Journal of the Mathematical Society of Japan Vol. 10, No. 1, January, 1958

On the genus of the alternating knot, I.

By Kunio MURASUGI

(Received Aug. 14, 1957)
(Revised Oct. 25, 1957)

F. Frankel and L. Pontrjagin [2] and H. Seifert [5] have given methods of construction of an orientable closed surface spanning a given knot i. e. having a given knot as a boundary. Seifert [5] has defined the genus $G(k)$ of the knot k as the minimum of the genera of orientable closed surfaces spanning k, whose existences are assured by [2] and [5]. Now let d be the degree of the Alexander polynomial of k. Seifert has proved that we have always

$$\frac{d}{2} \leqq G(k) \tag{1}$$

where the equality holds, if k is a torus knot, but there are also cases where the equality does not hold. (There are namely knots, whose Alexander polynomials are 1 and which are not equivalent to circles.)

In this paper, we shall show that the equality holds in (1) in certain classes of alternating knots (Theorem 1.1). For example, "alternierender Brezelknoten" of type $(p_1, p_2, \cdots, p_{2n+1})$, p_i being odd, i. e. alternating knots, whose projections have p_i crossing points on each arm and divide the plane into $\sum_{i=1}^{2n+1} p_i+2$ regions, of which $2n+2$ are "black", belong to these classes. It will be shown, at the same time, that for an alternating knot k of our classes, the orientable closed surface spanning k, whose genus is just equal to $G(k)$, is obtained by Seifert's construction.

§ 1. Main theorem.

Let k be a knot[1] and let K be an image of a regular projection[2] of k onto the plane E and let K be oriented by the orientation induced by that of k. Let K have n double points $c_1, c_2, \cdots, c_n$, called the *crossing points*. One of the two segments through a crossing point c_i passes under the other. It is called the *lower* segment at c_i and the other the *upper* segment. The

1) A knot means a polygonal simple closed (oriented) curve in Euclidean three dimensional space E^3.

2) See [3].

segments[3] of K connecting two consecutive crossing points are called *sides* of K. K divides E into $n+2$ regions $r_0, r_1, \cdots, r_{n+1}$, where we assume that r_0 is always an unbounded region. We can classify these regions into two classes, called "black" and "white" for convenience' sake, in such a way that each side is always a common boundary of a black and a white region, where r_0 belongs to a black class.

Let us assign to each crossing point c_i the *incidence number* $I(c_i)$, where $I(c_i)=+1$ or -1 according as the smaller rotation to make the lower segment coincide with the upper segment, the orientation of the segments being taken into account, is carried out in the black or in the white region (Fig. 1).

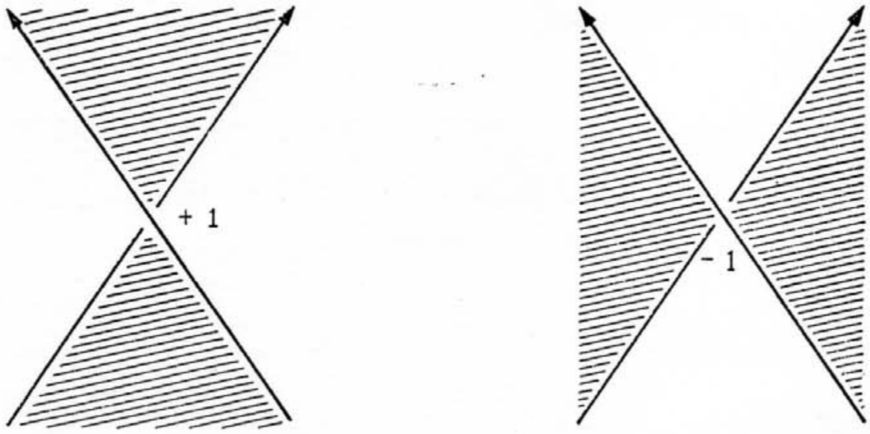

Fig. 1. (The parts drawn by the oblique lines represent the black regions)

Then the main theorem of our paper is the following

THEOREM 1.1. *For any alternating knot with a constant incidence number, the genus is exactly equal to one half of the degree of its Alexander polynomial.*

As a corollary of this theorem we have the following

COROLLARY 1.2. *Let k_1 and k_2 be alternating knots with constant incidence numbers. Then the degree of the Alexander polynomial of a product knot k_0 of k_1 and k_2 is exactly equal to double of the genus of k_0, where k_0 may not be an alternating knot and may not be of constant incidence numbers.*

COROLLARY 1.3. *The knots k_0, k_1, k_2 being as in Cor. 1.2, the genus of k_0 is equal to the sum of the genera of k_1 and k_2.*

REMARK. It was already shown by H. Schubert in [4] that the genus of the product knot is always equal to the sum of the genera of factors.

§ 2. Alexander polynomial and the genus of a knot.

Let us remember the definition of the Alexander polynomials defined in [1]. As in § 1 let us assume that there are n crossing points $c_1, c_2, \cdots, c_n$

3) Hereafter, a segment means generally a polygonal line.

in K and that K divides E into $n+2$ regions $r_0, r_1, \cdots, r_{n+1}$ and that these regions are classified into two classes, black and white.

To each region r_i an integer $I(r_i)$, called an *index* of r_i, is assigned. At each crossing point c_i, just four corners of four regions r_j, r_k, r_l and r_m, let us say, meet. Two corners among these four corners are marked with *dots* [1].

Now for each crossing point c_i, we shall write the following linear equation

$$c_i(r) = xr_j - xr_k + r_l - r_m = 0,$$

where c_i-corners[4] of r_j and r_k are dotted. We may assume, hereafter, that j, k, l and m are different from one another.[5]

Consider the matrix M, called the *L-matrix*, of the coefficients of these equations. M has n rows and $n+2$ columns, each row corresponding to a crossing point and each column corresponding to a region. If we denote the determinant of the square matrix obtained from M by striking out two columns corresponding to a pair of regions with consecutive indices p and $p+1$, by $\Delta_{p(p+1)}$, it follows[6]

$$\Delta_{p(p+1)} = \pm x^{r-p}\, \Delta_{r(r+1)}\,. \tag{2.1}$$

The *G.C.M.* of these determinants, freed from the factor x, is the *Alexander polynomial* of k. According to Alexander [1], we can assume that the signs of all the elements distinct from zero in the L-matrix M are positive, i.e. either x or 1.

Let us compute the genus of an orientable surface spanning k after the manner of H. Seifert [5].

Let us divide K into some loops,[7] called *standard loops*, in the same way as in [5]. Suppose that K is divided into m standard loops. Then the genus $G(k)$ of k is limited by[8]

$$G(k) \leqq \frac{n-m+1}{2}\,. \tag{2.2}$$

LEMMA 2.1. *For any alternating knot with a constant incidence number $I(c_i)$, the number m of the standard loops is either the number of the white or of the black regions according as $I(c_i)>0$ or $I(c_i)<0$.*

PROOF. We shall only prove Lemma in the case where $I(c_i)>0$. We shall prove that a standard loop L corresponds to a white region. To do

4) c_i-corner of r_j means the corner of r_j meeting at c_i.

5) In fact, it is impossible that $j=k$, or $k=l$, or $l=m$, or $m=j$. If $i=k$, we can transform K into K' which does not contain such a crossing point c_i. See [3].

6) See [1].

7) A loop means a simple closed curve.

8) See [5].

this we shall show that L will bound a white region W. Suppose that a point P moves positively along $\dot{W}$,[9] looking W on the left. When P arrives at a crossing point c_i, suppose it is always on the upper segment at c_i. Then the lower segment must be crossing under the upper segment from right to left, as $I(c_i)>0$. Thus P must turn to the left, and hence P must move positively along the boundary of a white region W', seeing it on the left again. It will be evident that $W=W'$. Thus P makes a round on $\dot{W}$, seeing W on the left. Consequently L bounds W. Furthermore it will be easily shown that two different standard loops do not bound the same white region.

If we assume that when P arrives at a crossing point, it is always on the lower segment, then we can prove Lemma in the same way as above.

In the same way, it will be proved that if $I(c_i)<0$, a standard loop will bound a black region.

q. e. d.

§ 3. L_0-matrix.

By Lemma 2.1 we can see that it is sufficient to prove Theorem 1.1 in the case where $I(c_i)>0$. Consequently we shall suppose, hereafter, that

(A) $I(c_i)>0$ for all i.

Hence the number m of standard loops is equal to the number of the white regions.

Lemma 3.1. *Under the assumption (A) the elements distinct from zero in the columns corresponding to the white regions are all x's or all* 1's.

Proof. It is sufficient to prove that the corners of a white region are either all dotted or all undotted. The proof of this fact is, however, contained in the proof of Lemma 2.1, taking notice of the dots of the corners.

q. e. d.

