in some cases it will be important to keep the distinction between an orthogonal symplectic $2n \times 2n$ matrix and the corresponding unitary $n \times n$ matrix.

and the multiplicities of λ *and* λ^{-1} *agree. If* ± 1 *is an eigenvalue of* Ψ *then it occurs with even multiplicity. Moreover,*

$$\Psi z = \lambda z, \quad \Psi z' = \lambda' z', \quad \lambda\lambda' \neq 1 \qquad \Longrightarrow \qquad \omega_0(z, z') = 0.$$

Proof: The first statement follows from the fact that Ψ^{T} is similar to Ψ^{-1}:

$$\Psi^{\mathrm{T}} = J_0 \Psi^{-1} J_0^{-1}.$$

Hence the total multiplicity of all eigenvalues not equal to 1 or -1 is even. Since the determinant is the product of all eigenvalues it follows from Lemma 1.14 that if -1 is an eigenvalue then it occurs with even multiplicity. Hence the eigenvalue 1 occurs with even multiplicity as well. The last statement follows from the identity

$$\lambda\lambda' \langle z', J_0 z \rangle = \langle \Psi z', J_0 \Psi z \rangle = \langle z', J_0 z \rangle. \qquad \square$$

Lemma 2.21 *If* $P = P^{\mathrm{T}} \in \mathrm{Sp}(2n)$ *is a symmetric, positive definite symplectic matrix then* $P^\alpha \in \mathrm{Sp}(2n)$ *for every real number* $\alpha > 0$.

Proof: By Lemma 1.14, it suffices to show that each P^α preserves the form ω_0, i.e. that

$$\omega_0(P^\alpha z, P^\alpha z') = \omega_0(z, z')$$

for all z and z'. To see this decompose $\mathbb{R}^{2n}$ into a direct sum of eigenspaces V_λ for P where $\lambda \in \sigma(P)$. Then V_λ is the eigenspace of P^α corresponding to the eigenvalue λ^α. Lemma 2.20 shows that if $\lambda\lambda' \neq 1$ then the spaces V_λ and $V_{\lambda'}$ are orthogonal with respect to the form ω_0. In particular, the form ω_0 vanishes on V_λ for $\lambda \neq 1$. Hence for $z \in V_\lambda$ and $z' \in V_{\lambda'}$ we have

$$\omega_0(P^\alpha z, P^\alpha z') = (\lambda\lambda')^\alpha \omega_0(z, z') = \omega_0(z, z').$$

Since every vector in $\mathbb{R}^{2n}$ is a sum of eigenvectors of P, the result now follows easily. $\square$

Proposition 2.22 *The unitary group* $\mathrm{U}(n)$ *is a maximal compact subgroup of* $\mathrm{Sp}(2n)$ *and the quotient* $\mathrm{Sp}(2n)/\mathrm{U}(n)$ *is contractible.*

Proof: First let us prove that the quotient $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible. Now, every matrix $\Psi \in \mathrm{Sp}(2n)$ can be uniquely decomposed as

$$\Psi = PQ,$$

where P is symmetric and positive definite and Q is orthogonal. By the preceding lemma

$$P = (\Psi\Psi^{\mathrm{T}})^{1/2}$$

is symplectic. Hence the map

$$\mathrm{Sp}(2n) \times [0,1] \to \mathrm{Sp}(2n) : (\Psi, t) \mapsto (\Psi\Psi^{\mathrm{T}})^{-t/2}\Psi$$

is a retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$.

To see that $\mathrm{U}(n)$ is a maximal compact subgroup, let $\mathrm{G} \subset \mathrm{Sp}(2n)$ be any compact subgroup. We must show that G is conjugate to a subgroup of $\mathrm{U}(n)$. To prove this, we choose a symmetric and positive definite matrix $P \in \mathrm{Sp}(2n)$ such that

$$\Psi^{\mathrm{T}} P \Psi = P \qquad \text{for} \qquad \Psi \in \mathrm{G}.$$

Such a matrix can be obtained by averaging the matrices $\Psi^{\mathrm{T}}\Psi$ over $\Psi \in \mathrm{G}$ using the Haar measure $C(\mathrm{G}, \mathbb{R}) \to \mathbb{R}$ for a compact Lie group.* Since $P^{1/2}$ is a symplectic matrix we obtain

$$\Psi \in \mathrm{G} \quad \Longrightarrow \quad P^{1/2}\Psi P^{-1/2} \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n).$$

This proves the proposition. □

Proposition 2.23 *The fundamental group of* $\mathrm{U}(n)$ *is isomorphic to the integers. The determinant map* $\det : \mathrm{U}(n) \to S^1$ *induces an isomorphism of fundamental groups.*

Proof: The determinant map $\det : \mathrm{U}(n) \to S^1$ is a fibration with fibre $\mathrm{SU}(n)$. Hence the homotopy exact sequence

$$\pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(S^1) \to \pi_0(\mathrm{SU}(n))$$

shows that $\pi_1(\mathrm{U}(n)) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

Here we have used the fact that $\mathrm{SU}(n)$ is simply connected. This is best seen by an induction argument. It obviously holds for $n = 1$. So suppose $n \geq 2$ and consider the map $\mathrm{SU}(n) \to S^{2n-1}$ that sends a matrix $U \in \mathrm{SU}(n)$ to its first column. This is a fibration with fibre $\mathrm{SU}(n-1)$. Hence there is an exact sequence

$$\pi_2(S^{2n-1}) \to \pi_1(\mathrm{SU}(n-1)) \to \pi_1(\mathrm{SU}(n)) \to \pi_1(S^{2n-1})$$

and this shows that if $\mathrm{SU}(n-1)$ is simply connected then so is $\mathrm{SU}(n)$. □

*Here $C(\mathrm{G}, \mathbb{R})$ denotes the space of continuous functions on G. Observe that P gives rise to an inner product g_P defined by $g_P(v, w) = w^{\mathrm{T}} P v$. Thus choosing P is equivalent to choosing a G-invariant inner product on the vector space $\mathbb{R}^{2n}$, which is compatible with ω_0 in the sense that it has the form $\omega_0(\cdot, J\cdot)$ for some ω_0-compatible almost complex structure J. A more elementary proof of this result is given in Section 2.5.

With these few lemmata we have obtained a great deal of information about symplectic matrices. In particular, any such matrix can be decomposed as a product of a positive definite symplectic matrix with a unitary matrix. The following exercises take these ideas somewhat further. More information about the structure of symplectic matrices and its implications for the stability of Hamiltonian flows may be found in the articles by Arnold and Givental [8] and Lalonde and McDuff [158] and the book by Ekeland [63].

Exercises

Exercise 2.24 **(i)** Show that if $\Psi \in \mathrm{Sp}(2n)$ is diagonalizable, it can be diagonalized by a symplectic matrix.
(ii) Deduce from Lemma 2.20 that the eigenvalues of $\Psi \in \mathrm{Sp}(2n)$ occur either in pairs $\lambda, 1/\lambda \in \mathbb{R}$, $\lambda, \bar{\lambda} \in S^1$ or in complex quadruplets

$$\lambda, \frac{1}{\lambda}, \bar{\lambda}, \frac{1}{\bar{\lambda}}.$$

(iii) Work out the conjugacy classes for matrices in Sp(2) and Sp(4): see [8] and [158]. □

Exercise 2.25 Use the argument of Proposition 2.22 to prove that the inclusion

$$\mathrm{O}(2n)/\mathrm{U}(n) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$$

of homogeneous spaces is a homotopy equivalence. Prove similarly that the inclusion

$$\mathrm{O}(2n)/\mathrm{U}(n) \hookrightarrow \mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$$

is a homotopy equivalence. We will see below that $\mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$ can be identified with the space of symplectic structures on $\mathbb{R}^{2n}$. Similarly, the homogeneous space $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$ can be identified with the space of complex structures on $\mathbb{R}^{2n}$. □

Exercise 2.26 Let $\mathrm{SP}(n, \mathbb{H})$ denote the group of quaternionic matrices $W \in \mathbb{H}^{n\times n}$ such that $W^*W = \mathbb{1}$. Prove that $\mathrm{SP}(n, \mathbb{H})$ is a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{C})$ and that the quotient $\mathrm{Sp}(2n, \mathbb{C})/\mathrm{SP}(n, \mathbb{H})$ is contractible. □

Exercise 2.27 Let

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}).$$

What is the relationship between $\det \Psi \in \mathbb{R}$ and $\det(X+iY) \in \mathbb{C}$? Compare the proof of Theorem 2.29. □

Exercise 2.28 The **Siegel upper half space** $\mathcal{S}_n$ is the space of complex symmetric matrices $Z = X + iY \in \mathbb{C}^{n\times n}$ with positive definite imaginary part Y. The symplectic group $\mathrm{Sp}(2n)$ acts on $\mathcal{S}_n$ via fractional linear transformations $\Psi_* : \mathcal{S}_n \to \mathcal{S}_n$ defined by

$$\Psi_* Z = (AZ + B)(CZ + D)^{-1}, \qquad \Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here we use the notation of Exercise 1.13. Prove that Ψ_* is well defined: if $Z \in \mathcal{S}_n$ then the matrix $CZ + D$ is invertible and $\Psi_* Z \in \mathcal{S}_n$. Prove that

$$\Psi_* \Phi_* Z = (\Psi\Phi)_* Z$$

for $\Phi, \Psi \in \mathrm{Sp}(2n)$ and $Z \in \mathcal{S}_n$. Prove that the action is transitive. Prove that

$$\Psi_*(i\mathbb{1}) = i\mathbb{1} \qquad \Longleftrightarrow \qquad \Psi \in \mathrm{U}(n).$$

Deduce that the map $\Psi \mapsto \Psi_*(i\mathbb{1})$ induces a diffeomorphism from the homogeneous space $\mathrm{Sp}(2n)/\mathrm{U}(n)$ to the Siegel upper half space $\mathcal{S}_n$. Thus the quotient $\mathrm{Sp}(2n)/\mathrm{U}(n)$ inherits the complex structure of $\mathcal{S}_n$. For more details see Siegel [258]. □

The Maslov index

It follows from Proposition 2.22 and Proposition 2.23 that the fundamental group of $\mathrm{Sp}(2n)$ is isomorphic to the integers. An explicit isomorphism $\pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$ is given by the Maslov index.*

Theorem 2.29 *There exists a unique functor μ, called the* **Maslov index,** *which assigns an integer $\mu(\Psi)$ to every loop*

$$\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$$

of symplectic matrices and satisfies the following axioms:

(homotopy) *Two loops in $\mathrm{Sp}(2n)$ are homotopic if and only if they have the same Maslov index.*

(product) *For any two loops $\Psi_1, \Psi_2 : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$ we have*

$$\mu(\Psi_1\Psi_2) = \mu(\Psi_1) + \mu(\Psi_2).$$

In particular, the constant loop $\Psi(t) \equiv \mathbb{1}$ has Maslov index 0.

***Warning:** In the following we shall use the notation $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$ for a loop $\Psi(t) = \Psi(t+1)$ of symplectic matrices. Sometimes we shall also use the same letter to denote an individual symplectic matrix. In each case it should be clear from the context which notation is used.

(direct sum) *If $n = n' + n''$ identify* $\mathrm{Sp}(2n') \oplus \mathrm{Sp}(2n'')$ *in the obvious way with a subgroup of* $\mathrm{Sp}(2n)$. *Then*

$$\mu(\Psi' \oplus \Psi'') = \mu(\Psi') + \mu(\Psi'').$$

(normalization) *The loop* $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{U}(1) \subset \mathrm{Sp}(2)$ *defined by* $\Psi(t) = e^{2\pi i t}$ *has Maslov index* 1.

Proof: Define the map $\rho : \mathrm{Sp}(2n) \to S^1$ by

$$\rho(\Psi) = \det(X + iY), \qquad \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = (\Psi\Psi^{\mathrm{T}})^{-1/2}\Psi \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n),$$

for $\Psi \in \mathrm{Sp}(2n)$. (Here Ψ is an individual matrix, not a loop.) The matrix $Q = (\Psi\Psi^{\mathrm{T}})^{-1/2}\Psi$ is the orthogonal part of Ψ in the polar decomposition $\Psi = PQ$. The **Maslov index** of the loop $\Psi(t) = \Psi(t+1) \in \mathrm{Sp}(2n)$ is the degree of the composition $\rho \circ \Psi : \mathbb{R}/\mathbb{Z} \to S^1$:

$$\mu(\Psi) = \deg \rho \circ \Psi.$$

In other words

$$\mu(\Psi) = \alpha(1) - \alpha(0)$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a lift of $\rho \circ \Psi$:

$$\det(X(t) + iY(t)) = e^{2\pi i\, \alpha(t)}.$$

The Maslov index is obviously an integer and depends only on the homotopy class of Ψ. By Proposition 2.22 and Proposition 2.23 the map $\rho : \mathrm{Sp}(2n) \to S^1$ induces an isomorphism of fundamental groups. This proves the homotopy axiom. The product axiom is obvious for loops of unitary matrices. Hence it follows from the homotopy axiom (every symplectic loop is homotopic to a unitary loop). The additivity and normalization axioms are obvious. This proves the existence part of the theorem. Uniqueness is left to the reader. □

We now sketch an alternative interpretation of the Maslov index as the intersection number of a loop in $\mathrm{Sp}(2n)$ with the **Maslov cycle**

$$\overline{\mathrm{Sp}_1}(2n)$$

of all symplectic matrices Ψ which satisfy $\det(B) = 0$ in the decomposition of Exercise 1.13. This set is a singular hypersurface of codimension 1 which admits a natural coorientation.* It is stratified by the rank of the matrix

*A coorientation is an orientation of the normal bundle.

