

Elements of Algebraic Topology

Munkres

Contents[†]

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[†]The sections marked with an asterisk can be omitted or postponed until needed. Chapter 2 can be omitted.

Homology Groups of a Simplicial Complex

A fundamental problem of topology is that of determining, for two spaces, whether or not they are homeomorphic. To show two spaces *are* homeomorphic, one needs to construct a continuous bijective map, with continuous inverse, mapping one space to the other. To show two spaces are not homeomorphic involves showing that such a map does not exist. To do that is often harder. The usual way of proceeding is to find some topological property (i.e., some property invariant under homeomorphisms) that is satisfied by one space but not the other. For example, the closed unit disc in $\mathbf{R}^2$ cannot be homeomorphic with the plane $\mathbf{R}^2$, because the closed disc is compact and the plane is not. Nor can the real line $\mathbf{R}$ be homeomorphic with $\mathbf{R}^2$, because deleting a point from $\mathbf{R}$ leaves a disconnected space remaining, while deleting a point from $\mathbf{R}^2$ does not.

Such elementary properties do not carry one very far in tackling homeomorphism problems. Classifying all compact surfaces up to homeomorphism, for instance, demands more sophisticated topological invariants than these. So does the problem of showing that, in general, $\mathbf{R}^n$ and $\mathbf{R}^m$ are not homeomorphic if $n \neq m$.

Algebraic topology originated in the attempts by such mathematicians as Poincaré and Betti to construct such topological invariants. Poincaré introduced a certain group, called the *fundamental group* of a topological space; it is by its definition a topological invariant. One can show fairly readily that a number of familiar spaces, such as the sphere, torus, and Klein bottle, have fundamental groups that are different, so these spaces cannot be homeomorphic ([Mu], Chapter 8). In fact, one can classify all compact surfaces using the fundamental group ([Ma], Chapter 4).

Betti, on the other hand, associated with each space a certain sequence of abelian groups called its *homology groups*. In this case, it was not at all obvious

that homeomorphic spaces had isomorphic homology groups, although it was eventually proved true. These groups can also be used to tackle homeomorphism problems; one advantage they possess is that they are often easier to compute than the fundamental group.

We shall begin our discussion of algebraic topology by studying the homology groups. Later on, we shall deal with other topological invariants, such as the cohomology groups and the cohomology ring.

There are several different ways of defining homology groups, all of which lead to the same results for spaces that are sufficiently "nice." The two we shall consider in detail are the *simplicial* and the *singular* groups. We begin with the simplicial homology groups, which came first historically. Both conceptually and computationally, they are concrete and down-to-earth. They are defined, however, only for particularly "nice" spaces (polyhedra), and it is hard work to prove their topological invariance. After that, we shall treat the singular homology groups, which were introduced as a generalization of the simplicial groups. They are defined for arbitrary spaces, and it is immediate from their definition that they are topological invariants. Furthermore, they are much more convenient for theoretical purposes than the simplicial groups. They are not as readily computable as the simplicial groups, but they agree with the simplicial homology groups when both are defined.

A third way of defining homology groups for arbitrary spaces is due to E. Čech. The Čech homology theory is still not completely satisfactory, but Čech cohomology theory is both important and useful. It will appear near the end of this book.

§1. SIMPLICES

Before defining simplicial homology groups, we must discuss the class of spaces for which they are defined, which is the class of all polyhedra. A polyhedron is a space that can be built from such "building blocks" as line segments, triangles, tetrahedra, and their higher dimensional analogues, by "gluing them together" along their faces. In this section, we shall discuss these basic building blocks; in the next, we shall use them to construct polyhedra.

First we need to study a bit of the analytic geometry of euclidean space.

Given a set $\{a_0, \dots, a_n\}$ of points of $\mathbb{R}^N$, this set is said to be **geometrically independent** if for any (real) scalars t_i , the equations

$$\sum_{i=0}^n t_i = 0 \quad \text{and} \quad \sum_{i=0}^n t_i a_i = 0$$

imply that $t_0 = t_1 = \dots = t_n = 0$.

It is clear that a one-point set is always geometrically independent. Simple

algebra shows that in general $\{a_0, \dots, a_n\}$ is geometrically independent if and only if the vectors

$$a_1 - a_0, \dots, a_n - a_0$$

are linearly independent in the sense of ordinary linear algebra. Thus two distinct points in $\mathbb{R}^N$ form a geometrically independent set, as do three non-collinear points, four non-coplanar points, and so on.

Given a geometrically independent set of points $\{a_0, \dots, a_n\}$, we define the *n-plane* P spanned by these points to consist of all points x of $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^n t_i a_i,$$

for some scalars t_i with $\sum t_i = 1$. Since the a_i are geometrically independent, the t_i are uniquely determined by x . Note that each point a_i belongs to the plane P .

The plane P can also be described as the set of all points x such that

$$x = a_0 + \sum_{i=1}^n t_i (a_i - a_0)$$

for some scalars $t_1, \dots, t_n$; in this form we speak of P as the "plane through a_0 parallel to the vectors $a_i - a_0$."

It is elementary to check that if $\{a_0, \dots, a_n\}$ is geometrically independent, and if w lies outside the plane that these points span, then $\{w, a_0, \dots, a_n\}$ is geometrically independent.

An **affine transformation** T of $\mathbb{R}^N$ is a map that is a composition of translations (i.e., maps of the form $T(x) = x + p$ for fixed p), and non-singular linear transformations. If T is an affine transformation, it is immediate from the definitions that T preserves geometrically independent sets, and that T carries the plane P spanned by $a_0, \dots, a_n$ onto the plane spanned by $Ta_0, \dots, Ta_n$.

Now the translation $T(x) = x - a_0$ carries P onto the vector subspace of $\mathbb{R}^N$ having $a_1 - a_0, \dots, a_n - a_0$ as a basis; if we follow T by a linear transformation of $\mathbb{R}^N$ carrying $a_1 - a_0, \dots, a_n - a_0$ to the first n unit basis vectors $\epsilon_1, \dots, \epsilon_n$ in $\mathbb{R}^N$, we obtain an affine transformation S of $\mathbb{R}^N$ such that $S(a_0) = 0$ and $S(a_i) = \epsilon_i$ for $i > 0$. The map S carries P onto the plane $\mathbb{R}^n \times 0$ of the first n coordinates in $\mathbb{R}^N$; it is thus clear why we call P a "plane of dimension n " in $\mathbb{R}^N$.

Definition. Let $\{a_0, \dots, a_n\}$ be a geometrically independent set in $\mathbb{R}^N$. We define the *n-simplex* σ spanned by $a_0, \dots, a_n$ to be the set of all points x of $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^n t_i a_i, \quad \text{where} \quad \sum_{i=0}^n t_i = 1$$

and $t_i \geq 0$ for all i . The numbers t_i are uniquely determined by x ; they are called the **barycentric coordinates** of the point x of σ with respect to $a_0, \dots, a_n$.

Example 1. In low dimensions, one can picture a simplex easily. A 0-simplex is a point, of course. The 1-simplex spanned by a_0 and a_1 consists of all points of the form

$$x = ta_0 + (1 - t)a_1$$

with $0 \leq t \leq 1$; this just the line segment joining a_0 and a_1 . Similarly, the 2-simplex σ spanned by a_0, a_1, a_2 equals the triangle having these three points as vertices. This can be seen most easily as follows: Assume $x \neq a_0$. Then

$$x = \sum_{i=0}^2 t_i a_i = t_0 a_0 + (1 - t_0) [(t_1/\lambda) a_1 + (t_2/\lambda) a_2]$$

where $\lambda = 1 - t_0$. The expression in brackets represents a point p of the line segment joining a_1 and a_2 , since $(t_1 + t_2)/\lambda = 1$ and $t_i/\lambda \geq 0$ for $i = 1, 2$.

Thus x is a point of the line segment joining a_0 and p . See Figure 1.1. Conversely, any point of such a line segment is in σ , as you can check. It follows that σ equals the union of all line segments joining a_0 to points of $a_1 a_2$; that is, σ is a triangle.

A similar proof shows that a 3-simplex is a tetrahedron.

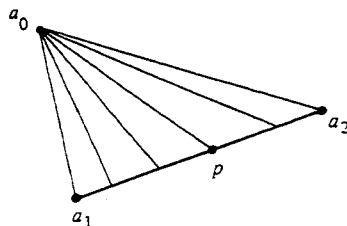


Figure 1.1

Let us list some basic properties of simplices. The proofs are elementary and are largely left as exercises.

Throughout, let P be the n -plane determined by the points of the geometrically independent set $\{a_0, \dots, a_n\}$, and let σ be the n -simplex spanned by these points. If $x \in \sigma$, let $\{t_i(x)\}$ be the barycentric coordinates of x ; they are determined uniquely by the conditions

$$x = \sum_{i=0}^n t_i a_i \quad \text{and} \quad \sum_{i=0}^n t_i = 1.$$

The following properties hold:

(1) The barycentric coordinates $t_i(x)$ of x with respect to $a_0, \dots, a_n$ are continuous functions of x .

(2) σ equals the union of all line segments joining a_0 to points of the simplex s spanned by $a_1, \dots, a_n$. Two such line segments intersect only in the point a_0 .

Recall now that a subset A of $\mathbb{R}^n$ is said to be **convex** if for each pair x, y of points of A , the line segment joining them lies in A .

(3) σ is a compact, convex set in $\mathbb{R}^n$, which equals the intersection of all convex sets in $\mathbb{R}^n$ containing $a_0, \dots, a_n$.

(4) Given a simplex σ , there is one and only one geometrically independent set of points spanning σ .

The points $a_0, \dots, a_n$ that span σ are called the **vertices** of σ ; the number n is called the **dimension** of σ . Any simplex spanned by a subset of $\{a_0, \dots, a_n\}$ is called a **face** of σ . In particular, the face of σ spanned by $a_1, \dots, a_n$ is called the face **opposite** a_0 . The faces of σ different from σ itself are called the **proper faces** of σ ; their union is called the **boundary** of σ and denoted $\text{Bd } \sigma$. The **interior** of σ is defined by the equation $\text{Int } \sigma = \sigma - \text{Bd } \sigma$; the set $\text{Int } \sigma$ is sometimes called an **open simplex**.

Since $\text{Bd } \sigma$ consists of all points x of σ such that at least one of the barycentric coordinates $t_i(x)$ is zero, $\text{Int } \sigma$ consists of those points of σ for which $t_i(x) > 0$ for all i . It follows that, given $x \in \sigma$, there is exactly one face s of σ such that $x \in \text{Int } s$, for s must be the face of σ spanned by those a_i for which $t_i(x)$ is positive.

(5) $\text{Int } \sigma$ is convex and is open in the plane P ; its closure is σ . Furthermore, $\text{Int } \sigma$ equals the union of all open line segments joining a_0 to points of $\text{Int } s$, where s is the face of σ opposite a_0 .

Let us recall here some standard notation. If x is in $\mathbb{R}^n$ and $x = (x_1, \dots, x_n)$, then the **norm** of x is defined by the equation

$$\|x\| = \left[\sum_{i=1}^n (x_i)^2 \right]^{1/2}.$$

The **unit n -ball** B^n is the set of all points x of $\mathbb{R}^n$ for which $\|x\| \leq 1$, and the **unit sphere** S^{n-1} is the set of points for which $\|x\| = 1$. The **upper hemisphere** E_+^{n-1} of S^{n-1} consists of all points x of S^{n-1} for which $x_n \geq 0$; while the **lower hemisphere** E_-^{n-1} consists of those points for which $x_n \leq 0$.

With these definitions, B^0 is a one-point space, B^1 equals the interval $[-1, 1]$, and S^0 is the two-point space $\{-1, 1\}$. The 2-ball B^2 is the unit disc in $\mathbb{R}^2$ centered at the origin; and S^1 is the unit circle.

(6) There is a homeomorphism of σ with the unit ball B^n that carries $\text{Bd } \sigma$ onto the unit sphere S^{n-1} .

We leave properties (1)–(5) as exercises, and prove (6). In fact, we shall prove a stronger result, which will be useful to us later.

Recall that if $w \in \mathbb{R}^n$, a **ray** $\mathcal{R}$ emanating from w is the set of all points of the form $w + tp$, where p is a fixed point of $\mathbb{R}^n - 0$ and t ranges over the non-negative reals.

Lemma 1.1. Let U be a bounded, convex, open set in $\mathbb{R}^n$; let $w \in U$.

(a) Each ray emanating from w intersects $\text{Bd } U = \bar{U} - U$ in precisely one point.

(b) There is a homeomorphism of $\bar{U}$ with B^n carrying $\text{Bd } U$ onto S^{n-1} .

Proof. (a) Given a ray $\mathcal{R}$ emanating from w , its intersection with U is convex, bounded, and open in $\mathcal{R}$. Hence it consists of all points of the form $w + tp$, where t ranges over a half-open interval $[0, a)$. Then $\mathcal{R}$ intersects $\bar{U} - U$ in the point $x = w + ap$.

Suppose $\mathcal{R}$ intersects $\bar{U} - U$ in another point, say y . Then x lies between w and y on the ray $\mathcal{R}$. Indeed, since $y = w + bp$ for some $b > a$, we have

$$x = (1 - t)w + ty,$$

where $t = a/b$. We rewrite this equation in the form

$$w = (x - ty)/(1 - t).$$

Then we choose a sequence y_n of points of U converging to y , and we define

$$w_n = (x - ty_n)/(1 - t).$$

See Figure 1.2. The sequence w_n converges to w , so that $w_n \in U$ for some n . But then since $x = tw_n + (1 - t)y_n$, the point x belongs to U , because U is convex. This fact contradicts our choice of x .

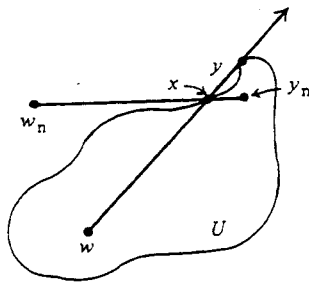


Figure 1.2

(b) Assume $w = 0$ for convenience. The equation $f(x) = x/\|x\|$ defines a continuous map f of $\mathbb{R}^n - 0$ onto S^{n-1} . By (a), f restricts to a bijection of $\text{Bd } U$ with S^{n-1} . Since $\text{Bd } U$ is compact, this restriction is a homeomorphism; let $g: S^{n-1} \rightarrow \text{Bd } U$ be its inverse. Extend g to a bijection $G: B^n \rightarrow \bar{U}$ by letting G map the line segment joining 0 to the point u of S^{n-1} linearly onto the line segment joining 0 to $g(u)$. Formally, we define

$$G(x) = \begin{cases} \|g(x/\|x\|)\| x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Continuity of G for $x \neq 0$ is immediate. Continuity at 0 is easy: If M is a bound for $\|g(x)\|$, then whenever $\|x - 0\| < \delta$, we have $\|G(x) - G(0)\| < M\delta$. $\square$

1. Verify properties (1)–(3) of simplices. [Hint: If T is the affine transformation carrying a_0 to 0 and a_i to e_i , then T carries the point

$$x = \sum_{i=0}^n t_i a_i$$

to the point $(t_1, \dots, t_n, 0, \dots, 0)$.]

2. Verify property (4) as follows:

- (a) Show that if $x \in \sigma$ and $x \neq a_0, \dots, a_n$, then x lies in some open line segment contained in σ . (Assume $n > 0$.)
- (b) Show that a_0 lies in no open line segment contained in σ , by showing that if $a_0 = tx + (1 - t)y$, where $x, y \in \sigma$ and $0 < t < 1$, then $x = y = a_0$.

3. Verify property (5).

4. Generalize property (2) as follows: Let σ be spanned by $a_0, \dots, a_n$. Let s be the face of σ spanned by $a_0, \dots, a_p$ (where $p < n$); and let t be the face spanned by $a_{p+1}, \dots, a_n$. Then t is called the face of σ opposite s .

- (a) Show that σ is the union of all line segments joining points of s to points of t , and two of these line segments intersect in at most a common end point.
- (b) Show that $\text{Int } \sigma$ is the union of all open line segments joining points of $\text{Int } s$ to points of $\text{Int } t$.

5. Let U be a bounded open set in $\mathbb{R}^n$. Suppose U is star-convex relative to the origin; this means that for each x in U , the line segment from 0 to x lies in U .

- (a) Show that a ray from 0 may intersect $\text{Bd } U$ in more than one point.
- *(b) Show by example that $\bar{U}$ need not be homeomorphic to B^n .

§2. SIMPLICIAL COMPLEXES AND SIMPLICIAL MAPS

Complexes in $\mathbb{R}^N$

Definition. A simplicial complex K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that:

- (1) Every face of a simplex of K is in K .
- (2) The intersection of any two simplexes of K is a face of each of them.

Example 1. The collection K_1 pictured in Figure 2.1, consisting of a 2-simplex and its faces, is a simplicial complex. The collection K_2 , consisting of two 2-simplices with an edge in common, along with their faces, is a simplicial complex; while the collection K_3 is not. What about K_4 ?

The following lemma is sometimes useful in verifying that a collection of simplices is a simplicial complex:

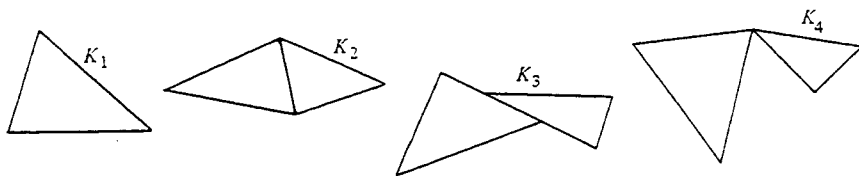


Figure 2.1

Lemma 2.1. *A collection K of simplices is a simplicial complex if and only if the following hold:*

- (1) *Every face of a simplex of K is in K .*
- (2') *Every pair of distinct simplices of K have disjoint interiors.*

Proof. First, assume K is a simplicial complex. Given two simplices σ and τ of K , we show that if their interiors have a point x in common, then $\sigma = \tau$. Let $s = \sigma \cap \tau$. If s were a proper face of σ , then x would belong to $\text{Bd } \sigma$, which it does not. Therefore, $s = \sigma$. A similar argument shows that $s = \tau$.

To prove the converse, assume (1) and (2'). We show that if the set $\sigma \cap \tau$ is nonempty, then it equals the face σ' of σ that is spanned by those vertices $b_0, \dots, b_m$ of σ that lie in τ . First, σ' is contained in $\sigma \cap \tau$ because $\sigma \cap \tau$ is convex and contains $b_0, \dots, b_m$. To prove the reverse inclusion, suppose $x \in \sigma \cap \tau$. Then $x \in \text{Int } s \cap \text{Int } t$, for some face s of σ and some face t of τ . It follows from (2') that $s = t$; hence the vertices of s lie in τ , so that by definition they are elements of the set $\{b_0, \dots, b_m\}$. Then s is a face of σ' , so that $x \in \sigma'$, as desired. $\square$

It follows from this lemma that if σ is a simplex, then the collection consisting of σ and its proper faces is a simplicial complex: Condition (1) is immediate; and condition (2') holds because for each point $x \in \sigma$, there is exactly one face s of σ such that $x \in \text{Int } s$.

Definition. If L is a subcollection of K that contains all faces of its elements, then L is a simplicial complex in its own right; it is called a **subcomplex** of K . One subcomplex of K is the collection of all simplices of K of dimension at most p ; it is called the **p -skeleton** of K and is denoted $K^{(p)}$. The points of the collection $K^{(0)}$ are called the **vertices** of K .

Definition. Let $|K|$ be the subset of $\mathbb{R}^N$ that is the union of the simplices of K . Giving each simplex its natural topology as a subspace of $\mathbb{R}^N$, we then topologize $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if $A \cap \sigma$ is closed in σ , for each σ in K . It is easy to see that this defines a topology on $|K|$, for this collection of sets is closed under finite unions and arbitrary intersections. The space $|K|$ is called the **underlying space** of K , or the **polytope** of K .

A space that is the polytope of a simplicial complex will be called a **poly-**

hedron. (We note that some topologists reserve this term for finite simplicial complexes.)

In general, the topology of $|K|$ is finer (larger) than the topology $|K|$ inherits as a subspace of $\mathbb{R}^N$: If A is closed in $|K|$ in the subspace topology, then $A = B \cap |K|$ for some closed set B in $\mathbb{R}^N$. Then $B \cap \sigma$ is closed in σ for each σ , so $B \cap |K| = A$ is closed in the topology of $|K|$, by definition.

The two topologies are different in general. (See Examples 2 and 3.) However, if K is finite, they are the same. For suppose K is finite and A is closed in $|K|$. Then $A \cap \sigma$ is closed in σ and hence closed in $\mathbb{R}^N$. Because A is the union of finitely many sets $A \cap \sigma$, the set A also is closed in $\mathbb{R}^N$.

Example 2. Let K be the collection of all 1-simplices in $\mathbb{R}$ of the form $[m, m+1]$, where m is an integer different from 0, along with all simplices of the form $[1/(n+1), 1/n]$ for n a positive integer, along with all faces of these simplices. Then K is a complex whose underlying space equals $\mathbb{R}$ as a set but not as a topological space. For instance, the set of points of the form $1/n$ is closed in $|K|$ but not in $\mathbb{R}$.

Example 3. Let K be the collection of 1-simplices $\sigma_1, \sigma_2, \dots$ and their vertices, where σ_i is the 1-simplex in $\mathbb{R}^2$ having vertices 0 and $(1, 1/i)$. See Figure 2.2. Then K is a simplicial complex. The intersection of $|K|$ with the open parabolic arc $\{(x, x^2) \mid x > 0\}$ is closed in $|K|$, because its intersection with each simplex σ_i is a single point. It is not closed in the topology $|K|$ derives from $\mathbb{R}^2$, however, because in that topology it has the origin as a limit point.

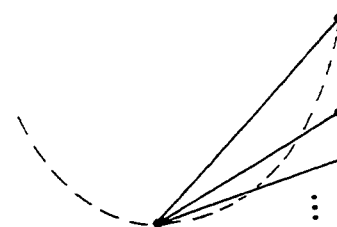


Figure 2.2

We prove some elementary topological properties of polyhedra.

Lemma 2.2. *If L is a subcomplex of K , then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.*

Proof. Suppose A is closed in $|L|$. If σ is a simplex of K , then $\sigma \cap |L|$ is the union of those faces s_i of σ that belong to L . Since A is closed in $|L|$, the set $A \cap s_i$ is closed in s_i and hence closed in σ . Since $A \cap \sigma$ is the finite union of the sets $A \cap s_i$, it is closed in σ . We conclude that A is closed in $|K|$.

Conversely, if B is closed in $|K|$, then $B \cap \sigma$ is closed in σ for each $\sigma \in K$, and in particular for each $\sigma \in L$. Hence $B \cap |L|$ is closed in $|L|$. $\square$

Lemma 2.3. *A map $f: |K| \rightarrow X$ is continuous if and only if $f|_{\sigma}$ is continuous for each $\sigma \in K$.*

Proof. If f is continuous, so is $f|_{\sigma}$ since σ is a subspace of K . Conversely, suppose each map $f|_{\sigma}$ is continuous. If C is a closed set of X , then $f^{-1}(C) \cap \sigma = (f|_{\sigma})^{-1}(C)$, which is closed in σ by continuity of $f|_{\sigma}$. Thus $f^{-1}(C)$ is closed in $|K|$ by definition. $\square$

Definition. If X is a space and if $\mathcal{C}$ is a collection of subspaces of X whose union is X , the topology of X is said to be **coherent** with the collection $\mathcal{C}$, provided a set A is closed in X if and only if $A \cap C$ is closed in C for each $C \in \mathcal{C}$. It is equivalent to require that U be open in X if and only if $U \cap C$ is open in C for each C .

In particular, the topology of $|K|$ is coherent with the collection of subspaces σ , for $\sigma \in K$.

The analogue of Lemma 2.3 holds for coherent topologies in general; a map $f: X \rightarrow Y$ is continuous if and only if $f|_C$ is continuous for each $C \in \mathcal{C}$.

Definition. If x is a point of the polyhedron $|K|$, then x is interior to precisely one simplex of K , whose vertices are (say) $a_0, \dots, a_n$. Then

$$x = \sum_{i=0}^n t_i a_i,$$

where $t_i > 0$ for each i and $\sum t_i = 1$. If v is an arbitrary vertex of K , we define the **barycentric coordinate** $t_v(x)$ of x with respect to v by setting $t_v(x) = 0$ if v is not one of the vertices a_i , and $t_v(x) = t_i$ if $v = a_i$.

For fixed v , the function $t_v(x)$ is continuous when restricted to a fixed simplex σ of K , since either it is identically zero on σ or equals the barycentric coordinate of x with respect to the vertex v of σ in the sense formerly defined. Therefore, $t_v(x)$ is continuous on $|K|$, by Lemma 2.3.

Lemma 2.4. $|K|$ is Hausdorff.

Proof. Given $x_0 \neq x_1$, there is at least one vertex v such that $t_v(x_0) \neq t_v(x_1)$. Choose r between these two numbers; then the sets $\{x \mid t_v(x) < r\}$ and $\{x \mid t_v(x) > r\}$ are the required disjoint open sets. $\square$

Lemma 2.5. If K is finite, then $|K|$ is compact. Conversely, if a subset A of $|K|$ is compact, then $A \subset |K_0|$ for some finite subcomplex K_0 of K .

Proof. If K is finite, then $|K|$ is a finite union of compact subspaces σ , and hence is compact. Now suppose A is compact and A does not lie in the polytope of any finite subcomplex of K . Choose a point $x_i \in A \cap \text{Int } \sigma_i$ whenever this set is nonempty. Then the set $B = \{x_i\}$ is infinite. Furthermore, every subset of B is closed, since its intersection with any simplex σ is finite. Being closed and discrete, B has no limit point, contrary to the fact that every infinite subset of a compact space has a limit point. $\square$

Three particular subspaces of $|K|$ are of special interest: the **star** of v in K , denoted by $\text{St } v$, or sometimes by $\text{St}(v, K)$, is the union of the interiors of those simplices of K that have v as a vertex. Its closure, denoted $\overline{\text{St } v}$, is called the **closed star** of v in K . It is the union of all simplices of K having v as a vertex, and is the polytope of a subcomplex of K . The set $\overline{\text{St } v} - \text{St } v$ is called the **link** of v in K and is denoted $\text{Lk } v$. See Figure 2.3.