On account of this Lemma we can replace the L-matrix M by the matrix M_0, whose elements distinct from zero in the columns corresponding to the white regions are all equal to 1. M_0 will be called the *L_0-matrix.*

Lemma 3.2. *Under the assumption (A) all the indices of the black regions are constant, say p, and then the indices of the white regions are either $p-1$ or $p+1$.*

Proof. Let two black regions B_1 and B_2, and two white regions W_1 and W_2, be four regions whose corners meet at a crossing point c_i. Among these four regions the c_i-corners of two regions, of which one is the black and the other the white, are dotted. Suppose that the c_i-corner of B_1 is

9) A dot over the symbol denotes the set of boundary points.

dotted. If the c_i-corner of W_1 is dotted, then the lower segment is oriented as we see W_1 and B_1 on the left. Since $I(c_i)=1$, the upper segment must be oriented as we see W_1 and B_2 on the left. Hence it follows $I(W_1)=p+1$, $I(W_2)=p-1$ and $I(B_2)=p$. Similarly if the c_i-corner of W_2 is dotted, then it follows $I(W_1)=p-1$, $I(W_2)=p+1$ and $I(B_2)=p$. In the case where the c_i-corner of B_2 is dotted, it will be shown in the same way that we have the same result. q. e. d.

From the proof of this Lemma, it follows

LEMMA 3.3. *The index of the white region with dotted corners is $p+1$ and the index of the other white region is $p-1$, provided that the index of the black region is p.*

From this Lemma it follows

LEMMA 3.4. *The elements distinct from zero in either column of two columns of the L_0-matrix M_0, which are corresponding to two regions with consecutive indices, are all 1's.*

Consequently, the following Lemma will be easily shown from Lemmas 3.2, 3.3 and 3.4.

LEMMA 3.5. *Any determinant $\Delta^0_{(p-1)p}$ or $\Delta^0_{p(p+1)}$ of the square matrix obtained from M_0 by striking out two columns corresponding to two regions with consecutive indices is uniquely determined, except for the sign.*

Hence, hereafter, we shall consider only $\Delta^0_{p(p+1)}$.

LEMMA 3.6.[10] *Under the assumption (A) there exist $2q\ (q>0)$ crossing points on the boundary of any black region B and the corners adjacent to the dotted (or undotted) corner of the black region are undotted (or dotted).*

PROOF. Suppose that $\dot{B}$ and the boundary of a white region W have a side s in common. Let us denote the end points of s by c_i and c_j. If c_i-corner and c_j-corner of B are both dotted, then either one of c_i-corner or c_j-corner of W is undotted and the other is dotted, which contradicts to Lemma 3.1. If two corners of B are both undotted, then c_i-corner of B' and c_j-corner of B'' are dotted, where B' and B'' are black regions meeting with B at c_i and c_j respectively. Then it is impossible that c_i-corner and c_j-corner of W are both dotted or both undotted. This is a contradiction. q. e. d.

§ 4. *L*-correspondence.

Consider the terms of the largest and the smallest degrees in the determinant $\Delta^0_{p(p+1)}$. Since $\Delta^0_{p(p+1)}$ is the determinant of the degree n and the elements of $m-1$ columns are either 0 or 1, it is the polynomial of the degree $n-m+1$ at most.

10) That the converse is also true, is pointed out by Prof. H. Terasaka.

Now let us assign to each crossing point c_i one of the four regions meeting at it such that

(C) Each one of the $n+2$ regions except certain two regions r_α and r_β with consecutive indices corresponds to one and only one of the crossing point lying on its boundary.

Such a correspondence will be called an *L^q-correspondence* if $n-m+1$ crossing points c_i correspond to $n-m+1$ black regions, of which the corners of the q black regions at the corresponding crossing points are dotted. Then we have clearly

Lemma 4.1. *An L^q-correspondence corresponds to a term x^q or $-x^q$ in $\Delta^0_{p(p+1)}$.*

Lemma 4.2. *Let σ be an L^{n-m+1}-correspondence*[11] *such that each crossing point corresponds to one and only one of the $n+2$ regions except for a pair of two adjacent regions r_α and r_β, and let τ be another L^{n-m+1}-correspondence which is obtained from σ by changing the correspondences in some crossing points. Denoting the terms in $\Delta^0_{p(p+1)}$ corresponding to σ and τ by εx^{n-m+1} and $\bar{\varepsilon} x^{n-m+1}$ respectively, it follows*

$$\varepsilon=\bar{\varepsilon},$$

where $\varepsilon, \bar{\varepsilon}=\pm 1$.

Proof. We can suppose that the columns of $\Delta^0_{p(p+1)}$ have been arranged so that i-th column corresponds to a black region B_i $(i=1, 2, \cdots, n-m+1)$ and j-th column corresponds to a white region $W_{j-n+m-1}$ $(j=n-m+2, \cdots, n)$.

Let us suppose that c_{j_λ} corresponds to B_λ $(\lambda=1, \cdots, n-m+1)$ and c_{j_ν} corresponds to $W_{\nu-n+m-1}$ $(\nu=n-m+2, \cdots, n)$ in σ. Then we can write

$$\varepsilon=\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}^{12)}.$$

In τ, if c_{k_λ} and c_{k_ν} correspond to B_λ and $W_{\nu-n+m-1}$ respectively, we can write

$$\bar{\varepsilon}=\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}.$$

Hence it is sufficient to prove that

$$\operatorname{sgn}\zeta=\operatorname{sgn}\begin{pmatrix} j_1 & j_2 & \cdots & j_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}=1.$$

Let ζ be represented as the product of r cycles $\zeta_1, \zeta_2, \cdots, \zeta_r$, which are mutually disjoint. Since $\operatorname{sgn}\zeta=(\operatorname{sgn}\zeta_1)(\operatorname{sgn}\zeta_2)\cdots(\operatorname{sgn}\zeta_r)$, it is sufficient to show that $\operatorname{sgn}\zeta_i=1$ for every i.

Let $\zeta_i=(s_1\cdots s_k)$. Now let us assign a chain L, called an *L-chain*, to ζ_i as follows. Take a point, called a *center*, in each region and fix it. Since

11) It will be shown in § 5 that there exists such a σ.

12) Sgn $P=1$ or -1 according as P is an even or an odd permutation.

both c_{s_1} and c_{s_2} lie on the boundary of a region r_{i_1}, say, we can join c_{s_1} and c_{s_2} with the center a_{i_1} of r_{i_1} by a segment l_1, in r_{i_1}. l_1 will be oriented in the direction from c_{s_1} to c_{s_2} through a_{i_1}. In the same way, we can join c_{s_2} and c_{s_3} with the center a_{i_2} of r_{i_2} by a segment l_2 in r_{i_2}, and so forth. We set $L=\bigcup_{i=1}^{k} l_i$. L is a loop. If L contains the centers of black regions, then we shall transform L into L_0 as follows. Suppose the interior[13] of L does not contain r_α and r_β. If L contains an oriented segment joining c_λ with c_μ through the center b of a black region B, denoted by $c_\lambda b c_\mu$, we replace it by a chain of the segments $c_\lambda w_1 c_\nu \cup c_\nu w_2 c_\xi \cup \cdots \cup c_\zeta w_l c_\mu$, where $c_\lambda, \cdots, c_\mu$ are the crossing points such that, a point P moving positively or negatively from c_λ to c_μ along $\dot{B}$ according as the orientation of E induced by L is positive or negative[14], passes $c_\lambda, \cdots, c_\mu$ in this order, and where $w_1, \cdots, w_l$ are the centers of the white regions which have the sides $c_\lambda c_\nu, \cdots, c_\zeta c_\mu$ with $\dot{B}$ in common. Thus we obtain a figure F. Let us transform F into L_0 with two following operations. (a) If F contains $c_i w_j c_k \cup c_k w_j c_m$, then we shall replace it by $c_i w_j c_m$. (b) If F contains $c_i w_j c_k \cup c_k w_j c_i$, we shall take it away. Thus F is transformed into a loop L_0.

Here we shall prove the following two facts.

LEMMA 4.3. *Let p_0 and q_0 be the numbers of the black and the white regions in the interior L_0^0 (or the exterior) of L_0 respectively and let s_0 be the number of the crossing points in L_0^0. Then*

$$s_0=p_0+q_0-1\,.$$

PROOF. Let t be the number of the centers of the white regions on L_0. Since t is equal to the number of the crossing points on L_0, $L_0^0 \cup L_0$ is divided into s_0+2t points, $2s_0+3t$ segments and p_0+q_0+t faces by the crossing points, the centers and the sides. Hence Euler's characteristic χ of $L_0^0 \cup L_0$ is given by

$$\chi=s_0+2t-(2s_0+3t)+p_0+q_0+t=-s_0+p_0+q_0\,.$$

On the other hand $\chi=1$, since $L_0^0 \cup L_0$ is homeomorphic to an 2-dimensional closed cell. Thus we have $s_0=p_0+q_0-1$. q. e. d.

LEMMA 4.4. *Let p_1, q_1 and s_1 be the numbers of the black, the white regions and the crossing points in the interior of L. Then denoting the number of the centers of the (white) regions lying on L_0 by k_0, it follows*

---

13) We may call either one of two sets E_1 and E_2 into which E is divided by L, the interior and the other the exterior. But hereafter, we assume that the interior of L, or generally a loop, means the bounded set among E_1 and E_2.