B, and so a generic loop will intersect only the highest stratum (where the rank of B is $n-1$) and all the intersections are transverse. More explicitly let

$$\Psi(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$$

be a path of symplectic matrices. A **crossing** is a number t such that $\Psi(t) \in \overline{\mathrm{Sp}}_1(2n)$. A crossing t is called **regular** if the **crossing form** $\Gamma(\Psi, t) : \ker B(t) \to \mathbb{R}$, defined by

$$\Gamma(\Psi, t)(y) = -\langle \dot{B}(t)y, D(t)y \rangle, \qquad y \in \ker B(t),$$

is nonsingular. At a regular crossing the **crossing index** is the signature (the number of positive minus the number of negative eigenvalues) of the crossing form. The Maslov index of a loop $\Psi(t)$ with only regular crossings can be defined by

$$\mu(\Psi) = \frac{1}{2} \sum_t \operatorname{sign} \Gamma(\Psi, t),$$

where the sum runs over all crossings. The usual arguments in differential topology (e.g. [208]) show that this number is a homotopy invariant, is therefore well defined for all loops, and satisfies the axioms of the above theorem. It follows that both our definitions of the Maslov index agree. For more details the reader may consult Robbin and Salamon [236].

2.3 Lagrangian subspaces

In this section we shall discuss Lagrangian subspaces in more detail. We denote by $\mathcal{L}(V, \omega)$ the set of Lagrangian subspaces of (V, ω) and abbreviate

$$\mathcal{L}(n) = \mathcal{L}(\mathbb{R}^{2n}, \omega_0).$$

In more explicit terms Lagrangian subspaces of $\mathbb{R}^{2n}$ are characterized as follows.

Lemma 2.30 *Let X and Y be real $n \times n$ matrices and define $\Lambda \subset \mathbb{R}^{2n}$ by*

$$\Lambda = \operatorname{range} Z, \qquad Z = \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{2.1}$$

Then $\Lambda \in \mathcal{L}(n)$ if and only if the matrix Z has rank n and

$$X^{\mathrm{T}}Y = Y^{\mathrm{T}}X.$$

In particular, the graph $\Lambda = \{(x, Ax) \mid x \in \mathbb{R}^n\}$ of a matrix $A \in \mathbb{R}^{n \times n}$ is Lagrangian if and only if A is symmetric.

Proof: Given two vectors $z = (Xu, Yu)$ and $z' = (Xu', Yu')$ in Λ we have $\omega_0(z, z') = u^{\mathrm{T}}(X^{\mathrm{T}}Y - Y^{\mathrm{T}}X)u'$. This proves the first assertion. The second assertion is the special case $X = \mathbb{1}$, $Y = A$. □

A matrix $Z \in \mathbb{R}^{2n\times n}$ of the form (2.1) which satisfies $X^{\mathrm{T}}Y = Y^{\mathrm{T}}X$ and has rank n is called a **Lagrangian frame**. If Z is a Lagrangian frame then its columns form an orthonormal basis of Λ if and only if the matrix $U = X + iY$ is unitary. In this case Z is called a **unitary Lagrangian frame**. In particular, the previous lemma shows that $\mathcal{L}(n)$ is a manifold of dimension $n(n+1)/2$. To see this, note that the space of symmetric $n \times n$ matrices can be identified with an open neighbourhood in $\mathcal{L}(n)$ of the horizontal Lagrangian

$$\Lambda_{\mathrm{hor}} = \left\{ z = (x, y) \in \mathbb{R}^{2n} \,|\, y = 0 \right\}.$$

The next lemma shows that any Lagrangian plane can be identified with Λ_{hor} via a linear symplectomorphism.

Lemma 2.31 **(i)** *If* $\Lambda \in \mathcal{L}(n)$ *and* $\Psi \in \mathrm{Sp}(2n)$ *then* $\Psi\Lambda \in \mathcal{L}(n)$.
(ii) *For any two Lagrangian subspaces* $\Lambda, \Lambda' \in \mathcal{L}(n)$ *there exists a symplectic matrix* $\Psi \in \mathrm{U}(n)$ *such that* $\Lambda' = \Psi\Lambda$.
(iii) *There is a natural isomorphism* $\mathcal{L}(n) = \mathrm{U}(n)/\mathrm{O}(n)$.

Proof: Statement (i) is obvious. To prove (ii) fix a Lagrangian subspace $\Lambda \subset \mathbb{R}^{2n}$ and choose a unitary frame of the form (2.1). Define the matrix

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

Then $\Psi \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ and $\Psi\Lambda_{\mathrm{hor}} = \Lambda$. This proves (ii). Statement (iii) holds because the unitary matrix $U = X + iY \in \mathrm{U}(n)$ determined by a unitary Lagrangian frame is uniquely determined by Λ up to right multiplication by a matrix in $\mathrm{O}(n)$. □

Exercises

Exercise 2.32 Prove that the orthogonal complement of a Lagrangian subspace $\Lambda \subset \mathbb{R}^{2n}$ with respect to the standard metric is given by $\Lambda^{\perp} = J_0\Lambda$. Deduce that if $u_1, \ldots, u_n$ is an orthonormal basis of Λ then the vectors $u_1, \ldots, u_n, J_0u_1, \ldots, J_0u_n$ form a basis for $\mathbb{R}^{2n}$ which is both symplectically standard and orthogonal. Relate this to the proof of Lemma 2.31. □

Exercise 2.33 State and prove the analogue of Lemma 2.31 for isotropic, symplectic and coisotropic subspaces. □

Exercise 2.34 Consider the vertical Lagrangian

$$\Lambda_{\text{vert}} = \left\{ z = (x, y) \in \mathbb{R}^{2n} \mid x = 0 \right\}.$$

Use Lemma 2.30 to show that $\mathcal{L}(n)$ is the disjoint union

$$\mathcal{L}(n) = \mathcal{L}_0(n) \cup \Sigma(n),$$

where $\mathcal{L}_0(n)$ can be identified with the affine space of symmetric $n \times n$ matrices and $\Sigma(n)$ consists of all Lagrangian subspaces which do not intersect Λ_{vert} transversally. The set $\Sigma(n)$ is called the **Maslov cycle** and is discussed further below. □

The Maslov index

Lemma 2.31 implies that the fundamental group of $\mathcal{L}(n)$ is isomorphic to the integers. An explicit homomorphism $\pi_1(\mathcal{L}(n)) \to \mathbb{Z}$ is given by the Maslov index.*

Theorem 2.35 *There exists a unique functor μ, called the Maslov index, which assigns an integer $\mu(\Lambda)$ to every loop $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$ of Lagrangian subspaces and satisfies the following axioms:*

(homotopy) *Two loops in $\mathcal{L}(n)$ are homotopic if and only if they have the same Maslov index.*

(product) *For any two loops $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(n)$ and $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$ we have*

$$\mu(\Psi\Lambda) = \mu(\Lambda) + 2\mu(\Psi).$$

In particular, a constant loop $\Lambda(t) \equiv \Lambda_0$ has Maslov index 0.

(direct sum) *If $n = n' + n''$ identify $\mathcal{L}(n') \oplus \mathcal{L}(n'')$ in the obvious way with a submanifold of $\mathcal{L}(n)$. Then*

$$\mu(\Lambda' \oplus \Lambda'') = \mu(\Lambda') + \mu(\Lambda'').$$

(normalization) *The loop $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(1)$, defined by*

$$\Lambda(t) = e^{\pi i t}\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2,$$

has Maslov index 1.

*__Warning:__ As before we shall use the notation $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$ for a loop $\Lambda(t) = \Lambda(t+1)$ of Lagrangian subspaces. Sometimes we shall also use the same letter Λ to denote an individual Lagrangian subspace. Again it should be clear from the context which notation is used.

Proof: Define the map $\rho : \mathcal{L}(n) \to S^1$ by[†]

$$\rho(\Lambda) = \det(U^2), \qquad \text{range}\begin{pmatrix} X \\ Y \end{pmatrix} = \Lambda, \qquad U = X + iY \in \mathrm{U}(n),$$

for $\Lambda \in \mathcal{L}(n)$. (Here Λ is an individual Lagrangian subspace, not a loop.) The **Maslov index** of the loop $\Lambda(t) = \Lambda(t+1) \in \mathcal{L}(n)$ is the degree of the composition $\rho \circ \Lambda : \mathbb{R}/\mathbb{Z} \to S^1$:

$$\mu(\Lambda) = \deg \rho \circ \Lambda.$$

In other words

$$\mu(\Lambda) = \alpha(1) - \alpha(0),$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a lift of $\rho \circ \Lambda$:

$$\det(X(t) + iY(t)) = e^{\pi i \alpha(t)}.$$

The Maslov index is an integer and depends only on the homotopy class of Λ. Conversely, assume that $\Lambda_0(t) = \Lambda_0(t+1)$ and $\Lambda_1(t) = \Lambda_1(t+1)$ are loops of Lagrangian subspaces with the same Maslov index $\mu(\Lambda_0) = \mu(\Lambda_1)$. By Lemma 2.31 we may assume without loss of generality that $\Lambda_j(0) = \Lambda_j(1) = \mathbb{R}^n \times \{0\}$. Choose lifts $U_j(t) = X_j(t) + iY_j(t) \in \mathrm{U}(n)$ as above such that $U_0(0) = U_1(0) = \mathbb{1}$. If necessary, we may alter $U_j(t)$ by right multiplication with a path of orthogonal matrices to obtain $U_j(1) = \pm\mathbb{1}$. Since $\mu(\Lambda_0) = \mu(\Lambda_1)$ we have $U_0(1) = U_1(1) = \pm\mathbb{1}$. Hence the unitary matrices $U(t) = U_1(t)U_0(t)^{-1}$ form a loop. Since $\mu(\Lambda_0) = \mu(\Lambda_1)$ the loop $\det U : S^1 \to S^1$ is contractible. By Proposition 2.23, U_0 is homotopic to U_1 and hence Λ_0 is homotopic to Λ_1.

Thus we have proved that our Maslov index, as defined above, satisfies the homotopy axiom. The product, direct sum, and normalization axioms are obvious. This proves the existence statement of the theorem. Uniqueness is left to the reader. □

Alternatively the Maslov index can be defined as the intersection number of the loop $\Lambda(t)$ with the **Maslov cycle**

$$\Sigma(n)$$

of all Lagrangian subspaces Λ which intersect the vertical $\{0\} \times \mathbb{R}^n$ non-transversally. This set is a singular hypersurface of $\mathcal{L}(n)$ of codimension 1 which admits a natural coorientation. It is stratified by the dimension of the intersection $\Lambda \cap \Lambda_{\mathrm{vert}}$. A generic loop will intersect only the highest

[†]Note the square in this formula. It is needed because we consider unoriented Lagrangian subspaces. Compare with the proof of Theorem 2.29 and Exercise 2.27.

stratum (where the intersection is 1-dimensional) and all the intersections will be transverse. More explicitly, let $\Lambda(t)$ be a path of Lagrangian planes represented by a lift $X(t) + iY(t) \in \mathrm{U}(n)$ of unitary Lagrangian frames. A **crossing** is a number t such that $\det(X(t)) = 0$. A crossing t is called **regular** if the **crossing form** $\Gamma(\Lambda, t) : \ker X(t) \to \mathbb{R}$, defined by

$$\Gamma(\Lambda, t)(u) = -\langle \dot{X}(t)u, Y(t)u \rangle,$$

is nonsingular. (See Fig. 2.1.)