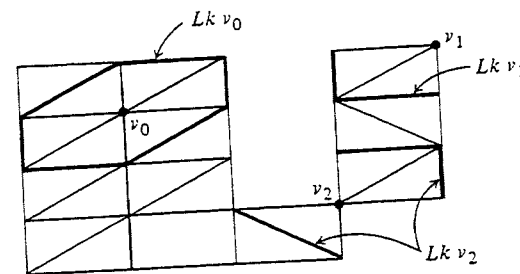


Figure 2.3

The set $\text{St } v$ is open in $|K|$, since it consists of all points x of $|K|$ such that $t_v(x) > 0$. Its complement is the union of all simplices of K that do not have v as a vertex; it is thus the polytope of a subcomplex of K . The set $\text{Lk } v$ is also the polytope of a subcomplex of K ; it is the intersection of $\overline{\text{St } v}$ and the complement of $\text{St } v$. The sets $\text{St } v$ and $\overline{\text{St } v}$ are easily seen to be path connected; the set $\text{Lk } v$ need not be connected, however.

Definition. A simplicial complex K is said to be **locally finite** if each vertex of K belongs only to finitely many simplices of K . Said differently, a complex K is locally finite if and only if each closed star $\overline{\text{St } v}$ is the polytope of a finite subcomplex of K .

Lemma 2.6. The complex K is locally finite if and only if the space $|K|$ is locally compact.

Proof. Suppose K is locally finite. Given $x \in |K|$, it lies in $\text{St } v$ for some vertex v of K . Since $\overline{\text{St } v}$ is a compact set, $|K|$ is locally compact. We leave the converse as an exercise. $\square$

Simplicial maps

Now we introduce the notion of a "simplicial map" of one complex into another.

Lemma 2.7. Let K and L be complexes, and let $f: K^{(0)} \rightarrow L^{(0)}$ be a map. Suppose that whenever the vertices $v_0, \dots, v_n$ of K span a simplex of K , the points $f(v_0), \dots, f(v_n)$ are vertices of a simplex of L . Then f can be extended to a continuous map $g: |K| \rightarrow |L|$ such that

$$x = \sum_{i=0}^n t_i v_i \quad \Rightarrow \quad g(x) = \sum_{i=0}^n t_i f(v_i).$$

We call g the (linear) **simplicial map** induced by the vertex map f .

Proof. Note that although the vertices $f(v_0), \dots, f(v_n)$ of L are not necessarily distinct, still they span a simplex τ of L , by hypothesis. When we “collect terms” in the expression for $g(x)$, it is still true that the coefficients are non-negative and their sum is 1; thus $g(x)$ is a point of τ . Hence g maps the n -simplex σ spanned by $v_0, \dots, v_n$ continuously to the simplex τ whose vertex set is $\{f(v_0), \dots, f(v_n)\}$.

The map g is continuous as a map of σ into τ , and hence as a map of σ into $|L|$. Then by Lemma 2.3, g is continuous as a map of $|K|$ into $|L|$. $\square$

We remark that the composite of simplicial maps is simplicial: Suppose $g: |K| \rightarrow |L|$ and $h: |L| \rightarrow |M|$ are simplicial maps. By definition, if $x = \sum t_i v_i$ (where the v_i are distinct vertices of $\sigma \in K$), then $g(x) = \sum t_i g(v_i)$.

Now this same formula would hold even if the v_i were not distinct, so long as $\{v_0, \dots, v_n\}$ is the vertex set of a simplex of K . For example, suppose

$$x = \sum_{i=0}^n t_i v_i,$$

where $t_i \geq 0$ for all i and $\sum t_i = 1$; and suppose that $v_0 = v_1$ and the vertices $v_1, \dots, v_n$ are distinct. Write

$$x = (t_0 + t_1)v_0 + t_2 v_2 + \dots + t_n v_n;$$

then by definition

$$\begin{aligned} g(x) &= (t_0 + t_1)g(v_0) + t_2 g(v_2) + \dots + t_n g(v_n) \\ &= \sum_{i=0}^n t_i g(v_i). \end{aligned}$$

Applying this remark to the present case, we note that even though the vertices $g(v_0), \dots, g(v_n)$ of L are not necessarily distinct, the following formula holds:

$$h(g(x)) = h(\sum t_i g(v_i)) = \sum t_i h(g(v_i)).$$

Therefore $h \circ g$ is a simplicial map, as claimed.

Lemma 2.8. Suppose $f: K^{(0)} \rightarrow L^{(0)}$ is a bijective correspondence such that the vertices $v_0, \dots, v_n$ of K span a simplex of K if and only if

$f(v_0), \dots, f(v_n)$ span a simplex of L . Then the induced simplicial map $g: |K| \rightarrow |L|$ is a homeomorphism.

The map g is called a **simplicial homeomorphism**, or an **isomorphism**, of K with L .

Proof. Each simplex σ of K is mapped by g onto a simplex τ of L of the same dimension as σ . We need only show that the linear map $h: \tau \rightarrow \sigma$ induced by the vertex correspondence f^{-1} is the inverse of the map $g: \sigma \rightarrow \tau$. And for that we note that if $x = \sum t_i v_i$, then $g(x) = \sum t_i f(v_i)$ by definition; whence

$$\begin{aligned} h(g(x)) &= h(\sum t_i f(v_i)) = \sum t_i f^{-1}(f(v_i)) \\ &= \sum t_i v_i = x. \quad \square \end{aligned}$$

Corollary 2.9. Let Δ^N denote the complex consisting of an N -simplex and its faces. If K is a finite complex, then K is isomorphic to a subcomplex of Δ^N for some N .

Proof. Let $v_0, \dots, v_N$ be the vertices of K . Choose $a_0, \dots, a_N$ to be geometrically independent points in $\mathbb{R}^N$, and let Δ^N consist of the N -simplex they span, along with its faces. The vertex map $f(v_i) = a_i$ induces an isomorphism of K with a subcomplex of Δ^N . $\square$

General simplicial complexes

Our insistence that a simplicial complex K must lie in $\mathbb{R}^N$ for some N puts a limitation on the cardinality of K and on the dimension of the simplices of K . We now remove these restrictions.

Let J be an arbitrary index set, and let $\mathbf{R}^J$ denote the J -fold product of $\mathbf{R}$ with itself. An element of $\mathbf{R}^J$ is a function from J to $\mathbf{R}$, ordinarily denoted in “tuple notation” by $(x_\alpha)_{\alpha \in J}$. The product $\mathbf{R}^J$ is of course a vector space with the usual component-wise addition and multiplication by scalars.

Let $\mathbf{E}^J$ denote the subset of $\mathbf{R}^J$ consisting of all points $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0$ for all but finitely many values of α . Then $\mathbf{E}^J$ is also a vector space under component-wise addition and multiplication of scalars. If ϵ_α is the map of J into $\mathbf{R}$ whose value is 1 on the index α and 0 on all other elements of J , then the set $\{\epsilon_\alpha \mid \alpha \in J\}$ is a basis for $\mathbf{E}^J$. (It is not, of course, a basis for $\mathbf{R}^J$.)

We call $\mathbf{E}^J$ **generalized euclidean space** and topologize it by the metric

$$|x - y| = \max \{|x_\alpha - y_\alpha|\}_{\alpha \in J}.$$

Everything we have done for complexes in $\mathbb{R}^N$ generalizes to complexes in $\mathbf{E}^J$. The space $\mathbf{E}^J$ is the union of certain of its finite-dimensional subspaces—namely, the subspaces spanned by finite subsets of the basis $\{\epsilon_\alpha \mid \alpha \in J\}$. Each such subspace is just a copy of $\mathbb{R}^N$ for some N . Any finite set $\{a_0, \dots, a_n\}$ of points of $\mathbf{E}^J$ lies in such a subspace; if they are independent, the simplex they span lies in the same subspace. Furthermore, the metric for $\mathbf{E}^J$ gives the usual

topology of each such subspace. Therefore, any finite collection of simplices in E^j lies in a copy of $\mathbb{R}^N$ for some N . All that we are really doing is to allow ourselves to deal with complexes for which the total collection of simplices cannot be fitted entirely into any one $\mathbb{R}^N$. Conceptually it may seem more complicated to work in E^j than in $\mathbb{R}^N$, but in practice it causes no difficulty at all.

We leave it to you to check that our results hold for complexes in E^j . We shall use these results freely henceforth.

Let us make one further comment. If K is a complex in $\mathbb{R}^N$, then each simplex of K has dimension at most N . On the other hand, if K is a complex in E^j , there need be no upper bound on the dimensions of the simplices of K . We define the **dimension** of K , denoted $\dim K$, to be the largest dimension of a simplex of K , if such exists; otherwise, we say that K has **infinite dimension**.

EXERCISES

1. Let K be a simplicial complex; let $\sigma \in K$. When is $\text{Int } \sigma$ open in $|K|$? When is σ open in $|K|$?
2. Show that in general, $\text{St } v$ and $\overline{\text{St } v}$ are path connected.
3. (a) Show directly that the polyhedron of Example 3 is not locally compact.
(b) Show that, in general, if a complex K is not locally finite, then the space $|K|$ is not locally compact.
4. Show that the polyhedron of Example 3 is not metrizable. [Hint: Show that first countability axiom fails.]
5. If $g: |K| \rightarrow |L|$ is a simplicial map carrying the vertices of σ onto the vertices of τ , show that g maps some face of σ homeomorphically onto τ .
6. Check that the proofs of Lemmas 2.1–2.8 apply without change to complexes in E^j , if one simply replaces $\mathbb{R}^N$ by E^j throughout.
7. Let K be a complex. Show that $|K|$ is metrizable if and only if K is locally finite. [Hint: The function

$$d(x, y) = \text{lub } |t_\sigma(x) - t_\sigma(y)|$$

is a metric for the topology of each finite subcomplex of K .]

8. Let K be a complex. Show that $|K|$ is normal. [Hint: If A is closed in $|K|$ and if $f: A \rightarrow [0, 1]$ is continuous, extend f step-by-step to $A \cup |K^{(p)}|$, using the Tietze theorem.]
9. Let K be a complex in $\mathbb{R}^N$. Show that $|K|$ is a subspace of $\mathbb{R}^N$ if and only if each point x of $|K|$ lies in an open set of $\mathbb{R}^N$ that intersects only finitely many simplices of K . Generalize to E^j .
10. Show that the collection of all simplices in $\mathbb{R}$ of the form $[1/(n+1), 1/n]$ for n a positive integer, along with their vertices, is a complex whose polytope is the subspace $(0, 1]$ of $\mathbb{R}$.

§3. ABSTRACT SIMPLICIAL COMPLEXES

In practice, specifying a polyhedron X by giving a collection of simplices whose union is X is not a very convenient way of dealing with specific polyhedra. One quickly gets immersed in details of analytic geometry; it is messy to specify all the simplices and to make sure they intersect only when they should. It is much easier to specify X by means of an “abstract simplicial complex,” a notion we now introduce.

Definition. An **abstract simplicial complex** is a collection $\mathcal{S}$ of finite non-empty sets, such that if A is an element of $\mathcal{S}$, so is every nonempty subset of A .

The element A of $\mathcal{S}$ is called a **simplex** of $\mathcal{S}$; its **dimension** is one less than the number of its elements. Each nonempty subset of A is called a **face** of A . The **dimension** of $\mathcal{S}$ is the largest dimension of one of its simplices, or is infinite if there is no such largest dimension. The **vertex set** V of $\mathcal{S}$ is the union of the one-point elements of $\mathcal{S}$; we shall make no distinction between the vertex $v \in V$ and the 0-simplex $\{v\} \in \mathcal{S}$. A subcollection of $\mathcal{S}$ that is itself a complex is called a **subcomplex** of $\mathcal{S}$.

Two abstract complexes $\mathcal{S}$ and $\mathcal{T}$ are said to be **isomorphic** if there is a bijective correspondence f mapping the vertex set of $\mathcal{S}$ to the vertex set of $\mathcal{T}$ such that $\{a_0, \dots, a_n\} \in \mathcal{S}$ if and only if $\{f(a_0), \dots, f(a_n)\} \in \mathcal{T}$.

Definition. If K is a simplicial complex, let V be the vertex set of K . Let $\mathcal{H}$ be the collection of all subsets $\{a_0, \dots, a_n\}$ of V such that the vertices $a_0, \dots, a_n$ span a simplex of K . The collection $\mathcal{H}$ is called the **vertex scheme** of K .

The collection $\mathcal{H}$ is a particular example of an abstract simplicial complex. It is in fact the crucial example, as the following theorem shows:

Theorem 3.1. (a) Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some simplicial complex K .

(b) Two simplicial complexes are linearly isomorphic if and only if their vertex schemes are isomorphic as abstract simplicial complexes.

Proof. Part (b) follows at once from Lemma 2.8. To prove (a), we proceed as follows: Given an index set J , let Δ^J be the collection of all simplices in E^j spanned by finite subsets of the standard basis $\{\epsilon_\alpha\}$ for E^j . It is easy to see that Δ^J is a simplicial complex; if σ and τ are two simplices of Δ^J , then their combined vertex set is geometrically independent and spans a simplex of Δ^J . We shall call Δ^J an “infinite-dimensional simplex.”

Now let $\mathcal{S}$ be an abstract complex with vertex set V . Choose an index set J large enough that there is an injective function $f: V \rightarrow \{\epsilon_\alpha\}_{\alpha \in J}$. (Let $J = V$ if you wish.) We specify a subcomplex K of Δ^J by the condition that for each

(abstract) simplex $\{a_0, \dots, a_n\} \in \mathcal{S}$, the (geometric) simplex spanned by $f(a_0), \dots, f(a_n)$ is to be in K . It is immediate that K is a simplicial complex and $\mathcal{S}$ is isomorphic to the vertex scheme of K ; f is the required correspondence between their vertex sets. $\square$

Definition. If the abstract simplicial complex $\mathcal{S}$ is isomorphic with the vertex scheme of the simplicial complex K , we call K a **geometric realization** of $\mathcal{S}$. It is uniquely determined up to a linear isomorphism.

Let us illustrate how abstract complexes can be used to specify particular simplicial complexes.

Example 1. Suppose we wish to indicate a simplicial complex K whose underlying space is homeomorphic to the cylinder $S^1 \times I$. (Here I denotes the closed unit interval $[0,1]$.) One way of doing so is to draw the picture in Figure 3.1, which specifies K as a collection consisting of six 2-simplices and their faces. Another way of picturing this same complex K is to draw the diagram in Figure 3.2. This diagram consists of two things: first, a complex L whose underlying space is a rectangle, and second, a particular labelling of the vertices of L (some vertices being given the same label). We shall consider this diagram to be a short-hand way of denoting the abstract complex $\mathcal{S}$ whose vertex set consists of the letters a, b, c, d, e , and f , and whose simplices are the sets $\{a, f, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{c, d, e\}$, $\{a, c, e\}$, $\{a, e, f\}$, along with their nonempty subsets. Of course, this abstract complex is isomorphic to the vertex scheme of the complex K pictured earlier, so it specifies precisely the same complex (up to linear isomorphism). That is, the complex K of Figure 3.1 is a geometric realization of $\mathcal{S}$.

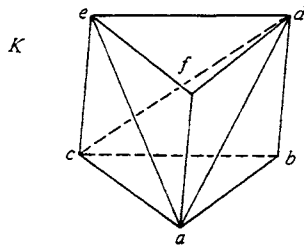


Figure 3.1

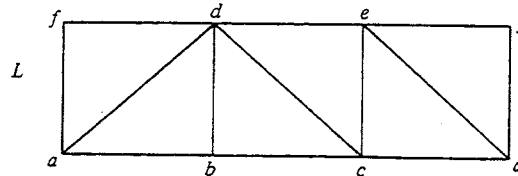


Figure 3.2

Let $f: L^{(0)} \rightarrow K^{(0)}$ be the map that assigns to each vertex of L the correspondingly labelled vertex of K . Then f extends to a simplicial map $g: |L| \rightarrow |K|$. Because the spaces are compact Hausdorff, g is a quotient map, or "pasting map." It identifies the right edge of $|L|$ linearly with the left edge of $|L|$. And of course this is the usual way one forms a cylinder from a rectangular piece of paper—one bends it around and pastes the right edge to the left edge!

Example 2. Now suppose we begin with a complex L and a labelling of its vertices. Consider for instance the same complex L with a different labelling of the vertices, as in Figure 3.3. Just as before, this diagram indicates a certain abstract complex $\mathcal{S}$,

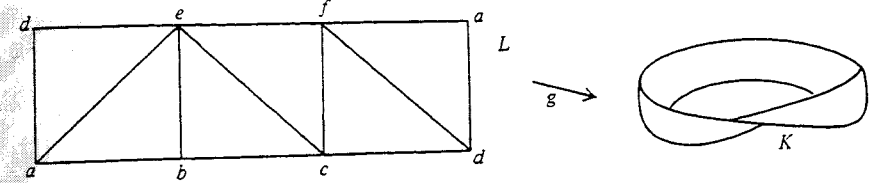


Figure 3.3

whose simplices one can list. Let K be a geometric realization of $\mathcal{S}$. As before, the vertices of K correspond to the letters $a, \dots, f$; we consider the linear simplicial map $g: |L| \rightarrow |K|$ that assigns to each vertex of L the correspondingly labelled vertex of K . Again, g is a quotient map; in this case it identifies the left edge of $|L|$ linearly with the right edge of $|L|$, but with a twist. The space $|K|$ is the one we call the **Möbius band**; it can be pictured in $\mathbb{R}^3$ as the familiar space indicated in Figure 3.3.

Example 3. The **torus** is often defined as the quotient space obtained from a rectangle by making the identifications pictured in Figure 3.4. If we wish to construct a complex whose underlying space is homeomorphic to the torus, we can thus obtain it by using the diagram in Figure 3.5. You can check that the resulting quotient map of L onto the geometric realization of this diagram carries out precisely the identifications needed to form the torus.

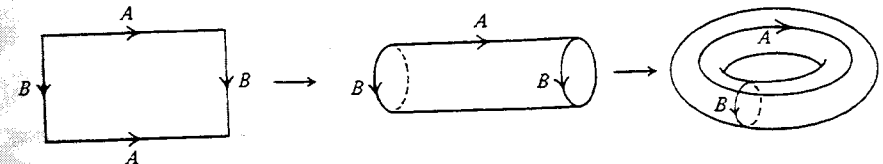


Figure 3.4

Example 4. Some care is required in general to make sure that the quotient map g does not carry out more identifications than one wishes. For instance, you may think that the diagram in Figure 3.6 determines the torus, but it does not. The quotient map in this case does more than paste opposite edges together, as you will see if you examine the diagram more closely.

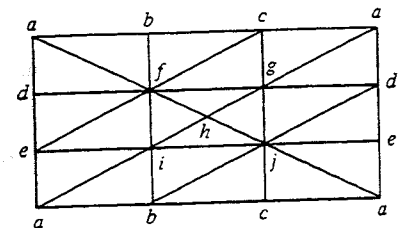


Figure 3.5

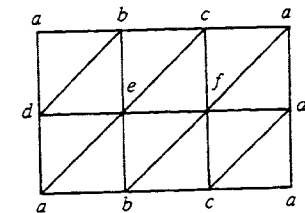


Figure 3.6

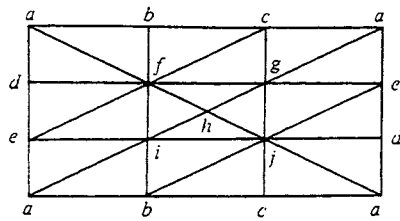


Figure 3.7

Example 5. The diagram in Figure 3.7 indicates an abstract complex whose underlying space is called the **Klein bottle**. It is the quotient space obtained from the rectangle by pasting the edges together as indicated in Figure 3.8. The resulting space cannot be imbedded in $\mathbb{R}^3$, but one can picture it in $\mathbb{R}^3$ by letting it "pass through itself."

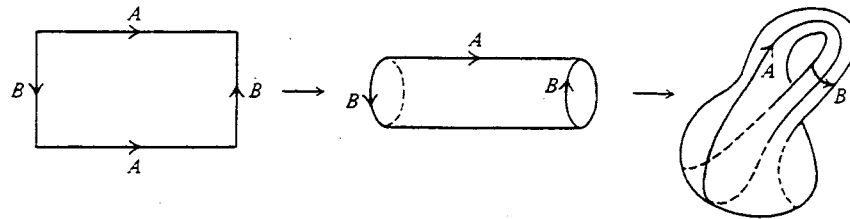


Figure 3.8

We now describe more carefully the process indicated in preceding examples: Given a finite complex L , a **labelling** of the vertices of L is a surjective function f mapping the vertex set of L to a set (called the **set of labels**). Corresponding to this labelling is an abstract complex $\mathcal{S}$ whose vertices are the labels and whose simplices consist of all sets of the form $\{f(v_0), \dots, f(v_n)\}$, where $v_0, \dots, v_n$ span a simplex of L . Let K be a geometric realization of $\mathcal{S}$. Then the vertex map of $L^{(0)}$ onto $K^{(0)}$ derived from f extends to a surjective simplicial map $g: |L| \rightarrow |K|$. We say that K is the **complex derived from the labelled complex L** , and we call g the **associated pasting map**.

Because $|L|$ is compact and $|K|$ is Hausdorff, g maps closed sets to closed sets; thus g is a closed quotient map. Of course, g can in general map a simplex of L onto a simplex of K of lower dimension. For instance, g might collapse all of L to a single point. We are more interested in cases where this does not occur. We are particularly interested in situations similar to those of the preceding examples. We now state a lemma giving conditions under which the general "pasting map" g behaves like those in our examples. First, we need a definition.

Definition. If L is a complex, a subcomplex L_0 of L is said to be a **full subcomplex** of L provided each simplex of L whose vertices belong to L_0 belongs to L_0 itself.

For example, the boundary of the rectangle L pictured in Figure 3.5 is a full subcomplex of L , but the boundary of the rectangle pictured in Figure 3.6 is not.

Lemma 3.2. Let L be a complex; let f be a labelling of its vertices; let $g: |L| \rightarrow |K|$ be the associated pasting map. Let L_0 be a full subcomplex of L . Suppose that whenever v and w are vertices of L having the same label:

- (1) v and w belong to L_0 .
- (2) $\overline{\text{St } v}$ and $\overline{\text{St } w}$ are disjoint.

Then $\dim g(\sigma) = \dim \sigma$ for all $\sigma \in L$. Furthermore, if $g(\sigma_1) = g(\sigma_2)$, then σ_1 and σ_2 must be disjoint simplices belonging to L_0 .

The proof is easy and is left as an exercise. In the usual applications of this lemma, $|L|$ is a polyhedral region in the plane or $\mathbb{R}^N$, and $|L_0|$ is the boundary of the region.

EXERCISES

1. The **projective plane P^2** is defined as the space obtained from the 2-sphere S^2 by identifying x with $-x$ for each $x \in S^2$.
 - (a) Show P^2 is homeomorphic with the space obtained from B^2 by identifying x with $-x$ for each $x \in S^1$.
 - (b) Show that the labelled complex L of Figure 3.9 determines a complex K whose space is homeomorphic to P^2 .

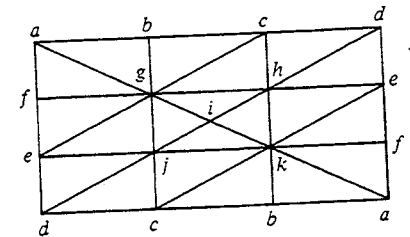


Figure 3.9

- (c) Describe the space determined by the labelled complex of Figure 3.10.

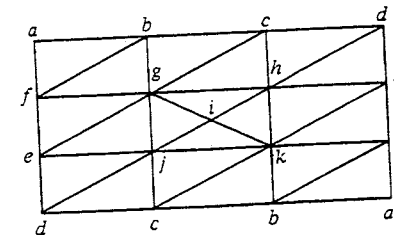


Figure 3.10

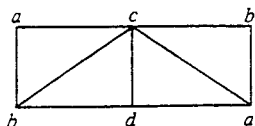


Figure 3.11

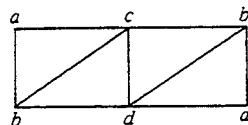


Figure 3.12

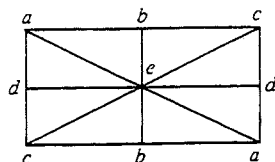


Figure 3.13

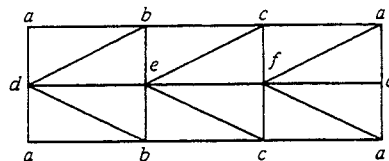


Figure 3.14

- Describe the spaces determined by the labelled complexes in Figures 3.11–3.14.
- Prove Lemma 3.2.
- Let S be a set with a partial order relation $\leq$. A standard technique in combinatorics is to associate with S the abstract complex $\mathcal{S}$ whose vertices are the elements of S and whose simplices are the finite simply-ordered subsets of S . Suppose one is given the partial order on $\{a_1, \dots, a_8\}$ generated by the following relations:

$$\begin{aligned} a_1 \leq a_3 \leq a_7 \leq a_8; & \quad a_1 \leq a_5 \leq a_7; \\ a_2 \leq a_6 \leq a_8; & \quad a_2 \leq a_5. \end{aligned}$$

Describe a geometric realization of $\mathcal{S}$.

§4. REVIEW OF ABELIAN GROUPS

In this section, we review some results from algebra that we shall be using—specifically, facts about abelian groups.

We write abelian groups additively. Then 0 denotes the neutral element, and $-g$ denotes the additive inverse of g . If n is a positive integer, then ng denotes the n -fold sum $g + \dots + g$, and $(-n)g$ denotes $n(-g)$.

We denote the group of integers by $\mathbb{Z}$, the rationals by $\mathbb{Q}$, and the complex numbers by $\mathbb{C}$.

