ON A CERTAIN SUBGROUP OF THE GROUP OF AN ALTERNATING LINK.*

By Kunio Murasugi.[1]

1. Introduction. Let $L \subset S^3$ be a tame oriented link of multiplicity μ. Let $G = (x_1, x_2, \cdots, x_n : r_1, r_2, \cdots, r_m)$ be a *Wirtinger* presentation of the group $G = \pi_1(S^3 - L)$ of L, and let θ be the homomorphism of G onto an infinite cyclic group $Z = (t:)$ defined by $\theta(x_i) = t$ for all i. Let H be the kernel of θ. In other words, H is the (normal) subgroup of G, an element of which is represented by a loop l in $S^3 - L$ such that the sum of the linking numbers of l with each component of L is zero. Clearly θ can be uniquely extended to the homomorphism $\tilde{\theta}$: $JG \to JZ$ of the integral group rings. Then, denoting the Jacobian matrix of $G = (x : r)$ by $\| \partial r_i / \partial x_j \|$, the g. c. d. of the determinants of all $(n-1) \times (n-1)$ minor of $\| \tilde{\theta}(\partial r_i / \partial x_j) \|$ is called the *reduced Alexander polynomial* of L, denoted by $\tilde{\Delta}(t)$. Cf. [1], [3]. Since $\tilde{\Delta}(t)$ is defined only to within an arbitrary factor $\pm t^{\lambda}$, we may assume that $\tilde{\Delta}(0) \neq 0$, unless $\tilde{\Delta}(t) = 0$.

The object of this paper is to show the following

Theorem 1.1. *Suppose L is an alternating link of multiplicity μ, and let d be the degree of its polynomial $\tilde{\Delta}(t)$. If $\tilde{\Delta}(0) = \pm 1$, then H is free of rank d.*

Since in the case $\mu = 1$, H coincides with the commutator subgroup $[G, G]$ and the reduced Alexander polynomial $\tilde{\Delta}(t)$ becomes the ordinary Alexander polynomial $\Delta(t)$ of the knot, Theorem 1.1 implies immediately the following:

Theorem 1.2. *Suppose K is an alternating knot. Then if $\Delta(0) = \pm 1$, $[G, G]$ of the knot group of K is free of rank d.*

Recently Neuwirth [6] and Rapaport [7] obtained some results on the commutator subgroup $[G, G]$ of the group G of a knot. Many of the arguments that were made in [6], [7] hold with slight modification in the case of the group of a link, if $[G, G]$ is replaced by H. Our proof of Theorem 1.1 based on these facts.

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Moreover, it should be noted that in Theorems 1.1 and 1.2, the hypothesis that L or K be alternating cannot be dropped [2].

In the following we do not distinguish exactly between knots and links. Therefore by a link (of multiplicity μ) is meant an ordinary knot or link according as $\mu = 1$ or > 1. Moreover, the reduced Alexander polynomial will be denoted by the same symbol $\Delta(t)$ as in the case $\mu = 1$.

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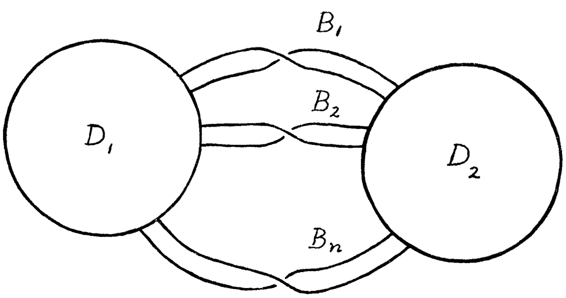
2. *s*-surfaces. Let us consider an orientable surface F in S^3, as is shown in Figure 1 below, consisting of two disks D_1, D_2 to which n bands $B_1, B_2, \cdots, B_n$ are attached. All B_i are twisted once in the same direction, and the bands are pairwise disjoint and do not link one another. Let us call F a *primitive s-surface* of type (n, ϵ), where $\epsilon = \pm 1$ according as the twisting is right-handed or left-handed. E.g. Figure 1 shows a primitive s-surface of type $(3, 1)$. The boundary $\dot{F}$ of F represents an elementary torus link of type $(2, n)$. In other words, F spans an elementary torus link.