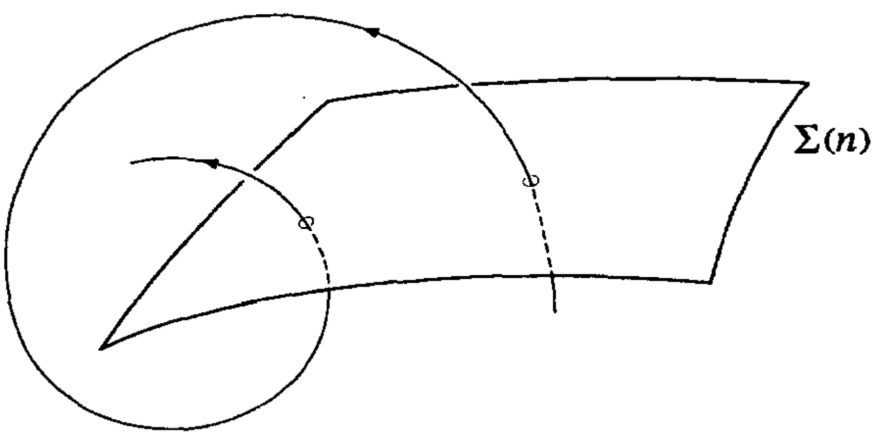

FIG. 2.1. The Maslov cycle

At a regular crossing the **crossing index** is the signature (the number of positive minus the number of negative eigenvalues) of the crossing form. The Maslov index of a loop $\Lambda(t) = \Lambda(t+1)$ with only regular crossings can be defined by

$$\mu(\Lambda) = \sum_t \operatorname{sign} \Gamma(\Lambda, t),$$

where the sum runs over all crossings. As in the case of symplectic matrices this definition satisfies the axioms of Theorem 2.35. For more details see [236].

Exercise 2.36 The Maslov index of a loop $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(V, \omega)$ of Lagrangian subspaces in a general symplectic vector space is defined as the Maslov index of the loop $t \mapsto \Psi^{-1}\Lambda(t) \in \mathcal{L}(n)$, where $\Psi : (\mathbb{R}^{2n}, \omega_0) \to (V, \omega)$ is a linear symplectomorphism. Show that this definition is independent of Ψ. Show that if one reverses the sign of ω the sign of the Maslov index reverses also. □

Exercise 2.37 Let $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(V, \omega)$ be a loop of linear symplectomorphisms. Prove that the corresponding loop $\Gamma_\Psi : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(V \times V, (-\omega) \times \omega)$ of Lagrangian graphs has twice the Maslov index $\mu(\Gamma_\Psi) = 2\mu(\Psi)$. ($\Gamma_\Psi$ is defined in Exercise 2.1.) □

2.4 The affine nonsqueezing theorem

An **affine symplectomorphism** of $\mathbb{R}^{2n}$ is a map $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of the form

$$\psi(z) = \Psi z + z_0,$$

where $\Psi \in \mathrm{Sp}(2n)$ and $z_0 \in \mathbb{R}^{2n}$. We denote by $\mathrm{ASp}(2n)$ the group of affine symplectomorphisms. The affine nonsqueezing theorem asserts that a ball in $\mathbb{R}^{2n}$ can only be embedded into a symplectic cylinder by an affine symplectomorphism if it has a smaller radius. As in Section 1.2, we denote the symplectic cylinder of radius $R > 0$ by

$$Z^{2n}(R) = B^2(R) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}.$$

Here the splitting $\mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2}$ is a symplectic one, i.e. the ball $B^2(R)$ corresponds to the coordinates (x_1, y_1).

Theorem 2.38 *Let $\psi \in \mathrm{ASp}(2n)$ be an affine symplectomorphism and assume that $\psi(B^{2n}(r)) \subset Z^{2n}(R)$. Then $r \leq R$.*

Proof: Assume without loss of generality that $r = 1$ and write $\psi(z) = \Psi z + z_0$. Let $u, v \in \mathbb{R}^{2n}$ be the columns of Ψ^{T} corresponding to the coordinates x_1 and y_1, and let $a, b \in \mathbb{R}$ be the corresponding coordinates of z_0. Then, since $\Psi \in \mathrm{Sp}(2n)$, we have

$$\omega_0(u, v) = 1$$

and the condition $\psi(B^{2n}(1)) \subset Z^{2n}(R)$ can be restated in the form

$$\sup_{|z|=1} \left((\langle u, z\rangle + a)^2 + (\langle v, z\rangle + b)^2 \right) \leq R^2.$$

But the left-hand side of this inequality is greater than or equal to 1 because $1 = \omega_0(u, v) \leq |u| \cdot |v|$ and hence either u or v has length greater than or equal to 1. Assume without loss of generality that $|u| \geq 1$ and choose $z = \pm u/|u|$. □

The nonsqueezing property can be formulated in a symplectically invariant way. A set $B \subset \mathbb{R}^{2n}$ is called a **linear symplectic ball** of **radius** r if it is linearly symplectomorphic to $B^{2n}(r)$. Similarly, a subset $Z \subset \mathbb{R}^{2n}$ is called a **linear symplectic cylinder** if there exists a symplectic matrix $\Psi \in \mathrm{Sp}(2n)$ and a number $R > 0$ such that $Z = \Psi Z^{2n}(R)$. It follows from Theorem 2.38 that for any such set Z the number $R > 0$ is a linear symplectic invariant. It is called the **radius** of Z. A matrix $\Psi \in \mathbb{R}^{2n \times 2n}$ is said to have the **linear nonsqueezing property** if for every linear symplectic ball B of radius r and every linear symplectic cylinder Z of radius R we have

$$\Psi B \subset Z \qquad \Longrightarrow \qquad r \leq R.$$

The following theorem shows that linear symplectomorphisms are characterized by the nonsqueezing property. More precisely, we must also include the case of **anti-symplectic** matrices Ψ which satisfy $\Psi^*\omega_0 = -\omega_0$.

Theorem 2.39 *Let $\Psi \in \mathbb{R}^{2n\times 2n}$ be a nonsingular matrix such that Ψ and Ψ^{-1} have the linear nonsqueezing property. Then Ψ is either symplectic or anti-symplectic.*

Proof: Assume that Ψ is neither symplectic nor anti-symplectic. Then neither is Ψ^{T} and so there exist vectors $u, v \in \mathbb{R}^{2n}$ such that

$$\omega_0(\Psi^{\mathrm{T}}u, \Psi^{\mathrm{T}}v) \neq \pm\omega_0(u, v).$$

Perturbing u and v slightly, and using the fact that Ψ is nonsingular, we may assume that $\omega_0(u, v) \neq 0$ and $\omega_0(\Psi^{\mathrm{T}}u, \Psi^{\mathrm{T}}v) \neq 0$. Moreover, replacing Ψ by Ψ^{-1} if necessary, we may assume that $|\omega_0(\Psi^{\mathrm{T}}u, \Psi^{\mathrm{T}}v)| < |\omega_0(u, v)|$. Now, by rescaling u if necessary, we obtain

$$0 < \lambda^2 = |\omega_0(\Psi^{\mathrm{T}}u, \Psi^{\mathrm{T}}v)| < \omega_0(u, v) = 1.$$

Hence there exist symplectic bases $u_1, v_1 \ldots, u_n, v_n$ and $u_1', v_1' \ldots, u_n', v_n'$ of $\mathbb{R}^{2n}$ such that

$$u_1 = u, \qquad v_1 = v, \qquad u_1' = \lambda^{-1}\Psi^{\mathrm{T}}u, \qquad v_1' = \pm\lambda^{-1}\Psi^{\mathrm{T}}v.$$

Denote by $\Phi \in \mathrm{Sp}(2n)$ the matrix which maps the standard basis $e_1, \ldots, f_n$ to $u_1, \ldots, v_n$ and by $\Phi' \in \mathrm{Sp}(2n)$ the matrix which maps $e_1, \ldots, f_n$ to $u_1', \ldots, v_n'$. Then the matrix

$$A = \Phi'^{-1}\Psi^{\mathrm{T}}\Phi$$

satisfies

$$Ae_1 = \lambda e_1, \qquad Af_1 = \pm\lambda f_1.$$

This implies that the transposed matrix A^{T} maps the unit ball $B^{2n}(1)$ to the cylinder $Z^{2n}(\lambda)$. But since $\lambda < 1$ this means that Ψ does not have the nonsqueezing property in contradiction to our assumption. This proves the theorem. □

The affine nonsqueezing theorem gives rise to the notion of the **linear symplectic width** of an arbitrary subset $A \subset \mathbb{R}^{2n}$, defined by

$$w_L(A) = \sup\left\{\pi r^2 \mid \psi(B^{2n}(r)) \subset A \;\text{ for some }\; \psi \in \mathrm{ASp}(\mathbb{R}^{2n})\right\}.$$

It follows from Theorem 2.38 that the linear symplectic width has the following properties:

(monotonicity) If $\psi(A) \subset B$ for some $\psi \in \mathrm{ASp}(2n)$ then $w_L(A) \leq w_L(B)$.

(conformality) $w_L(\lambda A) = \lambda^2 w_L(A)$.

(nontriviality) $w_L(B^{2n}(r)) = w_L(Z^{2n}(r)) = \pi r^2$.

The nontriviality axiom implies that w_L is a two-dimensional invariant. It is obvious from the monotonicity property that affine symplectomorphisms preserve the linear symplectic width. We shall prove that this property in fact characterizes symplectic and anti-symplectic linear maps.

Exercise 2.40 Prove that every anti-symplectic linear map has determinant $(-1)^n$. Prove that every anti-symplectic linear map preserves the linear symplectic width of subsets of $\mathbb{R}^{2n}$. **Hint:** The linear symplectic width $w_L(A)$ agrees with the maximal radius of a ball which can be mapped into A by an anti-symplectic affine transformation. □

Theorem 2.41 *Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map. Then the following are equivalent.*

(i) *Ψ preserves the linear symplectic width of ellipsoids centred at* 0.

(ii) *The matrix Ψ is either symplectic or anti-symplectic, i.e. $\Psi^*\omega_0 = \pm\omega_0$.*

Proof: We have seen above that (ii) implies (i). Hence assume (i). We prove that Ψ has the nonsqueezing property. To see this let B be a linear symplectic ball of radius r and Z be a linear symplectic cylinder of radius R such that

$$\Psi B \subset Z.$$

Then it follows from the monotonicity property of the linear symplectic width that

$$\pi r^2 = w_L(B) = w_L(\Psi B) \leq w_L(Z) = \pi R^2$$

and hence $r \leq R$. It also follows from (i) that Ψ must be nonsingular because otherwise the image of the unit ball under Ψ would have linear symplectic width zero. Moreover, Ψ^{-1} also satisfies (i) because $w_L(\Psi^{-1}E) = w_L(\Psi\Psi^{-1}E) = w_L(E)$ for every ellipsoid E which is centred at zero. Thus we have proved that both Ψ and Ψ^{-1} have the nonsqueezing property, and in view of Theorem 2.39 this implies that Ψ is either symplectic or anti-symplectic. □

We shall now study in more detail the action of the symplectic linear group on ellipsoids. The main tools for understanding this action are the affine nonsqueezing theorem and the following lemma about the simultaneous normalization of a symplectic form and an inner product. Note that this lemma provides an alternative proof of Theorem 2.3.

Lemma 2.42 *Let (V, ω) be a symplectic vector space and $g : V \times V \to \mathbb{R}$ be an inner product. Then there exists a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of V which* *is both g-orthogonal and ω-standard. Moreover, this basis can be chosen such that*

$$g(u_j, u_j) = g(v_j, v_j)$$

for all j.

Proof: Consider the vector space $V = \mathbb{R}^{2n}$ with the standard inner product $g = \langle \cdot, \cdot \rangle$ and assume that

$$\omega(z, w) = \langle z, Aw \rangle$$

is a nondegenerate skew-form. Then A is nondegenerate and $A^{\mathrm{T}} = -A$. Hence $iA \in \mathbb{C}^{2n \times 2n}$ is a Hermitian matrix and so the spectrum of A is purely imaginary and there exists an orthonormal basis of eigenvectors. Let the eigenvalues of A be $\pm i\alpha_j$ for $j = 1, \ldots, n$ with $\alpha_j > 0$ and choose eigenvectors $z_j = u_j + iv_j \in \mathbb{C}^{2n}$ such that

$$Az_j = i\alpha_j z_j, \qquad \bar{z}_j^{\mathrm{T}} z_k = \delta_{jk}.$$

Then we have $A\bar{z}_j = -i\alpha_j \bar{z}_j$ and hence

$$z_j^{\mathrm{T}} z_k = 0.$$

With $z_j = u_j + iv_j$ we obtain

$$Au_j = -\alpha_j v_j, \qquad Av_j = \alpha_j u_j$$

and

$$u_j^{\mathrm{T}} v_k = u_j^{\mathrm{T}} u_k = v_j^{\mathrm{T}} v_k = 0, \qquad j \neq k.$$

This implies

$$\omega(u_j, v_j) = u_j^{\mathrm{T}} A v_j = \alpha_j |u_j|^2 > 0$$

and similarly $\omega(u_j, v_k) = \omega(u_j, u_k) = \omega(v_j, v_k) = 0$ for $j \neq k$. Rescaling the vectors u_j and v_j, if necessary, we obtain the required symplectic orthogonal basis of $\mathbb{R}^{2n}$. □

The next lemma is a geometric interpretation of this result. Given an n-tuple $r = (r_1, \ldots, r_n)$ with $0 < r_1 \leq \cdots \leq r_n$ consider the closed ellipsoid

$$E(r) = \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} \left| \frac{z_i}{r_i} \right|^2 \leq 1 \right\}.$$

Lemma 2.43 *Given any ellipsoid*

$$E = \left\{ w \in \mathbb{R}^{2n} \,\middle|\, \sum_{i,j=1}^{2n} a_{ij} w_i w_j \le 1 \right\}$$

there is a symplectic linear transformation $\Psi \in \mathrm{Sp}(2n)$ *such that* $\Psi E = E(r)$ *for some* n*-tuple* $r = (r_1, \ldots, r_n)$ *with* $0 < r_1 \le \cdots \le r_n$. *Moreover, the numbers* r_j *are uniquely determined by* E.

