Math. Z. 166, 183–186 (1979)

Mathematische Zeitschrift

A Chain Level Transfer Homomorphism for PL Fibrations

Hans J. Munkholm*

Department of Mathematics, Princeton University, Princeton, N.J. 08540, U.S.A.
Matematisk Institut, Odense Universitet, Campusvej 55, DK-5320 Odense, Denmark

Let p: $E \to B$ be a Hurewicz fibration with compact fiber F, and h^* a generalized cohomology theory. It is well known, see e.g. [1] and [2], that p gives rise to a *transfer homomorphism* $\tau(p)^*$: $h^*(E) \to h^*(B)$. Also, if B has the homotopy type of a finite CW complex then $\tau(p)^*$ is induced by an S-map $\tau(p)$: $B^+ \to E^+$ ($+$ means addition of a base point); $\tau(p)$ is called a *transfer map.*

The simplest case of a transfer homomorphism occurs when h^* = ordinary cohomology and p: $E \to B$ is a finite sheeted covering. Then $\tau(p)^*$ is induced by the chain map (singular chains, say)

$$\tau(p)_\#: C(B) \to C(E)$$

having

$$\tau(p)_\#(\sigma) = \sum \tilde{\sigma} \tag{1}$$

where $\tilde{\sigma}$: $\Delta[s] \to E$ runs over all liftings of σ: $\Delta[s] \to B$, $\Delta[s]$ being the standard s-simplex.

In this note we show that when we deal with PL fibrations (i.e. Serre fibrations in the category of polyhedra and PL maps, compare [4]) having compact fibers then there is an equally trivial definition of a chain map

$$\tau(p)_\#: C(B) \to C(E)$$

which induces the transfer map on (co-)homology. Here C is the simplicial chain functor.

We have no specific applications in mind that cannot equally well be derived from the existing definitions. Thus our only reason for publishing the present note is the inherent simplicity (and consequent beauty) of the approach.

Let p: $E \to B$ be a PL map between polyhedra. Assume that E and B have been given triangulations for which p is simplicial. Simplices of E (and B) will be

* Partially supported by the Danish Natural Science Research Council

0025-5874/79/0166/0183/$01.00

denoted by e, e_i (and b, b_i). Also let B' be the standard barycentric subdivision of B and let E' be *some* barycentric subdivision of E for which p: $E' \to B'$ is simplicial (compare remark (1) p. 225 of [3]). The typical s-simplex β of B' has the form

$$\beta = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_s)$$

where $b_0 > b_1 > \ldots > b_s$ in B, and $\hat{}$ denotes barycenter. We shall let $C(B')$ be the simplicial chains on B'. To define $\tau(p)_\#$: $C(B') \to C(E')$ we note that when ε_0 is a vertex of E' with $p(\varepsilon_0) = \hat{b}_0$ then $\varepsilon_0 = \hat{e}_0$ for some simplex e_0 mapped *onto* b_0 by p. Now let $e_i = e_0 \cap p^{-1}(b_i)$. Then

$$\tau(\beta, \varepsilon_0) = (\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_s)$$

is a well defined s-simplex of E' which projects to β under p. We define $\tau(p)_\#(\beta)$ to be an alternating sum of these liftings of β with one summand for each simplex ϕ of $p^{-1}(\hat{b}_0) \subseteq E'$, to wit

$$\tau(p)_\#(\beta) = \sum_\phi (-1)^{|\phi|} \tau(\beta, \phi(0)) \tag{2}$$

Here $|\phi|$ is the dimension of ϕ and $\phi(0)$ is the 0^{th} vertex of ϕ.

Theorem. *Let p: $E \to B$ be a PL fibration with compact fibre F. Then*

(i) *$\tau(p)_\#$: $C(B') \to C(E')$ is a chain map*

(ii) *The composition $C(B') \xrightarrow{\tau(p)_\#} C(E') \xrightarrow{p_\#} C(B')$ is multiplication by $\chi(F)$*

(iii) *τ is natural w.r.t. inclusion of subcomplexes $A \subset B$.*

(iv) *With the standard diagonal approximation Δ the following diagram commutes*

$$\begin{array}{ccc}
C(B') & \xrightarrow{\tau(p)_\#} & C(E') \\
 & & \downarrow \Delta \\
\downarrow \Delta & & C(E') \otimes C(E') \\
 & & \downarrow 1 \otimes p_\# \\
C(B') \otimes C(B') & \xrightarrow[\tau(p)_\# \otimes 1]{} & C(E') \otimes C(B')
\end{array}$$

Remarks 1. Property (ii) is the chain level edition of the characteristic property for a transfer homomorphism.