Homomorphisms

If $f: G \rightarrow H$ is a homomorphism, the **kernel** of f is the subgroup $f^{-1}(0)$ of G , the **image** of f is the subgroup $f(G)$ of H , and the **cokernel** of f is the quotient group $H/f(G)$. We denote these groups by $\ker f$ and $\operatorname{im} f$ and $\operatorname{cok} f$, respectively. The map f is a monomorphism if and only if the kernel of f vanishes (i.e.,

equals the trivial group). And f is an epimorphism if and only if the cokernel of f vanishes; in this case, f induces an isomorphism $G/\ker f \cong H$.

Free abelian groups

An abelian group G is **free** if it has a **basis**—that is, if there is a family $\{g_\alpha\}_{\alpha \in J}$ of elements of G such that each $g \in G$ can be written uniquely as a finite sum

$$g = \sum n_\alpha g_\alpha,$$

with n_α an integer. Uniqueness implies that each element g_α has infinite order; that is, g_α generates an infinite cyclic subgroup of G .

More generally, if each $g \in G$ can be written as a finite sum $g = \sum n_\alpha g_\alpha$, but not necessarily uniquely, then we say that the family $\{g_\alpha\}$ **generates** G . In particular, if the set $\{g_\alpha\}$ is finite, we say that G is **finitely generated**.

If G is free and has a basis consisting of n elements, say $g_1, \dots, g_n$, then it is easy to see that every basis for G consists of precisely n elements. For the group $G/2G$ consists of all cosets of the form

$$(\sum \epsilon_i g_i) + 2G,$$

where $\epsilon_i = 0$ or 1; this fact implies that the group $G/2G$ consists of precisely 2^n elements. The number of elements in a basis for G is called the **rank** of G .

It is true more generally that if G has an infinite basis, any two bases for G have the same cardinality. We shall not use this fact.

A crucial property of free abelian groups is the following: If G has a basis $\{g_\alpha\}$, then any function f from the set $\{g_\alpha\}$ to an abelian group H extends uniquely to a homomorphism of G into H .

One specific way of constructing free abelian groups is the following: Given a set S , we define the **free abelian group G generated by S** to be the set of all functions $\phi: S \rightarrow \mathbb{Z}$ such that $\phi(x) \neq 0$ for only finitely many values of x ; we add two such functions by adding their values. Given $x \in S$, there is a characteristic function ϕ_x for x , defined by setting

$$\phi_x(y) = \begin{cases} 0 & \text{if } y \neq x, \\ 1 & \text{if } y = x. \end{cases}$$

The functions $\{\phi_x \mid x \in S\}$ form a basis for G , for each function $\phi \in G$ can be written uniquely as a finite sum

$$\phi = \sum n_x \phi_x,$$

where $n_x = \phi(x)$ and the summation extends over all x for which $\phi(x) \neq 0$. We often abuse notation and identify the element $x \in S$ with its characteristic function ϕ_x . With this notation, the general element of G can be written uniquely as a finite “formal linear combination”

$$\phi = \sum n_\alpha x_\alpha$$

of the elements of the set S .

If G is an abelian group, an element g of G has **finite order** if $ng = 0$ for some positive integer n . The set of all elements of finite order in G is a subgroup T of G , called the **torsion subgroup**. If T vanishes, we say G is **torsion-free**. A free abelian group is necessarily torsion-free, but not conversely.

If T consists of only finitely many elements, then the number of elements in T is called the **order** of T . If T has finite order, then each element of T has finite order; but not conversely.

Internal direct sums

Suppose G is an abelian group, and suppose $\{G_\alpha\}_{\alpha \in J}$ is a collection of subgroups of G , indexed bijectively by some index set J . Suppose that each g in G can be written uniquely as a finite sum $g = \sum g_\alpha$, where $g_\alpha \in G_\alpha$ for each α . Then G is said to be the **internal direct sum** of the groups G_α , and we write

$$G = \bigoplus_{\alpha \in J} G_\alpha.$$

If the collection $\{G_\alpha\}$ is finite, say $\{G_\alpha\} = \{G_1, \dots, G_n\}$, we also write this direct sum in the form $G = G_1 \oplus \dots \oplus G_n$.

If each g in G can be written as a finite sum $g = \sum g_\alpha$, but not necessarily uniquely, we say simply that G is the **sum** of the groups $\{G_\alpha\}$, and we write $G = \sum G_\alpha$, or, in the finite case, $G = G_1 + \dots + G_n$. In this situation, we also say that the groups $\{G_\alpha\}$ **generate** G .

If $G = \sum G_\alpha$, then this sum is direct if and only if the equation $0 = \sum g_\alpha$ implies that $g_\alpha = 0$ for each α . This in turn occurs if and only if for each fixed index α_0 , one has

$$G_{\alpha_0} \cap \left(\sum_{\alpha \neq \alpha_0} G_\alpha \right) = \{0\}.$$

In particular, if $G = G_1 + G_2$, then this sum is direct if and only if $G_1 \cap G_2 = \{0\}$.

The resemblance to free abelian groups is strong. Indeed, if G is free with basis $\{g_\alpha\}$, then G is the direct sum of the subgroups $\{G_\alpha\}$, where G_α is the infinite cyclic group generated by g_α . Conversely, if G is a direct sum of infinite cyclic subgroups, then G is a free abelian group.

If G is the direct sum of subgroups $\{G_\alpha\}$, and if for each α , one has a homomorphism f_α of G_α into the abelian group H , the homomorphisms $\{f_\alpha\}$ extend uniquely to a homomorphism of G into H .

Here is a useful criterion for showing G is a direct sum:

Lemma 4.1. *Let G be an abelian group. If G is the direct sum of the subgroups $\{G_\alpha\}$, then there are homomorphisms*

$$j_\beta: G_\beta \rightarrow G \quad \text{and} \quad \pi_\beta: G \rightarrow G_\beta$$

such that $\pi_\beta \circ j_\alpha$ is the zero homomorphism if $\alpha \neq \beta$, and the identity homomorphism if $\alpha = \beta$.

Conversely, suppose $\{G_\alpha\}$ is a family of abelian groups, and there are homomorphisms j_β and π_β as above. Then j_β is a monomorphism. Furthermore, if the groups $j_\alpha(G_\alpha)$ generate G , then G is their direct sum.

Proof. Suppose $G = \bigoplus G_\alpha$. We define j_β to be the inclusion homomorphism. To define π_β , write $g = \sum g_\alpha$, where $g_\alpha \in G_\alpha$ for each α ; and let $\pi_\beta(g) = g_\beta$. Uniqueness of the representation of g shows π_β is a well-defined homomorphism.

Consider the converse. Because $\pi_\alpha \circ j_\alpha$ is the identity, j_α is injective (and π_α is surjective). If the groups $j_\alpha(G_\alpha)$ generate G , every element of G can be written as a finite sum $\sum j_\alpha(g_\alpha)$, by hypothesis. To show this representation is unique, suppose

$$\sum j_\alpha(g_\alpha) = \sum j_\alpha(g'_\alpha).$$

Applying π_β , we see that $g_\beta = g'_\beta$. $\square$

Direct products and external direct sums

Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of abelian groups. Their **direct product** $\prod_{\alpha \in J} G_\alpha$ is the group whose underlying set is the cartesian product of the sets G_α , and whose group operation is component-wise addition. Their **external direct sum** G is the subgroup of the direct product consisting of all tuples $(g_\alpha)_{\alpha \in J}$ such that $g_\alpha = 0_\alpha$ for all but finitely many values of α . (Here 0_α is the zero element of G_α .) The group G is sometimes also called the "weak direct product" of the groups G_α .

The relation between internal and external direct sums is described as follows: Suppose G is the external direct sum of the groups $\{G_\alpha\}$. Then for each β , we define $\pi_\beta: G \rightarrow G_\beta$ to be projection onto the β th factor. And we define $j_\beta: G_\beta \rightarrow G$ by letting it carry the element $g \in G_\beta$ to the tuple $(g_\alpha)_{\alpha \in J}$, where $g_\alpha = 0_\alpha$ for all α different from β , and $g_\beta = g$. Then $\pi_\beta \circ j_\alpha = 0$ for $\alpha \neq \beta$, and $\pi_\beta \circ j_\beta$ is the identity. It follows that G equals the **internal** direct sum of the groups $G'_\alpha = j_\alpha(G_\alpha)$, where G'_α is isomorphic to G_α .

Thus the notions of internal and external direct sums are closely related. The difference is mainly one of notation. For this reason, we customarily use the notations

$$G = G_1 \oplus \dots \oplus G_n \quad \text{and} \quad G = \bigoplus G_\alpha$$

to denote either internal or external direct sums, relying on the context to make clear which is meant (if indeed, it is important). With this notation, one can for instance express the fact that G is free abelian of rank 3 merely by writing $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

If G_1 is a subgroup of G , we say that G_1 is a **direct summand** in G if there is a subgroup G_2 of G such that $G = G_1 \oplus G_2$. In this case, if H_i is a subgroup of G_i , for $i = 1, 2$, then the sum $H_1 + H_2$ is direct, and furthermore,

$$\frac{G}{H_1 \oplus H_2} \cong \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}.$$

In particular, if $G = G_1 \oplus G_2$, then $G/G_1 \cong G_2$.

Of course, one can have $G/G_1 \cong G_2$ without its following that $G = G_1 \oplus G_2$; that is, G_1 may be a subgroup of G without being a direct summand in G . For instance, the subgroup $n\mathbb{Z}$ of the integers is not a direct summand in $\mathbb{Z}$, for that would mean that

$$\mathbb{Z} \cong n\mathbb{Z} \oplus G_2$$

for some subgroup G_2 of $\mathbb{Z}$. But then G_2 is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, which is a group of finite order, while no subgroup of $\mathbb{Z}$ has finite order.

Incidentally, we shall denote the group $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n simply by $\mathbb{Z}/n$, in accordance with current usage.

The fundamental theorem of finitely generated abelian groups

There are actually two theorems that are important to us. The first is a theorem about subgroups of free abelian groups. We state it here, and give a proof in §11:

Theorem 4.2. *Let F be a free abelian group. If R is a subgroup of F , then R is also a free abelian group. If F has rank n , then R has rank $r \leq n$; furthermore, there is a basis $e_1, \dots, e_r$ for R and integers $t_1, \dots, t_k$ with $t_i > 1$ such that*

- (1) $t_1 e_1, \dots, t_k e_k, e_{k+1}, \dots, e_r$ is a basis for R .
- (2) $t_1 | t_2 | \dots | t_k$, that is, t_i divides t_{i+1} for all i .

The integers $t_1, \dots, t_k$ are uniquely determined by F and R , although the basis $e_1, \dots, e_r$ is not.

An immediate corollary of this theorem is the following:

Theorem 4.3 (The fundamental theorem of finitely generated abelian groups). *Let G be a finitely generated abelian group. Let T be its torsion subgroup.*

(a) *There is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.*

(b) *There are finite cyclic groups $T_1, \dots, T_k$, where T_i has order $t_i > 1$, such that $t_1 | t_2 | \dots | t_k$ and*

$$T = T_1 \oplus \dots \oplus T_k.$$

(c) *The numbers β and $t_1, \dots, t_k$ are uniquely determined by G .*

The number β is called the **beti number** of G ; the numbers $t_1, \dots, t_k$ are called the **torsion coefficients** of G . Note that β is the rank of the free abelian group $G/T \cong H$. The rank of the subgroup H and the orders of the subgroups T_i are uniquely determined, but the subgroups themselves are not.

Proof. Let S be a finite set of generators $\{g_i\}$ for G ; let F be the free abelian group on the set S . The map carrying each g_i to itself extends to a homomorphism carrying F onto G . Let R be the kernel of this homomorphism. Then $F/R \cong G$. Choose bases for F and R as in Theorem 4.2. Then

$$F = F_1 \oplus \dots \oplus F_n$$

where F_i is infinite cyclic with generator e_i ; and

$$R = t_1 F_1 \oplus \dots \oplus t_k F_k \oplus F_{k+1} \oplus \dots \oplus F_r.$$

We compute the quotient group as follows:

$$F/R \cong (F_1/t_1 F_1 \oplus \dots \oplus F_k/t_k F_k) \oplus (F_{r+1} \oplus \dots \oplus F_n).$$

Thus there is an isomorphism

$$f: G \rightarrow (\mathbb{Z}/t_1 \oplus \dots \oplus \mathbb{Z}/t_k) \oplus (\mathbb{Z} \oplus \dots \oplus \mathbb{Z}).$$

The torsion subgroup T of G must be mapped to the subgroup $\mathbb{Z}/t_1 \oplus \dots \oplus \mathbb{Z}/t_k$ by f , since any isomorphism preserves torsion subgroups. Parts (a) and (b) of the theorem follow. Part (c) is left to the exercises. $\square$

This theorem shows that any finitely generated abelian group G can be written as a finite direct sum of cyclic groups; that is,

$$G \cong (\mathbb{Z} \oplus \dots \oplus \mathbb{Z}) \oplus \mathbb{Z}/t_1 \oplus \dots \oplus \mathbb{Z}/t_k.$$

with $t_i > 1$ and $t_1 | t_2 | \dots | t_k$. This representation is in some sense a "canonical form" for G . There is another such canonical form, derived as follows:

Recall first the fact that if m and n are relatively prime positive integers, then

$$\mathbb{Z}/m \oplus \mathbb{Z}/n \cong \mathbb{Z}/mn.$$

It follows that any finite cyclic group can be written as a direct sum of cyclic groups whose orders are powers of primes. Theorem 4.3 then implies that for any finitely generated group G ,

$$G \cong (\mathbb{Z} \oplus \dots \oplus \mathbb{Z}) \oplus (\mathbb{Z}/a_1 \oplus \dots \oplus \mathbb{Z}/a_s)$$

where each a_i is a power of a prime. This is another canonical form for G , since the numbers a_i are uniquely determined by G (up to a rearrangement), as we shall see.

EXERCISES

1. Show that if G is a finitely generated abelian group, every subgroup of G is finitely generated. (This result does not hold for non-abelian groups.)
2. (a) Show that if G is free, then G is torsion-free.
(b) Show that if G is finitely generated and torsion-free, then G is free.

- (c) Show that the additive group of rationals $\mathbb{Q}$ is torsion-free but not free. [Hint: If $\{g_\alpha\}$ is a basis for $\mathbb{Q}$, let β be fixed and express $g_\beta/2$ in terms of this basis.]
3. (a) Show that if m and n are relatively prime, then $\mathbb{Z}/m \oplus \mathbb{Z}/n$ is cyclic of order mn .
 (b) If $G \cong \mathbb{Z}/18 \oplus \mathbb{Z}/36$, express G as a direct sum of cyclic groups of prime power order.
 (c) If $G \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$, find the torsion coefficients of G .
 (d) If $G \cong \mathbb{Z}/15 \oplus \mathbb{Z}/20 \oplus \mathbb{Z}/18$, find the invariant factors and the torsion coefficients of G .
4. (a) Let p be prime; let $b_1, \dots, b_k$ be non-negative integers. Show that if

$$G \cong (\mathbb{Z}/p)^{b_1} \oplus (\mathbb{Z}/p^2)^{b_2} \oplus \dots \oplus (\mathbb{Z}/p^k)^{b_k},$$

then the integers b_i are uniquely determined by G . [Hint: Consider the kernel of the homomorphism $f_i: G \rightarrow G$ that is multiplication by p^i . Show that f_1 and f_2 determine b_1 . Proceed similarly.]

- (b) Let $p_1, \dots, p_N$ be a sequence of distinct primes. Generalize (a) to a finite direct sum of terms of the form $(\mathbb{Z}/(p_i)^{k_i})^{b_{ik}}$, where $b_{ik} \geq 0$.
 (c) Verify (c) of Theorem 4.3. That is, show that the betti number, invariant factors, and torsion coefficients of a finitely generated abelian group G are uniquely determined by G .
 (d) Show that the numbers t_i appearing in the conclusion of Theorem 4.2 are uniquely determined by F and R .

§5. HOMOLOGY GROUPS

Now we are ready to define the homology groups. First we must discuss the notion of "orientation."

Definition. Let σ be a simplex (either geometric or abstract). Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim \sigma > 0$, the orderings of the vertices of σ then fall into two equivalence classes. Each of these classes is called an **orientation** of σ . (If σ is a 0-simplex, then there is only one class and hence only one orientation of σ .) An **oriented simplex** is a simplex σ together with an orientation of σ .

If the points $v_0, \dots, v_p$ are independent, we shall use the symbol

$$v_0 \dots v_p$$

to denote the simplex they span, and we shall use the symbol

$$[v_0, \dots, v_p]$$

to denote the oriented simplex consisting of the simplex $v_0 \dots v_p$ and the equivalence class of the particular ordering $(v_0, \dots, v_p)$.

Occasionally, when the context is clear, we shall use a single letter such as σ to denote either a simplex or an oriented simplex.

Example 1. We often picture an orientation of a 1-simplex by drawing an arrow on it. The oriented simplex $[v_0, v_1]$ is pictured in Figure 5.1; one draws an arrow pointing from v_0 to v_1 . An orientation of a 2-simplex is pictured by a circular arrow. The oriented simplex $[v_0, v_1, v_2]$ is indicated in the figure by drawing an arrow in the direction from v_0 to v_1 to v_2 . You can check that $[v_1, v_2, v_0]$ and $[v_2, v_0, v_1]$ are indicated by the same clockwise arrow. An arrow in the counterclockwise direction would indicate the oppositely oriented simplex.

Similarly, the oriented simplex $[v_0, v_1, v_2, v_3]$ is pictured by drawing a spiral arrow, as in the figure. The arrow in this picture is called a "right-hand screw"; if one curls the fingers of the right hand in the direction from v_0 to v_1 to v_2 , the thumb points toward v_3 . You can check that $[v_0, v_2, v_3, v_1]$, and each of the other ten orderings equivalent to these two, also give rise to right-hand screws. A "left-hand screw" is used to picture the opposite orientation.

These examples illustrate that our definition of orientation agrees with the intuitive geometric notions derived from vector calculus.

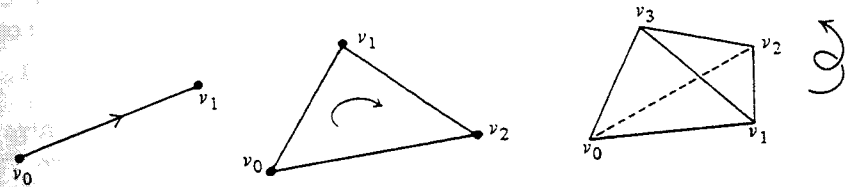


Figure 5.1

Definition. Let K be a simplicial complex. A p -chain on K is a function c from the set of oriented p -simplices of K to the integers, such that:

- (1) $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex.
 (2) $c(\sigma) = 0$ for all but finitely many oriented p -simplices σ .

We add p -chains by adding their values; the resulting group is denoted $C_p(K)$ and is called the **group of (oriented) p -chains** of K . If $p < 0$ or $p > \dim K$, we let $C_p(K)$ denote the trivial group.

If σ is an oriented simplex, the **elementary chain** c corresponding to σ is the function defined as follows:

$$\begin{aligned} c(\sigma) &= 1, \\ c(\sigma') &= -1 && \text{if } \sigma' \text{ is the opposite orientation of } \sigma, \\ c(\tau) &= 0 && \text{for all other oriented simplices } \tau. \end{aligned}$$

By abuse of notation, we often use the symbol σ to denote not only a simplex, or an oriented simplex, but also to denote the elementary p -chain c corresponding to the oriented simplex σ . With this convention, if σ and σ' are opposite orienta-

tions of the same simplex, then we can write $\sigma' = -\sigma$, because this equation holds when σ and σ' are interpreted as elementary chains.

Lemma 5.1. $C_p(K)$ is free abelian; a basis for $C_p(K)$ can be obtained by orienting each p -simplex and using the corresponding elementary chains as a basis.

Proof. The proof is straightforward. Once all the p -simplices of K are oriented (arbitrarily), each p -chain can be written uniquely as a finite linear combination

$$c = \sum n_i \sigma_i$$

of the corresponding elementary chains σ_i . The chain c assigns the value n_i to the oriented p -simplex σ_i , the value $-n_i$ to the opposite orientation of σ_i , and the value 0 to all oriented p -simplices not appearing in the summation. $\square$

The group $C_0(K)$ differs from the others, since it has a natural basis (since a 0-simplex has only one orientation). The group $C_p(K)$ has no "natural" basis if $p > 0$; one must orient the p -simplices of K in some arbitrary fashion in order to obtain a basis.

Corollary 5.2. Any function f from the oriented p -simplices of K to an abelian group G extends uniquely to a homomorphism $C_p(K) \rightarrow G$, provided that $f(-\sigma) = -f(\sigma)$ for all oriented p -simplices σ . $\square$

Definition. We now define a homomorphism

$$\partial_p : C_p(K) \rightarrow C_{p-1}(K)$$

called the **boundary operator**. If $\sigma = [v_0, \dots, v_p]$ is an oriented simplex with $p > 0$, we define

$$(*) \quad \partial_p \sigma = \partial_p [v_0, \dots, v_p] = \sum_{i=0}^p (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_p],$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array. Since $C_p(K)$ is the trivial group for $p < 0$, the operator ∂_p is the trivial homomorphism for $p \leq 0$.

We must check that ∂_p is well-defined and that $\partial_p(-\sigma) = -\partial_p \sigma$. For this purpose, it suffices to show that the right-hand side of (*) changes sign if we exchange two adjacent vertices in the array $[v_0, \dots, v_p]$. So let us compare the expressions for

$$\partial_p [v_0, \dots, v_j, v_{j+1}, \dots, v_p] \quad \text{and} \quad \partial_p [v_0, \dots, v_{j+1}, v_j, \dots, v_p].$$

For $i \neq j, j+1$, the i th terms in these two expressions differ precisely by a sign; the terms are identical except that v_j and v_{j+1} have been interchanged.

What about the i th terms for $i = j$ and $i = j+1$? In the first expression, one has

$$(-1)^j [\dots, v_{j-1}, \hat{v}_j, v_{j+1}, v_{j+2}, \dots] \\ + (-1)^{j+1} [\dots, v_{j-1}, v_j, \hat{v}_{j+1}, v_{j+2}, \dots].$$

In the second expression, one has

$$(-1)^j [\dots, v_{j-1}, \hat{v}_{j+1}, v_j, v_{j+2}, \dots] \\ + (-1)^{j+1} [\dots, v_{j-1}, v_{j+1}, \hat{v}_j, v_{j+2}, \dots].$$

Comparing, one sees these two expressions differ by a sign.

Example 2. For a 1-simplex, we compute $\partial_1 [v_0, v_1] = v_1 - v_0$. For a 2-simplex, one has

$$\partial_2 [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

And for a 3-simplex one has the formula

$$\partial_3 [v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

The geometric content of these formulas is pictured in Figure 5.2. If you remember the versions of Green's, Stokes', and Gauss' theorems you studied in calculus, these pictures should look rather familiar.

Example 3. Consider the 1-chain $\partial_2 [v_0, v_1, v_2]$ pictured in Figure 5.2. If you apply the operator ∂_1 to this 1-chain, you get zero; everything cancels out because each vertex appears as the initial point of one edge and as the end point of another edge. You can check that a similar cancellation occurs when you compute $\partial_2 \partial_3 [v_0, v_1, v_2, v_3]$.

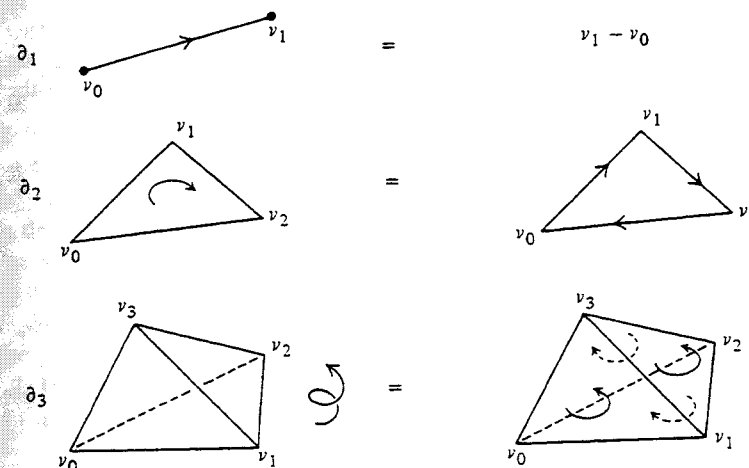


Figure 5.2

The computations discussed in Example 3 illustrate a general fact:

Lemma 5.3. $\partial_{p-1} \circ \partial_p = 0$.

Proof. The proof is straightforward. We compute

$$\begin{aligned}\partial_{p-1}\partial_p[v_0, \dots, v_p] &= \sum_{i=0}^p (-1)^i \partial_{p-1}[v_0, \dots, \hat{v}_i, \dots, v_p] \\ &= \sum_{j < i} (-1)^i (-1)^j [\dots, \hat{v}_j, \dots, \hat{v}_i, \dots] \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j-1} [\dots, \hat{v}_i, \dots, \hat{v}_j, \dots].\end{aligned}$$

The terms of these two summations cancel in pairs. $\square$

Definition. The kernel of $\partial_p: C_p(K) \rightarrow C_{p-1}(K)$ is called the group of p -cycles and denoted $Z_p(K)$ (for the German word "Zyklus"). The image of $\partial_{p+1}: C_{p+1}(K) \rightarrow C_p(K)$ is called the group of p -boundaries and is denoted $B_p(K)$. By the preceding lemma, each boundary of a $p+1$ chain is automatically a p -cycle. That is, $B_p(K) \subset Z_p(K)$. We define

$$H_p(K) = Z_p(K)/B_p(K),$$

and call it the p th homology group of K .

Let us compute a few examples.