Proof: Consider the inner product

$$g(v, w) = \sum_{i,j=1}^{2n} a_{ij} v_i w_j$$

on $\mathbb{R}^{2n}$. Then the ellipsoid E is given by

$$E = \{ w \in \mathbb{R}^{2n} \mid g(w, w) \le 1 \}.$$

By Lemma 2.42 there is a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of $\mathbb{R}^{2n}$ which is both symplectically standard and orthogonal for g. Moreover, we may assume that

$$g(u_j, u_j) = g(v_j, v_j) = \frac{1}{{r_j}^2}.$$

Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the symplectic linear transformation which takes the standard basis of $\mathbb{R}^{2n}$ to this new basis, i.e.

$$\Psi z = \sum_{j=1}^{n} (x_j u_j + y_j v_j)$$

for $z = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Then

$$g(\Psi z, \Psi z) = \sum_{j=1}^{n} \frac{{x_j}^2 + {y_j}^2}{{r_j}^2}$$

and hence $\Psi^{-1} E = E(r)$.

To prove uniqueness of the n-tuple $r_1 \le \cdots \le r_n$ consider the diagonal matrix

$$\Delta(r) = \mathrm{diag}(1/r_1^2, \ldots, 1/r_n^2, 1/r_1^2, \ldots, 1/r_n^2).$$

We must show that if there is a symplectic matrix Ψ such that

$$\Psi^{\mathrm{T}} \Delta(r) \Psi = \Delta(r')$$

then $r = r'$. Since $J_0\Psi^T = \Psi^{-1}J_0$, the above identity is equivalent to

$$\Psi^{-1}J_0\Delta(r)\Psi = J_0\Delta(r').$$

Hence $J_0\Delta(r)$ and $J_0\Delta(r')$ must have the same eigenvalues. But it is easy to check that the eigenvalues of $J_0\Delta(r)$ are $\pm i/{r_1}^2, \ldots, \pm i/{r_n}^2$. This proves the lemma. □

Remark 2.44 In the case $n = 1$ the existence statement of Lemma 2.43 asserts that every ellipse in $\mathbb{R}^2$ can be mapped into a circle by an area-preserving linear transformation. □

In view of Lemma 2.43 we define the **symplectic spectrum** of an ellipsoid E to be the unique n-tuple $r = (r_1, \ldots, r_n)$ with $0 < r_1 \leq \cdots \leq r_n$ such that E is linearly symplectomorphic to $E(r) = E(r_1, \ldots, r_n)$. The spectrum is invariant under linear symplectic transformations and, in fact, two ellipsoids in $\mathbb{R}^{2n}$, which are centred at 0, are linearly symplectomorphic if and only if they have the same spectrum. Moreover, the volume of an ellipsoid $E \subset \mathbb{R}^{2n}$ is given by

$$\mathrm{Vol}(E) = \int_E \frac{\omega_0^n}{n!} = \pi^n \prod_{j=1}^n {r_j}^2.$$

Ellipsoids with spectrum $(r, \ldots, r)$ are linear symplectic balls. The following theorem characterizes the linear symplectic width of an ellipsoid in terms of the spectrum.

Theorem 2.45 *Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centred at 0, with empty or nonempty interior. Then*

$$w_L(E) = \sup_{B \subset E} w_L(B) = \inf_{Z \supset E} w_L(Z),$$

where the supremum runs over all linear symplectic balls contained in E and the infimum runs over all linear symplectic cylinders containing E. If E has empty interior, then $w_L(E) = 0$ and, if E has nonempty interior and symplectic spectrum $0 < r_1 \leq \cdots \leq r_n$, then

$$w_L(E) = \pi {r_1}^2.$$

Proof: Assume first that E has empty interior. Then, by definition of the linear symplectic width, $w_L(E) = 0$. Moreover, there exists a linear symplectomorphism $\Psi \in \mathrm{Sp}(2n)$ such that that $\Psi E \subset \{0\} \times \mathbb{R}^{2n-1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. Hence, for every $\delta > 0$, the linear transformation $\Phi_\delta = \mathrm{diag}(1/\delta, \delta, 1, \ldots, 1)$ is symplectic and embeds ΨE into $Z^{2n}(\delta)$. This proves the lemma in the case of empty interior.

Now assume that E has nonempty interior and symplectic spectrum $0 < r_1 \le r_2 \le \cdots \le r_n$. Then there exists a symplectic matrix $\Psi \in \mathrm{Sp}(2n)$ such that $\Psi E = E(r_1, \ldots, r_n)$. Hence

$$\Psi^{-1} B^{2n}(r_1) \subset E \subset \Psi^{-1} Z^{2n}(r_1)$$

and so

$$\inf_{Z \supset E} w_L(Z) \le \pi {r_1}^2 \le \sup_{B \subset E} w_L(B).$$

Now suppose that B is a linear symplectic ball of radius r contained in E. Then $\Psi B \subset \Psi E \subset Z^{2n}(r_1)$ and so $r \le r_1$. Similarly, if Z is a linear symplectic cylinder of radius R containing E then $B^{2n}(r_1) \subset \Psi E \subset \Psi Z$ and so $r_1 \le R$. Hence

$$\sup_{B \subset E} w_L(B) \le \pi {r_1}^2 \le \inf_{Z \supset E} w_L(Z).$$

Since $w_L(E) = \sup_{B \subset E} w_L(B)$ this proves the theorem. □

Exercise 2.46 Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid and define the dual ellipsoid by

$$E^* = \{v \in \mathbb{R}^{2n} \mid \langle v, e \rangle \le 1 \ \forall e \in E\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. Prove that

$$E^{**} = E, \qquad (\Psi E)^* = (\Psi^{\mathrm{T}})^{-1} E^*,$$

for $\Psi \in \mathrm{Sp}(2n)$. Prove that the symplectic spectrum of E^* is given by $(1/r_n, \ldots, 1/r_1)$, where $(r_1, \ldots, r_n)$ is the symplectic spectrum of E. Deduce that the dual of a linear symplectic ball is again a linear symplectic ball. □

2.5 Complex structures

A **complex structure** on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathbb{1}$. With such a structure V becomes a complex vector space with multiplication by $i = \sqrt{-1}$ corresponding to J

$$\mathbb{C} \times V \to V : (s + it, v) \mapsto sv + tJv.$$

In particular, V is necessarily of even dimension over the reals. We denote the space of complex structures on V by $\mathcal{J}(V)$. The basic example is the automorphism

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

of $\mathbb{R}^{2n}$. As we saw above, if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the isomorphism $(x,y) \mapsto x+iy$ for $x,y \in \mathbb{R}^n$, then the matrix J_0 corresponds to multiplication by i. The following lemma shows that every complex structure is isomorphic to the standard complex structure J_0.

Proposition 2.47 *Let V be a $2n$-dimensional real vector space and let $J \in \mathcal{J}(V)$. Then there exists a vector space isomorphism $\Phi : \mathbb{R}^{2n} \to V$ such that*

$$J\Phi = \Phi J_0.$$

Proof: Let V^c denote the complexification of V and denote by $E^{\pm} = \ker(\mathbb{1} \pm iJ)) = \mathrm{range}\,(\mathbb{1} \mp iJ)$ the eigenspaces of J. Then $V^c = E^+ \oplus E^-$ and hence $\dim E^{\pm} = n$. Choose a basis $w_j = u_j + iv_j$, $j = 1, \ldots, n$, of E^+. Then the vectors $u_1, \ldots, u_n, v_1, \ldots, v_n$ form a basis of V and

$$Ju_j = -v_j, \qquad Jv_j = u_j.$$

The required transformation $\Phi : \mathbb{R}^{2n} \to V$ is given by

$$\Phi\zeta = \sum_{j=1}^{n} (\xi_j u_j - \eta_j v_j)$$

for $\zeta = (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$. □

Proposition 2.48 *The space $\mathcal{J}(\mathbb{R}^{2n})$ is diffeomorphic to the homogeneous space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$. This space has two components. The component $\mathcal{J}^+(\mathbb{R}^{2n})$ which contains J_0 is diffeomorphic to the homogeneous space $\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ and homotopy equivalent to $\mathrm{SO}(2n)/\mathrm{U}(n)$. It is the space of all complex structures on $\mathbb{R}^{2n}$ with fixed orientation. In the case $n = 2$ this space is homotopy equivalent to S^2.*

Proof: Define a map $\mathrm{GL}(2n,\mathbb{R}) \to \mathcal{J}(\mathbb{R}^{2n})$ by $A \mapsto A^{-1}J_0A$. By Proposition 2.47 this map is surjective and its kernel is $\mathrm{GL}(n;\mathbb{C})$. (Here we identify $\mathrm{GL}(n,\mathbb{C})$ with a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ as in Section 2.2.) Thus we have identified $\mathcal{J}(\mathbb{R}^{2n})$ with the homogeneous space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$. This space has two components distinguished by the determinant. By Exercise 2.25 the space $\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ is homotopy equivalent to the quotient $\mathrm{SO}(2n)/\mathrm{U}(n)$. Note also that $\mathcal{J}$ is homotopy equivalent to $\mathcal{J} \cap \mathrm{O}(2n,\mathbb{R})$.

Now consider the case $n = 2$ and let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis of $\mathbb{R}^4$. We claim that a matrix $J \in \mathcal{J} \cap \mathrm{SO}(4)$ is completely determined by the unit vector $Je_1 \in \{(x_1, x_2, y_1, y_2) \,|\, x_1 = 0\}$. To see this note that the vectors e_1, Je_1 form an orthonormal basis of a 2-plane $E \subset \mathbb{R}^4$. The matrix J is then completely determined by the fact that for any unit vector $v \in E^{\perp}$ the vectors v, Jv form an orthonormal basis of $E^{\perp}$, which is oriented so that the vectors e_1, Je_1, v, Jv form a positively oriented basis for $\mathbb{R}^4$. □

Compatible complex structures

Let (V, ω) be a symplectic vector space. A complex structure $J \in \mathcal{J}(V)$ is said to be **compatible with** ω if

$$\omega(Jv, Jw) = \omega(v, w)$$

for all $v, w \in V$ and

$$\omega(v, Jv) > 0$$

for all nonzero $v \in V$. If J is a compatible complex structure then

$$g_J(v, w) = \omega(v, Jw)$$

defines an inner product on V. Moreover, J is a g_J-skew-adjoint transformation of V, that is,

$$g_J(v, Jw) + g_J(Jv, w) = 0.$$

We denote the space of compatible complex structures on (V, ω) by $\mathcal{J}(V, \omega)$.

Exercise 2.49 Let (V, ω) be a symplectic vector space and J be a complex structure on V. Prove that the following are equivalent:
(i) J is compatible with ω.
(ii) The bilinear form $g_J : V \times V \to \mathbb{R}$ defined by

$$g_J(v, w) = \omega(v, Jw)$$

is symmetric, positive definite, and J-invariant.
(iii) The form $H : V \times V \to \mathbb{C}$ defined by

$$H(v, w) = \omega(v, Jw) + i\,\omega(v, w)$$

is complex linear in w, complex anti-linear in v, satisfies $H(w, v) = \overline{H(v, w)}$, and has a positive definite real part. Such a form is called a **Hermitian inner product** on (V, J). Note that here V is understood as a complex vector space with the complex structure given by J. Thus one must show that $H(v, Jw) = iH(v, w)$, etc. □

Our next aim is to show that the space $\mathcal{J}(V, \omega)$ is contractible. The next proposition gives two proofs of this important fact. The first is by a direct argument, and the second constructs a homotopy equivalence between $\mathcal{J}(V, \omega)$ and the space $\mathfrak{Met}(V)$ of all inner products (i.e. positive definite symmetric bilinear forms) which is obviously contractible.