2. Property (iv) is the dual of a chain level edition of the so called projection formula for the cohomology transfer homomorphism, vizually

$$\tau(p)^*(x \cdot p^* y) = \tau(p)^*(x) \cdot y, \quad x \in H^*(E),\ y \in H^*(B).$$

Proof. Properties (ii) and (iii) are trivially verified. The same is true for (iv) after we remark that the diagonal approximation has

$$\Delta(\hat{b}_0, \ldots, \hat{b}_s) = \sum_i (\hat{b}_0, \ldots, \hat{b}_i) \otimes (\hat{b}_i, \ldots, \hat{b}_s).$$

Let $\beta^{(i)}$ be the i^{th} face of β. One then has

$$\tau(\beta, \varepsilon)^{(i)} = \tau(\beta^{(i)}, \varepsilon)$$

whenever $i > 0$. Thus, to verify (i) we only need to establish the formula

$$(\tau(p)_\#(\beta))^{(0)} = \tau(p)_\#(\beta^{(0)}). \tag{3}$$

To that end we define the simplicial map $g = g^{(\beta)}: p^{-1}(\hat{b}_0) \to p^{-1}(\hat{b}_1)$ by the formula

$$g(\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_t) = (\hat{e}_0, \ldots, \hat{e}_t), \qquad e_i = f_i \cap p^{-1}(b_1);$$

here $\phi = (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_t)$ is a typical simplex of $p^{-1}(\hat{b}_0)$, i.e. $f_0 > f_1 > \ldots > f_t$ in E and $p(f_i) = b_0$ for all i. It is easily seen that g is indeed a simplicial map. Moreover, directly from the definitions one has

$$\tau(\beta, \varepsilon)^{(0)} = \tau(\beta^{(0)}, g(\varepsilon)) \tag{4}$$

for any vertex ε of $p^{-1}(\hat{b}_0)$. Using this, and the definition of $\tau(p)_\#$, we see that (3) follows from

$$\sum_{\phi \in g^{-1}(\psi)} (-1)^{|\phi|} = (-1)^{|\psi|}, \qquad \text{all } \psi \in p^{-1}(\hat{b}_1). \tag{5}$$

Now, for a given ψ, the subsets

$$g^{-1}(\hat{\psi}) \cap \phi, \qquad \phi \in g^{-1}(\psi)$$

form a decomposition of the polyhedron $g^{-1}(\hat{\psi})$ into convex cells with the dimension of $g^{-1}(\hat{\psi}) \cap \phi$ being $|\phi| - |\psi|$. Thus (5) is equivalent to

$$\chi(g^{-1}(\hat{\psi})) = 1 \qquad \text{for each } \psi \in p^{-1}(\hat{b}_1). \tag{6}$$

But Hatcher shows, see proposition 2.1 of [4], that each $g^{-1}(\hat{\psi})$ is contractible. Hence our proof is finished.

Remarks. 1. The above shows that $\tau(p)_\#$ is a chain map provided $\chi((g^{(\beta)})^{-1}(\hat{\psi})) = 1$ for each β in B' and each $\psi \in p^{-1}(\hat{b}_1) \subseteq E'$, $\beta = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_s)$. Hatcher, [4], shows that p is a PL fibration if and only if $(g^{(\beta)})^{-1}(\hat{\psi})$ is contractible for all ψ and β as above. Thus in our theorem we could state a weaker (but also less natural) hypothesis for p.

2. The above ideas could probably be used to give a new definition of the transfer homomorphism

$$\tau(p)^*: h^*(B) \to h^*(E)$$

for any *generalized* cohomology theory. However, such a definition would, most likely, not be any simpler, nor more elegant, than its existing competitors, so we have not pursued the idea.

Bibliography

1. Becker, J.C., Gottlieb, D.H.: Transfer maps for fibrations and duality. Compositio Math. **33**, 107–133 (1976)
2. Casson, A., Gottlieb, D.H.: Fibrations with compact fibres. Amer. J. Math. **99**, 159–189 (1977)
3. Cohen, M.M.: Simplicial structures and transverse cellularity. Ann. of Math. **85**, 218–245 (1967)
4. Hatcher, A.E.: Higher simple homotopy theory. Ann. of Math. **102**, 101–137 (1975)

Received June 29, 1978