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# CLASSIFICATION OF $(n-1)$-CONNECTED $2n$-MANIFOLDS

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We consider closed differential manifolds; our main interest being in their diffeomorphism classification. Using recent results of Smale [19], we obtain a set of diffeomorphism invariants which determine a manifold whose boundary is homeomorphic to a sphere. We then attempt to compute the conditions under which the boundary is diffeomorphic to a sphere, so that a disc can be attached to it to make the manifold closed. Our calculations give the form of the answer in general, and the complete answer for $3 \leqq n \leqq 8$.

The diffeomorphism invariants yield easily a full set of homotopy type invariants, and a large number of combinatorial invariants. Using all these, one can more easily follow the known examples of non-existence or non-uniqueness of differential structures on a given manifold under suitable conditions. We can also study the problem of unique factorisation of manifolds.

In a subsequent paper, the author intends to study the diffeomorphisms of the manifolds here obtained; in particular, to give a complete set of isotopy invariants of a diffeomorphism, and to consider more carefully the problem of actual diffeomorphism classification of closed $(n-1)$-connected $2n$-manifolds (which is not settled in this paper, even when our results are complete).

Throughout this paper it will be assumed that $n \geqq 3$. In the case $n = 2$, our arguments break down completely; for a discussion of what is known in that case (not substantially out of date) see Milnor [13]. The case $n = 1$ is well-known, and moreover our arguments are not valid without a great number of alterations. It is also to be understood throughout that all manifolds are oriented.

## The invariants of a presentation of $M$

We shall rely heavily on results of Smale [19], and adopt his notation for handlebodies. First recall [19, 1.1]:

PROPOSITION 1. *If $M^{2n}$ is an $(n-1)$-connected closed manifold, $n \geqq 3$, and if $N^{2n}$ is formed from $M^{2n}$ by removing the interior of a closed disc*

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$D^{2n}$ *imbedded in* $M^{2n}$, *then* $N \in \mathcal{H}(n)$.

Thus $N$ is completely determined by a presentation $\sigma = (D^{2n}; f_1, \cdots, f_r, n)$, which in this case is essentially an imbedding $f$ of $r$ copies of $\partial D^n \times D^n$ disjointly in $\partial D^{2n}$:

$$f\colon \bigcup_{i=1}^{r} (\partial D_i^n \times D^n) \to \partial D^{2n} \; . \tag{1}$$

Moreover, by the diffeotopy extension theorem [29], diffeotopic maps $f$ determine diffeomorphic manifolds $N$. Hence we seek a complete set of diffeotopy invariants for $f$.

We split this problem into two parts. First, we seek a set of diffeotopy invariants for $\bar{f} = f \,|\, \bigcup_{i=1}^{r} (\partial D_i^n \times 0)$. This problem also is solved by Smale [19, 4.1]. Write $c_{ij}$ for the linking number of $\bar{f}(\partial D_i^n \times 0)$, $\bar{f}(\partial D_j^n \times 0)$: two $(n-1)$-spheres disjointly imbedded in a $(2n-1)$-sphere. Then we know that

$$c_{ji} = (-1)^n c_{ij} \; . \tag{2}$$

We call $(c_{ij})$ satisfying this condition $n$-symmetric. Then we have

PROPOSITION 2. *The diffeotopy classes of* $\bar{f}$ *stand in* $(1-1)$ *correspondence with the n-symmetric integer matrices* $(c_{ij})$ *without diagonal elements.*

Next we must give diffeotopy invariants of $f$ corresponding to a given $\bar{f}$. Now it is easy to see that any $S^{n-1}$ differentiably imbedded in $S^{2n-1}$ has trivial normal bundle, so any tubular neighborhood is diffeomorphic to the product $S^{n-1} \times D^n$. By the tubular neighborhood theorem of Milnor, given any two tubular neighborhoods, there exists a diffeotopy carrying one onto the other so that the following is satisfied (after the diffeotopy):

$$t_2(x, y) = t_1(x, s(x) \cdot y) \tag{3}$$

where $t_1$, $t_2$ are the tubular neighborhoods, and $s\colon S^{n-1} \to SO_n$ a map. Thus diffeotopy classes of imbeddings of neighborhoods $f(S^{n-1} \times D^n)$ for given $f \,|\, S^{n-1} \times 0$ correspond to elements of $\pi_{n-1}(SO_n)$.

Since the $(n-1)$-spheres are disjoint, we may clearly take the neighborhoods now found, and the images of their diffeotopies, also disjoint. Thus we have our classification:

LEMMA 1. *The complete set of invariants of the presentation* $\sigma$ *of* $N$, *where equivalence of presentations is equivalence under a diffeomorphism of* $N^{2n}$ *diffeotopic to the identity, is*

(1) *An n-symmetric* $r \times r$ *integer matrix* $(c_{ij})$ *with* $c_{ii}$ *not defined,*

(2) *A set of* $r$ *elements of* $\pi_{n-1}(SO_n)$.

Of course, in this lemma, $N$ can be any element of $\mathcal{H}(n)$, not necessarily coming from a closed manifold $M$.

### Diffeomorphism invariants of $N$

We now look at $N$ in the abstract, to see which invariants we can find. In fact we obtain all the above, in a rather more natural manner.