Proposition 2.50 (i) *$\mathcal{J}(V, \omega)$ is homeomorphic to the space $\mathcal{P}$ of symmetric positive definite symplectic matrices.*

(ii) *There exists a continuous map* $r : \mathfrak{Met}(V) \to \mathcal{J}(V, \omega)$ *such that*

$$r(g_J) = J, \qquad r(\Phi^* g) = \Phi^* r(g)$$

for all $J \in \mathcal{J}(V, \omega)$, $g \in \mathfrak{Met}(V)$, $\Phi \in \mathrm{Sp}(V, \omega)$.
(iii) $\mathcal{J}(V, \omega)$ *is contractible.*

Proof: First note that either of the assertions (i) and (ii) implies (iii). We prove (i). By Theorem 2.3 we may assume $V = \mathbb{R}^{2n}$ and $\omega = \omega_0$. A matrix $J \in \mathbb{R}^{2n}$ is a compatible complex structure if and only if

$$J^2 = -\mathbb{1}, \qquad J^{\mathrm{T}} J_0 J = J_0, \qquad \langle v, -J_0 J v \rangle > 0 \ \ \forall v \neq 0.$$

The first two identities imply that

$$(J_0 J)^{\mathrm{T}} = -J^{\mathrm{T}} J_0 = J^{\mathrm{T}} J_0 J^2 = J_0 J.$$

Hence $P = -J_0 J$ is symmetric, positive definite and symplectic. Conversely, if a matrix P has these properties, then it is easy to check that $J = -J_0^{-1} P \in \mathcal{J}(\mathbb{R}^{2n}, \omega_0)$. Now it follows from Lemma 2.21 that $\mathcal{J}(V, \omega)$ is contractible.

We prove (ii). First observe that the very existence of a continuous map $r : \mathfrak{Met}(V) \to \mathcal{J}(V, \omega)$ with the stated properties implies the contractibility of $\mathcal{J}(V, \omega)$. In fact, the maps $f_t : \mathcal{J}(V, \omega) \to \mathcal{J}(V, \omega)$ given by

$$f_t(J) = r((1-t) g_{J_0} + t g_J), \qquad 0 \leq t \leq 1,$$

provide a homotopy connecting the constant map $f_0(J) = J_0$ to the identity $f_1(J) = J$.

Thus we just have to define the retraction r. To do this, let $g \in \mathfrak{Met}(V)$ and define the automorphism $A : V \to V$ by

$$\omega(v, w) = g(Av, w).$$

The identity $\omega(v, w) = -\omega(w, v)$ is equivalent to $g(Av, w) = -g(v, Aw)$. Therefore A is g-skew-adjoint. Hence, writing A^* for the g-adjoint of A, we find that $P = A^* A = -A^2$ is g-positive definite. It follows that there is a unique automorphism $Q : V \to V$ which is g-self-adjoint, g-positive definite, and satisfies

$$Q^2 = P.$$

(To see this, use the fact that P can be represented as a positive definite symmetric matrix with respect to a suitable basis for V. Details are given in Exercise 2.52 below.) It follows easily that the automorphism

$$J_g = Q^{-1} A$$

is a complex structure compatible with ω. Hence we define $r(g) = J_g$. The continuity of r follows from Exercise 2.52 .

If we begin with a metric g of the form g_J, then $A = J$ and $Q = \mathbb{1}$ and hence $r(g_J) = J$. Moreover, if g is replaced by $\Phi^* g(v, w) = g(\Phi v, \Phi w)$ where $\Phi \in \mathrm{Sp}(V, \omega)$, then A is replaced by $\Phi^{-1} A \Phi$ and hence $J_{\Phi^* g} = \Phi^{-1} J_g \Phi$. This proves that the map r has the required properties. □

The above result can be considered in terms of the following diagram:

$$\begin{array}{ccccccc}
\mathrm{U}(n) & \xrightarrow{\sim} & \mathrm{Sp}(2n) & \longrightarrow & \mathrm{Sp}(2n)/\mathrm{U}(n) & \xrightarrow{e} & \mathcal{J}(V, \omega) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow\uparrow r \\
\mathrm{SO}(2n) & \xrightarrow{\sim} & \mathrm{GL}^+(2n, \mathbb{R}) & \longrightarrow & \mathrm{GL}^+(2n, \mathbb{R})/\mathrm{SO}(2n) & \cong & \mathfrak{Met}
\end{array}$$

Statement (i) establishes the existence of the homeomorphism e while the proof of (ii) constructs the map r. Another way to prove that e is a bijection would be to define an action of $\mathrm{Sp}(2n)$ on $\mathcal{J}(V, \omega)$ with isotropy group $\mathrm{U}(n)$. To do this, one has to show that J is ω-compatible if and only if it is conjugate to J_0 by a symplectic matrix (see Exercise 2.52).

In many situations we do not need to work with compatible complex structures. For example, in order to ensure that the compactness theorems hold for J-holomorphic curves, it is enough that J satisfies only the second compatibility condition. Thus a complex structure $J \in \mathcal{J}(V)$ is called ω-**tame** if

$$\omega(v, Jv) > 0$$

for every nonzero vector $v \in V$. We denote the space of all ω-tame complex structures on V by $\mathcal{J}_\tau(V, \omega)$. Clearly, this is an open subset of the space of all complex structures on V. It is easy to see that the formula

$$g_J(v, w) = \frac{1}{2}(\omega(v, Jw) + \omega(w, Jv))$$

defines an inner product on V for every $J \in \mathcal{J}_\tau(V, \omega)$.

Proposition 2.51 *The space $\mathcal{J}_\tau(V, \omega)$ is contractible.*

We give two proofs of this result. The first appeared in Gromov [108] and is easy if you know some homotopy theory. The second is by direct calculation and is due to Sévennec [15, Chapter II]. Recall that $\Omega = \Omega(V)$ is the space of all symplectic forms on V and $\mathcal{J} = \mathcal{J}(V)$ is the space of all complex structures.

Proof 1: Consider the spaces

$$\mathcal{C}_\tau = \{(\omega, J) \in \Omega \times \mathcal{J} \mid J \in \mathcal{J}_\tau(V, \omega)\},$$

$$\mathcal{C} = \{(\omega, J) \in \Omega \times \mathcal{J} \mid J \in \mathcal{J}(V,\omega)\}.$$

It is easy to see that all the projections

$$\mathcal{C}_\tau \to \Omega, \qquad \mathcal{C}_\tau \to \mathcal{J}, \qquad \mathcal{C} \to \Omega, \qquad \mathcal{C} \to \mathcal{J}$$

are locally trivial fibrations. Moreover, the fibre of the map $\mathcal{C}_\tau \to \mathcal{J}$ is the set of all ω which tame J. This space is convex and hence contractible. Hence the projection $\mathcal{C}_\tau \to \mathcal{J}$ is a homotopy equivalence. A similar remark applies to the map $\mathcal{C} \to \mathcal{J}$. Hence, by considering the diagram

$$\begin{array}{ll} \mathcal{C} \hookrightarrow \mathcal{C}_\tau \\ \downarrow \swarrow & \\ \mathcal{J} & , \end{array}$$

where the vertical arrows are homotopy equivalences, we see that the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}_\tau$ is a homotopy equivalence. Now consider the diagram

$$\begin{array}{ll} \mathcal{C} \overset{\sim}{\hookrightarrow} \mathcal{C}_\tau \\ \downarrow \swarrow & \\ \Omega & . \end{array}$$

Since the projection $\mathcal{C} \to \Omega$ has contractible fibres, both projections to Ω are homotopy equivalences, and so the fibres of $\mathcal{C}_\tau \to \Omega$ are contractible. □

Proof 2: We may assume that $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Then $J = -J_0 Z$ is ω-tame if and only if

$$Z > 0, \quad Z^{-1} = J_0^{-1} Z J_0.$$

Here we write $Z > 0$ to mean that $\langle v, Zv\rangle > 0$ when $v \neq 0$ but the matrix Z is not required to be symmetric. To transform the second identity into something which is manageable we use the Cayley transform* $z \mapsto (1 - z)(1 + z)^{-1} = w$. As a map of the Riemann sphere this takes the positive half space $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ into the unit disc, interchanges the points $\pm i$, and transforms the involution $z \mapsto 1/z$ into the involution $w \mapsto -w$. Similarly, if we think of this as a map on matrices

$$W = F(Z) = (\mathbb{1} - Z)(\mathbb{1} + Z)^{-1},$$

it is easy to check that it is well defined on the half space $Z > 0$ and takes it onto the open unit disc $\{W \mid \|W\| < 1\}$. Moreover, it interchanges $\pm J_0$, and transforms the identity $Z^{-1} = J_0^{-1} Z J_0$ into

*This also appears in [8, I.4.2].

$$-W = J_0^{-1} W J_0.$$

But the set of such matrices W is convex and hence contractible. □

We note that the calculation in Proof 2 above can also be applied to show that $\mathcal{J}(V, \omega)$ is contractible. The only difference in this case is that Z is now symmetric, and this property is preserved by F.

Exercises

The following exercises explore further the interconnections between symplectic forms, inner products, and almost complex structures.

Exercise 2.52 **(i)** Prove the continuity of the map $r : \mathfrak{Met}(V) \to \mathcal{J}(V, \omega)$ in Proposition 2.50 as follows. If $V = \mathbb{R}^{2n}$ and $\omega = \omega_0$ then an inner product $g \in \mathfrak{Met}(\mathbb{R}^{2n})$ can be written in the form $g(v, w) = w^{\mathrm{T}} G v$, where $G \in \mathbb{R}^{2n \times 2n}$ is positive definite. The formula $\omega_0(v, w) = (J_0 v)^{\mathrm{T}} w = g(Av, w)$ determines the matrix $A = G^{-1} J_0$. Prove that the g-adjoint of A is represented by the matrix $A^* = G^{-1} A^{\mathrm{T}} G = -A$. Prove that the g-square root Q of the matrix $P = A^* A = -A^2 = G^{-1} {J_0}^{\mathrm{T}} G^{-1} J_0$ is given by

$$Q = G^{-1/2} \left(G^{-1/2} {J_0}^{\mathrm{T}} G^{-1} J_0 G^{-1/2} \right)^{1/2} G^{1/2}.$$

Deduce that the map $G \mapsto J_G = Q^{-1} G^{-1} J_0$ is continuous.

(ii) The algebra here is also just a reformulation of that in the proof of Lemma 2.42. Use the current methods to give an alternative proof of this result. **Hint:** Find a symplectic basis which is orthogonal with respect to both g and g_J, where $J = r(g)$.

(iii) Deduce from (ii) that a complex structure J is ω-compatible if and only if it has the form $J = \Psi^{-1} J_0 \Psi$ for some $\Psi \in \mathrm{Sp}(2n)$. **Hint:** Use a basis which is both g_J-orthogonal and ω_0-standard. □

Exercise 2.53 Here is yet another proof of the contractibility of $\mathcal{J}(V, \omega)$ taken from [15, Chapter II]. This proof illustrates in a clear geometric way the relationship between Lagrangian subspaces, complex structures, and inner products. Given a Lagrangian subspace $\Lambda_0 \in \mathcal{L}(V, \omega)$ there is a natural bijection

$$\mathcal{J}(V, \omega) \to \mathcal{L}_0(V, \omega, \Lambda_0) \times \mathcal{S}(\Lambda_0),$$

where $\mathcal{L}_0(V, \omega, \Lambda_0)$ is the space of all Lagrangian subspaces which intersect Λ_0 transversally and $\mathcal{S}(\Lambda_0)$ is the space of all positive definite quadratic forms on Λ_0. Note that, by Lemma 2.30, the space $\mathcal{L}_0(V, \omega, \Lambda_0)$ is contractible. The above correspondence is given by the map

$$J \mapsto (J\Lambda_0, \, g_J|_{\Lambda_0}),$$

where $g_J(v, w) = \omega(v, Jw)$ as above. Show that this is a bijection. □

Exercise 2.54 Let ω and g be given. Show that there is a basis for V which is both g-orthogonal and ω-standard if and only if there is a Lagrangian subspace Λ whose g-orthogonal complement $\Lambda^\perp$ is also Lagrangian. □

Exercise 2.55 Let $J \in \mathcal{J}(V,\omega)$. Prove that a subspace $\Lambda \subset V$ is Lagrangian with respect to ω if and only if $J\Lambda$ is the orthogonal complement of Λ with respect to the inner product g_J. Deduce that $\Lambda \in \mathcal{L}(V,\omega)$ if and only if $J\Lambda \in \mathcal{L}(V,\omega)$. □

Exercise 2.56 Suppose that J_t is a smooth family of complex structures on V depending on a parameter t. Prove that there exists a smooth family of isomorphisms $\Phi_t : \mathbb{R}^{2n} \to V$ such that $J_t\Phi_t = \Phi_t J_0$ for every t. □

Exercise 2.57 Prove that the real 2×2 matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies $J^2 = -1\!\!1$ if and only if $ad - bc = 1$ and $a = -d$. Deduce that J_0 and $-J_0$ lie in different components of $\mathcal{J}(\mathbb{R}^2)$. Prove that each component of $\mathcal{J}(\mathbb{R}^2)$ is contractible. □

Exercise 2.58 Let V be a $2n$-dimensional real vector space with complex structure J. Show that the space of all skew-forms ω which are compatible with J is convex. □

Exercise 2.59 A linear subspace $W \subset V$ is called **totally real** if it is of dimension n and

$$JW \cap W = \{0\}.$$

If $W \subset V$ is a totally real subspace show that the space of nondegenerate skew-forms $\omega : V \times V \to \mathbb{R}$ which are compatible with J and satisfy

$$W \in \mathcal{L}(V,\omega)$$

is naturally isomorphic to the space of inner products on W and hence is convex. □

2.6 Symplectic vector bundles

In this section we discuss the basic properties of symplectic vector bundles. For convenience, we will always assume that the base space M of the bundle is a smooth manifold, perhaps with boundary ∂M, but the theory can of course be developed in more generality. The main result is Theorem 2.62 which says that symplectic vector bundles are essentially the same as complex vector bundles. After sketching how this follows from the theory of classifying spaces, we give an explicit and elementary proof. We also give a complete proof of the existence of the first Chern class.