14) If the exterior of L does not contain r_α and r_β, P will move along $\dot{B}$ in the inverse direction.

$$k_0=k+\sum_{i=1}^{p_0-p_1}(2\lambda_i-1)-(s_0-s_1)-(q_0-q_1)\,, \tag{4.1}$$

where λ_i are positive integers.

PROOF. The number[15] of the centers of the regions on F is given by $k+\sum_{i=1}^{p_0-p_1}(2\lambda_i-1)$, since it increases by $2\lambda_i-1$ per a black region which is contained in the interior of L_0 and is not contained in the interior of L. But the number of the centers of the regions on L_0 is first decreased by s_0-s_1 by the operation (a) and again it is decreased by q_0-q_1 by the operation (b). Thus we have (4.1). q. e. d.

Now in our case it follows $s_1=p_1+q_1$ by the definition. Hence it follows from Lemmas 4.3 and 4.4

$$\begin{aligned} k_0&=k+\sum_{i=1}^{p_0-p_1}(2\lambda_i-1)-(s_0-s_1)-(q_0-q_1)\\ &=k+2\sum_{i=1}^{p_0-p_1}\lambda_i-(p_0-p_1)-(p_0+q_0-1-p_1-q_1)-(q_0-q_1)\\ &\equiv k+1 \pmod 2\,. \end{aligned}$$

While $k_0\equiv 0 \pmod 2$, as shown from the fact that if c-corner of a white region X is dotted (or undotted), then c-corner of the white region X' which is opposite to X over c is undotted (or dotted). Hence we obtain $k\equiv 1$ (mod 2), i. e. $\operatorname{sgn}\zeta_i=1$. q. e. d.

§ 5. Proof of theorem.

The subset G (or H) of E obtained by connecting the centers of all the black regions (or all the white regions) with the crossing points lying on their boundaries will be called the *graph* (or the *dual graph*) of K. The segments of G (or H) connecting two consecutive centers of the regions are called *sides* of G (or H). There is only one crossing point on each side. Denote by M_k the regions into which E is divided by G. M_k contains clearly only one white region. We can suppose that the indices k are so arranged that $\bigcup_{\lambda=1}^{r}\dot{M}_\lambda\cap\dot{M}_{r+1}$ contains at least one side on $\dot{M}_{r+1}$ for $r=1, 2, \cdots, n-m+1$.

Now let us prove the existence of an L^{n-m+1}-correspondence. To do this let us assign in the following way to each crossing point one of the $n+2$ regions except a pair of a white region r_α, contained in M_1, and a black region r_β adjacent to r_α.

15) A center lying on the part $c_i\, w_j\, c_k \cup c_k\, w_j\, c_m$ or $c_i\, w_j\, c_k \cup c_k\, w_j\, c_i$ of F is counted doubly.

First we shall assign $n-m+2$ black regions except r_β to $n-m+1$ crossing points. Let $\dot{M}_1$ consist of t sides $m_1, m_2, \cdots, m_t$, where m_i denotes the side connecting the center of B_i with that of B_{i+1} through a crossing point c_{ji} for $i=1, 2, \cdots, t$. (We put $B_{t+1}=B_1$.) We assume here that B_i does not coincide with B_j for any i, j, $(i\neq j)$. It is easily seen that if $B_i=B_j$ for some i, j, then k will be a product knot. We shall consider this case in the next section. We can assume without loss of generality that r_β is the black region B_1. Now, from the definition of the graph either of the c_1-corner or the c_t-corner of B_1 is dotted. Let the c_1-corner of B_1 be dotted. Then, since the c_1-corner of B_2 is undotted, the c_2-corner of B_2 is dotted. In general the c_i-corner of B_i is dotted. Hence we shall assign B_i to c_i for $i=2, \cdots, t$. If the c_t-corner of B_1 is dotted, we shall assign B_i to c_{i+1}. Next let us suppose that each of the black regions except r_β, whose center is on $\bigcup_{j=1}^{h} \dot{M}_j$, corresponds to one and only one crossing point such that the corner of this region at the corresponding crossing point is dotted. Then we shall assign the regions whose centers are on $\dot{M}_{h+1}$ to the crossing point as follows. Let $\dot{M}_{h+1}$ consist of s sides $m_1', m_2', \cdots, m_s'$ and let $m_1', m_2', \cdots, m_{h_1}', m_{h_2}', m_{h_2+1}', \cdots, m_{h_3}', \cdots, m_{h_{\lambda-1}}', m_{h_{\lambda-1}+1}', \cdots, m_{h_\lambda}'$ be contained in $\bigcup_{j=1}^{h} \dot{M}_j$, where m_j' denotes the side connecting the center of B_j' with that of B_{j+1}' through a crossing point c_j'. Then either of the c_{h_1}'-corner or the c_{h_1+1}'-corner of B_{h_1+1} is dotted. If c_{h_1}'-corner of B_{h_1+1} is dotted, then we shall assign B_{h_1+1} to c_{h_1}. In general, we shall assign B_{h_1+i+1} to c_{h_1+i} for $0\leqq i\leqq h_2-h_1-2$. If c_{h_1+1}-corner of B_{h_1+1} is dotted, then we shall assign B_{h_1+i+1} to c_{h_1+i+1}. In the same way we shall assign B_{h_l+j+1} to c_{h_l+j} or c_{h_l+j+1} for $l=2, \cdots, \lambda$ and $j=0, 1, \cdots, h_{l+1}-h_l-2$; $h_{\lambda+1}=s$.

Thus we obtain a correspondence such that each of the black regions except r_β corresponds to a crossing point on its boundary, where the corner of each region at corresponding point is dotted.

Finally we shall assign all white regions except r_α to the crossing points. To do this we shall consider a subset M of G, called the *semi-graph* with respect to the correspondence of the black regions. M is defined as a subset of G obtained from G by striking out the sides, where the crossing points on these sides do not correspond to any black regions. Then we have

LEMMA 5.1. *M is a tree, i.e. M is connected and does not contain a loop.*

PROOF. Set $M^h=M\cap \bigcup_{j=1}^{h} \dot{M}_j$, for $h=1, 2, \cdots, n-m+2$. Then it is obvious that $M^1=\dot{M}_1-m_1$ or $M^1=\dot{M}_1-m_t$ according as the c_1-corner of B_1 is dotted or undotted and that M^1 is a tree. Furthermore it follows from the definition of M that if M^h is connected then M^{h+1} is connected. Hence we shall see that $M=M^{n-m+2}$ will be connected by the induction. To prove the latter half of Lemma let us compute the Euler's characteristic χ of M. Since M is divided into $2(n-m+1)+1$ points and $2(n-m+1)$ segments by $n-m+1$ crossing points and $n-m+2$ centers of the black regions on M, we have $\chi=1$. Hence M does not contain a loop. q. e. d.

Now let N be the subset of the dual graph H obtained by striking out from H the sides meeting with M. Then

LEMMA 5.2. *N is a tree.*

PROOF. If N is decomposed into two components N_1 and N_2, where $N_1 \cap N_2 = \phi$, the sides $h_1, h_2, \cdots, h_t$ of N connecting N_1 with N_2 are meeting with the sides $g_1, g_2, \cdots, g_t$ of M respectively. Then $g_1 \cup \cdots \cup g_t$ is a loop, which contradicts to Lemma 5.1. Furthermore N does not contain a loop. For, if N contains a loop T, then the interior of T contains at least one black region B. Since M contains the center of B, $M \cap T \neq \phi$. Thus $M \cap N \neq \phi$, which is a contradiction. q. e. d.

Now we shall assign the white regions to the crossing points by means of N. Let $n_1, \cdots, n_\lambda$ be all the sides of N connecting the center of r_α with the centers of the white regions $W_1, \cdots, W_\lambda$ through the crossing points $c_{l_1}, \cdots, c_{l_\lambda}$ respectively. Then we shall assign W_i to c_{l_i}. Next, to the crossing points c_{p_j} on the sides n_j' of N, except n_i through the centers of W_i, we shall assign the regions W_j' which are opposite to W_i over c_{p_j}. Thus we shall obtain a correspondence such that each white region except r_α will correspond to one and only one crossing point on its boundary. For, we see from definition of N that each white region corresponds to a crossing point and moreover we see that if two crossing point correspond to one white region, then N would contain a loop. Thus we obtain

LEMMA 5.3. *There is an L^{n-m+1}-correspondence as stated in Lemma* 4.2.

Similarly it follows

LEMMA 5.4. *There is an L^0-correspondence σ_0 such that each crossing point corresponds to one and only one of the $n+2$ regions except a certain pair of two adjacent regions.*

PROOF. σ_0 will be constructed as follows. If a crossing point c corresponds to a black region B in an L^{n-m+1}-correspondence, then we assign to c a black region B', which is opposite to B over c. Since c-corner of B' is undotted, we shall obtain a correspondence such that $n-m+1$ crossing points c_i correspond to $n-m+1$ black regions whose c_i-corners are undotted. For the rest it will be shown in the same way as in the proof of Lemma 5.3, as the analogue of Lemma 4.2 holds for an L^0-correspondence. q. e. d.