Example 4. Consider the complex K of Figure 5.3, whose underlying space is the boundary of a square with edges e_1, e_2, e_3, e_4 . The group $C_1(K)$ is free abelian of rank 4; the general 1-chain c is of the form $\sum n_i e_i$. Computing $\partial_1 c$, we see that its value on the vertex v is $n_1 - n_2$. A similar argument, applied to the other vertices, shows that c is a cycle if and only if $n_1 = n_2 = n_3 = n_4$. One concludes that $Z_1(K)$ is infinite cyclic, and is generated by the chain $e_1 + e_2 + e_3 + e_4$. Since there are no 2-simplices in K , $B_1(K)$ is trivial. Therefore,

$$H_1(K) = Z_1(K) \simeq \mathbb{Z}.$$

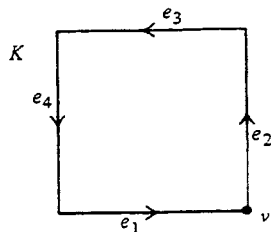


Figure 5.3

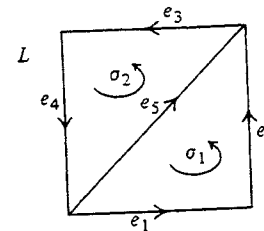


Figure 5.4

Example 5. Consider the complex L of Figure 5.4, whose underlying space is a square. The general 1-chain is of the form $\sum n_i e_i$. One reasons as before to conclude that this 1-chain is a cycle if and only if $n_1 = n_2$, $n_3 = n_4$, and $n_5 = n_3 - n_2$. One can assign values to n_2 and n_3 arbitrarily; then the others are determined. Therefore, $Z_1(L)$ is free abelian of rank 2; a basis consists of the chain $e_1 + e_2 - e_5$ obtained by taking $n_2 = 1$ and $n_3 = 0$, and the chain $e_3 + e_4 + e_5$ obtained by taking $n_2 = 0$ and $n_3 = 1$. The first equals $\partial_2 \sigma_1$ and the second equals $\partial_2 \sigma_2$. Therefore,

$$H_1(L) = Z_1(L)/B_1(L) = 0.$$

Likewise, $H_2(L) = 0$; the general 2-chain $m_1 \sigma_1 + m_2 \sigma_2$ is a cycle if and only if $m_1 = m_2 = 0$.

These examples may begin to give you a feeling for what the homology groups mean geometrically. Only by computing many more examples can one begin to "see" what a homology class is. Our hope is that after you get a feeling for what homology means geometrically, you will begin to believe what is at the moment far from clear—that the homology groups of a complex K actually depend only on the underlying space $|K|$.

Now let us consider another example. It involves a complex having more simplices than those in the preceding examples. In general, as the number of simplices increases, calculating the group of cycles Z_p and the group of boundaries B_p becomes more tedious. We can short-cut some of these calculations by avoiding calculating these groups and proceeding directly to a calculation of the homology groups H_p .

We deal here only with the groups $H_p(K)$ for $p > 0$, postponing discussion of the group $H_0(K)$ to §7.

We need some terminology. We shall say that a chain c is **carried by** a subcomplex L of K if c has value 0 on every simplex that is not in L . And we say that two p -chains c and c' are **homologous** if $c - c' = \partial_{p+1} d$ for some $p+1$ chain d . In particular, if $c = \partial_{p+1} d$, we say that c is **homologous to zero**, or simply that c **bounds**.

Example 6. Consider the complex M indicated in Figure 5.5, whose underlying space is a square. Instead of computing the group of 1-cycles specifically, we reason

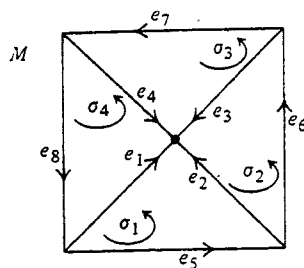


Figure 5.5

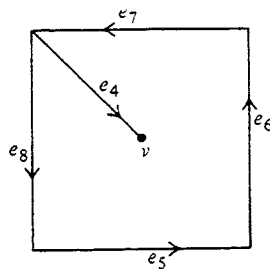


Figure 5.6

as follows: Given a 1-chain c , let a be the value of c on e_1 . Then by direct computation, the chain

$$c_1 = c + \partial_2(a\sigma_1)$$

has value 0 on the oriented simplex e_1 . Intuitively speaking, by modifying c by a boundary, we have "pushed it off e_1 ." We then "push c_1 off e_2 " in a similar manner, as follows: Let b be the value of c_1 on e_2 ; then the chain

$$c_2 = c_1 + \partial_2(b\sigma_2)$$

has value 0 on the oriented simplex e_2 . It also has value 0 on e_1 , since e_1 does not appear in the expression for $\partial_2\sigma_2$. Now letting d denote the value of c_2 on e_3 , one sees that

$$c_3 = c_2 + \partial_2(d\sigma_3)$$

has value 0 on e_3 and on e_2 and on e_1 . We have thus "pushed c off" all of e_1 , e_2 , and e_3 . Said differently, we have proved the following result:

Given a 1-chain c , it is homologous to a chain c_3 that is carried by the subcomplex of M pictured in Figure 5.6.

Now if c happens to be a cycle, then c_3 is also a cycle; it follows that the value of c_3 on the simplex e_4 must be 0. (Otherwise, ∂c_3 would have non-zero value on the center vertex v .) Thus every 1-cycle of M is homologous to a 1-cycle carried by the boundary of the square. By the same argument as used before, such a cycle must be some multiple of the cycle $e_5 + e_6 + e_7 + e_8$. And this cycle bounds; indeed, it clearly equals $\partial(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)$. Thus $H_1(M) = 0$, as expected.

The fact that $H_2(M) = 0$ is easy; one sees readily (as before) that $\sum m_i \sigma_i$ is a cycle if and only if $m_i = 0$ for all i .

Note that the homology groups of M are the same as the homology groups of the complex L of Example 5. This fact lends some plausibility to our remark (yet to be proved) that the homology groups of a complex depend only on its underlying space.

EXERCISES

- Let $\mathcal{S}$ be the abstract complex consisting of the 1-simplices $\{v_0, v_1\}$, $\{v_1, v_2\}$, $\dots$, $\{v_{n-1}, v_n\}$, $\{v_n, v_0\}$ and their vertices. If K is a geometric realization of $\mathcal{S}$, compute $H_1(K)$.

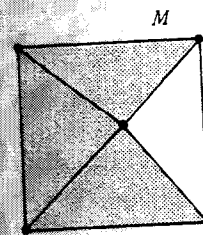


Figure 5.7

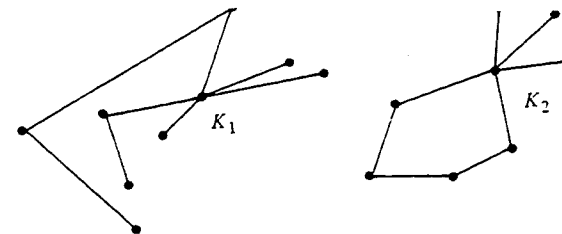


Figure 5.8

- Consider the complex M pictured in Figure 5.7; it is the union of three triangles and a line segment. Compute the homology groups $H_1(M)$ and $H_2(M)$.
- A 1-dimensional complex is called a tree if its 1-dimensional homology vanishes. Is either of the complexes pictured in Figure 5.8 a tree?
- Let K be the complex consisting of the proper faces of a 3-simplex. Compute $H_1(K)$ and $H_2(K)$.
- For what values of i is it true that

$$H_i(K^{(p)}) \cong H_i(K)?$$

- An "infinite p -chain" on K is a function c from the oriented p -simplices of K to the integers such that $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex. We do not require that $c(\sigma) = 0$ for all but finitely many oriented simplices. Let $C_p^\infty(K)$ denote the group of infinite p -chains. It is abelian, but it will not in general be free.

(a) Show that if K is locally finite, then one can define a boundary operator

$$\partial_p^\infty : C_p^\infty(K) \rightarrow C_{p-1}^\infty(K)$$

by the formula used earlier, and Lemma 5.3 holds. The resulting groups

$$H_p^\infty(K) = \ker \partial_p^\infty / \text{im } \partial_{p+1}^\infty$$

are called the **homology groups based on infinite chains**.

- Let K be the complex whose space is $\mathbb{R}$ and whose vertices are the integers. Show that

$$H_1(K) = 0 \quad \text{and} \quad H_1^\infty(K) \cong \mathbb{Z}.$$

- Let $\mathcal{S}$ be the abstract complex whose simplices consist of the sets $\{im, m\}$, $\{im, -m\}$, and $\{m, -m\}$ for all positive integers m , along with their faces. If K is a geometric realization of $\mathcal{S}$, compute $H_1(K)$ and $H_1^\infty(K)$.

§6. HOMOLOGY GROUPS OF SURFACES

If K is a finite complex, then the chain group $C_p(K)$ has finite rank, so the cycle group $Z_p(K)$ also has finite rank. Then $H_p(K)$ is finitely generated, so that the fundamental theorem of abelian groups applies. The betti number and torsion

coefficients of $H_p(K)$ are called, classically, the betti number of K and torsion coefficients of K in dimension p . The fact that these numbers are topological invariants of $|K|$ will be proved in Chapter 2.

In former times, much attention was paid to *numerical* invariants, not only in topology, but also in algebra and other parts of mathematics. Nowadays, mathematicians are likely to feel that homology *groups* are the more important notion, and one is more likely to study properties of these groups than to compute their numerical invariants. Nevertheless, it is still important in many situations to compute the homology groups specifically—that is, to find the betti numbers and torsion coefficients for a given space.

One of the greatest virtues of the *simplicial* homology groups is that it is in fact possible to do precisely this. In §11 we shall prove a theorem to the effect that, for a finite complex K , the homology groups are effectively computable. This means that there is a specific algorithm for finding the betti numbers and torsion coefficients of K .

In the present section, we shall compute the betti numbers and torsion coefficients of the compact surfaces. The techniques we shall use may seem a bit awkward and ad hoc in nature. But, in fact, they are effective on a large class of spaces. In a later section, when we study CW complexes, we shall return to these techniques and show they are part of a systematic approach to computing homology groups.

Convention. For convenience in notation, we shall henceforth delete the dimensional subscript p on the boundary operator ∂_p , and rely on the context to make clear which of these operators is intended.

We shall compute the homology of the torus, the Klein bottle, and several other spaces that can be constructed from a rectangle L by identifying its edges appropriately. Thus we begin by proving certain facts about L itself.

Lemma 6.1. Let L be the complex of Figure 6.1, whose underlying space is a rectangle. Let $\text{Bd } L$ denote the complex whose space is the boundary of the rectangle. Orient each 2-simplex σ_i of L by a counterclockwise arrow. Orient the 1-simplices arbitrarily. Then:

- (1) Every 1-cycle of L is homologous to a 1-cycle carried by $\text{Bd } L$.

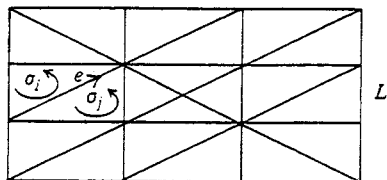


Figure 6.1

(2) If d is a 2-chain of L and if ∂d is carried by $\text{Bd } L$, then d is a multiple of the chain $\Sigma \sigma_i$.

Proof. The proof of (2) is easy. If σ_i and σ_j have an edge e in common, then ∂d must have value 0 on e . It follows that d must have the same value on σ_i as it does on σ_j . Continuing this process, we see that d has the same value on every oriented 2-simplex σ_i .

To prove (1), we proceed as in Example 6 of the preceding section. Given a 1-chain c of L , one “pushes it off” the 1-simplices, one at a time. First, one shows that c is homologous to a 1-chain c_1 carried by the subcomplex pictured in Figure 6.2. Then one shows that c_1 is in turn homologous to a 1-chain c_2 carried by the subcomplex of Figure 6.3. Finally, one notes that in the case where the original chain c is a cycle, then the chain c_2 is also a cycle. It follows that c_2 must be carried by $\text{Bd } L$, for otherwise c_2 would have a non-zero coefficient on one or more of the vertices $v_1, \dots, v_5$. $\square$

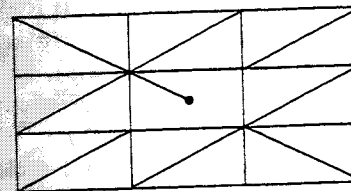


Figure 6.2

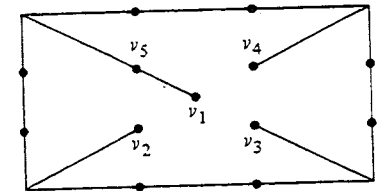


Figure 6.3

Theorem 6.2. Let T denote the complex represented by the labelled rectangle L of Figure 6.4; its underlying space is the torus. Then:

$$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H_2(T) \cong \mathbb{Z}.$$

Orient each 2-simplex of L counterclockwise; use the induced orientation of the 2-simplices of T ; let γ denote their sum. Let

$$w_1 = [a, b] + [b, c] + [c, a],$$

$$z_1 = [a, d] + [d, e] + [e, a].$$

Then γ generates $H_2(T)$ and w_1 and z_1 represent a basis for $H_1(T)$.

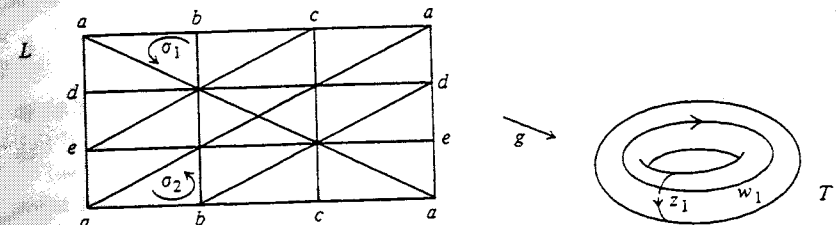


Figure 6.4

Proof. Let $g: |L| \rightarrow |T|$ be the pasting map; let $A = g(|\text{Bd } L|)$. Then A is homeomorphic to a space that is the union of two circles with a point in common. (Such a space is called a **wedge of two circles**.) Orient the 1-simplices of T arbitrarily. Because g makes identifications only among simplices of $\text{Bd } L$, the arguments we gave earlier in proving Lemma 6.1 apply verbatim to prove the following:

- (1) Every 1-cycle of T is homologous to a 1-cycle carried by A .
- (2) If d is a 2-chain of T and if ∂d is carried by A , then d is a multiple of γ .

However, in the complex T , two further results hold:

- (3) If c is a 1-cycle of T carried by A , then c is of the form $nw_1 + mz_1$.
- (4) $\partial\gamma = 0$.

The proof of (3) is easy, given the fact that A is just the 1-dimensional complex pictured in Figure 6.5. The proof of (4) is similarly direct: It is clear that $\partial\gamma$ has value 0 on every 1-simplex of T not in A . One checks directly that it also has value 0 on each 1-simplex in A . The elementary chain $[a, b]$, for instance, appears in the expression for $\partial\sigma_1$ with value -1 and in the expression for $\partial\sigma_2$ with value $+1$, so that $\partial\gamma$ has value 0 on $[a, b]$. (See Figure 6.4.)

Using results (1)–(4), we can compute the homology of T . Every 1-cycle of T is homologous to a 1-cycle of the form $c = nw_1 + mz_1$, by (1) and (3). Such a cycle bounds only if it is trivial: For if $c = \partial d$ for some d , then (2) applies to show that $d = p\gamma$ for some p ; since $\partial\gamma = 0$ by (4), we have $c = \partial d = 0$. We conclude that

$$H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z};$$

and the (cosets of the) 1-cycles w_1 and z_1 form a basis for the 1-dimensional homology.

To compute $H_2(T)$, note that by (2) any 2-cycle d of T must be of the form $p\gamma$ for some p . Each such 2-chain is in fact a cycle, by (4), and there are no 3-chains for it to bound. We conclude that

$$H_2(T) \simeq \mathbb{Z},$$

and this group has as generator the 2-cycle γ . $\square$

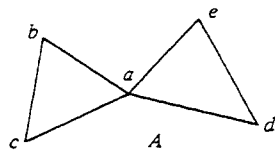


Figure 6.5

Theorem 6.3. Let S denote the complex represented by the labelled rectangle of Figure 6.6; its underlying space is the Klein bottle. Then

$$H_1(S) \simeq \mathbb{Z} \oplus \mathbb{Z}/2 \quad \text{and} \quad H_2(S) = 0.$$

The torsion element of $H_1(S)$ is represented by the chain z_1 , and a generator for the group $H_1(S)$ modulo torsion is represented by w_1 , where

$$w_1 = [a, b] + [b, c] + [c, a],$$

$$z_1 = [a, d] + [d, e] + [e, a].$$

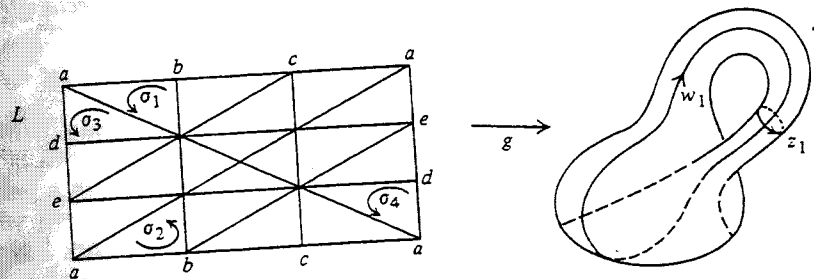


Figure 6.6

Proof. Let $g: |L| \rightarrow |S|$ be the pasting map. Let $A = g(|\text{Bd } L|)$; as before, it is the wedge of two circles. Orient the 2-simplices of S as before; let γ be their sum. Orient the 1-simplices of S arbitrarily. Note that (1) and (2) of the preceding proof hold; neither involve the particular identifications on the boundary. Because A is the wedge of two circles, (3) holds as well. The final condition is different, however; one has $\partial\gamma = 2z_1$.

This equation follows by direct computation. For example, $[a, b]$ appears in $\partial\sigma_1$ with coefficient -1 and in $\partial\sigma_2$ with coefficient $+1$, while $[a, d]$ appears in both $\partial\sigma_3$ and $\partial\sigma_4$ with coefficient $+1$.

Putting these facts together, we compute the homology of S : Every 1-cycle of S is homologous to a cycle of the form $c = nw_1 + mz_1$, by (1) and (3). If $c = \partial d$ for some d , then $d = p\gamma$ by (2); whence $\partial d = 2pz_1$. Thus $nw_1 + mz_1$ bounds if and only if m is even and n is zero. We conclude that

$$H_1(S) \simeq \mathbb{Z} \oplus \mathbb{Z}/2.$$

The cycle z_1 represents the torsion element, and w_1 represents a generator of the infinite cyclic group $H_1(S)/T_1(S)$.

To compute $H_2(S)$, note that any 2-cycle d of S must be of the form $p\gamma$ by (2); since $p\gamma$ is not a cycle, by (4), we have

$$H_2(S) = 0. \quad \square$$

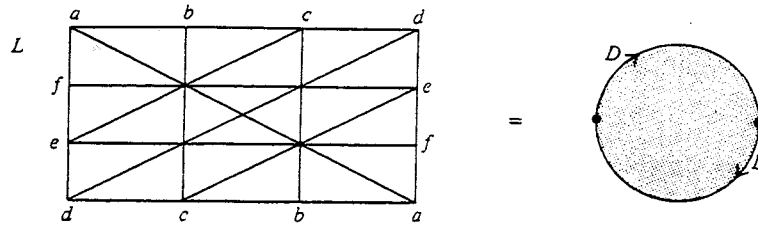


Figure 6.7

Theorem 6.4. Let P^2 be the complex indicated by the labelled rectangle of Figure 6.7; its underlying space is called the **projective plane**. Then

$$H_1(P^2) \cong \mathbb{Z}/2 \quad \text{and} \quad H_2(P^2) = 0.$$

Proof. Let $g: |L| \rightarrow |P^2|$ be the pasting map. Let $A = g(|Bd L|)$; it is homeomorphic to a circle. Let γ be as before; let

$$z_1 = [a, b] + [b, c] + [c, d] + [d, e] + [e, f] + [f, a].$$

Conditions (1) and (2) hold as before. The additional results, which are easy to verify, are the following:

(3) Every 1-cycle carried by A is a multiple of z_1 .

(4) $\partial\gamma = -2z_1$.

From these facts, we conclude that

$$H_1(P^2) \cong \mathbb{Z}/2 \quad \text{and} \quad H_2(P^2) = 0.$$

The non-zero element of H_1 is represented by the cycle z_1 . $\square$

Definition. The **connected sum** of P^2 with itself is defined as the space obtained from two copies of the projective plane by deleting a small open disc from each, and pasting together the pieces that remain, along their free edges. We denote this space by $P^2 \# P^2$.

The space $P^2 \# P^2$ can be represented as a quotient space of a rectangle, obtained by pasting its edges together in the manner indicated in Figure 6.8. (Note that if you cut the rectangle along the dotted line C , you have two projective planes with an open disc removed from each, as indicated in the figure.)

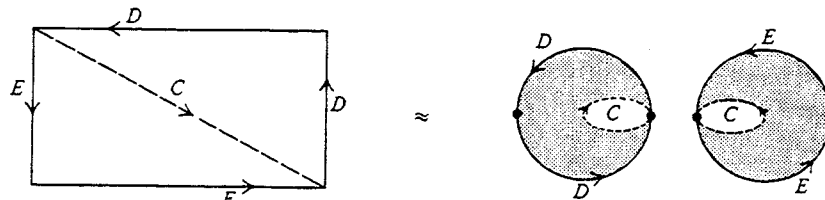


Figure 6.8

Theorem 6.5. Let $P^2 \# P^2$ be the connected sum of two projective planes. Then

$$H_1(P^2 \# P^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \quad \text{and} \quad H_2(P^2 \# P^2) = 0.$$

Proof. We represent $P^2 \# P^2$ by the same rectangle L as before, with an appropriate vertex-labelling. In this case, the complex $A = g(|Bd L|)$ is again the wedge of two circles. Let w_1 be the 1-cycle "running across the top edge" and let z_1 be the 1-cycle "running down the left edge" in Figure 6.8, in the directions indicated. Conditions (1) and (2) of Theorem 6.2 hold. Conditions (3) and (4) are the following:

(3) Every 1-cycle carried by A is of the form $nw_1 + mz_1$.

(4) $\partial\gamma = 2w_1 + 2z_1$.

It is then clear that $H_2(P^2 \# P^2) = 0$. But how can one compute H_1 ? Some diddling is needed. We want to compute the quotient of the group G having w_1 and z_1 as basis, by the subgroup H generated by $2(w_1 + z_1)$. For this purpose, we need to choose bases for the two groups that are "compatible," as in our basic theorem on subgroups of free abelian groups (Theorem 4.2). In this case, it is easy to see what is needed: we need to choose $w_1 + z_1$ to be one of the basis elements for G . Can we do this? Of course; $\{w_1, w_1 + z_1\}$ will serve as a basis for G just as well as $\{w_1, z_1\}$ does. (One can express each set in terms of the other: $w_1 = w_1$ and $z_1 = -(w_1) + (w_1 + z_1)$.) If we use this basis for G , computation is easy:

$$H_1(P^2 \# P^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2,$$

the torsion element is represented by $w_1 + z_1$, and w_1 represents a generator of H_1/T_1 .

(We remark that w_1 is not the only cycle representing a generator of H_1/T_1 . The cycle z_1 does just as well; so does the cycle $2w_1 + 3z_1$, as well as many others. For $\{z_1, w_1 + z_1\}$ and $\{2w_1 + 3z_1, w_1 + z_1\}$ are other bases for G , as you can check.) $\square$

The astute reader might notice that the answers here are the same as for the Klein bottle. This is no accident, for the two spaces are in fact homeomorphic. Figure 6.9 indicates the proof.

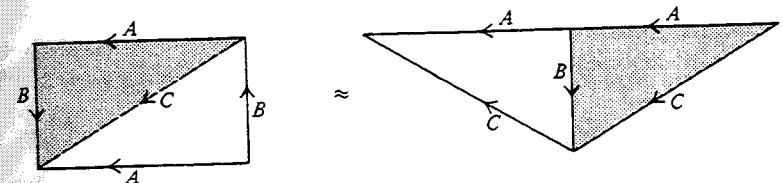


Figure 6.9

By now we have worked enough examples that you should be able to compute the homology groups of the general compact surface. We leave this computation to the exercises.

EXERCISES

1. Let w_1 and z_1 be the cycles on the Klein bottle pictured in Figure 6.6. Show that $w_1 + z_1$ represents a generator of the infinite cyclic group $H_1(S)/T_1(S)$.
2. The connected sum $T \# T$ of two tori is obtained by deleting an open disc from each of two disjoint tori and gluing together the pieces that remain, along their boundaries. It can be represented as a quotient space of an octagonal region in the plane by making identifications on the boundary as indicated in Figure 6.10. (Splitting this octagon along the dotted line gives two tori with open discs deleted.)
 - (a) Construct a complex K whose underlying space is $T \# T$ by an appropriate vertex-labelling of the complex L pictured in Figure 6.11.
 - (b) Compute the homology of $T \# T$ in dimensions 1 and 2 by following the pattern of Theorem 6.2. Specifically, let A be the image of $\text{Bd } L$ under the quotient map; then A is a wedge of four circles. Orient each 2-simplex of L counterclockwise; let γ be the sum of the correspondingly oriented simplices of K . Show first that every 1-cycle of K is homologous to one carried by A . Then show that every 2-chain of K whose boundary is carried by A is a multiple of γ . Complete the computation by analyzing the 1-cycles carried by A , and by computing $\partial\gamma$.

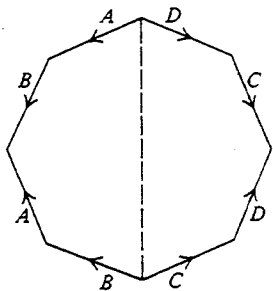


Figure 6.10

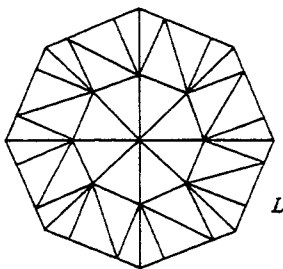


Figure 6.11

3. Represent the 4-fold connected sum $P^2 \# P^2 \# P^2 \# P^2$ by an appropriate labelling of the complex L of Figure 6.11. Compute its homology in dimensions 1 and 2.
4. (a) Define the n -fold connected sum $X_n = T \# \dots \# T$ of tori, and compute its homology in dimensions 1 and 2.
 (b) Define the n -fold connected sum $Y_n = P^2 \# \dots \# P^2$ of projective planes, and compute its homology in dimensions 1 and 2.