A **symplectic vector bundle** (E, ω) over a manifold M is a real vector bundle $\pi : E \to M$ which is provided with a symplectic bilinear form ω_q on each fibre E_q, which varies smoothly with $q \in M$. These forms ω_q fit together to give a smooth section ω of the exterior power $E^* \wedge E^*$ of the dual bundle $E^* = \mathrm{Hom}(E, \mathbb{R})$. This form ω is a nondegenerate skew-symmetric bilinear form on E which we will call a **symplectic bilinear form**. Two symplectic vector bundles (E_1, ω_1) and (E_2, ω_2) are isomorphic if and only if there is a linear map $\Psi : E_1 \to E_2$ such that $\Psi^*(\omega_2) = \omega_1$.

Example 2.60 If $E \to M$ is any vector bundle, the sum $E \oplus E^*$ is a symplectic bundle with form Ω_{can} given by

$$\Omega_{\mathrm{can}}(v_0 \oplus v_1^*, w_0 \oplus w_1^*) = w_1^*(v_0) - v_1^*(w_0). \qquad \Box$$

Exercise 2.61 Prove that such a symplectic vector bundle is locally symplectically trivial. This means that every point $q \in M$ has a neighbourhood U such that $(\pi^{-1}(U), \omega)$ is isomorphic to $(U \times \mathbb{R}^{2n}, \omega_0)$ by a map ψ which lifts the identity map of U. **Hint:** Construct sections which form a symplectic basis in each fibre. Compare with Exercise 2.11. □

It follows from Exercise 2.61 that the structure group of a symplectic vector bundle (E, ω) can be reduced from $\mathrm{GL}(2n; \mathbb{R})$ to $\mathrm{Sp}(2n)$. This means that E can be constructed by patching together disjoint pieces $U_\alpha \times \mathbb{R}^{2n}$ using transition functions $U_\alpha \cap U_\beta \to \mathrm{Sp}(2n)$. According to the theory of classifying spaces (see, for example, [210] and [136]) isomorphism classes of symplectic bundles over the base M correspond to homotopy classes of maps from M to the classifying space $B\mathrm{Sp}(2n)$. But $B\mathrm{Sp}(2n)$ is homotopy equivalent to $B\mathrm{U}(n)$ by Proposition 2.22. Hence each symplectic bundle has a complex structure which is well defined up to homotopy. Moreover, this complex structure characterizes the isomorphism class of the bundle in the following sense.

Theorem 2.62 *The symplectic vector bundles (E_1, ω_1) and (E_2, ω_2) are isomorphic if and only if their underlying complex bundles are isomorphic.*

Here is a more direct proof of this result. As in the case of vector spaces, a **complex structure** on a vector bundle $E \to M$ is an automorphism J of E such that $J^2 = -\mathbb{1}$ (think of J as multiplication by i). This complex structure is said to be **compatible** with ω if the induced complex structure J_q on the fibre E_q is compatible with ω_q for all $q \in M$. For any such compatible pair the bilinear form $g_J : E \times E \to \mathbb{R}$ defined by

$$g_J(v, w) = \omega(v, Jw)$$

is symmetric and positive definite. A triple (ω, J, g) with these properties is called a **Hermitian structure** on E. The following proposition asserts that every symplectic vector bundle admits a Hermitian structure.

Proposition 2.63 *Let $E \to M$ be a $2n$-dimensional vector bundle.*
(i) *For every symplectic bilinear form on E there exists an almost complex structure J which is compatible with ω. The space $\mathcal{J}(E, \omega)$ of such complex structures is contractible.*
(ii) *Let J be a complex structure on E. Then there exists a symplectic bilinear form ω which is compatible with J. The space of such forms is contractible.*

Proof: The corresponding statements when M is a single point are proved in Proposition 2.50 and Exercise 2.52 above. The global statements are proved by the same arguments. In particular, the existence of a compatible complex structure on E follows from the existence of an inner product on E via the map $r : \mathfrak{Met}(V) \to \mathcal{J}(V, \omega)$ of Proposition 2.50 with $V = E_q$. Statement (ii) follows from Exercise 2.58. The details are left as an exercise for the reader. □

Proof of Theorem 2.62: Let J_1 and J_2 be compatible complex structures on the symplectic vector bundles (E_1, ω_1) and (E_2, ω_2). Assume first that there exists a bundle isomorphism $\Psi : E_1 \to E_2$ such that

$$\Psi^* \omega_2 = \omega_1.$$

Then the complex structures J_1 and $\Psi^* J_2$ are both compatible with ω_1. Hence it follows from Proposition 2.63 (i) that there exists a smooth family of complex structures

$$J_t : E_1 \to E_1$$

which are compatible with ω_1 and connect $J_0 = \Psi^* J_2$ to J_1. By Exercise 2.56 there exists a smooth family of bundle isomorphisms $\Phi_t : E_1 \to E_1$ such that

$$\Phi_t{}^* J_t = J_1.$$

Thus the bundle isomorphism $\Psi \circ \Phi_0 : E_1 \to E_2$ intertwines J_1 and J_2. Conversely, a similar argument using Proposition 2.63 (ii) and Corollary 2.4 shows that if the complex vector bundles (E_1, J_1) and (E_2, J_2) are isomorphic then so are the symplectic vector bundles (E_1, ω_1) and (E_2, ω_2). □

Similar results hold for pairs (E, F), where E is a symplectic bundle and F is a subbundle with fibres which are symplectic or Lagrangian subspaces. The next exercise formulates this precisely in the case when F is a Lagrangian subbundle over the boundary ∂M of M.

Exercise 2.64 Let $E \to M$ be a $2n$-dimensional vector bundle with complex structure J and $F \to \partial M$ be an n-dimensional totally real subbundle. This means

$$J_q F_q \cap F_q = \{0\}$$

for all $q \in \partial M$. Prove that there exists a symplectic bilinear form ω which is compatible with J and satisfies $F_q \in \mathcal{L}(E_q, \omega_q)$ for $q \in \partial M$. Prove that the space of such forms is contractible. □

Trivializations

A trivialization of a bundle E is an isomorphism from E to the trivial bundle which preserves the structure under consideration. For example, a symplectic trivialization preserves the symplectic structure, and a complex trivialization is an isomorphism of complex vector bundles. It follows from Theorem 2.62 that these notions are essentially the same. For example, a symplectic bundle (E, ω) is symplectically trivial if and only if, given any compatible complex structure J on E, the complex vector bundle (E, J) is trivial as a complex bundle. It is therefore convenient to combine these notions and consider unitary trivializations. The results we develop now will be used to define the first Chern class in the next subsection.

A **unitary trivialization** of a Hermitian vector bundle E is a smooth map

$$M \times \mathbb{R}^{2n} \to E : (q, \zeta) \mapsto \Phi(q)\zeta$$

which transforms ω, J and g to the standard structures on $\mathbb{R}^{2n}$:

$$\Phi^* J = J_0, \qquad \Phi^* \omega = \omega_0, \qquad \Phi^* g = g_0,$$

where $g_0(\xi, \eta) = \langle \xi, \eta \rangle$ for $\xi, \eta \in \mathbb{R}^{2n}$. A **unitary trivialization along a curve** $\gamma : [0, 1] \to M$ is a unitary trivialization of the pull-back bundle $\gamma^* E$.

Lemma 2.65 *Let $E \to M$ be a vector bundle with Hermitian structure (ω, J, g). Let $\gamma : [0, 1] \to M$ be a smooth curve with unitary isomorphisms $\Phi_0 : \mathbb{R}^{2n} \to E_{\gamma(0)}$ and $\Phi_1 : \mathbb{R}^{2n} \to E_{\gamma(1)}$ at the endpoints. Then there exists a unitary trivialization $\Phi(t) : \mathbb{R}^{2n} \to E_{\gamma(t)}$ of $\gamma^* E$ such that $\Phi(0) = \Phi_0$ and $\Phi(1) = \Phi_1$.*

Proof: We first prove that such a trivialization exists on some interval $0 \le t < \varepsilon$. Choose $s_{j0} \in E_{\gamma(0)}$ such that

$$\Phi_0 \zeta = \sum_j s_{j0} \zeta_j$$

for $\zeta \in \mathbb{R}^{2n}$. We must construct $2n$ sections $s_j(t) \in E_{\gamma(t)}$ which satisfy

$$\langle s_j, s_k \rangle = \delta_{jk}, \qquad \omega(s_j, s_{j+n}) = 1$$

and $\omega(s_j, s_k) = 0$ for all other values of j and k. Here $\langle s_j, s_k \rangle = g(s_j, s_k)$ denotes the inner product. To construct these sections choose a Riemannian connection ∇ on E (see Kobayashi and Nomizu [145] or Donaldson and Kronheimer [61]) and choose $\tilde{s}_j(t) \in E_{\gamma(t)}$ to be parallel:

$$\nabla \tilde{s}_j = 0, \qquad \tilde{s}_j(0) = s_{j0}.$$

Then for small t the first n vectors $\tilde{s}_1(t), \ldots, \tilde{s}_n(t)$ are linearly independent over $\mathbb{C}$. Now use Gram–Schmidt over the complex numbers to obtain a unitary basis:

$$s_k = \frac{\tilde{s}_k}{|\tilde{s}_k|} - \sum_{j=1}^{k-1} \frac{\langle s_j, \tilde{s}_k \rangle}{|\tilde{s}_k|} s_j - \sum_{j=1}^{k-1} \frac{\omega(s_j, \tilde{s}_k)}{|\tilde{s}_k|} J s_j, \qquad s_{k+n} = J s_k.$$

This works for small time intervals. Now cover the interval $[0, 1]$ by finitely many intervals over which a unitary trivialization exists and use a patching argument on overlaps. More precisely, given two trivializations Φ_1 and Φ_2 over an interval (a, b) choose a smooth path of unitary matrices $\Psi : (a, b) \to \mathrm{U}(n) \subset \mathrm{Sp}(2n)$ such that $\Psi(t) = \mathbb{1}$ for t near a and $\Psi(t) = \Phi_1(t)^{-1}\Phi_2(t)$ for t near b. Then $\Phi(t) = \Phi_1(t)\Psi(t)$ agrees with Φ_1 near a and with Φ_2 near b. □

Proposition 2.66 *A Hermitian vector bundle $E \to \Sigma$ over a compact Riemann surface Σ with nonempty boundary $\partial\Sigma$ admits a unitary trivialization.*

Proof: The proof of the proposition is by induction over the number

$$k(\Sigma) = 2g(\Sigma) + \ell(\Sigma),$$

where $\ell(\Sigma) \geq 1$ is the number of boundary components and $g(\Sigma) \geq 0$ is the genus. If $k(\Sigma) = 1$ then Σ is diffeomorphic to the unit disc and in this case the statement is a parametrized version of Lemma 2.65: trivialize along rays starting at the origin.

Suppose the statement has been proved for $k(\Sigma) \leq m$ and let $k(\Sigma) = m + 1$. Then Σ can be decomposed as $\Sigma = \Sigma_1 \cup_C \Sigma_2$ such that $k(\Sigma_1) = m$ and Σ_2 is diffeomorphic to the unit disc with two holes. (See Fig. 2.2.)