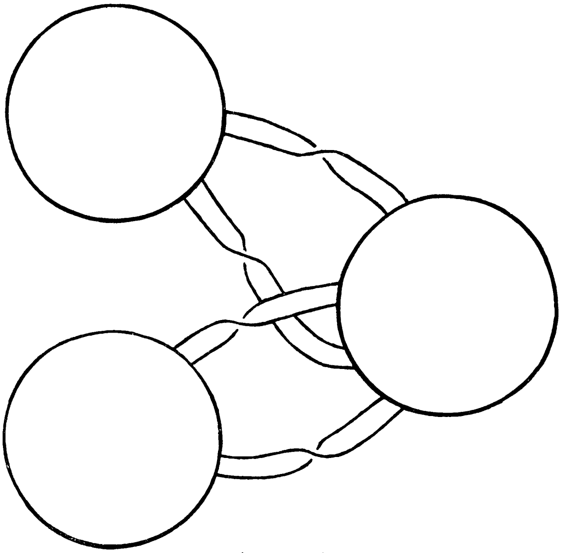
(Fig. 1)

Consider two primitive s-surfaces F and F' in S^3 of type (n, ϵ) and (m, η). Take two disks, D_1 and D_1' say, from each F and F' and identify them so that the resulting orientable surface $\tilde{F} = F \cup F'$ spans a link, and that $\tilde{F} - F$ and $\tilde{F} - F'$ are *separated,* i.e. there exists a 2-sphere S in S^3 such that $S \cap \tilde{F} = D_1$ $(= D_1')$ and each component of $S^3 - S$ contains points of $\tilde{F} - D_1$. $\tilde{F}$ consists of three disks and $n + m$ disjoint bands. $\tilde{F}$ will be called an *s-surface.* Similarly, given two s-surfaces, we can construct an orientable surface spanning a link by identifying two disks, one from each of the s-surfaces. In general, by an *s-surface* is meant an orientable surface obtained from a number of primitive s-surfaces by identifying their disks in this manner.

An identification of this kind leads to the product (in the sense of [4], [9]) of two given elementary torus links. Namely we have

(2.1) *Any product of two elementary torus links of type* $(2, m)$ *and* $(2, n)$ *is represented as the boundary of an s-surface with* $m + n$ *bands and three disks.*

As another example, Figure 2 shows an identification of two primitive s-surfaces of type $(2, 1)$ and $(2, -1)$ that produces a surface spanning the figure-eight knot.



(Fig. 2)

3. Alternating links.

LEMMA 3.1. *Any alternating link* L *for which* $\Delta(0) = \pm 1$, *can be spanned by an s-surface.*

Proof. Let p be a regular projection of L into S^2. Suppose that $p(L)$ is connected and *alternating* [1]. $p(L)$ possesses an orientation inherited from that of L. Now $p(L)$ is decomposed into a number of oriented circuits called *Seifert circuits,* in such a way that no two Seifert circuits have any side of $p(L)$ in common [1], [5]. A Seifert circuit C is said to be the *first* type if C bounds a 2-cell in $S^2 - p(L)$. Otherwise C is of the *second* type. Let us suppose that there are m Seifert circuits $C_1, \cdots, C_m$ of the second type. Since the underlying space $|C_i|$ of C_i is a simple closed curve in S^2, it divides S^2 into simply connected domains $|C_i|^\alpha$ and $|C_i|^\beta$. Let

$$R(\gamma_1, \cdots, \gamma_m) = \mathrm{Cl}(|C_1|^{\gamma_1} \cap \cdots \cap |C_m|^{\gamma_m}),$$

γ_i being α or β. Then only $m+1$ of the sets $R(\gamma_1, \cdots, \gamma_m)$ are non-empty. Let them be $R_1, \cdots, R_{m+1}$. R_i consists of some Seifert circuits of the second type, and two different domains R_i and R_j have at most one Seifert circuit of the second type in common.

Now $p^{-1}(p(L) \cap R_i)$ consists of some simple arcs in S^3. Their end points can be joined together in such a way that a special alternating link L_i results and $p(L_i) = p(L) \cap R_i$. We then write $L = L_1 * \cdots * L_{m+1}$. Since L is a link for which $\Delta(0) = \pm 1$, each L_i is a product of elementary torus links (Theorem 3.28 in [5]). Therefore, by (2.1), L_i is spanned by an s-surface F_i. Suppose R_i and R_j have a Seifert circuit C_k of the second type in common. Let r and s be the number of Seifert circuits contained in R_i and R_j. Then F_i and F_j contain just r and s disks, each of whose boundaries corresponds to a Seifert circuit. Let D and D' be disks of F_i and F_j corresponding to a common Seifert circuit C_k. Then we can identify D and D' in such a way that the resulting s-surface $F_i \cup F_j$ spans a link $L_i * L_j$. Thus, by performing the identification m times in this way, we obtain the required s-surface spannning the link $L = L_1 * \cdots * L_{m+1}$.