LEMMA 5.5. *If K is of m standard loops, the Alexander polynomial of k is a polynomial of degree $n-m+1$.*

PROOF. It follows from Lemma 4.2, 5.3 and 5.4.

PROOF OF THEOREM 1.1.

Denote the genus of k by $G(k)$. If K is of m standard loops, then $G(k) \leqq \frac{n-m+1}{2}$. Thus $2G(k) \leqq n-m+1=d$, where d denotes the degree of the Alexander polynomial of k. On the other hand $d \leqq 2G(k)$. Hence it follows $d=2G(k)$. Thus the proof of Theorem 1.1 is completed.

§ 6. Proof of corollary 1.2.

It is well known[16] that the Alexander polynomial of k_0 is the product of those of k_1 and k_2. Hence $d_0=d_1+d_2$, where d_i denote the degrees of the Alexander polynomials of k_i. Let K_i be the images of the regular projections of k_i onto E. From the assumption, there is a circle C on E such that C meets with K_0 at only two points P and Q, where P and Q are not crossing points and these lie on two sides of the boundary of a region r_k. C divides E into two parts E_1 and E_2, and $s=C \cap r_k$ divides r_k into two regions r_k' and r_k''. Let E_1 and E_2 contain r_k' and r_k'' respectively. Let $(E_1 \cap K_0) \cup s=K_1'$ and $(E_2 \cap K_0) \cup s=K_2'$. Since K_i' are equivalent to K_i, we shall write K_i instead of K_i'. Denoting the number of the crossing points and that of the standard loops of K_i by n_i and m_i respectively, the genera $G(k_i)$ are given by

$$G(k_i)=\frac{n_i-m_i+1}{2} \qquad \text{for } i=1,2. \tag{6.1}$$

Now it is obvious that

$$n_0=n_1+n_2. \tag{6.2}$$

To compute m_0, let us classify the regions into which E is divided by K_i, into two classes, called *black* and *white*, where the unbounded region always belongs to the black class. Then it is easy to show that

$$m_0=m_1+m_2-1. \tag{6.3}$$

Hence it follows from (6.1), (6.2) and (6.3) that

$$\begin{aligned} 2G(k_0) &\leqq n_0-m_0+1 \\ &=n_1+n_2-(m_1+m_2-1)+1 \\ &=(n_1-m_1+1)+(n_2-m_2+1) \\ &=2G(k_1)+2G(k_2) \\ &=d_1+d_2 \\ &=d_0. \end{aligned}$$

16) For example, see [1].

Since $d_0 \leqq 2G(k_0)$, we have $d_0 = 2G(k_0)$. q. e. d

Corollary 1.3 is immediately obtained from Corollary 1.2.

Hôsei University.

References

[1] J. W. Alexander, Topological invariants of knots and links, Trans. Amer. Math. Soc., **30** (1928), pp. 275–306.

[2] F. Frankel und L. Pontrjagin, Ein Knotensatz mit Anwendung auf die Dimensionstheorie, Math. Ann., **102** (1930).

[3] K. Reidemeister, Knotentheorie, Julius Springer, (1932).

[4] H. Schubert, Die eindeutige Zerlegbarkeit eines Knotens in Primknoten, S. Ber. Heidelberg. Akad. Wiss., (1949).

[5] H. Seifert, Über das Geschlecht von Knoten, Math. Ann., **110** (1935), pp. 571–592.

Journal of the Mathematical Society of Japan Vol. 10, No. 3, July, 1958

On the genus of the alternating knot II.

By Kunio MURASUGI

(Received Oct. 25, 1957)
(Revised May 12, 1958)

Let k be a knot and let $G(k)$ be the genus of k as defined by Seifert [6]. Let $\Delta(x)=a_0+a_1x+a_2x^2+\cdots+a_{2t}x^{2t}$ be the Alexander polynomial of k. Then Seifert has proved in [6] that we have always

$$t\leqq G(k). \tag{1}$$

In a previous paper [3], we proved that the equality holds in (1) for a knot in a special class of alternating knots. In the present paper we shall show that the equality holds in (1) for all alternating knots (Theorem 4.1). It was also shown in [3] that, for an alternating knot k of the class considered in that paper, the orientable surface spanning k, whose genus is equal to $G(k)$, is obtained by Seifert's construction [6]. It will be shown that this is the case for every alternating knot.

Furthermore we shall show that $\Delta(x)$ is "alternating" for an alternating knot k (Theorem 4.4).

From this theorem, we can immediately deduce the well-known fact that a knot 8_{19} in [2] is not equivalent with an alternating knot. Throughout this paper we shall use the same notations as in [3].

§ 1. Preliminaries.

Let k be a polygonal oriented knot in the 3-sphere S^3 and let S^2 be a 2-sphere in S^3, which does not meet k. Let K be an image of a regular projection of k into S^2.

Let K have n crossing points $c_1, c_2, \cdots, c_n$. Then K divides S^2 into $n+2$ regions $r_0, r_1, \cdots, r_{n+1}$, which are classified into two classes, called "black" or "white", in such a way that every side of K is the common boundary of black and white regions. (Whenever we speak of the classification of regions in "black" and "white", we always mean a classification of this nature.) As is well-known, an integer $I(r_i)$, called the index of r_i, corresponds to each region r_i. We have

LEMMA 1.1. *For two regions r_i and r_j of the same colour, we have*

$$I(r_i)\equiv I(r_j) \qquad (\text{mod } 2)$$

and conversely.

This is proved by the same method as used in the proof of Lemma 3.2 in [3].

Each corner of the two of the four regions[1] meeting at a crossing point c_i is marked with a dot, and we can assume that the signs of the elements distinct from zero in any column of the L-matrices are positive, i. e. either x or 1 (cf. [1], [3]).

§ 2. The loops of the first and of the second kind.

Let us divide K into some oriented loops, called the *standard loops*, in the same way as in [6].

DEFINITION 2.1. If a standard loop L bounds a region r_i, we say L is of the *first kind* and r_i is the region bounded by L. Otherwise L is of the *second kind.*

LEMMA 2.2. *The corners of the regions bounded by a loop L of the first kind are either all dotted or all undotted.*

This is proved in the same way as in Lemma 3.1 in [3].

Conversely it is obvious that

LEMMA 2.3. *If the corners of a region r_i are either all dotted or all undotted, $\dot{r}_i$ is a loop of the first kind.*

Let m be the number of the loops of the second kind of K. The case $m=0$ has been treated in [3]. In the following we assume $m\geqq 1$.

Now let us deform the loops of the second kind into the following loops. Let the loops C_i and C_j of the second kind have a crossing point c. Let ε be a sufficiently small positive number, and a, b and d, e the points of intersection of the circle (in S^2) of radius ε with the center c with C_i and C_j respectively (Fig. 1).

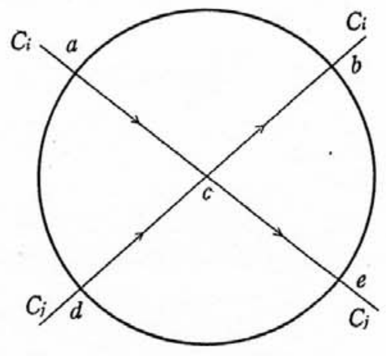

Fig. 1

Then we replace the parts $ac\cup cb$ and $dc\cup ce$ of C_i and C_j by the disjoint arcs ab and de respectively. If we perform this operation at each crossing

1) We may assume that these regions are different from one another. See the note 5) in [3].

point of two loops of the second kind, then we obtain m disjoint loops. These m disjoint loops will be called hereafter loops of the second kind, and if we need to consider the loops of the second kind in the older sense of Def. 2.1, we shall mention it expressly. Then it is obvious that

LEMMA 2.4. *m loops of the second kind divide S^2 into $m+1$ domains*[2] $E_0, E_1, \cdots, E_m$.