There are two cases. Either C has one component and

$$g(\Sigma_1) = g(\Sigma), \qquad \ell(\Sigma_1) = \ell(\Sigma) - 1$$

or C has two components and

$$g(\Sigma_1) = g(\Sigma) - 1, \qquad \ell(\Sigma_1) = \ell(\Sigma) + 1.$$

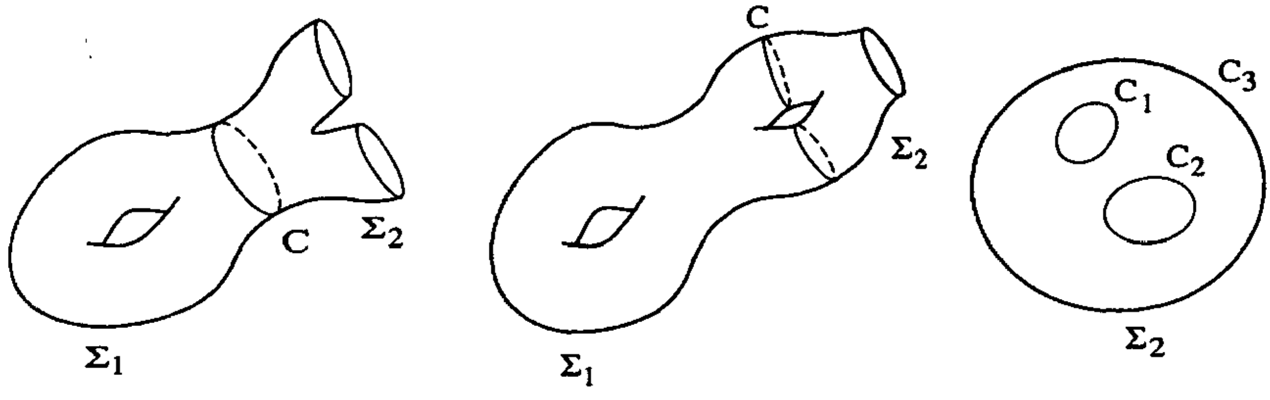

FIG. 2.2. Decomposing Σ

In either case $k(\Sigma_1) = k(\Sigma) - 1$.

By the induction hypothesis the bundle E admits a trivialization over Σ_1. We must prove that any trivialization of E over C extends to a trivialization over Σ_2. Denote

$$\partial\Sigma_2 = C_1 \cup C_2 \cup C_3,$$

where C_1 and C_2 are the (interior) boundaries of the two holes and C_3 is the (exterior) boundary of the disc. If $C = C_1 \cup C_2$ connect the two circles C_1 and C_2 by a straight line. By Lemma 2.65 extend the trivialization over C to that line. Then use Lemma 2.65 along rays to extend the trivialization to Σ. If $C = C_1$ choose first any trivialization over C_2 and then proceed as above. □

Exercise 2.67 Define the notion 'symplectic trivialization'. Show that a Hermitian bundle has a unitary trivialization if and only if its underlying symplectic bundle has a symplectic trivialization. □

Exercise 2.68 Prove that the space of paths $\Psi : [0,1] \to \mathrm{Sp}(2n)$ of symplectic matrices satisfying

$$\Psi(1) = \Psi(0)^{-1}$$

has two components. Deduce that up to isomorphism there are precisely two symplectic vector bundles (of every given dimension) over the real projective plane $\mathbb{R}P^2$. **Hint:** Think of $\mathbb{R}P^2$ as the 2-disc with opposite points on the boundary identified. □

First Chern class

Since the set of isomorphism classes of symplectic vector bundles coincides with the set of isomorphism classes of complex vector bundles, symplectic vector bundles have the same characteristic classes as complex vector bundles, namely the Chern classes. In this subsection we establish the existence of the first Chern class. This is an element of the integral 2-dimensional cohomology of the base manifold. For bundles over 2-dimensional bases, the first Chern class c_1 is completely described by the first Chern number, which is the value taken by c_1 on the fundamental 2-cycle of the base. Therefore we will begin by describing this integer invariant for symplectic vector bundles over compact oriented Riemann surfaces without boundary. It can be defined axiomatically as follows.

Theorem 2.69 *There exists a unique functor* c_1*, called the* **first Chern number**, *that assigns an integer* $c_1(E) \in \mathbb{Z}$ *to every symplectic vector bundle* E *over a compact oriented Riemann surface* Σ *without boundary and satisfies the following axioms.*

(naturality) *Two symplectic vector bundles* E *and* E' *over* Σ *are isomorphic iff they have the same dimension and the same Chern number.*

(functoriality) *For any smooth map* $\phi : \Sigma' \to \Sigma$ *of oriented Riemann surfaces and any symplectic vector bundle* $E \to \Sigma$

$$c_1(\phi^* E) = \deg(\phi) \cdot c_1(E).$$

(additivity) *For any two symplectic vector bundles* $E_1 \to \Sigma$ *and* $E_2 \to \Sigma$

$$c_1(E_1 \oplus E_2) = c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2).$$

(normalization) *The Chern number of the tangent bundle of* Σ *is*

$$c_1(T\Sigma) = 2 - 2g,$$

where g *is the genus.*

Remark 2.70 **(i)** It follows from the axioms that the first Chern number vanishes if and only if the bundle is trivial. Hence the first Chern number $c_1(E)$ can be viewed as an obstruction for the bundle E to admit a symplectic trivialization.
(ii) If E is a symplectic vector bundle over any manifold M then the first Chern number assigns an integer $c_1(f^*E)$ to every smooth map $f : \Sigma \to M$ from a compact oriented Riemann surface without boundary to M. We will see in Exercise 2.78 that this integer depends only on the homology class of f. Thus the first Chern number generalizes to a homomorphism $H_2(M;\mathbb{Z}) \to \mathbb{Z}$. This gives rise to a cohomology class $c_1(E) \in H^2(M; x\mathbb{Z})/\text{torsion}$. There is in fact a natural choice of a lift of this class to $H^2(M;\mathbb{Z})$, also denoted by $c_1(E)$, which is called the **first Chern class**. We shall not discuss this lift in detail, but only remark that in the case of a line bundle $L \to M$ the class $c_1(L) \in H^2(M;\mathbb{Z})$ is Poincaré dual to the homology class determined by the zero set of a generic section.

(iii) It is customary to define the Chern class as an invariant for *complex* vector bundles. It follows from Theorem 2.62 that our definition agrees with the usual one. □

We will prove Theorem 2.69 by giving an explicit definition of the first Chern number and checking that the axioms are satisfied. Given a compact oriented Riemann surface Σ without boundary choose a splitting

$$\Sigma = \Sigma_1 \cup_C \Sigma_2$$

such that $\partial\Sigma_1 = \partial\Sigma_2 = C$. Orient the 1-manifold C as the boundary of Σ_1: a vector $v \in T_qC$ is positively oriented if $\{\nu(q), v\}$ is a positively oriented basis of $T_q\Sigma$, where $\nu : C \to T\Sigma$ is a normal vector field along C which points out of Σ_1.

Now let E be a symplectic vector bundle over Σ and choose symplectic trivializations

$$\Sigma_k \times \mathbb{R}^{2n} \to E : (q, \zeta) \mapsto \Phi_k(q)\zeta$$

of E over Σ_1 and Σ_2. The **overlap map** $\Psi : C \to \mathrm{Sp}(2n)$ is defined by

$$\Psi(q) = \Phi_1(q)^{-1}\Phi_2(q)$$

for $q \in C$. Consider the map $\rho : \mathrm{Sp}(2n) \to S^1$ defined by

$$\rho(\Psi) = \det(X + iY), \qquad \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = (\Psi\Psi^{\mathrm{T}})^{-1/2}\Psi$$

as in Theorem 2.29. We shall prove that the first Chern number of E is the degree of the composition $\rho \circ \Psi : C \to S^1$:

$$\mathrm{c}_1(E) = \deg \rho \circ \Psi.$$

In other words, the first Chern number is the sum of the Maslov indices of the loops $\Psi \circ \gamma_j : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2n)$

$$c_1(E) = \sum_{j=1}^{\ell} \mu(\Psi \circ \gamma_j),$$

where ℓ is the number of components of C and each component is parametrized by a loop $\gamma_j : \mathbb{R}/\mathbb{Z} \to C$ such that $\dot\gamma_j(t)$ is positively oriented. A direct application of this definition is given in Example 2.74 below. The next lemma is required to prove that with this definition the first Chern number is independent of the choices, i.e. that the degree of $\rho \circ \Psi$ is independent of the splitting and of the trivializations.

Lemma 2.71 *Let Σ be a compact oriented Riemann surface with nonempty boundary. A smooth map $\Psi : \partial\Sigma \to \mathrm{Sp}(2n)$ extends to Σ if and only if*

$$\deg \rho \circ \Psi = 0.$$

Proof: First assume that Ψ extends to a smooth map $\Sigma \to \mathrm{Sp}(2n)$. Then the composition $\rho \circ \Psi : \partial\Sigma \to S^1$ extends to a smooth map $\Sigma \to S^1$ and hence must have degree zero.

Now assume that $\partial\Sigma = C_0 \cup C_1$, where both C_0 and C_1 are nonempty. We prove that any smooth map $\Psi : C_0 \to \mathrm{Sp}(2n)$ extends over Σ. This is obvious in the case of the cylinder $\Sigma = S^1 \times [0,1]$. The case where Σ is the disc with two holes and C_1 is the outer boundary can be reduced to that of the cylinder by extending the map $\Psi : C_0 \to \mathrm{Sp}(2n)$ over a line which connects the two holes. The general case is proved by decomposing a Riemann surface as in the proof of Proposition 2.66.

Conversely, assume $\deg \rho \circ \Psi = 0$. By what we just proved the map $\Psi : \partial\Sigma \to \mathrm{Sp}(2n)$ extends over $\Sigma - B$, where B is a disc. By the first part of the proof the resulting loop $\partial B \to \mathrm{Sp}(2n)$ has Maslov index zero. By Theorem 2.29 this loop is contractible. □

Proof of Theorem 2.69: We prove that the degree of $\rho \circ \Psi$ as defined above is independent of the choice of the splitting and the trivialization and satisfies the axioms of Theorem 2.69. By Lemma 2.71 the degree of $\rho \circ \Psi$ is independent of the choice of the trivialization. We prove that it is independent of the choice of the splitting. Consider a threefold splitting

$$\Sigma = \Sigma_{32} \cup_{C_2} \Sigma_{21} \cup_{C_1} \Sigma_{10}$$

and choose trivializations Φ_{31} and Φ_{20} of E over

$$\Sigma_{31} = \Sigma_{32} \cup_{C_2} \Sigma_{21}, \qquad \Sigma_{20} = \Sigma_{21} \cup_{C_1} \Sigma_{10}.$$

(See Fig. 2.3.)

The corresponding overlap map is

$$\Psi_{21}(q) = \Phi_{31}(q)^{-1}\Phi_{20}(q)$$

for $q \in \Sigma_{21}$. The restriction $\Psi_1 = \Psi_{21}|C_1$ corresponds to the splitting $\Sigma = \Sigma_{31} \cup_{C_1} \Sigma_{10}$ and $\Psi_2 = \Psi_{21}|C_2$ to the splitting $\Sigma = \Sigma_{32} \cup_{C_2} \Sigma_{20}$. By Lemma 2.71 both curves Ψ_1 and Ψ_2 with the induced orientations of $C_1 = \partial\Sigma_{10}$ and of $C_2 = \partial\Sigma_{20}$ have the same Maslov index. This proves the statement for two splittings along disjoint curves C_1 and C_2. The general case can be reduced to this one by choosing a third splitting.

We prove naturality. Fix a splitting $\Sigma = \Sigma_1 \cup_C \Sigma_2$, where C has only one component. Choose trivializations of E and E' over Σ_1 and Σ_2 and

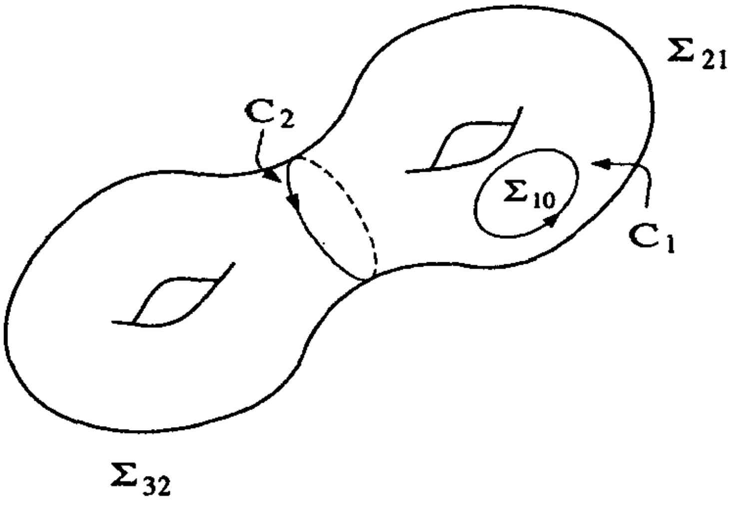

FIG. 2.3. A threefold splitting

let $\Psi : C \to \mathrm{Sp}(2n)$ and $\Psi' : C \to \mathrm{Sp}(2n)$ be the corresponding loops of symplectic matrices. Since Ψ and Ψ' have the same Maslov index the loop $\Psi'\Psi^{-1} : C \to \mathrm{Sp}(2n)$ extends to a smooth map $\Sigma_1 \to \mathrm{Sp}(2n)$. Use this map to construct the required symplectic isomorphism $E \to E'$.