4. Proofs of theorems. Let S be an s-surface in S^3 and let U be an open regular neighborhood (in the sense of [11]) of the interior of S in S^3. U is the union of two surfaces $S^\#$ and S^b whose intersection is $\dot{S}$ and which span $\dot{S}$. Then the inclusion maps $\phi^\#: S^\# \to S^3 - U$ and $\phi^b: S^b \to S^3 - U$ induce homomorphisms

$$\phi^\#_*: \pi_1(S^\#) \to \pi_1(S^3 - U)$$

and

$$\phi^b_*: \pi_1(S^b) \to \pi_1(S^3 - U).$$

We claim, moreover, the following:

LEMMA 4.1. *$\phi^\#_*$ and ϕ^b_* are isomorphisms onto.*

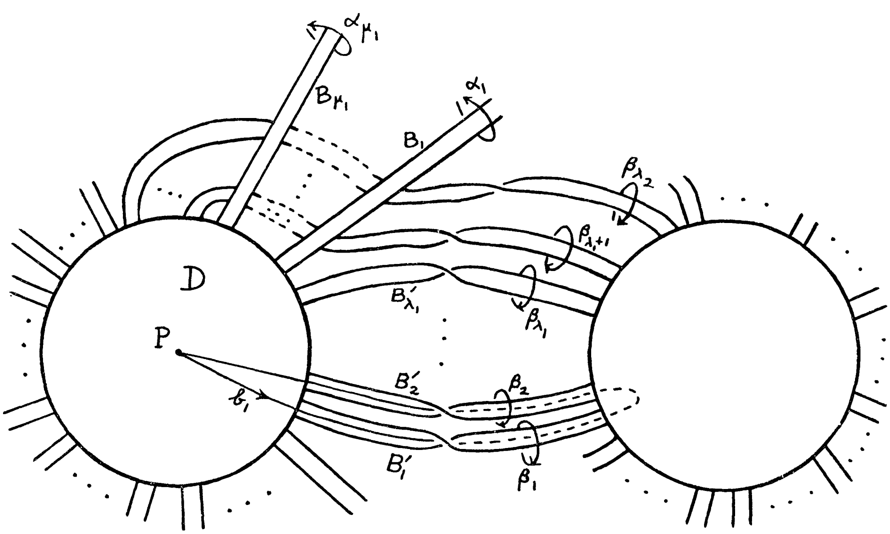
Proof. Proof need be made only for $\phi^\#_*$. If S is a primitive s-surface of type (n, ϵ), then $\phi^\#_*$ is clearly an isomorphism onto, and it is easy to show that $\pi_1(S^\#)$ and $\pi_1(S^3 - U)$ are free groups of the same rank $n-1$, and are freely generated by $b_1, \cdots, b_{n-1}$ and $\beta_1, \cdots, \beta_{n-1}$ respectively. Each b_i is represented by a simple closed curve on $S^\#$ that passes through two adjacent bands of $S^\#$, and each β_i is represented by a loop winding once around a band.

Now let S be an s-surface obtained from an s-surface F_1 and a primitive

s-surface F_2 of type (n, ϵ) by identifying their disks. For the sake of simplicity we suppose $\epsilon = 1$. Let U_i be open regular neighborhoods of the interiors of F_i in S^3. Because of the properties of the identification, we may take $U_1 \cup U_2$ as an open regular neighborhood U of S. Then $S^\# = F_1{}^\# \cup F_2{}^\#$, $S^b = F_1{}^b \cup F_2{}^b$, where $F_i{}^\#$, $F_i{}^b$ have the meaning analogous to that of $S^\#$, S^b. We may assume inductively that

(4.1) (1) The inclusion map $\psi_i{}^\#: F_i{}^\# \to S^3 - U_i$ induces an isomorphism onto between their fundamental groups, $(i = 1, 2)$, and

(2) $\pi_1(F_1{}^\#)$ and $\pi_1(S^3 - U_1)$ are free groups of rank r, generated by $a_1, \cdots, a_r$ and $\alpha_1, \cdots, \alpha_r$ respectively.