LEMMA 2.5. *Let E_j be a domain bounded by some loops, $C_{j_1}, \cdots, C_{j_\nu}$, of the second kind: $\dot{E}_j = C_{j_1} \cup \cdots \cup C_{j_\nu}$. Then the regions $r_\xi, \cdots, r_\eta$ contained in E_j having some sides in common with C_{j_i} have the same index (depending on j_i).*

Furthermore we have

LEMMA 2.6. *The regions contained in the domain E_j can be classified in black and white, and in such a way that the regions having some sides in common with $\dot{E}_j$ have the same colour, say white. All these white regions have then the same index, say p, and the black regions have loops of the first kind as boundaries. Then indices of the black regions are either $p-1$ or $p+1$.*

PROOF. Let $\dot{E}_j = C_{j_1} \cup \cdots \cup C_{j_\nu}$, where C_{j_i}, $i=1, 2, \cdots, \nu$, are loops of the second kind, and $r_\xi, \cdots, r_\eta$ the regions contained in E_j such that each of $\dot{r}_\xi, \cdots, \dot{r}_\eta$ has some sides with C_{j_1} in common. By Lemma 2.5, we have $I(r_\xi) = \cdots = I(r_\eta)$, so that, if we classify $r_0, \cdots, r_{n+1}$ in black and white as said above, $r_\xi, \cdots, r_\eta$ have the same colour, say white, by Lemma 1.1. Let us fix this classification, and let r_λ be one of black regions contained in E_j such that $\dot{r}_\lambda$ has some sides in common with one of $\dot{r}_\xi, \cdots, \dot{r}_\eta$. Then $\dot{r}_\lambda$ is a loop of the first kind, because, if a common side of $\dot{r}_\lambda$ and $\dot{r}_\xi$, say, were a part of a loop of the second kind, r_λ would not be contained in E_j. Now let r_μ be a black region in E_j opposite to r_λ over a crossing point c_k. Then $\dot{r}_\mu$ is also a loop of the first kind (Fig. 2).

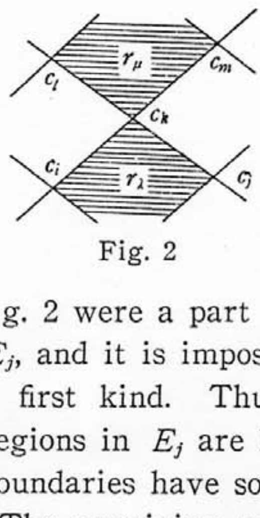

Fig. 2

In fact, if $c_m c_k \cup c_k c_l$ in Fig. 2 were a part of a loop of the second kind, r_μ would not be contained in E_j, and it is impossible that $c_m c_k$ and $c_k c_l$ belong to two different loops of the first kind. Thus it is easily seen that the boundaries of all the black regions in E_j are loops of the first kind. Hence the regions in E_j, whose boundaries have some sides in common with any one of $C_{j_1}, \cdots, C_{j_\nu}$, are white. The remaining part is easy to prove.

2) A domain is connected and is an open subset of S^2.

Hereafter we shall use almost exclusively the classification in black and white of the regions contained in each domain E_j and consider the classification of all regions $r_0, \cdots, r_{n+1}$ only in exceptional cases.

§ 3. Sign of the domain.

Hitherto the numbering of the domains $E_0, E_1, \cdots, E_m$ and loops of the second kind $C_1, C_2, \cdots, C_m$ was made arbitrarily. Now we introduce some rules on the numbering.

There is at least one loop of the second kind such that one of the two parts, into which S^2 is divided by it, does not contain other loops of the second kind. Let us fix one of these loops and denote it by C_1. We denote the domain bounded by C_1 which does not contain any loop of the second kind by E_0, and the domain bounded by C_1 and other loops of the second kind by E_1. Let the domains bounded by loops of the second kind other than E_0, E_1 be numbered arbitrarily. They will be denoted by $E_2, \cdots, E_m$. We define the *outer boundary* C_i of $E_i, i=2, \cdots, m$, as follows. C_i is one of the loops of the second kind bounding E_i such that the following holds: C_i divides S^2 into two parts, one of which contains E_0 and the other E_i. It is clear that the loops of the second kind and the domains bounded by them are thus numbered consistently.

Now let us take a point e_i from each E_i for $i=0, 1, \cdots, m$, and fix it. Let l_{ij} be an arc connecting e_i with e_j not crossing over any crossing point and not touching any loop of the second kind. We shall now define an *intersection number* $I(l_{ij}, C_h, q)$ for a point q at which l_{ij} meets C_h.

DEFINITION 3.1. $I(l_{ij}, C_h, q)=+1$, or -1 according as l_{ij} crosses over C_h at q from the right to the left or from the left to the right with reference to the orientation of C_h. We set $I(l_{ij}, C_h)=\sum_q I(l_{ij}, C_h, q)$. If l_{ij} and C_h are disjoint, we set $I(l_{ij}, C_h)=0$. Then set $e_{ij}=\sum_h I(l_{ij}, C_h)$.

It is easily shown that

LEMMA 3.2. *e_{ij} is uniquely determined by e_i and e_j independently of the choice of l_{ij}.*

Hence we may assume that l_{ij} meets C_h at most at one point for every i, j and h. We can easily show that

$$e_{ij}=-e_{ji} \tag{3.1}$$

$$e_{ij}=e_{ih}+e_{hj} \qquad 0 \leqq i, j, h \leqq m . \tag{3.2}$$

DEFINITION 3.3. We shall call the *sign* of E_j $(j=1, 2, \cdots, m)$ positive or negative according as $I(l_{0j}, C_j, C_j \cap l_{0j})=1$ or -1. The sign of E_0 is defined as the same as that of E_1.

We may assume without loss of generality that $E_0, \cdots, E_q$ are positive and

$E_{d+1},\cdots, E_m$ are negative, where $d\geqq 1$. (We have only to change the orientation of the knot and change the numbering of $E_2,\cdots, E_m$, if necessary.) Let us put $\min_i I(r_i)=p-1$ and $\max_j I(r_j)=p+h+1$. We may suppose $h\geqq 1$.[3)]

LEMMA 3.4. *The regions with the maximal and minimal indices are the black regions and the corners of the former are all dotted.*

PROOF. Suppose that the region r_{i_0} with the maximal index $p+h+1$ is white. Let r_{i_0} be contained in E_t. Then E_t is positive. For otherwise, the index of a white region in E_s, which is a domain separated from E_t by C_t, will be $p+h+2$, which is a contradiction. Furthermore E_t must contain black regions. For otherwise, there would exist a positive domain E_u whose outer boundary would be $\subset\dot{E}_t$. Hence a white region in E_u would be of the index $p+h+2$, which is a contradiction. Consequently, E_t must contain a black region with the index $p+h+2$. This contradicts the assumption. Hence r_{i_0} is a black region. It will be easily shown that the corners of r_{i_0} are all dotted.

In the same way, we shall see that a region with the minimal index is black, q. e. d.

REMARK. More generally, we obtain the following Lemma in the same way as above.

LEMMA 3.5. $\max_i I(r_i)-\min_j I(r_j)=\max_{0\leqq i,j\leqq m} e_{ij}+2.$

As this lemma will not be used in following sections, the proof is omitted.

§ 4. Statement of the main theorems.

As mentioned in the introduction, our main theorems are the following:

THEOREM 4.1. *The genus of an alternating knot is exactly one half of the degree of its Alexander polynomial.*

This will be proved in § 7. Hence follows in the same way as in § 8 [3]

THEOREM 4.2. *The genus of the product knot*[4)] k_0 *of the two alternating knots* k_1 *and* k_2 *is exactly one half of the degree of its Alexander polynomial.*

Since the Alexander polynomial of k_0 is the product of those of k_1 and k_2, we have

COROLLARY 4.3.[5)] *The genus of* k_0 *is equal to the sum of the genera of* k_1 *and* k_2.

Furthermore, we have

THEOREM 4.4. *If k is an alternating knot, then its Alexander polynomial is*

3) If $h=0$, there is no loop of the second kind. This case was considered in [3].
4) k_0 may not be alternating.
5) This fact is already shown by H. Schubert in [5] for all knots.

of the form

$$\Delta(x)=a_0-a_1x+a_2x^2-\cdots+(-1)^t a_t x^t+\cdots+a_{2t}x^{2t},$$

where $a_i\geqq 0$, *and in particular,* a_0, a_t *and* $a_{2t}\neq 0$, *and* $a_i=a_{2t-i}$ *for* $i=0,1,\cdots,2t$.

§ 5. Preparations for the proofs of theorems.