We prove functoriality. Choose a regular value $q \in \Sigma$ of $\phi : \Sigma' \to \Sigma$ and cut out a small neighbourhood $B \subset \Sigma$ of q. Then $\phi^{-1}(B)$ consists of $d = \deg(\phi)$ discs $B_1', \ldots, B_d' \subset \Sigma'$. Let $\Psi_j' : \partial B_j' \to \mathrm{Sp}(2n)$ be the corresponding parametrized loops of symplectic matrices arising from trivializations of ϕ^*E over B_j' and $\Sigma' - \cup_j B_j'$. These loops all have the same Maslov index $\deg \rho \circ \Psi_j' = c_1(E)$. Hence $c_1(\phi^*E) = d \cdot c_1(E)$.

Additivity follow from the identities

$$\det(U_1 \oplus U_2) = \det(U_1 \otimes U_2) = \det(U_1)\det(U_2)$$

for unitary matrices $U_j \in \mathrm{U}(n_j)$. The axiom of normalization is left as an exercise and so is the proof of uniqueness. □

Chern–Weil theory

The first Chern class can also expressed as a curvature integral. We explain this in the case of a line bundle $L \to \Sigma$ over a compact oriented Riemann surface Σ. Let L be equipped with a Hermitian structure and consider the circle bundle (or unitary frame bundle) $\pi : P \to \Sigma$ of all vectors in L of length 1:

$$P = \{(z, v) \mid z \in \Sigma,\ v \in L_z,\ |v| = 1\}.$$

The circle $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ acts on this bundle in the obvious way. We denote this action by $P \times S^1 \to P : (p, \lambda) \mapsto p \cdot \lambda$. For $i\theta$ in the Lie algebra $i\mathbb{R} = \mathrm{Lie}(S^1)$ we denote the induced vertical vector field on P by

$$p \cdot i\theta = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{i\theta t}$$

for $p \in P$. A **connection** on P is a 1-form $A \in \Omega^1(P, i\mathbb{R})$ with values in $i\mathbb{R} = \mathrm{Lie}(S^1)$ which is invariant under the action of S^1 and canonical in the vertical direction, namely

$$A_{p\cdot\lambda}(v \cdot \lambda) = A_p(v), \qquad A_p(p \cdot i) = i$$

for $p \in P$, $v \in T_pP$, and $\lambda \in S^1$. Since the group is abelian, the curvature of a connection 1-form A agrees with the differential dA. This differential is invariant and horizontal in the sense that $dA_p(p \cdot i, v) = 0$ for all $v \in T_pP$. Any such form descends to the base Σ, i.e.

$$dA = \pi^*\tau, \qquad \tau \in \Omega^2(\Sigma, i\mathbb{R}).$$

The following result is a simple case of Chern–Weil theory.

Theorem 2.72 *Every connection 1-form A on P satisfies*

$$c_1(P) = \frac{i}{2\pi}\int_\Sigma \tau, \qquad dA = \pi^*\tau.$$

Moreover, for every 2-form $\tau \in \Omega^2(\Sigma, i\mathbb{R})$ with $\int_\Sigma \tau = -2\pi i c_1(P)$ there exists a connection 1-form A on P with $dA = \pi^*\tau$.

Proof: Decompose $\Sigma = \Sigma_1 \cup_C \Sigma_2$ and choose sections

$$s_j : \Sigma_j \to P.$$

These give rise to trivializations $\Sigma_j \times \mathbb{C} \to L : (z, \zeta) \mapsto s_j(z)\zeta$ and, according to the discussion after Theorem 2.69, the first Chern number $c_1(L) = c_1(P) \in \mathbb{Z}$ is the degree of the loop $\gamma : C \to S^1$ defined by

$$s_1(z)\gamma(z) = s_2(z), \qquad z \in C,$$

where C is oriented as the boundary of Σ_1. Now the connection A pulls back to 1-forms $\alpha_j = s_j^*A \in \Omega^1(\Sigma_j, i\mathbb{R})$ and we have

$$\begin{aligned}
\int_\Sigma \tau &= \int_{\Sigma_1} d\alpha_1 + \int_{\Sigma_2} d\alpha_2 \\
&= \int_C (\alpha_1 - \alpha_2) \\
&= -\int_C \gamma^{-1} d\gamma \\
&= -2\pi i \deg(\gamma).
\end{aligned}$$

The last identity is a simple exercise. Since $c_1(L) = c_1(P) = \deg(\gamma)$ this proves the first statement of the theorem. To prove the second statement let $\tau' \in \Omega^2(\Sigma)$ be any 2-form such that $\int_\Sigma \tau' = -2\pi i c_1(P)$. Then $\tau' - \tau$ is exact and hence $\tau' - \tau = d\alpha$ for some 1-form $\alpha \in \Omega^1(\Sigma, i\mathbb{R})$. It follows that $A' = A + \pi^*\alpha$ is a connection 1-form with curvature $dA' = \pi^*\tau'$. □

Chern class and Euler class

Let $L \to \Sigma$ be a complex line bundle (or symplectic bundle of rank 2) over an oriented Riemann surface Σ. Then the first Chern number can also be interpreted as the self-intersection number of the zero section of L. More precisely, let $s : \Sigma \to L$ be a section of L which is transverse to the zero section. Then s has finitely many zeros and at each zero z the intersection index $\iota(z, s)$ is defined to be ± 1 according to whether or not the linearized map $Ds(z) = \pi \circ ds(z) : T_z\Sigma \to L_z$ is orientation preserving or orientation reversing. The following theorem shows that the sum of these indices agrees with the first Chern number of L. In other words $c_1(L)$ is the obstruction to the existence of a nonvanishing section of L and so coincides with the **Euler class** of L.

Theorem 2.73 *If the section $s : \Sigma \to L$ is transverse to the zero section then the first Chern number of L is given by*

$$c_1(L) = \sum_{s(z)=0} \iota(z, s). \tag{2.2}$$

Proof: The proof is an exercise with hint. Cut out a small neighbourhood U of the zero set of s and use s to trivialize the bundle L over the complement $\Sigma' = \Sigma - U$. Use a different method to trivialize Σ over U and then compare the two trivializations over the boundary of U (which consists of finitely many circles). □

Examples

We begin by working out the first Chern class of the normal bundle $\nu_{\mathbb{C}P^1}$ to the line $\mathbb{C}P^1$ in the complex projective plane $\mathbb{C}P^2$. We will use the notation of Example 4.21 and, for simplicity, will work in the complex context, choosing complex trivializations rather than symplectic ones. By Theorem 2.62, this will make no difference to the final result.

Example 2.74 Consider the line $\{[z_0 : z_1 : 0]\}$ in $\mathbb{C}P^2$. It is covered by the two sets $\Sigma_1 = \{[1 : z_1 : 0] \mid |z_1| \le 1\}$ and $\Sigma_2 = \{[z_0 : 1 : 0] \mid |z_0| \le 1\}$, which intersect in the circle $C = \{[1 : e^{2\pi i\theta} : 0]\}$. Note that θ is a positively oriented coordinate for C. It is not hard to see that the fibre of $\nu_{\mathbb{C}P^1}$ over the point $[1 : z_1 : 0] \in \Sigma_1$ embeds into $\mathbb{C}P^2$ as the set $\{[1 : z_1 : w] \mid w \in \mathbb{C}\}$. Therefore we may choose the complex trivializations

$$\Phi_1 : \Sigma_1 \times \mathbb{C} \to \nu_{\mathbb{C}P^1}, \quad \Phi_1([1 : z_1 : 0], w) = [1 : z_1 : w],$$

$$\Phi_2 : \Sigma_2 \times \mathbb{C} \to \nu_{\mathbb{C}P^1}, \quad \Phi_2([z_0 : 1 : 0], w) = [z_0 : 1 : w].$$

The map $\Phi_1^{-1}\Phi_2(z) : \mathbb{C} \to \mathbb{C}$ is the composite

$$w \mapsto [1/z : 1 : w] = [1 : z : zw] \mapsto zw,$$

and so induces the map

$$C \to U(1) : \theta \mapsto e^{2\pi i\theta}.$$

This shows that $c_1(\nu_{\mathbb{C}P^1}) = 1$. □

Exercise 2.75 Use the formula (2.2) to calculate the first Chern number $c_1(\nu_{\mathbb{C}P^1})$ of the normal bundle $\nu_{\mathbb{C}P^1}$ of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. □

Exercise 2.76 Let $L \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ be the incidence relation:

$$\begin{aligned} L &= \{(z, \ell) \mid z \in \ell\} \\ &= \{(z_1, \dots, z_n; [w_1 : \dots : w_n]) \mid w_j z_k = w_k z_j \ \forall\, j, k\}. \end{aligned}$$

The projection $\mathrm{pr} : L \to \mathbb{C}P^{n-1}$ gives L the structure of a complex line bundle over $\mathbb{C}P^{n-1}$. Show that when $n = 2$ the first Chern number of the restriction $L|_{\mathbb{C}P^1}$ is -1, and hence calculate $c_1(L)$ for arbitrary n. Another approach to this calculation is given in Lemma 7.1. □

Exercise 2.77 Prove that every symplectic vector bundle over a Riemann surface decomposes as a direct sum of 2-dimensional symplectic vector bundles. You can either prove this directly, or use the naturality axiom in Theorem 2.69. □

Exercise 2.78 **(i)** Suppose that $E \to \Sigma$ is a symplectic vector bundle over an oriented Riemann surface Σ that extends over a compact oriented 3-manifold Y with boundary $\partial Y = \Sigma$. Prove that the restriction $E|_\Sigma$ has Chern class zero. **Hint:** Use part (i) of Remark 2.70 and consider a section s as in Theorem 2.73.
(ii) Use (i) above and Exercise 2.77 to substantiate the claim made in Remark 2.70 that the Chern class $c_1(f^*E)$ depends only on the homology class of f. Here the main problem is that when $f_*([\Sigma])$ is null-homologous the 3-chain that bounds it need not be representable by a 3-manifold. However, its singularities can be assumed to have codimension 2 and so the proof of (i) goes through. □

Exercise 2.79 Prove that every symplectic vector bundle $E \to \Sigma$ that admits a Lagrangian subbundle can be symplectically trivialized. **Hint:** Use the proof of Theorem 2.69 to show that $c_1(E) = 0$. □

3

SYMPLECTIC MANIFOLDS

This is a foundational chapter, and everything in it (except perhaps Section 3.4 on contact structures) is needed to understand later chapters. The first section contains elementary definitions and first examples of symplectic manifolds. The second section is devoted to Darboux's theorem. As Gromov notes in [110], symplectic geometry is a curious mixture of the 'hard' and the 'soft'. Some situations are flexible and there are no nontrivial invariants, while other situations are rigid. In this chapter we deal with 'soft' phenomena, the fact that in symplectic geometry there are no local invariants. The classical formulation of this principle is known as Darboux's theorem: all symplectic forms are locally diffeomorphic. However there are many other related results, such as Moser's stability theorem and various versions of the symplectic neighbourhood theorem. We prove these in Sections 3.2 and 3.3 using Moser's homotopy method. The chapter ends with a brief discussion of contact geometry. This is the odd-dimensional analogue of symplectic geometry, and there are many connections between the two subjects.

3.1 Basic concepts

Throughout we will assume that M is a C^∞-smooth manifold, which (unless specific mention is made to the contrary) has no boundary. Very often, M will also be compact.

A **symplectic structure** on a smooth manifold M is a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. Nondegeneracy means that each tangent space (T_qM, ω_q) is a symplectic vector space. The manifold M is necessarily of even dimension $2n$ and, by Corollary 2.5, the n-fold wedge product

$$\omega \wedge \ldots \wedge \omega$$

never vanishes. Thus M is orientable.

The first example of a symplectic manifold is $\mathbb{R}^{2n}$ itself with the standard symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

which was defined in Section 1.1. Another basic example is the 2-sphere with its standard area form. If we think of S^2 as the unit sphere

$$\Phi_2 : \Sigma_2 \times \mathbb{C} \to \nu_{\mathbb{C}P^1}, \quad \Phi_2([z_0 : 1 : 0], w) = [z_0 : 1 : w].$$

The map $\Phi_1^{-1}\Phi_2(z) : \mathbb{C} \to \mathbb{C}$ is the composite

$$w \mapsto [1/z : 1 : w] = [1 : z : zw] \mapsto zw,$$

and so induces the map

$$C \to U(1) : \theta \mapsto e^{2\pi i\theta}.$$

This shows that $c_1(\nu_{\mathbb{C}P^1}) = 1$. □

Exercise 2.75 Use the formula (2.2) to calculate the first Chern number $c_1(\nu_{\mathbb{C}P^1})$ of the normal bundle $\nu_{\mathbb{C}P^1}$ of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. □

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