[It is a standard theorem that every compact surface is homeomorphic to one of the spaces in the following list:

$$S^2; X_1, X_2, \dots; Y_1, Y_2, \dots$$

(See [Ma].) Once we have proved that the homology groups are topological invariants, it then follows from the computations of this exercise that no two of these surfaces are homeomorphic.]

5. Compute the homology of $T \# P^2$. To which of the surfaces listed in Exercise 4 must it be homeomorphic? Can you construct the homeomorphism?
6. (a) Compute the homology in dimensions 1 and 2 of the quotient space indicated in Figure 6.12. We call this space the "5-fold dunce cap."
 (b) Define analogously the " k -fold dunce cap" and compute its homology.
7. Compute the homology of the space indicated in Figure 6.13.
8. Given finitely generated abelian groups G_1 and G_2 , with G_2 free, show there is a finite 2-dimensional complex K such that $|K|$ is connected, $H_1(K) \cong G_1$, and $H_2(K) \cong G_2$.

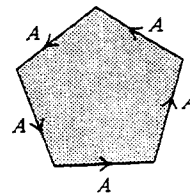


Figure 6.12

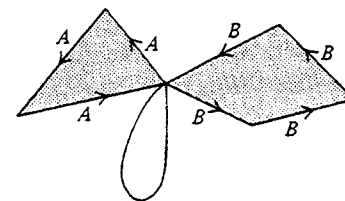


Figure 6.13

§7. ZERO-DIMENSIONAL HOMOLOGY

We have not yet computed any zero-dimensional homology group. In this section, we shall show that this group has a simple topological interpretation that makes its computation trivial.

We prove the following theorem:

Theorem 7.1. *Let K be a complex. Then the group $H_0(K)$ is free abelian. If $\{v_a\}$ is a collection consisting of one vertex from each component of $|K|$, then the homology classes of the chains v_a form a basis for $H_0(K)$.*

Proof. Step 1. If v and w are vertices of K , let us define $v \sim w$ if there is a sequence

$$a_0, \dots, a_n$$

of vertices of K such that $v = a_0$ and $w = a_n$, and $a_i a_{i+1}$ is a 1-simplex of K for

each i . This relation is clearly an equivalence relation. Given v , define

$$C_v = \bigcup \{St\ w \mid w \sim v\}.$$

We show that the sets C_v are the components of $|K|$.

Note first that C_v is open because it is a union of open sets. Furthermore, $C_v = C_{v'}$ if $v \sim v'$.

Second, we show C_v is connected, in fact, path connected. Given v , let $w \sim v$ and let x be a point of $St\ w$. Choose a sequence $a_0, \dots, a_n$ of vertices of K , as before. Then the broken line path with successive vertices $a_0, \dots, a_n, x$ lies in C_v . For $a_i \sim v$ by definition, so that $St\ a_i \subset C_v$, and in particular, the line segment $a_i a_{i+1}$ lies in C_v . Similarly, the line segment $a_n x$ lies in $St\ a_n$, which is contained in C_v . Hence C_v is path connected.

Third, we show that distinct sets C_v and $C_{v'}$ are disjoint. Suppose x is a point of $C_v \cap C_{v'}$. Then $x \in St\ w$ for some w equivalent to v , and $x \in St\ w'$ for some w' equivalent to v' . Since x has positive barycentric coordinates with respect to both w and w' , some simplex of K has w and w' as vertices. Then ww' must be a 1-simplex of K , so $w \sim w'$. It follows that $v \sim v'$, so that the two sets C_v and $C_{v'}$ are the same.

Being connected, open, and disjoint, the sets C_v are necessarily the components of $|K|$. Note that each is the space of a subcomplex of K ; each simplex of K (being connected) lies entirely in one component of $|K|$.

Step 2. Now we prove the theorem. Let $\{v_\alpha\}$ be a collection of vertices containing one vertex v_α from each component C_α of $|K|$. Given a vertex w of K , it belongs to some component of $|K|$, say C_α . By hypothesis, $w \sim v_\alpha$, so there is a sequence $a_0, \dots, a_n$ of vertices of K , as before, leading from v_α to w . The 1-chain

$$[a_0, a_1] + [a_1, a_2] + \dots + [a_{n-1}, a_n]$$

has as its boundary the 0-chain $a_n - a_0 = w - v_\alpha$. Thus the 0-chain w is homologous to the 0-chain v_α . We conclude that every chain in K is homologous to a linear combination of the elementary 0-chains v_α .

We now show that no non-trivial chain of the form $c = \sum n_\alpha v_\alpha$ bounds. Suppose $c = \partial d$ for some 1-chain d . Since each 1-simplex of K lies in a unique component of $|K|$, we can write $d = \sum d_\alpha$, where d_α consists of those terms of d that are carried by C_α . Since $\partial d = \sum \partial d_\alpha$ and ∂d_α is carried by C_α , we conclude that $\partial d_\alpha = n_\alpha v_\alpha$. It follows that $n_\alpha = 0$ for each α . For let $\epsilon: C_0(K) \rightarrow \mathbb{Z}$ be the homomorphism defined by setting $\epsilon(v) = 1$ for each vertex v of K . Then $\epsilon(\partial[v, w]) = \epsilon(w - v) = 1 - 1 = 0$ for any elementary 1-chain $[v, w]$. As a result, $\epsilon(\partial d) = 0$ for every 1-chain d . In particular, $0 = \epsilon(\partial d_\alpha) = \epsilon(n_\alpha v_\alpha) = n_\alpha$. $\square$

For some purposes, it is convenient to consider another version of 0-dimensional homology. We consider that situation now.

Definition. Let $\epsilon: C_0(K) \rightarrow \mathbb{Z}$ be the surjective homomorphism defined by $\epsilon(v) = 1$ for each vertex v of K . Then if c is a 0-chain, $\epsilon(c)$ equals the sum of the values of c on the vertices of K . The map ϵ is called an **augmentation map** for $C_0(K)$. We have just noted that $\epsilon(\partial d) = 0$ if d is a 1-chain. We define the **reduced homology group** of K in dimension 0, denoted $\tilde{H}_0(K)$, by the equation

$$\tilde{H}_0(K) = \ker \epsilon / \text{im } \partial_1$$

(If $p > 0$, we let $\tilde{H}_p(K)$ denote the usual group $H_p(K)$.)

The relation between reduced and ordinary homology is as follows:

Theorem 7.2. The group $\tilde{H}_0(K)$ is free abelian, and

$$\tilde{H}_0(K) \oplus \mathbb{Z} \cong H_0(K).$$

Thus $\tilde{H}_0(K)$ vanishes if $|K|$ is connected. If $|K|$ is not connected, let $\{v_\alpha\}$ consist of one vertex from each component of $|K|$; let α_0 be a fixed index. Then the homology classes of the chains $v_\alpha - v_{\alpha_0}$, for $\alpha \neq \alpha_0$, form a basis for $\tilde{H}_0(K)$.

Proof. Given a 0-chain c , it is homologous to a 0-chain of the form $c' = \sum n_\alpha v_\alpha$; and the chain c' bounds only if $n_\alpha = 0$ for all α . Now if $c \in \ker \epsilon$, then $\epsilon(c) = \epsilon(c') = \epsilon(\sum n_\alpha v_\alpha) = \sum n_\alpha = 0$. If $|K|$ has only one component, this means that $c' = 0$. If $|K|$ has more than one component, it implies that c' is a linear combination of the 0-chains $v_\alpha - v_{\alpha_0}$. $\square$

EXERCISE

1. (a) Let G be an abelian group and let $\phi: G \rightarrow \mathbb{Z}$ be an epimorphism. Show that G has an infinite cyclic subgroup H such that

$$G = (\ker \phi) \oplus H.$$

[Hint: Define a homomorphism $\psi: \mathbb{Z} \rightarrow G$ such that $\phi \circ \psi$ is the identity; let $H = \text{im } \psi$.]

- (b) Show that if $\phi: C_0(K) \rightarrow \mathbb{Z}$ is any epimorphism such that $\phi \circ \partial_1 = 0$, then

$$H_0(K) \cong (\ker \phi) / (\text{im } \partial_1) \oplus \mathbb{Z}.$$

§8. THE HOMOLOGY OF A CONE

Now we compute the homology of the n -simplex and of its boundary. A convenient way of doing this is to introduce the notion of a *cone*.

Definition. Suppose that K is a complex in E^j , and w is a point of E^j such that each ray emanating from w intersects $|K|$ in at most one point. We define

the cone on K with vertex w to be the collection of all simplices of the form $wa_0 \dots a_p$, where $a_0 \dots a_p$ is a simplex of K , along with all faces of such simplices. We denote this collection $w * K$.

We show that $w * K$ is a well-defined complex, and it contains K as a subcomplex; K is often called the **base** of the cone.

First we show that the set $\{w, a_0, \dots, a_p\}$ is geometrically independent: If w were in the plane P determined by $a_0, \dots, a_p$, we could consider the line segment joining w to an interior point x of $\sigma = a_0 \dots a_p$. The set $\text{Int } \sigma$, being open in P , would contain an interval of points on this line segment. But the ray from w through x intersects $|K|$ in only one point, by hypothesis.

We now show that $w * K$ is a complex. The simplices of $w * K$ are of three types: simplices $a_0 \dots a_p$ of K , simplices of the form $wa_0 \dots a_p$, and the 0-simplex w . A pair of simplices of the first type have disjoint interiors because K is a complex. The open simplex $\text{Int}(wa_0 \dots a_p)$ is the union of all open line segments joining w to points of $\text{Int}(a_0 \dots a_p)$; no two such open simplices can intersect because no ray from w contains more than one point of $|K|$. For the same reason, simplices of the first and second types have disjoint interiors.

Example 1. If K_s is the complex consisting of the simplex $\sigma = v_0 \dots v_n$ and its faces, then $K_s = v_0 * K$, where s is the face of σ opposite v_0 . Thus every simplex of positive dimension is a cone.

Example 2. If K is the complex in $\mathbb{R}^2$ consisting of the intervals $[n, n+1] \times 0$ on the x -axis and their vertices, and if w is a point on the y -axis different from the origin, then $w * K$ is the complex illustrated in Figure 8.1. Although $|K|$ is a subspace of $\mathbb{R}^2$, $|w * K|$ is not a subspace of $\mathbb{R}^2$. (See Exercise 9 of §2.)

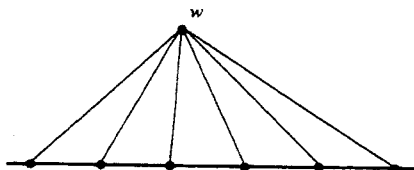


Figure 8.1

One particularly useful consequence of the cone construction is the following:

Lemma 8.1. Let U be a bounded convex open set in $\mathbb{R}^n$; let $w \in U$. If K is a finite complex such that $|K| = \bar{U} - U$, then $w * K$ is a finite complex such that $|w * K| = \bar{U}$.

Proof. It follows at once from Lemma 1.1 that each ray emanating from w intersects $|K|$ in precisely one point, and that $\bar{U}$ is the union of all line segments joining w to points of $|K|$. $\square$

Given K , we note that any two cones $w * K$ and $z * K$ over K are isomorphic. The vertex map that carries each vertex of K to itself and carries w to z induces an isomorphism of $w * K$ with $z * K$.

Note also that for a complex K in $\mathbb{R}^N$, there may be no point w in $\mathbb{R}^N$ such that the cone complex $w * K$ can be formed. However, we can always consider K as a complex in $\mathbb{R}^N \times 0 \subset \mathbb{R}^{N+1}$; then the point $w = (0, \dots, 0, 1) \in \mathbb{R}^{N+1}$ will do. A similar remark applies to a complex in E^J .

Now we compute the homology of a cone, and show that it vanishes in positive dimensions. For this purpose, we shall introduce a certain bracket operation that will also be useful later.

Definition. Let $w * K$ be a cone. If $\sigma = [a_0, \dots, a_p]$ is an oriented simplex of K , let $[w, \sigma]$ denote the oriented simplex $[w, a_0, \dots, a_p]$ of $w * K$. This operation is well-defined; exchanging two vertices in the array $[a_0, \dots, a_p]$ results in exchanging two vertices in the array $[w, a_0, \dots, a_p]$. More generally, if

$$c_p = \sum n_i \sigma_i$$

is a p -chain of K , we define

$$[w, c_p] = \sum n_i [w, \sigma_i].$$

This bracket operation is a homomorphism carrying $C_p(K)$ into $C_{p+1}(w * K)$.

We compute readily from the boundary formula:

$$\partial[w, \sigma] = \begin{cases} \sigma - w & \text{if } \dim \sigma = 0, \\ \sigma - [w, \partial\sigma] & \text{if } \dim \sigma > 0. \end{cases}$$

This leads to the following more general formulas:

$$(*) \quad \begin{aligned} \partial[w, c_0] &= c_0 - \epsilon(c_0)w, \\ \partial[w, c_p] &= c_p - [w, \partial c_p] \quad \text{if } p > 0. \end{aligned}$$

Theorem 8.2. If $w * K$ is a cone, then for all p ,

$$\tilde{H}_p(w * K) = 0.$$

In general, a complex whose reduced homology vanishes in all dimensions is said to be **acyclic**.

Proof. The reduced homology of $w * K$ vanishes in dimension 0, because $|w * K|$ is connected. Consider the case $p > 0$. Let z_p be a p -cycle of $w * K$; we show that z_p bounds. Let us write

$$z_p = c_p + [w, d_{p-1}],$$

where c_p consists of those terms of z_p that are carried by K , and d_{p-1} is a chain of K . We show that

$$z_p - \partial[w, c_p] = 0;$$

then our result is proved. By direct computation,

$$\begin{aligned} z_p - \partial[w, c_p] &= c_p + [w, d_{p-1}] - c_p + [w, \partial c_p] \\ &= [w, e_{p-1}], \end{aligned}$$

where $e_{p-1} = d_{p-1} + \partial c_p$ is a chain of K . Now since z_p is a cycle,

$$0 = \begin{cases} e_{p-1} - \epsilon(e_{p-1})w & \text{if } p = 1, \\ e_{p-1} - [w, \partial e_{p-1}] & \text{if } p > 1. \end{cases}$$

Now the portion of this chain carried by K is e_{p-1} ; therefore, $e_{p-1} = 0$. We conclude that

$$z_p - \partial[w, c_p] = [w, e_{p-1}] = 0,$$

as desired. $\square$

Theorem 8.3. Let σ be an n -simplex. The complex K_σ consisting of σ and its faces is acyclic. If $n > 0$, let Σ^{n-1} denote the complex whose polytope is $\text{Bd } \sigma$. Orient σ . Then $\tilde{H}_{n-1}(\Sigma^{n-1})$ is infinite cyclic and is generated by the chain $\partial\sigma$; furthermore, $\tilde{H}_i(\Sigma^{n-1}) = 0$ for $i \neq n-1$.

Proof. Because K_σ is a cone, it is acyclic. Let us compare the chain groups of K_σ and Σ^{n-1} ; they are equal except in dimension n :

$$\begin{array}{ccccccc} C_n(K_\sigma) & \xrightarrow{\partial_n} & C_{n-1}(K_\sigma) & \xrightarrow{\partial_{n-1}} & \dots & \rightarrow & C_0(K_\sigma) \xrightarrow{\epsilon} \mathbb{Z} \\ & & \parallel & & & & \parallel \\ 0 & \xrightarrow{\partial'_n} & C_{n-1}(\Sigma^{n-1}) & \xrightarrow{\partial'_{n-1}} & \dots & \rightarrow & C_0(\Sigma^{n-1}) \rightarrow \mathbb{Z} \end{array}$$

It follows at once that $\tilde{H}_i(\Sigma^{n-1}) = \tilde{H}_i(K_\sigma) = 0$ for $i \neq n-1$. Let us compute the homology in dimension $n-1$. First take the case $n > 1$. One has

$$\begin{aligned} H_{n-1}(\Sigma^{n-1}) &= Z_{n-1}(\Sigma^{n-1}), \text{ because there are no } n-1 \text{ boundaries,} \\ &= \ker \partial_{n-1} \\ &= \text{im } \partial_n, \text{ because } H_n(K_\sigma) = 0. \end{aligned}$$

Now $C_n(K_\sigma)$ is infinite cyclic and is generated by σ . Hence $\text{im } \partial_n$ is cyclic and is generated by $\partial_n \sigma$; it is infinite because $C_{n-1}(K_\sigma)$ has no torsion.

The argument for $n = 1$ is similar, except that ∂_{n-1} is replaced by ϵ throughout. $\square$

EXERCISE

- Let K be a complex; let $w_0 * K$ and $w_1 * K$ be two cones on K whose polytopes intersect only in $|K|$.
 - Show that $(w_0 * K) \cup (w_1 * K)$ is a complex; it is called a **suspension** of K and denoted $S(K)$.

- Using the bracket notation, define $\phi: C_p(K) \rightarrow C_{p+1}(S(K))$ by the equation

$$\phi(c_p) = [w_0, c_p] - [w_1, c_p].$$

Show that ϕ induces a homomorphism

$$\phi_*: \tilde{H}_p(K) \rightarrow \tilde{H}_{p+1}(S(K)).$$

- Show ϕ_* is an isomorphism when K consists of the proper faces of a 2-simplex.

We will see later that ϕ_* is an isomorphism in general (see §25).

§9. RELATIVE HOMOLOGY

Suppose K_0 is a subcomplex of the complex K . In many of the applications of topology, it is convenient to consider what are called the *relative* homology groups of K modulo K_0 . We introduce them briefly here and compute some examples, postponing a more complete discussion to Chapter 3.

If K_0 is a subcomplex of the complex K , then the chain group $C_p(K_0)$ can be considered to be a subgroup of the chain group $C_p(K)$ in the natural way. Formally, if c_p is a chain on K_0 (that is, a function on the oriented simplices of K_0), one extends it to a chain on K by letting its value be zero on each oriented p -simplex of K not in K_0 . When we write c_p as a linear combination of oriented p -simplices of K_0 , we need merely to "consider" these simplices as belonging to K in order to "consider" c_p as a chain of K .

Definition. If K_0 is a subcomplex of K , the quotient group $C_p(K)/C_p(K_0)$ is called the group of **relative chains of K modulo K_0** , and is denoted by $C_p(K, K_0)$.

Note that the group $C_p(K, K_0)$ is free abelian. Indeed, if we orient the p -simplices of K so as to obtain a basis for $C_p(K)$, the subcollection consisting of the oriented p -simplices of K_0 is a basis for $C_p(K_0)$. Then the quotient $C_p(K)/C_p(K_0)$ is free, for it has as basis all cosets of the form

$$\{\sigma_i\} = \sigma_i + C_p(K_0),$$

where σ_i is a p -simplex of K that is *not* in K_0 .

The boundary operator $\partial: C_p(K_0) \rightarrow C_{p-1}(K_0)$ is just the restriction of the boundary operator on $C_p(K)$. We use the same symbol to denote both these homomorphisms, when no confusion will result. This homomorphism induces a homomorphism

$$C_p(K, K_0) \rightarrow C_{p-1}(K, K_0)$$

of the relative chain groups, which we also denote by ∂ . As before, it satisfies the equation $\partial \circ \partial = 0$. We let

$$Z_p(K, K_0) = \ker \partial : C_p(K, K_0) \rightarrow C_{p-1}(K, K_0),$$

$$B_p(K, K_0) = \text{im } \partial : C_{p+1}(K, K_0) \rightarrow C_p(K, K_0),$$

$$H_p(K, K_0) = Z_p(K, K_0) / B_p(K, K_0).$$

These groups are called, respectively, the group of relative p -cycles, the group of relative p -boundaries, and the relative homology group in dimension p , of K modulo K_0 .

Note that a relative p -chain, which is a coset $c_p + C_p(K_0)$, is a relative cycle if and only if ∂c_p is carried by K_0 . Furthermore, it is a relative boundary if and only if there is a $p+1$ chain d_{p+1} of K such that $c_p - \partial d_{p+1}$ is carried by K_0 .

Example 1. Let K consist of an n -simplex and its faces; let K_0 be the set of proper faces of K . Then the group $C_p(K, K_0)$ vanishes except when $p = n$, in which case it is infinite cyclic. It follows that

$$H_i(K, K_0) = 0 \quad \text{for } i \neq n,$$

$$H_n(K, K_0) \cong \mathbb{Z}.$$

Example 2. Let K be a complex and let K_0 consist of a single vertex v of K . Using the results of §7, one sees readily that $H_0(K, v)$ is free abelian; one obtains a basis for $H_0(K, v)$ by choosing one vertex from each component of $|K|$ other than the component containing v . Then $H_0(K, v) \cong \tilde{H}_0(K)$.

It is not hard to show that $H_p(K, v) \cong H_p(K)$ for $p > 0$; see the exercises.

Example 3. Let K be the complex indicated in Figure 9.1, whose underlying space is a square. Let K_0 be the subcomplex whose space is the boundary of the square. It is easy to see that the 2-chain $\sum m_i \sigma_i$ represents a relative cycle of K modulo K_0 if and only if $m_1 = m_2 = m_3 = m_4$. Since there are no boundaries in dimension 2,

$$H_2(K, K_0) \cong \mathbb{Z}$$

and the chain $\gamma = \sum \sigma_i$ represents a generator.

We showed in Example 6 of §5 that any 1-chain c of K is homologous to a 1-chain c_1 carried by $K_0 \cup e_4$. Now if c represents a relative 1-cycle (so that ∂c

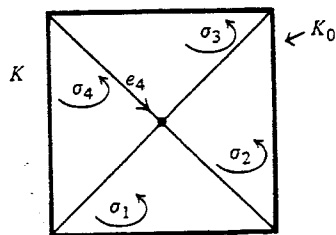


Figure 9.1

carried by K_0), $\partial c_1 = \partial c$ is also carried by K_0 . We conclude that ∂c_1 must be zero, whence c_1 is actually carried by K_0 .

$$H_1(K, K_0) = 0.$$

Comparing this computation with that of Example 1 lends some plausibility to the statement (yet to be proved) that the relative homology groups are topological invariants.

Example 4. Let K be the complex indicated in Figure 9.2. Its underlying space is called an **annulus**. Let K_0 denote the 1-dimensional complex whose space is the union of the inner and outer edges of K . We compute the homology of K modulo K_0 .

First, $H_0(K, K_0) = 0$ because the relative chain group itself vanishes in dimension 0. To compute H_1 and H_2 , one first verifies three facts:

- If c is a 1-chain of K , then c is homologous to a 1-chain of K that is carried by the subcomplex M pictured in Figure 9.3.
- Orient each 2-simplex of K counterclockwise. If d is any 2-chain of K such that ∂d is carried by M , then $d = m\gamma$, where γ is the sum of all the oriented 2-simplices of K .
- $\partial\gamma$ is carried by K_0 .

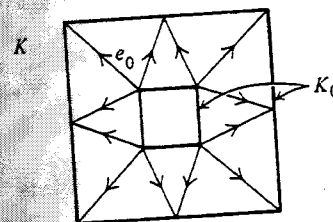


Figure 9.2

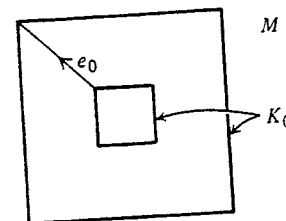


Figure 9.3

The computation then proceeds as follows: Let e_0 be the oriented 1-simplex pictured in Figure 9.2; then e_0 represents a relative 1-cycle of K modulo K_0 , because ∂e_0 lies in K_0 . It follows from (i) that any relative 1-cycle $\{c\}$ is relatively homologous to a multiple of $\{e_0\}$. Furthermore, no such relative cycle bounds. For suppose $ne_0 = \partial d$ is carried by K_0 , for some 2-chain d of K . Then ∂d is carried by M , whence by (ii), $d = m\gamma$ for some integer m . But $\partial d = m\partial\gamma$ is carried by K_0 , by (iii), so that $n = 0$. We conclude that

$$H_1(K, K_0) \cong \mathbb{Z},$$

and the relative cycle e_0 represents a generator.

A similar argument, using (ii) and (iii), shows that

$$H_2(K, K_0) \cong \mathbb{Z},$$

and the relative cycle γ represents a generator.

Students often visualize the relative homology group $H_p(K, K_0)$ as representing the homology of the quotient space $X = |K|/|K_0|$ obtained by collapsing

$|K_0|$ to a point p , modulo that point. Assuming X is homeomorphic to a polyhedron (so its simplicial homology is defined) this is in fact correct, but the proof is not easy. (See Lemma 70.1 and the exercises of §39.)

Roughly speaking, the relative homology group $H_p(K, K_0)$ depends only on the part of K lying outside or on the boundary of K_0 ; it "ignores" the part of K lying inside K_0 . We express this fact formally in the following theorem:

Theorem 9.1 (Excision theorem). *Let K be a complex; let K_0 be a subcomplex. Let U be an open set contained in $|K_0|$, such that $|K| - U$ is the polytope of a subcomplex L of K . Let L_0 be the subcomplex of K whose polytope is $|K_0| - U$. Then inclusion induces an isomorphism*

$$H_p(L, L_0) \cong H_p(K, K_0).$$

We think of $(|L|, |L_0|)$ as having been formed by "excising away" the open set U from $|K|$ and $|K_0|$. See Figure 9.4.

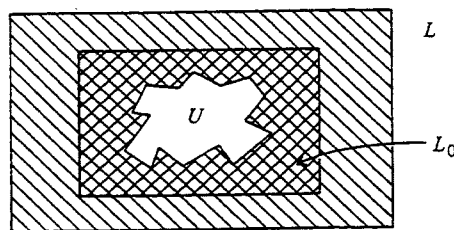


Figure 9.4

Proof. Consider the composite map ϕ ,

$$C_p(L) \rightarrow C_p(K) \rightarrow C_p(K)/C_p(K_0),$$

which is inclusion followed by projection. Then ϕ is surjective, because $C_p(K)/C_p(K_0)$ has as basis all cosets $\{\sigma_i\}$ for σ_i not in K_0 , and L contains all such simplices σ_i . The kernel of ϕ is precisely $C_p(L_0)$. Thus ϕ induces an isomorphism

$$C_p(L)/C_p(L_0) \cong C_p(K)/C_p(K_0),$$

for all p . Since the boundary operator is preserved under this isomorphism, it follows that $H_p(L, L_0) \cong H_p(K, K_0)$. $\square$

This elementary fact has some useful consequences we shall consider later.