Let Δ_{pq} be the determinant of the matrix obtained by striking out from the L-matrix of K two columns corresponding to two regions with indices p and q. Since $\Delta_{(q+1)q}=\pm x^{r-q}\Delta_{(r+1)r}$, the determinant of the smallest degree with respect to x among the determinants of the forms $\Delta_{(s+1)s}$ is $\Delta_{(p+h+1)(p+h)}$. Hence the Alexander polynomial $\Delta(x)$ of k is

(5.1) $$\Delta(x)=\pm x^{-\mu}\Delta_{(p+h+1)(p+h)},$$

where μ is a non-negative integer. Now the determinant of the matrix obtained by striking out from the L-matrix two columns corresponding to two adjacent white regions r_α and r_β contained in E_0 and E_1 respectively, may be denoted by $\Delta_{(p+q+1)(p+q)}$, with a suitable q, $0\leqq q\leqq h-1$, and we have

(5.2) $$\Delta_{(p+h+1)(p+h)}=\pm x^{q-h}\Delta_{(p+q+1)(p+q)}.$$

If λ denotes the number of the black regions with all dotted corners, then we have

(5.3) $$\Delta_{(p+q+1)(p+q)}=x^{\lambda}\Delta^0_{(p+q+1)(p+q)}.$$ [6]

Hence, from (5.1), (5.2) and (5.3), we have

(5.4) $$\pm x^{h+\mu-\lambda-q}\Delta(x)=\Delta^0_{(p+q+1)(p+q)}.$$

Consequently, the proof of the main theorem will be complete if only we prove the following

LEMMA 5.1. $\Delta^0_{(p+q+1)(p+q)}$ *has terms of the degrees* $\sum_{i=0}^{m}w_i-m+d-2$ *and* $d-1$, *where* w_i *denotes the number of the white regions in* E_i, *and where* $d+1$ *is the number of the positive domains.*

In fact, it will follow that $h+\mu-q-\lambda\leqq d-1$ and $h+\mu-q-\lambda+2t\geqq\sum_{i=0}^{m}w_i-m+d-2$, where $2t$ is the degree of $\Delta(x)$. Hence $2t\geqq\sum_{i=0}^{m}w_i-m+d-2-(h+\mu-q-\lambda)\geqq\sum_{i=0}^{m}w_i-m+d-2-(d-1)=\sum_{i=0}^{m}w_i-m-1$. On the other hand, we have $2t\leqq 2G(k)\leqq n-(\sum_{i=0}^{m}b_i+m)+1=(\sum_{i=0}^{m}w_i+\sum_{i=0}^{m}b_i-2)-(\sum_{i=0}^{m}b_i+m)+1=\sum_{i=0}^{m}w_i-m-1$, where $G(k)$ denotes the genus of k and b_i denotes the number of the black regions in E_i. Therefore we have $t=G(k)$.

6) See [3].

§ 6. Preparations for the proofs of theorems, continued.

Let us denote the white regions in E_i by $W_{i,1},\cdots,W_{i,h_i}$, and the black regions in E_i by $B_{i,1},\cdots,B_{i,l_i}$. Let $K_i=\bigcup_{\lambda=1}^{h_i} W_{i,\lambda}\bigcup_{\mu=1}^{l_i} B_{i,\mu}$.

DEFINITION 6.1. A crossing point such that at least two of four regions meeting at it are contained in E_i is called a crossing point which is *contained in* K_i (or simply *in* K_i).

Hereafter a side of K_i will mean a segment of K_i connecting two consecutive crossing points *in* K_i. Then K_i may be regarded as an image of the regular projection of a link[7] into S^2, and we have clearly

LEMMA 6.2. *K_i are alternating.*

Since there is no loop of the second kind in K_i, lemmas obtained in [3] hold for K_i with slight modifications. Consequently it follows in the same way as in Lemma 3.6 in [3]

LEMMA 6.3. *The corners of the black regions in E_i are either all dotted or all undotted. And the corners adjacent to the dotted (or undotted) corners of the white regions in E_i are undotted (or dotted). We shall say that the c-corner and c′-corner of a region are adjacent, if two crossing points c and c′ are connected by a side of K_i.*

Let c be a crossing point on C_i not contained in K_i and let a region r_j in E_i be one of the four regions meeting at c. Then it will be easily shown that

LEMMA 6.4. *The c-corner of r_j is either dotted or undotted according as E_i is positive or negative.*

LEMMA 6.5. *Let $\bar{s}$ be the number of the crossing points in K_i, $\bar{p}$ the number of the regions in E_i and let $\dot{E}_i$ consist of the $\bar{q}$ loops of the second kind. Then*

$$\bar{s}=\bar{p}+\bar{q}-2\,.$$

PROOF. The number of the sides of K_i is given by $2\bar{s}$. Since $\bar{s}$ crossing points and $2\bar{s}$ sides divide $\bar{E}_i$[8] into $\bar{s}$ points, $2\bar{s}$ segments and $\bar{p}$ faces, Euler's characteristic χ of $\bar{E}_i$ is given by $\chi=\bar{s}-2\bar{s}+\bar{p}=-\bar{s}+\bar{p}$. On the other hand, $\chi=-\bar{q}+2$, since $\bar{E}_i$ is homeomorphic to a 2-sphere with $\bar{q}$ holes. Thus we have $\bar{s}=\bar{p}+\bar{q}-2$, q. e. d.

LEMMA 6.6. *Let σ be an L^t-correspondence[9] such that each crossing point*

---

7) A link means a figure composed of a finite number of the disjoint knots in S^3. We can define the standard loops of the first and of the second kind for an image of the regular projection of a link in the same way as for a knot.

8) A bar over the symbol denotes the closure of the set.

9) In the next section, we shall show that there exists such a σ. See [3] for the definition of an L^t-correspondence.

corresponds to one and only one of the $n+2$ *regions except for a pair of two adjacent regions* r_α *and* r_β *contained in* E_0 *and* E_1 *respectively. Then at least one region in* E_i *must correspond by* σ *to a crossing point on* C_i *not contained in* K_i *for* $i=2,\cdots,m$.

PROOF. If E_i is bounded by the outer boundary C_i alone, this lemma is true by Lemma 6.5. Now let us suppose that E_i is bounded by $l+1$ loops $C_{i_1},\cdots,C_{i_l}$ and C_i, and the lemma is true for domains $E_{i_1},\cdots,E_{i_{l-1}}$ and E_{i_l}. That is, let us suppose that $t_{i_h}(\geqq 1)$ regions in E_{i_h} correspond to crossing points not contained in K_{i_h}. Since the number of the crossing points in K_i is larger than the number of the regions in E_i by $l-1$, $\sum_{h=1}^{l} t_{i_h}-l+1$ $(\geqq 1)$ regions in E_i must correspond to the crossing points not contained in K_i. Thus at least one region in E_i must correspond to a crossing point on C_i not contained in K_i, q. e. d.

In the special case where $\bar{l}=\sum_{i=0}^{m} w_i-m+d-2$, it follows

LEMMA 6.7. w_i+b_i-1 *regions in* E_i *correspond to the crossing points in* K_i *for* $i=2,\cdots,m$.

PROOF. Let us suppose that t_i (>1) regions in E_i correspond to the crossing points on C_i not contained in K_i. If E_i is negative, t_i (white) regions in E_i correspond to the crossing points at which these regions have undotted corners. On the other hand, if E_i is positive, t_j (white) regions in E_j, which is separated from E_i by C_i, correspond to the crossing points at which these regions have undotted corners. Thus in all cases it is impossible that σ is an $L^{\Sigma w_i-m+d-2}$-correspondence, since at least one white region in every E_i for $i=d+1,\cdots,m$, corresponds to a crossing point on C_i not contained in K_i at which this region has undotted corner, q. e. d.

LEMMA 6.8. *Let* σ *be an* $L^{\bar{l}}$*-correspondence and let* τ *be another* $L^{\bar{l}}$*-correspondence,* $\bar{l}=\sum_{i=0}^{m} w_i-m+d-2$, *such that the following property* (P) *holds:*

(P) σ *and* τ *are defined on the same set of regions, and each of* σ, τ *assigns each region of this set to some crossing point, the correspondence between the regions and crossing points defined by* σ *and* τ *being allowed to be entirely different.*

Then denoting the terms in $\Delta^0_{(p+q+1)(p+q)}$ *corresponding to* σ *and* τ *by* $\varepsilon x^{\bar{l}}$ *and* $\bar{\varepsilon} x^{\bar{l}}$ *respectively, it follows*

$$\varepsilon=\bar{\varepsilon}.$$

PROOF. Let L_h be the closed and oriented L-chain corresponding to a cyclic permutation ζ_h as used in the proof of Lemma 4.2 in [3]. To show $\operatorname{sgn} \zeta_h=1$, it is sufficient to show that the number of the centers of regions on L_h is odd.

First we shall show that if L_h crosses over the outer boundary of a domain, then it will cross over the boundary in just two places. In fact, let us suppose that L_h crosses over C_i at least at four crossing points. If L_h goes over C_i into E_i through some two crossing points, we see from Lemma 6.6 that these crossing points are not contained in K_i and these correspond to some two regions in E_i, which contradicts Lemma 6.7. Moreover it follows from the above fact that L_h does not cross over C_1.

Next we shall show the following

LEMMA 6.9. *Let T_h be any L-chain and $T_h \cap E_j = T^1 \cup \cdots \cup T^p$ and*

$$T^i = c_{i,1}x_{i,1}c_{i,2} \cup c_{i,2}x_{i,2}c_{i,3} \cup \cdots \cup c_{i,\lambda_i}x_{i,\lambda_i}c_{i,\lambda_i+1}\ ^{10)} \qquad for \quad i=1,\cdots,p,$$

where $x_{i,1},\cdots,x_{i,\lambda_i}$ are the centers of the regions in T^i and $c_{i,1},\cdots,c_{i,\lambda_i+1}$ are the crossing points in T^i. Let t_i denote the number of the centers of the regions in T^i.