(Fig. 3)

Now $\pi_1(S^\#)$ is a free product of two free groups $\pi_1(F_1{}^\#)$ and $\pi_1(F_2{}^\#)$ *amalgamated* on the subgroup $\pi_1(F_1{}^\# \cap F_2{}^\#)$. Since this subgroup is trivial, $\pi_1(S^\#)$ is the free group freely generated by $a_1, \cdots, a_r$ and $b_1, \cdots, b_{n-1}$. On the other hand, since $\pi_1(S^3 - U)$ is a *free product* of two free groups $\pi_1(S^3 - U_1)$ and $\pi_1(S^3 - U_2)$, $\pi_1(S^3 - U)$ is a free group freely generated by $\alpha_1, \cdots, \alpha_r$ and $\beta_1, \cdots, \beta_{n-1}$.

Let D be an identified disk of S and suppose that the bands attached to D are arranged in cyclic order as follows: $B'_1, \cdots, B'_{\lambda_1}$ $B_1, \cdots, B_{\mu_1}$, $B'_{\lambda_1+1}, \cdots, B'_{\lambda_2}$, $\cdots, B'_{\lambda_{l-1}+1}, \cdots, B'_{\lambda_l}$, $B_{\mu_{l-1}+1}, \cdots, B_{\mu_l}$, where B_i and B'_j are bands of F_1 and F_2.

Let α_i $(i=1,\cdots,\mu_l)$ and β_j $(j=1,2,\cdots,\lambda_l=n-1)$ be elements represented by loops winding once around B_i and B_j'. We may assume, moreover, without loss of generality, that there is a regular projection of L whose image in the neighborhood of D will be shown in Figure 3. (We have only to deform L isotopically, if necessary.)

Then by choosing the base point P of $\pi_1(S^{\#})$ in $F_1^{\#}\cap F_2^{\#}$, the induced homomorphism $\phi^{\#}{}_*$ is given by

$$(4.2)\qquad \phi^{\#}{}_*(a_i)=\psi^{\#}{}_{1*}(a_i),\ i=1,2,\cdots,r,$$

$$\phi^{\#}{}_*(b_j)=\begin{cases}\beta_j, & \text{for } 1\leqq j\leqq\lambda_1-1,\\ (A_{k-1}{}^{-1}A_{k-2}{}^{-1}\cdots A_1{}^{-1})\beta_j(A_1\cdots A_k), & \text{for } j=\lambda_k,\ k=1,\cdots,l-1,\\ (A_k{}^{-1}\cdots A_1{}^{-1})\beta_j(A_1\cdots A_k), & \text{for } \lambda_k+1\leqq j\leqq\lambda_{k+1}-1,\\ & k=1,\cdots,l-1.\end{cases}$$

where $A_h=\alpha_{\mu_{h-1}+1}\alpha_{\mu_{h-1}+2}\cdots\alpha_{\mu_h}$, for $h=1,\cdots,l-1$, $\mu_0=0$, and $A_0=1$. Thus since by the assumption of induction $\psi^{\#}{}_{1*}$ is an isomorphism onto, it follows that $\phi^{\#}{}_*$ is also an isomorphism onto.

As remarked in § 1, Lemma 4.1, and a slight modification of Theorem 1 in [6] imply that

LEMMA 4.2. *H is a free group.*

Moreover, if H is free, then a slight modification of results of Theorem 1 and Theorem 3, Corollary in [7] also show that the rank of H must be the degree of $\Delta(t)$ and $\Delta(0)=\pm 1$. Thus we obtain the following

THEOREM 4.3. *Let S be an s-surface and let d be the degree of the reduced Alexander polynomial $\Delta(t)$ of a link $\dot{S}$. Then H is free of rank d and $\Delta(0)=\pm 1$.*

Theorems 1.1 and 1.2 follow from Lemma 3.1 and Theorem 4.3.

Remark 1. Any s-surface S is one of the surfaces with minimum genus spanning the oriented link $\dot{S}$. Because, let h be the genus of S. Since H is free and inclusion maps $S^{\#}\to(S^3-U)$ and $S^b\to(S^3-U)$ induce isomorphisms onto, the proof of Theorem 1 in [6] can be modified to hold in the case of the group of a link. Thus the rank d of H must be $2h+\mu-1$. On the other hand, since $d\leqq 2g+\mu-1$, g denoting the *genus* of $\dot{S}$, we see $d\leqq 2g+\mu-1\leqq 2h+\mu-1=d$. Therefore $g=h$. d is equal to $N-M+1$, where N and M denote the number of bands and disks of S.

Remark 2. Generally the converse of Theorem 4.3 does not hold. For example, 9_{44} in [8] cannot be spanned by any s-surface, but its group G and polynomial $\Delta(t)$ satisfy the conclusion in Theorem 4.3.

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