EXERCISES

1. Let K be the complex pictured in Figure 9.2; let K_1 be its "outer edge." Compute $H_i(K)$ and $H_i(K, K_1)$.

2. Let K be a complex whose underlying space is the Möbius band; let K_0 be its "edge." Compute $H_i(K)$ and $H_i(K, K_0)$. [Hint: See Example 2 of §3. Here the "edge" consists of the line segments ab, bc, cd, de, ef , and fa .]
3. Show that if K is a complex and v is a vertex of K , then $H_i(K, v) \cong \tilde{H}_i(K)$ for all i . [Hint: Care is needed when $i = 1$.]
4. Describe $H_0(K, K_0)$ in general.
5. Let $|K|$ be the torus, represented by a labelled rectangle in the usual way; let K_0 be the subcomplex represented by the top edge of the rectangle. Compute $H_i(K, K_0)$. (See Figure 6.4; $|K_0|$ is the union of the line segments ab, bc , and ca .)
6. Let K be a 2-dimensional complex; let σ be a 2-simplex of K ; let K_0 be the subcomplex whose space is $|K| - \text{Int } \sigma$. Compute $H_i(K, K_0)$.

*§10. HOMOLOGY WITH ARBITRARY COEFFICIENTS

There is one further version of homology that we shall mention here, although we shall not study it in detail until Chapter 6. It arises when one introduces an arbitrary abelian group as "coefficient group."

Let G be an abelian group. Let K be a simplicial complex. A p -chain of K with coefficients in G is a function c_p from the oriented p -simplices of K to G that vanishes on all but finitely many p -simplices, such that

$$c_p(\sigma') = -c_p(\sigma)$$

if σ' and σ are opposite orientations of the same simplex. Two chains are added by adding their values. The resulting group is denoted $C_p(K; G)$.

If σ is an oriented simplex and if $g \in G$, we use $g\sigma$ to denote the elementary chain whose value is g on σ , $-g$ on the opposite orientation of σ , and 0 on all other oriented simplices. In this notation, $g(-\sigma) = (-g)\sigma$, where $-\sigma$ as usual denotes σ with the opposite orientation. If one orients all the p -simplices of K , then each chain c_p can be written uniquely as a finite sum

$$c_p = \sum g_i \sigma_i$$

of elementary chains. Thus $C_p(K; G)$ is the direct sum of subgroups isomorphic to G , one for each p -simplex of K .

The boundary operator $\partial : C_p(K; G) \rightarrow C_{p-1}(K; G)$ is defined easily by the formula

$$\partial(g\sigma) = g(\partial\sigma)$$

where $\partial\sigma$ is the ordinary boundary, defined earlier. As before, $\partial \circ \partial = 0$; and we define $Z_p(K; G)$ to be the kernel of the homomorphism

$$\partial : C_p(K; G) \rightarrow C_{p-1}(K; G),$$

$B_{p-1}(K; G)$ to be its image, and

$$H_p(K; G) = Z_p(K; G)/B_p(K; G).$$

These groups are called, respectively, the cycles, boundaries, and homology of X with coefficients in G .

Of course, one can also study *relative* homology with coefficients in G . The details are clear. The groups in question are denoted by $H_p(K, K_0; G)$.

We are not going to do much concerning homology with general coefficients for some time to come. But you should be aware of its existence at an early stage, for it is often useful.

Example 1. One group that is frequently used as a coefficient group is the group $\mathbb{Z}/2$ of integers mod 2. Let us calculate the homology of the torus and the Klein bottle using these coefficients. The argument given in §6 goes through essentially unchanged to show that

$$\begin{aligned} H_1(T; \mathbb{Z}/2) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2, \\ H_2(T; \mathbb{Z}/2) &\cong \mathbb{Z}/2. \end{aligned}$$

For the Klein bottle S , the argument goes through but with some changes. One has the results

$$\begin{aligned} H_1(S; \mathbb{Z}/2) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2, \\ H_2(S; \mathbb{Z}/2) &\cong \mathbb{Z}/2. \end{aligned}$$

For with $\mathbb{Z}/2$ coefficients, the basic 2-chain γ , which is the sum of the 2-simplices of S , has boundary zero. (In the group $\mathbb{Z}/2$, one has $2 = 1 + 1 = 0$.)

Note that homology with $\mathbb{Z}/2$ coefficients is inadequate to distinguish between T and S .

Example 2. Let us compute the homology of the torus and Klein bottle using the rational numbers $\mathbb{Q}$ as coefficients. The same arguments as before apply, but the end results are different. For the torus, one has

$$H_1(T; \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}, \quad H_2(T; \mathbb{Q}) \cong \mathbb{Q}.$$

For the Klein bottle, one has

$$H_1(S; \mathbb{Q}) \cong \mathbb{Q}, \quad H_2(S; \mathbb{Q}) = 0.$$

For with $\mathbb{Q}$ coefficients, the cycle z_1 bounds the chain $\frac{1}{2}\gamma$.

EXERCISES

1. Compute the homology of P^2 with $\mathbb{Z}/2$ and $\mathbb{Q}$ coefficients.
2. Show that homology with $\mathbb{Q}$ coefficients suffices to distinguish among S^2 and the connected sums $P^2 \# \dots \# P^2$ and $T \# \dots \# T$.
3. Let S be the Klein bottle; compute the homology of S with $\mathbb{Z}/3$ and $\mathbb{Z}/4$ coefficients.
4. Compute the homology of the k -fold dunce cap with $\mathbb{Z}/n$ coefficients and with $\mathbb{Q}$ coefficients. (See Exercise 6 of §6.)
5. Let (K, K_0) be the pair consisting of the Möbius band and its edge. Compute $H_i(K, K_0; \mathbb{Z}/2)$ and $H_i(K, K_0; \mathbb{Q})$. (See Exercise 2 of §9.)

*§11. THE COMPUTABILITY OF HOMOLOGY GROUPS

We have computed the homology groups of some familiar spaces, such as the sphere and the torus and the Klein bottle. Now we ask the question whether one can in fact compute homology groups in general. For finite complexes, the answer is affirmative. In this section, we present an explicit algorithm for carrying out the computation.

First, we prove a basic theorem giving a "normal form" for homomorphisms of finitely generated free abelian groups. The proof is constructive in nature. One corollary is the theorem about subgroups of free abelian groups that we stated earlier as Theorem 4.2. A second corollary is a theorem concerning standard bases for free chain complexes. And a third corollary gives our desired algorithm for computing the homology groups of a finite complex.

First, we need two lemmas with which you might already be familiar.

Lemma 11.1. *Let A be a free abelian group of rank n . If B is a subgroup of A , then B is free abelian of rank $r \leq n$.*

Proof. We may without loss of generality assume that B is a subgroup of the n -fold direct product $\mathbb{Z}^n = \mathbb{Z} \times \dots \times \mathbb{Z}$. We construct a basis for B as follows:

Let $\pi_i: \mathbb{Z}^n \rightarrow \mathbb{Z}$ denote projection on the i th coordinate. For each $m \leq n$, let B_m be the subgroup of B defined by the equation

$$B_m = B \cap (\mathbb{Z}^m \times 0).$$

That is, B_m consists of all $x \in B$ such that $\pi_i(x) = 0$ for $i > m$. In particular, $B_n = B$. Now the homomorphism

$$\pi_m: B_m \rightarrow \mathbb{Z}$$

carries B_m onto a subgroup of $\mathbb{Z}$. If this subgroup is trivial, let $x_m = 0$; otherwise, choose $x_m \in B_m$ so that its image $\pi_m(x_m)$ generates this subgroup. We assert that the non-zero elements of the set $\{x_1, \dots, x_n\}$ form a basis for B .

First, we show that for each m , the elements $x_1, \dots, x_m$ generate B_m . (Then, in particular, the elements $x_1, \dots, x_n$ generate B .) It is trivial that x_1 generates B_1 ; indeed if d is the integer $\pi_1(x_1)$, then

$$x_1 = (d, 0, \dots, 0)$$

and B_1 consists of all multiples of this element.

Assume that $x_1, \dots, x_{m-1}$ generate B_{m-1} ; let $x \in B_m$. Now $\pi_m(x) = k\pi_m(x_m)$ for some integer k . It follows that

$$\pi_m(x - kx_m) = 0,$$

so that $x - kx_m$ belongs to B_{m-1} . Then

$$x - kx_m = k_1x_1 + \dots + k_{m-1}x_{m-1}$$

by the induction hypothesis. Hence $x_1, \dots, x_m$ generate B_m .

Second, we show that for each m , the non-zero elements in the set $\{x_1, \dots, x_m\}$ are independent. The result is trivial when $m = 1$. Suppose it true for $m - 1$. Then we show that if

$$\lambda_1 x_1 + \dots + \lambda_m x_m = 0,$$

then it follows that for each i , $\lambda_i = 0$ whenever $x_i \neq 0$; independence follows.

Applying the map π_m , we derive the equation

$$\lambda_m \pi_m(x_m) = 0.$$

From this equation, it follows that either $\lambda_m = 0$ or $x_m = 0$. For if $\lambda_m \neq 0$, then $\pi_m(x_m) = 0$, whence the subgroup $\pi_m(B_m)$ is trivial and $x_m = 0$ by definition. We conclude two things:

$$\lambda_m = 0 \quad \text{if} \quad x_m \neq 0,$$

$$\lambda_1 x_1 + \dots + \lambda_{m-1} x_{m-1} = 0.$$

The induction hypothesis now applies to show that for $i < m$,

$$\lambda_i = 0 \quad \text{whenever} \quad x_i \neq 0. \quad \square$$

For later use, we generalize this result to arbitrary free abelian groups:

Lemma 11.2. *If A is a free abelian group, any subgroup B of A is free.*

Proof. The proof given for the finite case generalizes, provided we assume that the basis for A is indexed by a well-ordered set J having a largest element. (And the well-ordering theorem, which is equivalent to the axiom of choice, tells us this assumption is justified.)

We begin by assuming A equals a direct sum of copies of $\mathbb{Z}$; that is, A equals the subgroup of the cartesian product $\mathbb{Z}^J$ consisting of all tuples $(n_\alpha)_{\alpha \in J}$ such that $n_\alpha = 0$ for all but finitely many α . Then we proceed as before.

Let B be a subgroup of A . Let B_β consist of those elements x of B such that $\pi_\alpha(x) = 0$ for $\alpha > \beta$. Consider the subgroup $\pi_\beta(B_\beta)$ of $\mathbb{Z}$; if it is trivial define $x_\beta = 0$, otherwise choose $x_\beta \in B_\beta$ so $\pi_\beta(x_\beta)$ generates the subgroup.

We show first that the set $\{x_\alpha \mid \alpha \leq \beta\}$ generates B_β . This fact is trivial if β is the smallest element of J . We prove it in general by transfinite induction. Given $x \in B_\beta$, we have

$$\pi_\beta(x) = k\pi_\beta(x_\beta)$$

for some integer k . Hence $\pi_\beta(x - kx_\beta) = 0$. Consider the set of those indices α for which $\pi_\alpha(x - kx_\beta) \neq 0$. (If there are none, $x = kx_\beta$ and we are through.) All of these indices are less than β , because x and x_β belong to B_β . Furthermore, this set of indices is *finite*, so it has a largest element γ , which is less than β . But this means that $x - kx_\beta$ belongs to B_γ , whence by the induction hypothesis $x - kx_\beta$ can be written as a linear combination of elements x_α with each $\alpha \leq \gamma$.

Second, we show that the non-zero elements in the set $\{x_\alpha \mid \alpha \leq \beta\}$ are inde-

pendent. Again, this fact is trivial if β is the smallest element of J . In general, suppose

$$\lambda_{\alpha_1} x_{\alpha_1} + \dots + \lambda_{\alpha_k} x_{\alpha_k} + \lambda_\beta x_\beta = 0,$$

where $\alpha_i < \beta$. Applying π_β , we see that

$$\lambda_\beta \pi_\beta(x_\beta) = 0.$$

As before, it follows that either $\lambda_\beta = 0$ or $x_\beta = 0$. We conclude that

$$\lambda_\beta = 0 \quad \text{if} \quad x_\beta \neq 0,$$

and

$$\lambda_{\alpha_1} x_{\alpha_1} + \dots + \lambda_{\alpha_k} x_{\alpha_k} = 0.$$

The induction hypothesis now implies that $\lambda_{\alpha_i} = 0$ whenever $x_{\alpha_i} \neq 0$. $\square$

We now prove our basic theorem. First we need a definition.

Definition. Let G and G' be free abelian groups with bases $a_1, \dots, a_n$ and $a'_1, \dots, a'_m$, respectively. If $f: G \rightarrow G'$ is a homomorphism, then

$$f(a_i) = \sum_{j=1}^m \lambda_{ij} a'_j$$

for unique integers λ_{ij} . The matrix (λ_{ij}) is called the **matrix of f** relative to the given bases for G and G' .

Theorem 11.3. *Let G and G' be free abelian groups of ranks n and m , respectively; let $f: G \rightarrow G'$ be a homomorphism. Then there are bases for G and G' such that, relative to these bases, the matrix of f has the form*

$$B = \left[\begin{array}{ccc|ccc} b_1 & & 0 & & & \\ & \ddots & & & & 0 \\ & & & & & \\ 0 & & b_l & & & \\ \hline & & & & & \\ & 0 & & & 0 & \end{array} \right]$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \dots \mid b_l$.

This matrix is in fact uniquely determined by f (although the bases involved are not). It is called a **normal form** for the matrix of f .

Proof. We begin by choosing bases in G and G' arbitrarily. Let A be the matrix of f relative to these bases. We shall give shortly a procedure for modify-

ing these bases so as to bring the matrix into the normal form described. It is called "the reduction algorithm." The theorem follows. $\square$

Consider the following "elementary row operations" on an integer matrix A :

- (1) Exchange row i and row k .
- (2) Multiply row i by -1 .
- (3) Replace row i by $(\text{row } i) + q(\text{row } k)$, where q is an integer and $k \neq i$.

Each of these operations corresponds to a change of basis in G' . The first corresponds to an exchange of a'_i and a'_k . The second corresponds to replacing a'_i by $-a'_i$. And the third corresponds to replacing a'_k by $a'_k - qa'_i$, as you can readily check.

There are three similar "column operations" on A that correspond to changes of basis in G .

We now show how to apply these six operations to an arbitrary matrix A so as to reduce it to our desired normal form. We assume A is not the zero matrix, since in that case the result is trivial.

Before we begin, we note the following fact: If c is an integer that divides each entry of the matrix A , and if B is obtained from A by applying any one of these elementary operations, then c also divides each entry of B .

The reduction algorithm

Given a matrix $A = (a_{ij})$ of integers, not all zero, let $\alpha(A)$ denote the smallest non-zero element of the set of numbers $|a_{ij}|$. We call a_{ij} a **minimal entry** of A if $|a_{ij}| = \alpha(A)$.

The reduction procedure consists of two steps. The first brings the matrix to a form where $\alpha(A)$ is as small as possible. The second reduces the dimensions of the matrix involved.

Step 1. We seek to modify the matrix by elementary operations so as to decrease the value of the function α . We prove the following:

If the number $\alpha(A)$ fails to divide some entry of A , then it is possible to decrease the value of α by applying elementary operations to A ; and conversely.

The converse is easy. If the number $\alpha(A)$ divides each entry of A , then it will divide each entry of any matrix B obtained by applying elementary operations to A . In this situation, it is not possible to reduce the value of α by applying elementary operations.

To prove the result itself, we suppose a_{ij} is a minimal entry of A that fails to divide some entry of A . If the entry a_{ij} fails to divide some entry a_{kj} in its column, then we perform a division, writing

$$\frac{a_{kj}}{a_{ij}} = q + \frac{r}{a_{ij}},$$

where $0 < |r| < |a_{ij}|$. Signs do not matter here; q and r may be either positive or negative. We then replace $(\text{row } k)$ of A by $(\text{row } k) - q(\text{row } i)$. The result is to replace the entry a_{kj} in the k th row and j th column of A by $a_{kj} - qa_{ij} = r$. The value of α for this new matrix is at most $|r|$, which is less than $\alpha(A)$.

A similar argument applies if a_{ij} fails to divide some entry in its row.

Finally, suppose a_{ij} divides each entry in its row and each entry in its column, but fails to divide the entry a_{st} , where $s \neq i$ and $t \neq j$. Consider the following four entries of A :

$$\begin{array}{cccc} a_{ij} & \cdots & a_{it} \\ \vdots & & \vdots \\ a_{sj} & \cdots & a_{st} \end{array}$$

Because a_{ij} divides a_{sj} , we can by elementary operations bring the matrix to the form where the entries in these four places are as follows:

$$\begin{array}{cccc} a_{ij} & \cdots & a_{it} \\ \vdots & & \vdots \\ 0 & \cdots & a_{st} + la_{it} \end{array}$$

If we then replace $(\text{row } i)$ of this matrix by $(\text{row } i) + (\text{row } s)$, we are back in the previous situation, where a_{ij} fails to divide some entry in its row.

Step 2. At the beginning of this step, we have a matrix A whose minimal entry divides every entry of A .

Apply elementary operations to bring a minimal entry of A to the upper left corner of the matrix and to make it positive. Because it divides all entries in its row and column, we can apply elementary operations to make all the other entries in its row and column into zeros. Note that at the end of this process, the entry in the upper left corner divides all entries of the matrix.

One now begins Step 1 again, applying it to the smaller matrix obtained by ignoring the first row and first column of our matrix.

Step 3. The algorithm terminates either when the smaller matrix is the zero matrix or when it disappears. At this point our matrix is in normal form. The only question is whether the diagonal entries $b_1, \dots, b_l$ successively divide one another. But this is immediate. We just noted that at the end of the first application of Step 2, the entry b_1 in the upper left corner divides all entries of the matrix. This fact remains true as we continue to apply elementary operations. In particular, when the algorithm terminates, b_1 must divide each of $b_2, \dots, b_l$.

A similar argument shows b_2 divides each of $b_3, \dots, b_l$. And so on.

It now follows immediately from Exercise 4 of §4 that the numbers $b_1, \dots, b_l$ are uniquely determined by the homomorphism f . For the number l of non-zero entries in the matrix is just the rank of the free abelian group $f(G) \subset G'$. And those numbers b_i that are greater than 1 are just the torsion coefficients $t_1, \dots, t_k$ of the quotient group $G'/f(G)$.

Applications of the reduction algorithm

Now we prove the basic theorem concerning subgroups of free abelian groups, which we stated in §4.

Proof of Theorem 4.2. Given a free abelian group F of rank n , we know from Lemma 11.1 that any subgroup R is free of rank $r \leq n$. Consider the inclusion homomorphism $j: R \rightarrow F$, and choose bases $a_1, \dots, a_r$ for R and $e_1, \dots, e_n$ for F relative to which the matrix of j is in the normal form of the preceding theorem. Because j is a monomorphism, this normal form has no zero columns. Thus $j(a_i) = b_i e_i$ for $i = 1, \dots, r$, where $b_i \geq 1$ and $b_1 | b_2 | \dots | b_r$. Since $j(a_i) = a_i$, it follows that $b_1 e_1, \dots, b_r e_r$ is a basis for R . $\square$

Now we prove the "standard basis theorem" for free chain complexes.

Definition. A chain complex $\mathcal{C}$ is a sequence

$$\dots \rightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \rightarrow \dots$$

of abelian groups C_i and homomorphisms ∂_i , indexed with the integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all p . The p th homology group of $\mathcal{C}$ is defined by the equation

$$H_p(\mathcal{C}) = \ker \partial_p / \operatorname{im} \partial_{p+1}.$$

If $H_p(\mathcal{C})$ is finitely generated, its betti number and torsion coefficients are called the betti number and torsion coefficients of $\mathcal{C}$ in dimension p .

Theorem 11.4 (Standard bases for free chain complexes). Let $\{C_p, \partial_p\}$ be a chain complex; suppose each group C_p is free of finite rank. Then for each p there are subgroups U_p, V_p, W_p of C_p such that

$$C_p = U_p \oplus V_p \oplus W_p,$$

where $\partial_p(U_p) \subset W_{p-1}$ and $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. Furthermore, there are bases for U_p and W_{p-1} relative to which $\partial_p: U_p \rightarrow W_{p-1}$ has a matrix of the form

$$B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_l \end{bmatrix},$$

where $b_i \geq 1$ and $b_1 | b_2 | \dots | b_l$.

Proof. Step 1. Let

$$Z_p = \ker \partial_p \quad \text{and} \quad B_p = \operatorname{im} \partial_{p+1}.$$

Let W_p consist of all elements c_p of C_p such that some non-zero multiple of c_p

belongs to B_p . It is a subgroup of C_p , and is called the group of weak boundaries. Clearly

$$B_p \subset W_p \subset Z_p \subset C_p.$$

(The second inclusion uses the fact that C_p is torsion-free, so that the equation $mc_p = \partial_{p+1}d_{p+1}$ implies that $\partial_p c_p = 0$.) We show that W_p is a direct summand in Z_p .

Consider the natural projection

$$Z_p \rightarrow H_p(\mathcal{C}) \rightarrow H_p(\mathcal{C})/T_p(\mathcal{C}),$$

where $T_p(\mathcal{C})$ is the torsion subgroup of $H_p(\mathcal{C})$. The kernel of this projection is W_p ; therefore, $Z_p/W_p \cong H_p/T_p$. The latter group is finitely generated and torsion-free, so it is free. If $c_1 + W_p, \dots, c_k + W_p$ is a basis for Z_p/W_p , and $d_1, \dots, d_l$ is a basis for W_p , then it is straightforward to check that $c_1, \dots, c_k, d_1, \dots, d_l$ is a basis for Z_p . Then $Z_p = V_p \oplus W_p$, where V_p is the group with basis $c_1, \dots, c_k$.

Step 2. Suppose we choose bases $e_1, \dots, e_n$ for C_p , and $e'_1, \dots, e'_m$ for C_{p-1} , relative to which the matrix of $\partial_p: C_p \rightarrow C_{p-1}$ has the normal form

$$\begin{array}{c} e'_1 \dots e'_l \quad e'_{l+1} \dots e'_m \\ \left[\begin{array}{cc|cc} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_l & \\ \hline & & & 0 \\ & 0 & & 0 \end{array} \right] \end{array}$$

where $b_i \geq 1$ and $b_1 | b_2 | \dots | b_l$. Then the following hold:

- (1) $e_{l+1}, \dots, e_n$ is a basis for Z_p .
- (2) $e'_1, \dots, e'_l$ is a basis for W_{p-1} .
- (3) $b_1 e'_1, \dots, b_l e'_l$ is a basis for B_{p-1} .

We prove these results as follows: Let c_p be the general p -chain. We compute its boundary; if

$$c_p = \sum_{i=1}^n a_i e_i, \quad \text{then} \quad \partial_p c_p = \sum_{i=1}^l a_i b_i e'_i.$$

To prove (1), we note that since $b_i \neq 0$, the p -chain c_p is a cycle if and only if $a_i = 0$ for $i = 1, \dots, l$. To prove (3), we note that any $p-1$ boundary $\partial_p c_p$ lies in the group generated by $b_1 e'_1, \dots, b_l e'_l$; since $b_i \neq 0$, these elements are inde-

pendent. Finally, we prove (2). Note first that each of $e'_1, \dots, e'_l$ belongs to W_{p-1} , since $b_i e'_i = \partial e_i$. Conversely, let

$$c_{p-1} = \sum_{i=1}^m d_i e'_i$$

be a $p-1$ chain and suppose $c_{p-1} \in W_{p-1}$. Then c_{p-1} satisfies an equation of the form

$$\lambda c_{p-1} = \partial_p c_p = \sum_{i=1}^l a_i b_i e'_i$$

for some $\lambda \neq 0$. Equating coefficients, we see that $\lambda d_i = 0$ for $i > l$, whence $d_i = 0$ for $i > l$. Thus $e'_1, \dots, e'_l$ is a basis for W_{p-1} .

Step 3. We prove the theorem. Choose bases for C_p and C_{p-1} , as in Step 2. Define U_p to be the group spanned by $e_1, \dots, e_l$; then

$$C_p = U_p \oplus Z_p.$$

Using Step 1, choose V_p so that $Z_p = V_p \oplus W_p$. Then we have a decomposition of C_p such that $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. The existence of the desired bases for U_p and W_{p-1} follows from Step 2. $\square$

Note that W_p and $Z_p = V_p \oplus W_p$ are uniquely determined subgroups of C_p . The subgroups U_p and V_p are not uniquely determined, however.

Theorem 11.5. *The homology groups of a finite complex K are effectively computable.*

Proof. By the preceding theorem, there is a decomposition

$$C_p(K) = U_p \oplus V_p \oplus W_p$$

where $Z_p = V_p \oplus W_p$ is the group of p -cycles and W_p is the group of weak p -boundaries. Now

$$H_p(K) = Z_p/B_p \cong V_p \oplus (W_p/B_p) \cong (Z_p/W_p) \oplus (W_p/B_p).$$

The group Z_p/W_p is free and the group W_p/B_p is a torsion group; computing $H_p(K)$ thus reduces to computing these two groups.