(a) *If all c_{i_μ} are contained in K_j, then it follows*

$$\sum t_i \equiv p+1 \qquad (\text{mod } 2).$$

(b) *If c_{11} and c_{p,λ_p+1} are not contained in K_j and others are all contained in K_j, then it follows*

$$\sum t_i \equiv p \qquad (\text{mod } 2).$$

(c) *If $x_{i,1},\cdots,x_{i,\lambda_i}$ are all the centers of the black regions for some i, then t_i is odd or even according as the $c_{i,1}$-corner of $r_{i,1}$ and the c_{i,λ_i+1}-corner of r_{i,λ_i} are either all dotted (or undotted) or not, where r_h denotes the black region in E_j with the center x_h.*

PROOF of (a). In the same way as in Lemma 4.2 in [3], we have $\sum t_i + p \equiv 1$ (mod 2), which is equivalent to (a).

PROOF of (b). Let us transform T^i into T_0^i as constructed in the proof of Lemma 4.2 in [3]. Here, in particular, we transform $c_{11}x_{11}c_{12}$ and $c_{p,\lambda_p}x_{p,\lambda_p}c_{p,\lambda_p+1}$ into the chains $c'_{11}y_{11}c'_{12} \cup c'_{12}y_{12}c'_{13} \cup \cdots \cup c'_{1\mu}y_{1\mu}c_{12}$ and $c_{p,\lambda_p}z_{p1}c''_{p1} \cup c''_{p1}z_{p2}c''_{p2} \cup \cdots \cup c''_{p,\nu-1}z_p c''_{p\nu}$, respectively, where $c'_{1\xi}$ and $c''_{p\eta}$ are crossing points on the boundaries of the white regions r_{11} and r_{p,λ_p} respectively and c'_{11} and $c''_{p\nu}$ are contained in K_j and lie on C_h, and $y_{1\xi}$ and $z_{p\eta}$ are the centers of the black regions whose boundaries have the sides $c'_{1\xi}c'_{1,\xi+1}$ and $c''_{p,\eta-1}c''_{p,\eta}$ with r_{11} and r_{p,λ_p} in common, respectively, for $\xi=1,2,\cdots,\mu$, $\eta=1,2,\cdots,\nu$ and $c'_{1,\mu+1}=c_{12}$, $c''_{p,0}=c_{p,\lambda_p}$. Let $\bar{w}_1$ be the number of the white regions and $\bar{b}_1$ the number of the black regions, which are contained in a domain D in E_j bounded by $T^1,\cdots,T^p$ and the parts $C^0,C^1,\cdots,C^p$ of $C_j,C_{j_1},\cdots,C_{j_p}$, which are contained in $\dot{E}_j$. Let $\bar{s}_1$ be the number of the crossing points in $D \cap K_j$. Similarly let $\bar{w}_0$ and $\bar{b}_0$ be the numbers of the white and the black regions in D_0 respectively,

10) For the notation see the proof of Lemma 4.2 in [3].

which is bounded by $T_0^1,\cdots,T_0^p,C_0^0,C^1,\cdots,C^p$, where C_0^0 is the curve connecting c'_{11} with $c''_{p\nu}$ on C^0 or on the complement of C^0 with respect to C_i, and $\bar{s}_0$ be the number of the crossing points in $D_0\cap K_j$. Then denoting the number of the centers of the white regions on T^i by u_i, we have $\bar{w}_0=\bar{w}_1+\sum_{i=1}^{p}u_i$. Let $\bar{b}_0=\bar{b}_1+\bar{k}$. Then, since $\bar{s}_1=\bar{b}_1+\bar{w}_1$ by the definition, it follows $\bar{s}_0=\bar{b}_0+\bar{w}_0-1$ $=\bar{s}_1+\sum u_i+\bar{k}-1$.[11] Moreover since one of μ and ν is odd and the other even, we can write $\mu+\nu-2=2\gamma-1$. Hence denoting the number of the centers of the regions in $\bigcup_{i=1}^{p}T_0{}^i$ by t_0, we have[12]

$$t_0=\sum_{i=1}^{p}t_i+\sum_{i=1}^{u_1-1}(2\lambda_{i1}-1)+\sum_{j=2}^{p-1}\sum_{i=1}^{u_j}(2\lambda_{ij}-1)+\sum_{i=1}^{u_p-1}(2\lambda_{ip}-1)+2\gamma-1-(\bar{s}_0-\bar{s}_1+\bar{k})$$

$$\equiv\sum_{i=1}^{p}t_i-(u_1-1)-\sum_{j=2}^{p-1}u_j-(u_p-1)-1-(\sum_{i=1}^{p}u_i+2\bar{k}-1) \qquad (\mathrm{mod}\ 2)$$

$$\equiv\sum_{i=1}^{p}t_i \qquad (\mathrm{mod}\ 2) \qquad (\lambda_{ij}\ \text{intgers}).$$

On the other hand, since $t_0\equiv p$ (mod 2), we have $\sum_{i=1}^{p}t_i\equiv p$ (mod 2).

PROOF of (c). If the c-corner of the black region r_i is dotted, then the c-corner of the black region r_j which is opposite to r_i over c is undotted and conversely. From this, (c) is immediately proved.

Thus Lemma 6.9 is proved.

Now we shall prove Lemma 6.8.

Let L_h be divided into $L_h=\bigcup_{i=1}^{p_1}L_i^{(0)}\cup L^{(1)}$, where all $L_i^{(0)}$ are connected and contained in only one domain $\bar{E}_{h_1}$, and $\bigcup_{i=1}^{p_1}L_i^{(0)}\cap C_h=\phi$[13] and $L^{(1)}=L_h-\bigcup_{i=1}^{p_1}L_i^{(0)}$. Now denoting the number of the centers of regions in $L_j^{(i)}$ by $t_j^{(i)}$, we have, by Lemma 6.9 (a),

$$\sum_{i=1}^{p_1}t_i^{(0)}\equiv p_1+1 \qquad (\mathrm{mod}\ 2).$$

Next consider $L^{(1)}$. $L^{(1)}$ consists of p_1 L-chains $L_1^{(1)},\cdots,L_{p_1}^{(1)}$, whose end points are on the outer boundaries $C_{l,1},\cdots,C_{l,p_1}$ and are not contained in $K_{l,1},\cdots,K_{l,p_1}$, respectively. Let $L_1^{(1)}$ be divided into $L_1^{(1)}=\bigcup_{i=1}^{p_{11}}L_i^{(11)}\cup L^{(110)}$, where all $L_i^{(11)}$ are contained in a domain $\bar{E}_{l,i}$ and $L^{(110)}=L_1^{(1)}-\bigcup_{i=1}^{p_{11}}L_i^{(11)}$. Then by Lemma 6.9 (b), we have

$$\sum t_i^{(11)}\equiv p_{11} \qquad (\mathrm{mod}\ 2).$$

11) See (4.3) in [3].

12) See (4.1) in [3].

13) ϕ denotes the empty set.

Defining $t_l^{(ij)}$ and p_{1j} in the same way as above, we have

$$\sum_l \sum_{j=1}^{p_1} t_l^{(1j)} \equiv \sum_{j=1}^{p_1} p_{1j} \qquad (\text{mod } 2).$$

Moreover dividing $L^{(110)}$ into some L-chains and computing $t_j^{(11h)}$ and p_{11h} in the same way as above, we have

$$\sum_j \sum_h t_j^{(11h)} \equiv \sum_h p_{11h} \qquad (\text{mod } 2).$$

Since the above decomposition will finish after a finite number of steps, the number t of the centers of the regions in L_h will finally be given by

$$t = \sum t_i^{(0)} + \sum_{j,l} t_l^{(1j)} + \sum t_j^{(1**)} + \cdots + \sum t_j^{(1**\cdots*)}$$

$$\equiv p_1 + 1 + \sum p_{1j} + \sum p_{1**} + \cdots + \sum p_{1*\cdots*}.$$

On the other hand, $p_1 + \sum p_{1j} + \cdots + \sum p_{1*\cdots*}$ is even by Lemma 6.9 (c). Hence we have $t \equiv 1$ (mod 2). Thus Lemma 6.8 is proved.

§ 7. Proof of Theorem 4.1.

In this section, we shall show that there exists an $L^{\bar{t}}$-correspondence, where $\bar{t} = \sum_{i=0}^{m} w_i - m + d - 2$.

Let G_j be the graph[14] of K_j. Denote the regions into which G_j divides S^2 by M_{ji}. Then, if we regard the complement of E_j as the black regions, then we see clearly that each M_{ji} contains one and only one black region. We can suppose that the indices i, j are so arranged that M_{j1} contains C_j for $j=1,\cdots,m$, and M_{01} contains C_1, and $(\bigcup_{i=1}^{\lambda-1} \dot{M}_{ji}) \cap \dot{M}_{j\lambda}$ must contain at least one side of $\dot{M}_{j\lambda}$.