Let us choose bases for the chain groups $C_p(K)$ by orienting the simplices of K , once and for all. Then consider the matrix of the boundary homomorphism $\partial_p: C_p(K) \rightarrow C_{p-1}(K)$ relative to this choice of bases; the entries of this matrix will in fact have values in the set $\{0, 1, -1\}$. Using the reduction algorithm described earlier, we reduce this matrix to normal form. Examining Step 2 of the preceding proof, we conclude from the results proved there the following facts about this normal form:

- (1) The rank of Z_p equals the number of zero columns.
- (2) The rank of W_{p-1} equals the number of non-zero rows.
- (3) There is an isomorphism

$$W_{p-1}/B_{p-1} \cong \mathbb{Z}/b_1 \oplus \mathbb{Z}/b_2 \oplus \dots \oplus \mathbb{Z}/b_l.$$

Thus the normal form for the matrix of $\partial_p: C_p \rightarrow C_{p-1}$ gives us the torsion coefficients of K in dimension $p-1$; they are the entries of the matrix that are greater than 1. This normal form also gives us the rank of Z_p . On the other hand, the normal form for $\partial_{p+1}: C_{p+1} \rightarrow C_p$ gives us the rank of W_p . The difference of these numbers is the rank of Z_p/W_p —that is, the betti number of K in dimension p . $\square$

EXERCISES

1. Show that the reduction algorithm is not needed if one wishes merely to compute the betti numbers of a finite complex K ; instead all that is needed is an algorithm for determining the rank of a matrix. Specifically, show that if A_p is the matrix of $\partial_p: C_p(K) \rightarrow C_{p-1}(K)$ relative to some choice of basis, then

$$\beta_p(K) = \text{rank } C_p(K) - \text{rank } A_p - \text{rank } A_{p+1}.$$

2. Compute the homology groups of the quotient space indicated in Figure 11.1. [Hint: First check whether all the vertices are identified.]
3. Reduce to normal form the matrix

$$\begin{bmatrix} 2 & 6 & 4 \\ 4 & -7 & 4 \\ 4 & 8 & 4 \end{bmatrix}.$$

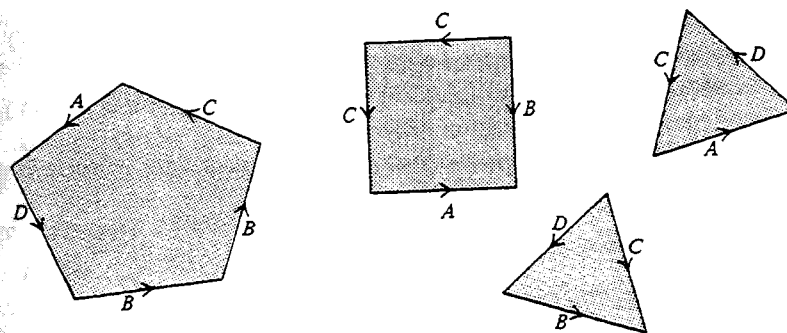


Figure 11.1

§12. HOMOMORPHISMS INDUCED BY SIMPLICIAL MAPS

If f is a simplicial map of $|K|$ into $|L|$, then f maps each p -simplex σ_i of K onto a simplex τ_i of L of the same or lower dimension. We shall define a homomorphism of p -chains that carries a formal sum $\sum m_i \sigma_i$ of oriented p -simplices of K onto the formal sum $\sum m_i \tau_i$ of their images. (We delete from the latter sum those simplices τ_i whose dimension is less than p .) This map in turn induces a homomorphism of homology groups, as we shall see.

As a general notation, we shall use the phrase

“ $f: K \rightarrow L$ is a simplicial map”

to mean that f is a continuous map of $|K|$ into $|L|$ that maps each simplex of K linearly onto a simplex of L . Thus f maps each vertex of K to a vertex of L , and it equals the simplicial map induced by this vertex map, as defined in §2.

Definition. Let $f: K \rightarrow L$ be a simplicial map. If $v_0 \dots v_p$ is a simplex of K , then the points $f(v_0), \dots, f(v_p)$ span a simplex of L . We define a homomorphism $f_\# : C_p(K) \rightarrow C_p(L)$ by defining it on oriented simplices as follows:

$$f_\#([v_0, \dots, v_p]) = \begin{cases} [f(v_0), \dots, f(v_p)] & \text{if } f(v_0), \dots, f(v_p) \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

This map is clearly well-defined; exchanging two vertices in the expression $[v_0, \dots, v_p]$ changes the sign of the right side of the equation. The family of homomorphisms $\{f_\#\}$, one in each dimension, is called the **chain map induced by the simplicial map f** .

Properly speaking, one should use dimensional subscripts to distinguish these homomorphisms, denoting the map in dimension p by $(f_\#)_p : C_p(K) \rightarrow C_p(L)$. Normally, however, we shall omit the subscript, relying on the context to make the situation clear, just as we do with the boundary operator ∂ .

In a similar vein, we shall use the symbol ∂ to denote the boundary operators in both K and L , in order to keep the notation from becoming cumbersome. If it is necessary to distinguish them, we can use the notations ∂_K and ∂_L .

Lemma 12.1. *The homomorphism $f_\#$ commutes with ∂ ; therefore $f_\#$ induces a homomorphism $f_* : H_p(K) \rightarrow H_p(L)$.*

Proof. We need to show that

$$(*) \quad \partial f_\#([v_0, \dots, v_p]) = f_\#(\partial[v_0, \dots, v_p]).$$

Let τ be the simplex of L spanned by $f(v_0), \dots, f(v_p)$. We consider various cases.

Case 1. $\dim \tau = p$. In this case, the vertices $f(v_0), \dots, f(v_p)$ are distinct and the result follows at once from the definitions of $f_\#$ and ∂ .

Case 2. $\dim \tau \leq p - 2$. In this case, the left side of (*) vanishes because $f(v_0), \dots, f(v_p)$ are not distinct, and the right side vanishes because for each i , at least two of the points $f(v_0), \dots, f(v_{i-1}), f(v_{i+1}), \dots, f(v_p)$ are the same.

Case 3. $\dim \tau = p - 1$. In this case, we may assume the vertices so ordered that $f(v_0) = f(v_1)$, and $f(v_1), \dots, f(v_p)$ are distinct. Then the left side of (*) vanishes. The right side has only two non-zero terms; it equals

$$[f(v_1), f(v_2), \dots, f(v_p)] - [f(v_0), f(v_2), \dots, f(v_p)].$$

Since $f(v_0) = f(v_1)$, these terms cancel each other, as desired.

The homomorphism $f_\#$ carries cycles to cycles, since the equation $\partial c_p = 0$ implies that $\partial f_\#(c_p) = f_\#(\partial c_p) = 0$. And $f_\#$ carries boundaries to boundaries, since the equation $c_p = \partial d_{p+1}$ implies that $f_\#(c_p) = f_\#(\partial d_{p+1}) = \partial f_\#(d_{p+1})$. Thus $f_\#$ induces a homomorphism $f_* : H_p(K) \rightarrow H_p(L)$ of homology groups. $\square$

Theorem 12.2. (a) *Let $i: K \rightarrow K$ be the identity simplicial map. Then $i_* : H_p(K) \rightarrow H_p(K)$ is the identity homomorphism.*

(b) *Let $f: K \rightarrow L$ and $g: L \rightarrow M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$; that is, the following diagram commutes:*

$$\begin{array}{ccc} H_p(K) & \xrightarrow{(g \circ f)_*} & H_p(M) \\ f_* \searrow & & \nearrow g_* \\ & H_p(L) & \end{array}$$

Proof. It is immediate from the definition that $i_\#$ is the identity and $(g \circ f)_\# = g_\# \circ f_\#$, as you can check. The theorem follows. $\square$

This theorem expresses what are called the “functorial properties” of the induced homomorphism. This phrase will be defined formally later when we discuss *categories* and *functors*. For the present, we point out simply that the operator H_p assigns to each simplicial complex an abelian group, and the operator $*$ assigns to each simplicial map of one complex into another, a homomorphism of the corresponding abelian groups. Because (a) and (b) hold, we say that $(H_p, *)$ is a “functor” from the “category” of simplicial complexes and simplicial maps to the “category” of abelian groups and homomorphisms.

Lemma 12.3. *The chain map $f_\#$ preserves the augmentation map ϵ ; therefore, it induces a homomorphism f_* of reduced homology groups.*

Proof. Let $f: K \rightarrow L$ be simplicial. Then $\epsilon f_\#(v) = 1$ and $\epsilon(v) = 1$ for each vertex v of K . Thus $\epsilon \circ f_\# = \epsilon$. This equation implies that $f_\#$ carries the kernel of $\epsilon_K : C_0(K) \rightarrow \mathbb{Z}$ into the kernel of $\epsilon_L : C_0(L) \rightarrow \mathbb{Z}$, and thus induces a homomorphism $f_* : \tilde{H}_0(K) \rightarrow \tilde{H}_0(L)$. $\square$

Example 1. Consider the complexes K and T indicated in Figure 12.1. Their underlying spaces are the circle and torus, respectively. Now $H_1(K) \cong \mathbb{Z}$; let us

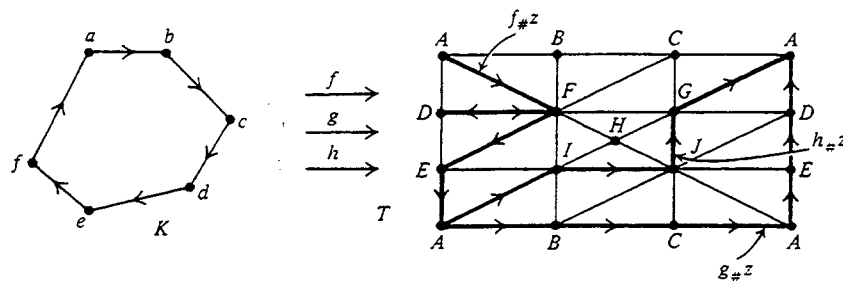


Figure 12.1

use the cycle z indicated in the figure as a generator. Similarly, $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$; let us use the cosets of the cycles $w_1 = [A, B] + [B, C] + [C, A]$ and $z_1 = [A, D] + [D, E] + [E, A]$ as a basis.

Now consider the simplicial maps:

$$\begin{array}{lll} f: a \rightarrow A & g: a \rightarrow A & h: a \rightarrow A \\ b \rightarrow F & b \rightarrow B & b \rightarrow I \\ c \rightarrow D & c \rightarrow C & c \rightarrow J \\ d \rightarrow D & d \rightarrow A & d \rightarrow G \\ e \rightarrow F & e \rightarrow E & e \rightarrow G \\ f \rightarrow E & f \rightarrow D & f \rightarrow A \end{array}$$

You can check that $f_\#(z)$ is homologous to z_1 , that $g_\#(z)$ equals $w_1 - z_1$, and that $h_\#(z)$ is homologous to $w_1 - z_1$. Thus $g_\#$ and $h_\#$ are equal as homomorphisms of 1-dimensional homology. They are also equal on 0-dimensional homology.

In general, a given homomorphism can be induced by quite different simplicial maps, as the preceding example shows. This fact leads us to consider the general question: Under what conditions do two simplicial maps induce the same homomorphism of homology groups? Answering this question involves an important technique, one we shall use many times. So we begin by explaining the underlying motivation.

Given simplicial maps $f, g: K \rightarrow L$, we wish to find conditions under which $f_\#(z)$ and $g_\#(z)$ are homologous for each cycle $z \in Z_p(K)$. Said differently, we want to find conditions under which there is a function (commonly called D) that assigns to each p -cycle z of K , a $p+1$ chain Dz of L , such that

$$\partial Dz = g_\#(z) - f_\#(z).$$

Let us consider an example in which this is possible. Suppose K is the boundary of a triangle, and L consists of the sides of a triangular prism, as pictured in Figure 12.2. Suppose f and g are the simplicial maps that carry K onto the two ends of the prism, respectively. If z is the 1-cycle generating $H_1(K)$, as indicated in the figure, it is quite easy to find a 2-chain Dz whose boundary is

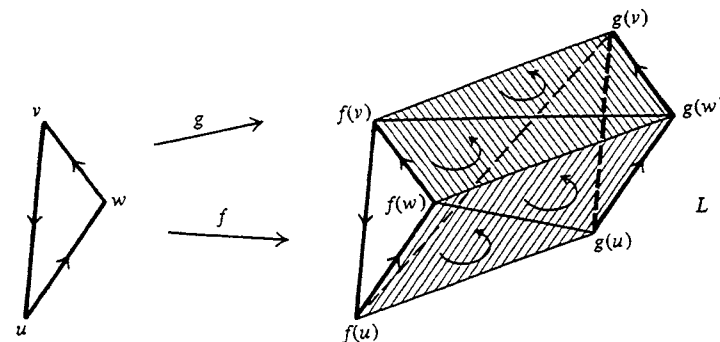


Figure 12.2

$g_\#(z) - f_\#(z)$; we let Dz equal the sum of all the 2-simplices of L , oriented appropriately. You can check that this 2-chain does the job.

In general, it is awkward to define D just for cycles, since that requires us to compute the group of cycles. What proves more satisfactory is to define D as a homomorphism on the entire chain group, because then one can define D one simplex at a time. How might that procedure work in the present example? It is fairly clear how to proceed. Given a vertex v of K , we define Dv to be the edge of L that leads from $f(v)$ to $g(v)$, as indicated in Figure 12.3. And given the oriented 1-simplex σ of K , we define $D\sigma$ to be the sum of the two oriented simplices that are heavily shaded in the figure, which form one side of the prism. We proceed similarly for the other simplices of K . Since z is the sum of the oriented edges of K , the chain Dz will be the sum of the oriented 2-simplices of L , just as before, and $\partial Dz = g_\#(z) - f_\#(z)$, as desired.

One can ask what sort of formula holds for $\partial D\sigma$ when σ is an arbitrary chain. The answer is clear from the definition. Given the oriented simplex σ , we compute $\partial D\sigma = g_\#(\sigma) - Dv - f_\#(\sigma) + Dw$; see Figure 12.3 for verification. That is,

$$(*) \quad \partial(D\sigma) = g_\#(\sigma) - f_\#(\sigma) - D(\partial\sigma).$$

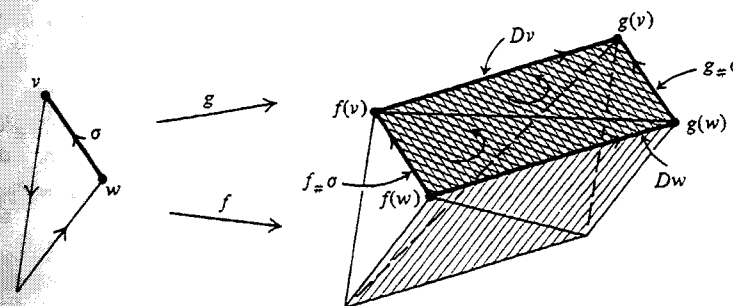


Figure 12.3

This same formula holds for the other simplices of K , as you can check.

This formula expresses the crucial algebraic property of the homomorphism D . It is the formula at which we were aiming; we make it part of the following definition.

Definition. Let $f, g: K \rightarrow L$ be simplicial maps. Suppose that for each p , one has a homomorphism

$$D: C_p(K) \rightarrow C_{p+1}(L)$$

satisfying the equation

$$\partial D + D\partial = g_\# - f_\#.$$

Then D is said to be a **chain homotopy** between $f_\#$ and $g_\#$.

We have omitted dimensional subscripts in this formula; the following diagram may make the maps involved clearer:

$$\begin{array}{ccc} & & C_{p+1}(L) \\ & \nearrow D_p & \downarrow \partial_{p+1} \\ C_p(K) & \xrightarrow[(g_\#)_p]{(f_\#)_p} & C_p(L) \\ \downarrow \partial_p & & \nearrow D_{p-1} \\ C_{p-1}(K) & & \end{array}$$

With subscripts, the formula becomes

$$\partial_{p+1}D_p + D_{p-1}\partial_p = (g_\#)_p - (f_\#)_p.$$

This formula is more precise, but messier. We shall customarily omit dimensional subscripts.

The importance of chain homotopies comes from the following theorem:

Theorem 12.4. *If there is a chain homotopy between $f_\#$ and $g_\#$, then the induced homomorphisms f_* and g_* , for both reduced and ordinary homology, are equal.*

Proof. Let z be a p -cycle of K . Then

$$g_\#(z) - f_\#(z) = \partial Dz + D\partial z = \partial Dz + 0,$$

so $g_\#(z)$ and $f_\#(z)$ are in the same homology class. Thus $g_*(\{z\}) = f_*(\{z\})$, as desired. $\square$

We still want to find conditions on two simplicial maps f and g under which the induced homomorphisms f_* and g_* are equal. We have reduced this problem to the problem of finding conditions under which one can construct a chain

homotopy between $f_\#$ and $g_\#$. Here is one set of conditions under which this is possible:

Definition. Given two simplicial maps $f, g: K \rightarrow L$, these maps are said to be **contiguous** if for each simplex $v_0 \dots v_p$ of K , the points

$$f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$$

span a simplex τ of L . (The simplex τ may be of any dimension from 0 to $2p + 1$, depending on how many of these points are distinct.)

Roughly speaking, this condition says that f and g are "fairly close"; one can move the simplex $f(\sigma)$ to the simplex $g(\sigma)$ across some possibly larger simplex τ of which both are faces.

Theorem 12.5. *If $f, g: K \rightarrow L$ are contiguous simplicial maps, then there is a chain homotopy between $f_\#$ and $g_\#$.*

Proof. The argument we give here is a standard one; you should master it.

For each simplex $\sigma = v_0 \dots v_p$ of K , let $L(\sigma)$ denote the subcomplex of L consisting of the simplex whose vertex set is $\{f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)\}$, and its faces. We note the following facts:

- (1) $L(\sigma)$ is nonempty, and $\tilde{H}_i(L(\sigma)) = 0$ for all i .
- (2) If s is a face of σ , then $L(s) \subset L(\sigma)$.
- (3) For each oriented simplex σ , the chains $f_\#(\sigma)$ and $g_\#(\sigma)$ are carried by $L(\sigma)$.

Using these facts, we shall construct the required chain homotopy $D: C_p(K) \rightarrow C_{p+1}(L)$, by induction on p . For each σ , the chain $D\sigma$ will be carried by $L(\sigma)$.

Let $p = 0$; let v be a vertex of K . Because $f_\#$ and $g_\#$ preserve augmentation, $\epsilon(g_\#(v) - f_\#(v)) = 1 - 1 = 0$. Thus $g_\#(v) - f_\#(v)$ represents an element of the reduced homology group $\tilde{H}_0(L(v))$. Because this group vanishes, we can choose a 1-chain Dv of L carried by the subcomplex $L(v)$ such that

$$\partial(Dv) = g_\#(v) - f_\#(v).$$

Then $\partial Dv + D\partial v = \partial Dv + 0 = g_\#(v) - f_\#(v)$, as desired. Define D in this way for each vertex of K .

Now suppose D is defined in dimensions less than p , such that for each oriented simplex s of dimension less than p , the chain Ds is carried by $L(s)$, and such that

$$\partial Ds + D\partial s = g_\#(s) - f_\#(s).$$

Let σ be an oriented simplex of dimension p . We wish to define $D\sigma$ so that $\partial(D\sigma)$ equals the chain

$$c = g_\#(\sigma) - f_\#(\sigma) - D\partial\sigma.$$

Note that c is a well-defined chain; $D\partial\sigma$ is defined because $\partial\sigma$ has dimension $p-1$. Furthermore, c is a cycle, for we compute

$$\begin{aligned}\partial c &= \partial g_{\#}(\sigma) - \partial f_{\#}(\sigma) - \partial D(\partial\sigma) \\ &= \partial g_{\#}(\sigma) - \partial f_{\#}(\sigma) - [g_{\#}(\partial\sigma) - f_{\#}(\partial\sigma) - D\partial(\partial\sigma)],\end{aligned}$$

applying the induction hypothesis to the $p-1$ chain $\partial\sigma$. Using the fact that $\partial \circ \partial = 0$, we see that $\partial c = 0$.

Finally, we note that c is carried by $L(\sigma)$: Both $g_{\#}(\sigma)$ and $f_{\#}(\sigma)$ are carried by $L(\sigma)$, by (3). To show $D\partial\sigma$ is carried by $L(\sigma)$, note that the chain $\partial\sigma$ is a sum of oriented faces of σ . For each such face s , the chain Ds is carried by $L(s)$, and $L(s) \subset L(\sigma)$ by (2). Thus $D\partial\sigma$ is carried by $L(\sigma)$.

Since c is a p -cycle carried by $L(\sigma)$, and since $H_p(L(\sigma)) = 0$, we can choose a $p+1$ chain $D\sigma$ carried by $L(\sigma)$ such that

$$\partial D\sigma = c = g_{\#}(\sigma) - f_{\#}(\sigma) - D\partial\sigma.$$

We then define $D(-\sigma) = -D(\sigma)$. We repeat this process for each p -simplex σ of K ; then we have the required chain homotopy D in dimension p . The theorem follows. $\square$

Some students are bothered by the fact that constructing the chain homotopy D involves arbitrary choices. They would be happier if there were a definite formula for D . Unfortunately, there is no such neat formula, because basically there are many possible chain homotopies between $f_{\#}$ and $g_{\#}$, and there is no reason for preferring one over the other.

To illustrate this fact, let us consider the preceding proof in a particular case. At the first step of the proof, when v is a vertex, the chain Dv is in fact uniquely determined. Since Dv is to be a chain carried by $L(v)$, then necessarily

$$\begin{aligned}Dv &= 0 & \text{if } f(v) = g(v), \\ Dv &= [f(v), g(v)] & \text{if } f(v) \neq g(v).\end{aligned}$$

In the first case, $L(v)$ is a vertex; and in the second case, it consists of a 1-simplex and its faces.

But at the very next step of the proof, choices can arise. Let $\sigma = vw$ be a 1-simplex, and suppose the points $f(v), f(w), g(v), g(w)$ are all distinct, so they span a 3-simplex τ , as indicated in Figure 12.4. The chains Dv and Dw are indicated in the figure. How shall we define $D\sigma$? There are two obvious choices for a 2-chain $D\sigma$ whose boundary is the 1-chain $g_{\#}(\sigma) - f_{\#}(\sigma) - D\partial\sigma$ pictured in Figure 12.5. One choice would consist of the front two faces of τ , oriented as indicated; the other choice would consist of the back two faces, appropriately oriented. There is no reason to prefer one choice to the other; we must simply choose one arbitrarily.

Application to relative homology

Let K_0 be a subcomplex of K , and let L_0 be a subcomplex of L . Let $f: K \rightarrow L$ be a simplicial map that carries each simplex of K_0 into a simplex of

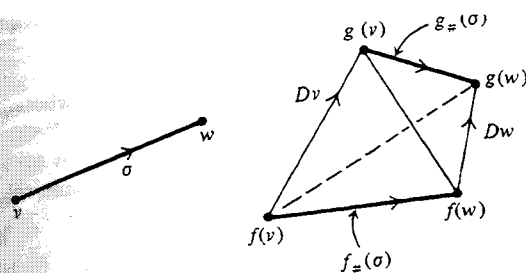


Figure 12.4

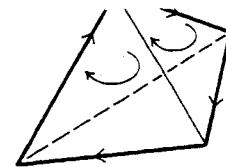


Figure 12.5

L_0 . We often express this by the phrase

" $f: (K, K_0) \rightarrow (L, L_0)$ is a simplicial map."

In this case, it is immediate that $f_{\#}$ maps $C_p(K_0)$ into $C_p(L_0)$, so that one has an induced map (also denoted $f_{\#}$)

$$f_{\#}: C_p(K, K_0) \rightarrow C_p(L, L_0).$$

This map commutes with ∂ and thus induces a homomorphism

$$f_{\#}: H_p(K, K_0) \rightarrow H_p(L, L_0).$$

The functorial properties stated in Theorem 12.2 carry over immediately to relative homology.

Definition. Let $f, g: (K, K_0) \rightarrow (L, L_0)$ be two simplicial maps. We say f and g are **contiguous as maps of pairs** if for each simplex $\sigma = v_0 \dots v_p$ of K , the points

$$f(v_0), \dots, f(v_p), g(v_0), \dots, g(v_p)$$

span a simplex of L , and if $\sigma \in K_0$, they span a simplex of L_0 .

Theorem 12.6. Let $f, g: (K, K_0) \rightarrow (L, L_0)$ be contiguous as maps of pairs. Then there is for all p a homomorphism

$$D: C_p(K, K_0) \rightarrow C_{p+1}(L, L_0)$$

such that $\partial D + D\partial = g_{\#} - f_{\#}$. It follows that $f_{\#}$ and $g_{\#}$ are equal as maps of relative homology groups.

Proof. The chain homotopy D constructed in the preceding proof automatically maps $C_p(K_0)$ into $C_{p+1}(L_0)$. For if $\sigma \in K_0$, the complex $L(\sigma)$ is by definition a subcomplex of L_0 . Given σ , the chain $D\sigma$ is carried by $L(\sigma)$; therefore, D maps $C_p(K_0)$ into $C_{p+1}(L_0)$. Then D induces the required homomorphism of the relative chain groups. $\square$

EXERCISES

1. Check the assertions made in Example 1.
2. Consider the complex K indicated by the vertex labelling in Figure 12.6, whose underlying space is the torus. Although this complex is slightly different from the one considered earlier, the computation of the homology of the torus remains the same. Let γ be the sum of all the 2-simplices of K , oriented counter-clockwise; then γ generates $H_2(K)$. The cycles

$$w_1 = [a, b] + [b, c] + [c, a] \quad \text{and} \quad z_1 = [a, d] + [d, e] + [e, a]$$

generate $H_1(K)$.

- (a) Define simplicial maps f, g, h, k of K with itself such that

$$\begin{array}{llll} f: a \rightarrow a & g: a \rightarrow a & h: a \rightarrow d & k: a \rightarrow p \\ & b \rightarrow d & b \rightarrow c & b \rightarrow p \\ & c \rightarrow e & c \rightarrow b & c \rightarrow q \\ & d \rightarrow c & d \rightarrow d & d \rightarrow d \\ & e \rightarrow b & e \rightarrow e & e \rightarrow d \end{array}$$

- (b) Compute the values of f_* , g_* , h_* , and k_* on the homology classes $\{w_1\}$, $\{z_1\}$, and $\{\gamma\}$.

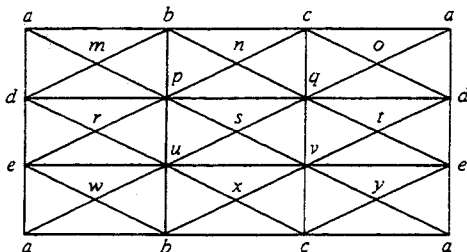


Figure 12.6

3. Use the complex in Figure 12.6, but with the letters d and e on the right-hand edge reversed, to represent the Klein bottle S . Let w_1 and z_1 be as before.
 - (a) Define a simplicial map $f: S \rightarrow S$ such that $f_*(\{w_1\}) = \{z_1\}$.
 - (b) Show there is no simplicial map $g: S \rightarrow S$ such that $g_*(\{z_1\}) = \{w_1\}$.
4. Let K be the torus, as in Exercise 2; let γ be the 2-cycle indicated there. Let L be the complex consisting of the proper faces of the 3-simplex having vertices A, B, C, D ; and let γ' be the 2-cycle $\partial[A, B, C, D]$ of L .
 - (a) Show that any map of the vertices of K to the vertices of L induces a simplicial map of K to L .
 - (b) Let f be the simplicial map carrying m and r to A , p to B , b and u to C , and all other vertices to D . Compute the induced homomorphism $f_*: H_2(K) \rightarrow H_2(L)$, in terms of γ and γ' .

- (c) Let g be the simplicial map agreeing with f on the vertices of K except that $g(r) = C$. Compute g_* .
 - (d) Let h agree with g on the vertices of K except that $h(u) = A$. Compute h_* .
5. Let $f, g: (K, K_0) \rightarrow (L, L_0)$ be simplicial maps. Show that if f and g are contiguous as maps of K into L , and if L_0 is a full subcomplex of L , then f and g are contiguous as maps of pairs.

§13. CHAIN COMPLEXES AND ACYCLIC CARRIERS

Many of the definitions and constructions we have made within the context of simplicial complexes also occur in more general situations. We digress at this point to discuss this more general context. We shall use these results many times later on. Proofs are left as exercises.

First we define the algebraic analogues to our chain groups. (We mentioned them earlier in §11.)

Definition. A **chain complex** $\mathcal{C}$ is a family $\{C_p, \partial_p\}$ of abelian groups C_p and homomorphisms

$$\partial_p: C_p \rightarrow C_{p-1},$$

indexed with the integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all p .

If $C_p = 0$ for $p < 0$, then $\mathcal{C}$ is said to be a **non-negative** chain complex. If C_p is a free abelian group for each p , then $\mathcal{C}$ is called a **free** chain complex. The group

$$H_p(\mathcal{C}) = \ker \partial_p / \text{im } \partial_{p+1}$$

is called the **p th homology group** of the chain complex $\mathcal{C}$.

If $\mathcal{C}$ is a non-negative chain complex, an **augmentation** for $\mathcal{C}$ is an epimorphism $\epsilon: C_0 \rightarrow \mathbb{Z}$ such that $\epsilon \circ \partial_1 = 0$. The **augmented chain complex** $\{\mathcal{C}, \epsilon\}$ is the chain complex obtained from $\mathcal{C}$ by adjoining the group $\mathbb{Z}$ in dimension -1 and using ϵ as the boundary operator in dimension 0. The homology groups of the augmented chain complex are called the **reduced homology groups** of the original chain complex $\mathcal{C}$, relative to the augmentation ϵ . They are denoted either $H_i(\{\mathcal{C}, \epsilon\})$ or $\tilde{H}_i(\mathcal{C})$. It follows readily that $\tilde{H}_p(\mathcal{C}) = H_p(\mathcal{C})$ for $p \neq 0$, and

$$H_0(\mathcal{C}) \simeq \tilde{H}_0(\mathcal{C}) \oplus \mathbb{Z}.$$

(See the exercises.)

An arbitrary chain complex $\mathcal{D}$ is said to be **acyclic** if $H_i(\mathcal{D}) = 0$ for all i . In particular, the augmented chain complex $\{\mathcal{C}, \epsilon\}$ is acyclic if $H_i(\{\mathcal{C}, \epsilon\}) = \tilde{H}_i(\mathcal{C}) = 0$ for all i , or equivalently if $H_0(\mathcal{C}) \simeq \mathbb{Z}$ and $H_i(\mathcal{C}) = 0$ for $i \neq 0$.

Definition. Let $\mathcal{C} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'_p\}$ be chain complexes. A **chain map** $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ is a family of homomorphisms

$$\phi_p : C_p \rightarrow C'_p$$

such that $\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p$ for all p .

A chain map $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ induces a homomorphism

$$(\phi_*)_p : H_p(\mathcal{C}) \rightarrow H_p(\mathcal{C}').$$

Furthermore, the following hold:

- (1) The identity map i of $\mathcal{C}$ is a chain map, and $(i_*)_p$ is the identity map of $H_p(\mathcal{C})$.
- (2) If $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ and $\psi : \mathcal{C}' \rightarrow \mathcal{C}''$ are chain maps, then $\psi \circ \phi$ is a chain map, and $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

If $\{\mathcal{C}, \epsilon\}$ and $\{\mathcal{C}', \epsilon'\}$ are augmented chain complexes, the chain map $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ is said to be **augmentation-preserving** if $\epsilon' \circ \phi_0 = \epsilon_0$. If we extend ϕ to the (-1) -dimensional groups by letting it equal the identity map of $\mathbb{Z}$, then ϕ is called a **chain map of augmented chain complexes**. It follows that an augmentation-preserving chain map ϕ induces a homomorphism $\phi_* : H_p(\mathcal{C}) \rightarrow H_p(\mathcal{C}')$ of reduced homology groups.

Example 1. The chain complex

$$\mathcal{C}(K, K_0) = \{C_p(K, K_0), \partial_p\}$$

defined in §9 is called the **oriented chain complex of the simplicial pair** (K, K_0) . It is both free and non-negative, but, in general, it does not have an augmentation. (For example, in Example 4 of §9 the entire group $C_0(K, K_0)$ vanishes, so there can be no surjective map $C_0(K, K_0) \rightarrow \mathbb{Z}$.) If K_0 is empty, then the complex $\mathcal{C}(K, \emptyset) = \mathcal{C}(K)$ has a standard augmentation, defined by $\epsilon(v) = 1$ for each vertex v of K , as we have seen.

Example 2. If $f : K \rightarrow L$ is a simplicial map of simplicial complexes K and L , then f_* is an augmentation-preserving chain map of $\mathcal{C}(K)$ into $\mathcal{C}(L)$. However, in general there exist augmentation-preserving chain maps $\phi : \mathcal{C}(K) \rightarrow \mathcal{C}(L)$ that are not induced by simplicial maps.

Definition. If $\phi, \psi : \mathcal{C} \rightarrow \mathcal{C}'$ are chain maps, then a **chain homotopy** of ϕ to ψ is a family of homomorphisms

$$D_p : C_p \rightarrow C'_{p+1}$$

such that

$$\partial'_{p+1} D_p + D_{p-1} \partial_p = \psi_p - \phi_p$$

for all p .

Henceforth we shall normally omit subscripts on boundary operators and

chain maps and chain homotopies. The preceding formula, for instance, then assumes the more familiar form $\partial' D + D \partial = \psi - \phi$.

Definition. A chain map $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ is called a **chain equivalence** if there is a chain map $\phi' : \mathcal{C}' \rightarrow \mathcal{C}$ such that $\phi' \circ \phi$ and $\phi \circ \phi'$ are chain homotopic to the identity maps of $\mathcal{C}$ and $\mathcal{C}'$, respectively. We call ϕ' a **chain-homotopy inverse** to ϕ .

We list several properties of chain homotopies; proofs are left to the exercises.

- (1) Chain homotopy is an equivalence relation on the set of chain maps from $\mathcal{C}$ to $\mathcal{C}'$.
- (2) Composition of chain maps induces a well-defined composition operation on chain-homotopy classes.
- (3) If ϕ and ψ are chain homotopic, then they induce the same homomorphism in homology.
- (4) If ϕ is a chain equivalence, with chain-homotopy inverse ϕ' , then ϕ_* and $(\phi')_*$ are homology isomorphisms that are inverse to each other.
- (5) If $\phi : \mathcal{C} \rightarrow \mathcal{C}'$ and $\psi : \mathcal{C}' \rightarrow \mathcal{C}''$ are chain equivalences, then $\psi \circ \phi$ is a chain equivalence.

Now we investigate what all this means in the special case of augmented chain complexes.

Lemma 13.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be non-negative chain complexes. Let $\phi, \psi : \mathcal{C} \rightarrow \mathcal{C}'$ be chain maps; let D be a chain homotopy between them. Suppose $\mathcal{C}$ and $\mathcal{C}'$ are augmented by ϵ and ϵ' , respectively. If ϕ preserves augmentation, so does ψ . If we extend ϕ and ψ to be the identity in dimension -1 , and extend D to be zero in dimension -1 , then D is a chain homotopy between the extended chain maps.

Proof. If $c_0 \in C_0$, we have

$$\partial D c_0 = \phi(c_0) - \psi(c_0)$$

because $\partial c_0 = 0$. Then

$$0 = \epsilon'(\partial D c_0) = \epsilon' \phi(c_0) - \epsilon' \psi(c_0) = \epsilon(c_0) - \epsilon' \psi(c_0),$$

as desired. Furthermore, we have in dimension 0 the equation

$$D \epsilon(c_0) + \partial D(c_0) = \phi(c_0) - \psi(c_0),$$

because $D \epsilon(c_0) = 0$, and we have in dimension -1 the equation

$$\epsilon'(D(1)) = \phi(1) - \psi(1)$$

because both sides vanish. Thus D is a chain homotopy between the extended chain maps. $\square$

Lemma 13.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be non-negative chain complexes. Let $\phi: \mathcal{C} \rightarrow \mathcal{C}'$ be a chain equivalence with chain-homotopy inverse ϕ' . Suppose $\mathcal{C}$ and $\mathcal{C}'$ are augmented by ϵ, ϵ' respectively. If ϕ preserves augmentation, so does ϕ' . Furthermore, ϕ and ϕ' are chain-homotopy inverses as maps of augmented complexes. Therefore, they induce inverse isomorphisms in reduced homology.

Proof. If D' is a chain homotopy between $\phi \circ \phi'$ and the identity, then in dimension 0,

$$\partial' D' c'_0 = \phi \phi' (c'_0) - c'_0.$$

so

$$0 = \epsilon' (\partial' D' c'_0) = \epsilon' \phi \phi' (c'_0) - \epsilon' (c'_0) = \epsilon \phi' (c'_0) - \epsilon' (c'_0).$$

Thus ϕ' preserves augmentation. The remainder of the statement follows from Lemma 13.1. $\square$

These definitions set up the general algebraic framework into which the oriented simplicial chain groups fit as a special case. Now we seek to put into this general context the method by which in the preceding section we constructed a chain homotopy. As motivation, we first consider the case where the maps ϕ and ψ are general chain maps, but the chain groups are the familiar oriented simplicial chain groups. Later, we consider general chain complexes and general chain maps.

Definition. Let K and L be simplicial complexes. An **acyclic carrier** from K to L is a function Φ that assigns to each simplex σ of K , a subcomplex $\Phi(\sigma)$ of L such that:

- (1) $\Phi(\sigma)$ is nonempty and acyclic.
- (2) If s is a face of σ , then $\Phi(s) \subset \Phi(\sigma)$.

If $f: C_p(K) \rightarrow C_q(L)$ is a homomorphism, we say that f is **carried by Φ** if for each oriented p -simplex σ of K , the chain $f(\sigma)$ is carried by the subcomplex $\Phi(\sigma)$ of L .

Theorem 13.3 (Acyclic carrier theorem, geometric version). Let Φ be an acyclic carrier from K to L .

(a) If ϕ and ψ are two augmentation-preserving chain maps from $\mathcal{C}(K)$ to $\mathcal{C}(L)$ that are carried by Φ , there exists a chain homotopy D of ϕ to ψ that is also carried by Φ .

(b) There exists an augmentation-preserving chain map from $\mathcal{C}(K)$ to $\mathcal{C}(L)$ that is carried by Φ .

Proof. Part (a) of the theorem is proved by copying the proof of Theorem 12.5; one simply replaces the subcomplex $L(\sigma)$ by the subcomplex $\Phi(\sigma)$ throughout. To prove (b), we proceed as follows: For each vertex v of K , define $\phi(v)$ to be a 0-chain c of $\Phi(v)$ such that $\epsilon(c) = 1$. This we can do because $\Phi(v)$

is nonempty, so $\epsilon: \Phi(v) \rightarrow \mathbb{Z}$ is surjective. (In fact, we can simply choose $\phi(v)$ to be a vertex of $\Phi(v)$.) Then ϕ preserves augmentation, and

$$\partial \phi(v) = 0 = \phi(\partial v).$$

Let $\sigma = [v, w]$ be an oriented 1-simplex of K . The chain $c = \phi(\partial \sigma)$ is well-defined because $\partial \sigma$ is a 0-chain and ϕ has been defined in dimension 0. Furthermore, c is carried by $\Phi(\sigma)$; for $\phi(\partial \sigma)$ is carried by $\Phi(v)$ and $\Phi(w)$, both of which are contained in $\Phi(\sigma)$ by (2). Finally,

$$\epsilon(c) = \epsilon \phi(\partial \sigma) = \epsilon(\partial \sigma) = 0,$$

because ϕ preserves augmentation. Hence c represents an element of $\tilde{H}_0(\Phi(\sigma))$. Because $\Phi(\sigma)$ is acyclic (by (1)), we can choose a 1-chain carried by $\Phi(\sigma)$ whose boundary is c . We denote this 1-chain by $\phi(\sigma)$; then

$$\partial \phi(\sigma) = c = \phi(\partial \sigma).$$

For the induction step, let $p > 1$. Assume that if $\dim s < p$, then $\phi(s)$ is defined and $\partial \phi(s) = \phi \partial(s)$. Let σ be an oriented simplex of dimension p ; we seek to define $\phi(\sigma)$. The chain $c = \phi(\partial \sigma)$ is a well-defined $p-1$ chain. It is carried by $\Phi(\sigma)$, since $\Phi(\partial \sigma)$ is carried by the union of the complexes $\Phi(s_i)$, where s_i ranges over the $p-1$ faces of σ , and each of these complexes is contained in $\Phi(\sigma)$. Furthermore, c is a cycle since

$$\partial c = \partial \phi(\partial \sigma) = \phi \partial(\partial \sigma) = 0.$$

Here we apply the induction hypothesis to the $p-1$ chain $\partial \sigma$. Since $\Phi(\sigma)$ is acyclic, we can choose $\phi(\sigma)$ to be a p -chain carried by $\Phi(\sigma)$ such that $\partial \phi(\sigma) = c$. Then $\partial \phi(\sigma) = \phi \partial(\sigma)$, as desired.

The theorem follows by induction. $\square$

With this proof as motivation, we now formulate an even more general version of this theorem. In this form, all geometry disappears and only algebra remains!

Definition. Let $\mathcal{C} = \{C_p, \partial_p\}$ be a chain complex. A **subchain complex** $\mathcal{D}$ of $\mathcal{C}$ is a chain complex whose p th chain group is a subgroup of C_p , and whose boundary operator in each dimension p is the restriction of ∂_p .

Definition. Let $\{\mathcal{C}, \epsilon\} = \{C_p, \partial_p, \epsilon\}$ be an augmented chain complex. Suppose $\mathcal{C}$ is free; let $\{\sigma_p^\alpha\}$ be a basis for C_p , as α ranges over some index set J_p . Let $\{\mathcal{C}', \epsilon'\} = \{C'_p, \partial'_p, \epsilon'\}$ be an arbitrary augmented chain complex. An **acyclic carrier** from $\mathcal{C}$ to $\mathcal{C}'$, relative to the given bases, is a function Φ that assigns to each basis element σ_p^α , a subchain complex $\Phi(\sigma_p^\alpha)$ of $\mathcal{C}'$, satisfying the following conditions:

- (1) The chain complex $\Phi(\sigma_p^\alpha)$ is augmented by ϵ' and is acyclic.
- (2) If σ_{p-1}^β appears in the expression for $\partial_p \sigma_p^\alpha$ in terms of the preferred basis for C_{p-1} , then $\Phi(\sigma_{p-1}^\beta)$ is a subchain complex of $\Phi(\sigma_p^\alpha)$.

A homomorphism $f: C_p \rightarrow C'_q$ is said to be **carried** by Φ if $f(\sigma_p^\alpha)$ belongs to the q -dimensional group of the subchain complex $\Phi(\sigma_p^\alpha)$ of $\mathcal{C}'$, for each α .

Theorem 13.4 (Acyclic carrier theorem, algebraic version). *Let $\mathcal{C}$ and $\mathcal{C}'$ be augmented chain complexes; let $\mathcal{C}$ be free. Let Φ be an acyclic carrier from $\mathcal{C}$ to $\mathcal{C}'$, relative to some set of preferred bases for $\mathcal{C}$. Then there is an augmentation-preserving chain map $\phi: \mathcal{C} \rightarrow \mathcal{C}'$ carried by Φ , and any two such are chain homotopic; the chain homotopy is also carried by Φ .*

Proof. The proof of this theorem is just a jazzed-up version of the preceding proof. The requirement that the restriction of ϵ' give an augmentation for $\Phi(\sigma_p^\alpha)$ means that ϵ' must map the 0-dimensional group of this chain complex onto $\mathbb{Z}$; this corresponds to the requirement in the earlier version that $\Phi(\sigma)$ be nonempty for all σ . $\square$

Application: Ordered simplicial homology

The algebraic version of the acyclic carrier theorem may seem unnecessarily abstract to you. But it is indeed useful. Let us give one application now; it is a theorem that we will use later when we study singular homology. This theorem involves a new way of defining the simplicial homology groups, using *ordered* simplices rather than *oriented* simplices.

Let K be a simplicial complex. An **ordered p -simplex** of K is a $p+1$ tuple $(v_0, \dots, v_p)$ of vertices of K , where v_i are vertices of a simplex of K but need not be distinct. (For example, if uv is a 1-simplex of K , then (v, w, w, v) is an ordered 3-simplex of K .)

Let $C'_p(K)$ be the free abelian group generated by the ordered p -simplices of K ; it is called the group of **ordered p -chains** of K . As usual, we shall identify the ordered p -simplex $(v_0, \dots, v_p)$ with the elementary p -chain whose value is 1 on this ordered simplex and 0 on all other ordered simplices. Then every element of $C'_p(K)$ can be written uniquely as a finite linear combination, with integral coefficients, of ordered p -simplices. We define $\partial'_p: C'_p(K) \rightarrow C'_{p-1}(K)$ by the formula

$$\partial'_p(v_0, \dots, v_p) = \sum_{i=0}^p (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_p).$$

Then ∂'_p is a well-defined homomorphism; and one checks as before that $\partial'_p \circ \partial'_{p+1} = 0$.

The chain complex $\mathcal{C}'(K) = \{C'_p(K), \partial'_p\}$ is called the **ordered chain complex** of K . It is augmented by defining $\epsilon'(v) = 1$ for every vertex v of K . Although much too huge to be useful for computational purposes, this chain complex is sometimes quite convenient for theoretical purposes. Its homology is, surprisingly enough, isomorphic to the simplicial homology of K , in a natural way. We outline a proof of this fact, leaving the details to you.

Lemma 13.5. *If $w * K$ is a cone over the complex K , then $w * K$ is acyclic in ordered homology.*

Proof. Define

$$D: C'_p(w * K) \rightarrow C'_{p+1}(w * K)$$

for $p \geq 0$ by the equation

$$D((v_0, \dots, v_p)) = (w, v_0, \dots, v_p).$$

Note that it is irrelevant here whether any of the v_i are equal to w . Let $c_p \in C'_p(w * K)$. We compute

$$\partial'_1 Dc_0 = c_0 - \epsilon'(c_0)w,$$

$$\partial'_{p+1} Dc_p = c_p - D\partial'_p c_p \quad \text{if } p > 0.$$

The lemma follows: If c_p is a cycle and $p > 0$, then $c_p = \partial'_{p+1} Dc_p$; if c_0 is a 0-chain lying in $\ker \epsilon'$, then $c_0 = \partial'_1 Dc_0$. $\square$

Theorem 13.6. *Choose a partial ordering of the vertices of K that induces a linear ordering on the vertices of each simplex of K . Define $\phi: C'_p(K) \rightarrow C_p(K)$ by letting*

$$\phi([v_0, \dots, v_p]) = (v_0, \dots, v_p)$$

if $v_0 < v_1 < \dots < v_p$ in the given ordering. Define $\psi: C'_p(K) \rightarrow C_p(K)$ by the equation

$$\psi((w_0, \dots, w_p)) = \begin{cases} [w_0, \dots, w_p] & \text{if the } w_i \text{ are distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ and ψ are augmentation-preserving chain maps that are chain-homotopy inverses.

If K_0 is a subcomplex of K , then ϕ and ψ induce chain maps of the relative chain complexes that are chain-homotopy inverses.

The proof is an application of the acyclic carrier theorem; it is left as an exercise.

It follows from this theorem that oriented and ordered homology are isomorphic in a rather "natural" way. To explain what we mean by "naturality," we need to consider how a simplicial map acts in ordered homology.

Definition. Let $f: K \rightarrow L$ be a simplicial map. Define $f'_\#: \mathcal{C}'(K) \rightarrow \mathcal{C}'(L)$ by the rule

$$f'_\#((v_0, \dots, v_p)) = (f(v_0), \dots, f(v_p)).$$

It is easy to check that $f'_\#$ is a chain map, easier in fact than in the oriented case,

for we need not worry whether or not vertices are distinct. Clearly $f_{\#}$ preserves augmentation. If f maps K_0 to L_0 , then $f'_{\#}$ induces a chain map

$$f'_{\#} : \mathcal{C}'(K, K_0) \rightarrow \mathcal{C}'(L, L_0),$$

and a corresponding homomorphism in homology.

Theorem 13.7. *Let $f : (K, K_0) \rightarrow (L, L_0)$ be a simplicial map. Let ϕ and ψ be as in the preceding theorem. Then the following diagram commutes:*

$$\begin{array}{ccc} H_i(K, K_0) & \xrightarrow{f_{\#}} & H_i(L, L_0) \\ \uparrow \psi_{\#} & & \uparrow \psi_{\#} \\ H_i(\mathcal{C}'(K, K_0)) & \xrightarrow{f'_{\#}} & H_i(\mathcal{C}'(L, L_0)). \end{array}$$

Similarly, $\phi_{\#} \circ f_{\#} = f'_{\#} \circ \phi_{\#}$.

Proof. One checks directly from the definition that $f_{\#} \circ \psi = \psi \circ f'_{\#}$. Thus the diagram already commutes on the chain level, so it commutes on the homology level as well. It is not true that $\phi \circ f_{\#} = f'_{\#} \circ \phi$, since ϕ depends on a particular ordering of vertices. However, because $\phi_{\#}$ is the inverse of $\psi_{\#}$, it is true that $\phi_{\#} \circ f_{\#} = f'_{\#} \circ \phi_{\#}$. $\square$

EXERCISES

1. If $\{\mathcal{C}, \epsilon\}$ is an augmented chain complex, show that $\tilde{H}_{-1}(\mathcal{C}) = 0$ and

$$\tilde{H}_0(\mathcal{C}) \oplus \mathbb{Z} \cong H_0(\mathcal{C}).$$

(See the exercises of §7.) Conclude that $\{\mathcal{C}, \epsilon\}$ is acyclic if and only if $H_p(\mathcal{C})$ is infinite cyclic for $p = 0$ and vanishes for $p \neq 0$.

2. Check properties (1)–(5) of chain homotopies. Only (2) and (5) require care.
3. Prove (a) of Theorem 13.3.
4. Consider Example 1 of §12. Show that although the maps g and h are not contiguous, there is nevertheless a chain homotopy between $g_{\#}$ and $h_{\#}$, as follows:
 - (a) Define an acyclic carrier Φ from K to L carrying both $g_{\#}$ and $h_{\#}$.
 - (b) Define a specific chain homotopy between $g_{\#}$ and $h_{\#}$ that is carried by Φ .
 - (c) Define a chain map $\phi : \mathcal{C}(K) \rightarrow \mathcal{C}(L)$ carried by Φ that is not induced by a simplicial map; define a chain homotopy between ϕ and $h_{\#}$.
5. Check the details of the proof of Theorem 13.4.
6. Prove Theorem 13.6 as follows:
 - (a) Show ϕ and ψ are augmentation-preserving chain maps, and show that $\psi \circ \phi$ equals the identity map of $\mathcal{C}(K)$.
 - (b) Define an acyclic carrier from $\mathcal{C}'(K)$ to $\mathcal{C}'(L)$ that carries both $\phi \circ \psi$ and the identity map.

Topological Invariance of the Homology Groups

In the preceding chapter, we defined a function assigning to each simplicial complex K a sequence of abelian groups called its homology groups. We now prove that these groups depend only on the underlying topological space of K .

The way to approach this problem is to study continuous maps of one polyhedron to another, and what such maps do to the homology groups. We show that a continuous map $h : |K| \rightarrow |L|$ of polyhedra induces, in a rather natural way, a homomorphism $h_{\#} : H_p(K) \rightarrow H_p(L)$ of the homology groups of the corresponding simplicial complexes. Constructing this induced homomorphism will prove to be a reasonably arduous task.

It turns out that when h is a homeomorphism of topological spaces, then $h_{\#}$ is an isomorphism of groups. The topological invariance of the simplicial homology groups follows.

It is not hard to see intuitively why there should be such an induced homomorphism. If one thinks of a homology class as a geometric object, it seems fairly reasonable that its image under h should be a well-defined homology class. A closed loop on the torus T , for instance, is mapped by $h : T \rightarrow X$ into a closed loop in X . But to make this idea algebraically precise requires some effort.

We already know that a simplicial map $f : |K| \rightarrow |L|$ induces a homomorphism $f_{\#}$ of homology groups. This chapter is devoted to showing that an arbitrary continuous map h can be approximated (in a suitable sense) by a simplicial map f , and that the resulting induced homomorphism depends only on the map h , not on the particular approximation chosen.