Let r_α and r_β be a pair of two adjacent white regions in E_0 and E_1 respectively. Then we can assign each one of the $w_0 + w_1$ white regions in E_0 and E_1 except for r_α and r_β to one and only one crossing point lying on its boundary by means of the graphs G_0 and G_1 in the same way as in [3], where the corner of the region at the corresponding crossing point is dotted. Let P_0 and P_1 denote the semi-graph of G_0 and G_1 with respect to the correspondences of the white regions in E_0 and E_1 respectively. Then P_0 and P_1 are disjoint and these are trees. Now let $\dot{E}_1 = C_1 \cup C_{i_1} \cup \cdots \cup C_{i_{j_1}}$. Then we have

LEMMA 7.1. *In each E_{i_λ}, there exists a region r_{i_λ}, say, whose center is on a*

---

14) The graph (or the dual graph) of K means the totality of the segments connecting the centers of the white (or the black) regions with the crossing points lying on their boundaries.

side m_{i_λ} *in* $M_{i_\lambda,0}$, *and each* $\dot{r}_{i_\lambda}$ *contains at least one crossing point* c_{i_λ} *which is not contained in* K_{i_λ}.

PROOF. If there does not exist such a region in E_{i_μ}, then P_1 would contain the boundary of $M_{1\lambda}$, in which E_{i_μ} would be contained.

Furthermore we have

LEMMA 7.2. *We can so choose these crossing points* c_{i_λ} *that they are different from each other.*

PROOF. If $c_{i_\mu}=c_{i_\nu}$ for some μ, ν, i.e. if there is only one crossing point which is not contained in K_{i_μ} and K_{i_ν}, there would be $M_{1\xi}$ and $M_{1\eta}$, in which E_{i_μ} and E_{i_ν} would be contained, and P_1 would contain a loop $\dot{M}_{1\xi}\cup\dot{M}_{1\eta}-(\dot{M}_{1\xi}\cap\dot{M}_{1\eta})$.

Now we can assign each one $r_{i_\lambda,j}$ of the w_{i_λ} white regions in E_{i_λ} except for the regions r_{i_λ}, whose existence is assured in Lemma 7.1, to only one crossing point contained in K_{i_λ} which lies on $\dot{r}_{i_\lambda,j}$ by means of the graphs G_{i_λ}, where the corners of the regions at the corresponding crossing points are dotted. Let P_{i_λ} denote the semi-graph of G_{i_λ} with respect to the correspondence of the white regions in E_{i_λ}. Then P_{i_λ} are the trees and these are mutually disjoint. In the same way, we obtain

LEMMA 7.3. *In each* E_i, *there is one white region* r_i, *say, whose center is on a side of* M_{i0} *and there exists on* $\dot{r}_i$ *at least one crossing point* c_i, *say, not contained in* K_i. *And these crossing points are different from each other.*

Let P_i be the semi-graph of G_i with respect to the correspondence of all the white regions except for r_i in E_i. P_i are mutually disjoint.

Now we shall prove the existence of an L^l-correspondence. This will be performed if we can assign each one of the $m-1$ white regions r_i and the $\sum_{i=0}^{m} b_i$ black regions to one and only one crossing point. To do this, we shall first assign r_i (in E_i) to a crossing point c_i obtained by Lemma 7.3. Next, to obtain a correspondence between the black regions in each E_i and the crossing points, we shall apply the proof of Lemma 5.3 in [3] to our case. We regard the region r_i and the connected component, which contains E_0, in the complement of E_i as r_α and r_β respectively and we consider the subset Q_i, disjoint to P_i, of the dual graph H_i of K_i. Then we can assign also black regions to the crossing points on its boundaries by means of Q_i. Thus we obtain the required correspondence. Thus we have

LEMMA 7.4. *There is an* $L^{\Sigma w_i-m+d-2}$*-correspondence* σ *as stated in Lemma* 6.7.

Similarly, it follows

LEMMA 7.5. *There is an* L^{d-1}*-correspondence.*

From Lemmas 7.4 and 7.5, we have Lemma 5.7. Thus the proof of Theorem 4.1 is completed.

§ 8. Proof of Theorem 4.4.

We can slightly extend Lemma 6.8 as follows.

LEMMA 8.1. *Let σ be an $L^{\bar{t}}$-correspondence and τ an $L^{\bar{s}}$-correspondence, $d-1\leqq\bar{t}$, $\bar{s}\leqq\sum_{i=0}^{m} w_i-m+d-2$, which have the property* (P) *as stated in Lemma* 6.8. *If the terms in $\Delta^0_{(p+q+1)(p+q)}$ corresponding to σ and τ are denoted by $\varepsilon x^{\bar{t}}$ and $\bar{\varepsilon}x^{\bar{s}}$, where $\varepsilon, \bar{\varepsilon}=\pm1$, then $\varepsilon=\bar{\varepsilon}$ or $\varepsilon=-\bar{\varepsilon}$ according as $\bar{t}\equiv\bar{s}$* (mod 2) *or not.*

PROOF. We can assume without loss of generality that $\bar{t}=\sum_{i=0}^{m} w_i-m+d-2$. First we shall prove this lemma in the case where $m=0$ and $d=1$, i. e. $\bar{t}=w_0-1$. We may suppose that n crossing points $c_1, c_2, \cdots, c_n$ correspond to n regions $r_1, r_2, \cdots, r_n$ respectively, of which first w_0-1 regions are white, by σ. Let c_{ji} correspond to r_i by τ for $i=1, \cdots, n$ and let us assume that c_{jh}-corner of r_h are dotted for $h=1, \cdots, \bar{s}$ and c_{jl}-corner of r_l are undotted for $l=\bar{s}+1, \cdots, w_0-1$. Then, to prove Lemma 8.1, it is sufficient to show that

$$\operatorname{sgn}\zeta=\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}. \tag{8.1}$$

Let ζ be represented as the product of some cyclic permutations $\zeta_1, \zeta_2, \cdots, \zeta_r$, which are mutually disjoint.

Let $\zeta_1=(y_1\cdots y_h)$, $1\leqq y_1, \cdots, y_h\leqq n$. Consider an oriented L-chain, L corresponding to ζ_1. Let us assume that L_1 contains t_1 centers of white regions, of which α_1 centers lie on the segments of L_1 oriented as proceeding from the dotted corner to the undotted corner. Then we shall transform L_1 into L_0 which does not contain the centers of white regions, in the same way as in the proof of Lemma 4.2 in [3]. Let p_1 be the number of the white regions, q_1 the number of the black regions and let s_1 the number of crossing points, which are contained in the interior[15] of L_1. Then we have $s_1=p_1+q_1$. On the other hand, the number of the white regions contained in the interior $\widetilde{L}_0$ of L_0 is given by p_1+t_1. Denoting the number of the black regions contained in $\widetilde{L}_0$ by $q_1+\bar{w}_1$, the number of the crossing points contained in $\widetilde{L}_0$ is given by $s_0=q_1+p_1+t_1+\bar{w}_1-1=s_1+\bar{w}_1+t_1-1$. If the number of the centers of the regions lying on L_0 is denoted by h_0, then it follows

$$h_0=h+\sum_{i=1}^{t_1-\alpha_1}(2\lambda_i-1)+\sum_{j=1}^{\alpha_1}2(\mu_j-1)-(s_0-s_1+\bar{w}_1)$$

15) The interior of L_1 means the parts in which L_0 is not contained, between two parts into which S^2 are divided by L_1.

$$=h+2\sum\lambda_i-(t_1-\alpha_1)+2\sum(\mu_j-1)-(2\overline{w}_1+t_1-1)$$

$$\equiv h+\alpha_1+1 \qquad (\text{mod } 2) \quad (\lambda_i, \mu_j \text{ being positive integers}).$$

Thus we have $h\equiv\alpha_1+1$, since $h_0\equiv 0$ (mod 2). Hence we have $\operatorname{sgn}\zeta_1=(-1)^{\alpha_1}$. In the same way, we have $\operatorname{sgn}\zeta_i=(-1)^{\alpha_i}$, where α_i are defined in the same way as α_1. Since $\sum\alpha_i=w_0-1-\bar{s}$, it follows $\operatorname{sgn}\zeta=\prod_{i=1}^{r}\operatorname{sgn}\zeta_i=\prod_{i=1}^{r}(-1)^{\alpha_i}=(-1)^{w_0-1-\bar{s}}$.

To prove this lemma in this case where $m>0$, we may compute the numbers of the centers on the chains, into which L_h is divided, in the same way as in the proof of Lemma 6.8. Since we can accomplish this computation in the same way as above, we shall omit the detail.

From this lemma and the fact that $\Delta(-1)$ is always odd, Theorem 4.4 is easily proved.

Hôsei University.

References

[1] J. W. Alexander, Topological invariants of knots and links, Trans. Amer. Math. Soc., **30** (1928), 275-306.

[2] J. W. Alexander and G. B. Briggs, On types of knotted curves, Ann. of Math., **28** (1927), 563-586.

[3] K. Murasugi, On the genus of the alternating knot, J. Math. Soc. Japan, **10** (1958), 94-105.

[4] K. Reidemeister, Knotentheorie, Julius Springer, (1932).

[5] H. Schubert, Die eindeutige Zerlegbarkeit eines Knotens in Primknoten, S. Ber. Heidelberg. Akad., Wiss., (1949).

[6] H. Seifert, Über das Geschlecht von Knoten, Math. Ann., **110** (1935), 571-592.