Write $H = H_n(N)$. This is a free abelian group of rank $r$, so $r$ is invariant. Intersection numbers define an $n$-symmetric bilinear form $H \otimes H \to Z$, where $Z$ is the group of integers. Now as $N$ is $(n-1)$-connected, we have the Hurewicz isomorphism $\pi_n(N) = H_n(N) = H$. Then by theorems of Haefliger [5], for $n \geqq 3$, every element of $\pi_n(N)$ can be represented by a differentiable imbedding $S^n \to N$, and for $n \geqq 4$, two such imbeddings which are homotopic are diffeotopic. Hence for $n \geqq 4$ an element of $H$ defines a unique (up to diffeomorphism) imbedding of $S^n$ in $N$, and so a corresponding normal bundle, which is classified by an element of $\pi_{n-1}(SO_n)$. Thus we have a map $\alpha$: $H \to \pi_{n-1}(SO_n)$. For $n = 3$, the map is still well-defined, since $\pi_2(SO_3)$ consists only of the zero element.

Thus we have a free abelian group $H$ of rank $r$, an $n$-symmetric bilinear form $H \otimes H \to Z$ (which we write simply as the product), and a map $\alpha$: $H \to \pi_{n-1}(SO_n)$. We assert that, from this, all the invariants of Lemma 1 can be recovered.

First recall that $N$ is constructed as follows. We take a disc $D^{2n}$, a map $f$: $\bigcup_{i=1}^r \partial D_i^n \times D^n \to \partial D^{2n}$ with the given invariants, attach $\bigcup_{i=1}^r D_i^n \times D^n$ using $f$, and round the corners. Now $\partial D_i^n \times 0 \subset \partial D^{2n}$ bounds a disc $D_i'^n$ in $D^{2n}$. It also bounds $D_i^n \times 0$. These two discs give us a sphere $S_i^n$, which we may take to be differentiably imbedded. Then since the classes of the $D_i^n \times 0$ in $H_n(N^{2n}, D^{2n})$ form a free basis, so do those of the $S_i^n$ in $H_n(N) = H$. The intersections of $S_i^n$ and $S_j^n$ in $N$ are just those of $D_j'^n$, $D_j'^n$ in $D^{2n}$: their number is by definition the linking number of $S_i^{n-1}$, $S_j^{n-1}$ in $\partial D^{2n}$. Hence we recover the $c_{ij}$: if $e_i$ is the class of $S_i^n$ in $N$, we have $c_{ij} = e_i \cdot e_j$. Finally, $\alpha(e_i)$ is the characteristic class of the normal bundle of $S_i^n$, so if the tubular neighborhood $t_1$ in (3) is the neighborhood of $S_i^{n-1}$ in $\partial D^{2n}$ deduced from one of $D_i'^n$ in $D^{2n}$, since $t_2$ is deduced from the obvious tubular neighborhood of $D_i^n \times 0$ in $D_i^n \times D^n$, we have, by the definition of the characteristic class of a bundle, that the map $s$ of (3) represents $\alpha(e_i)$, so the $r$ invariants in $\pi_{n-1}(SO_n)$ are simply the $\alpha(e_i)$.

We note that a presentation of $N$ determines not merely the above structures on $H$, but also a basis for $H$. It will be useful to have the following theorem. The proof follows without difficulty by the methods of Smale, so we shall only outline it.

THEOREM 1. *Suppose $N \in \mathcal{H}(n)$, and choose a basis for $H_n(N)$. Then $N$ has a presentation corresponding to this basis.*

PROOF. Let $D^{2n}$ be a fixed disc in the interior of $N^{2n}$, and let $K^{2n}$ denote the closure of $N^{2n} - D^{2n}$. Since all these spaces are $(n-1)$-connected, we have

$$H_n(N^{2n}) \cong H_n(N^{2n}, D^{2n}) \cong H_n(K^{2n}, \partial D^{2n}) \cong \pi_n(K^{2n}, \partial D^{2n}) .$$

We represent the elements of our chosen basis by maps $\bar{f}_i\colon (D^n, \partial D^n) \to (K^{2n}, \partial D^{2n})$; by results of Whitney, we may suppose that each of these is an imbedding, with $\bar{f}_i(D^n)$ transverse to $\partial D^{2n}$ along their intersection $\bar{f}_i(\partial D^n)$. We may further suppose that the images of any two $\bar{f}_i$ meet in a finite number of points; but these can be 'pushed away' across the boundary into $D^{2n}$ by methods of [30], since $(M^{2n}, D^{2n})$ is 2-connected.

We now have disjointly imbedded discs, in the right homology classes. We can extend the maps $\bar{f}_i$ to tubular neighborhoods $f_i\colon (D^n, \partial D^n) \times D^n \to (K^{2n}, \partial D^{2n})$. Of course, the restrictions of these to the boundary will give a presentation. Let $N'$ be the union of $D^{2n}$ with the images of the $f_i$; then $N'$ is a submanifold of $N$ (we can round the corners if we wish) and the injection of $N'$ in $N$ is a homotopy equivalence.

Write $L$ for the closure of $N - N'$. This has two boundary components, $\partial N$ and $\partial N'$, both $(n-2)$-connected $(2n-1)$-manifolds. It is easy enough to compute the homology groups (cf., Lemma 3 below) and deduce that each of these is a deformation retract of $L$, which thus defines a $J$-equivalence (or "$h$-cobordism") between them. Now by a result of [17], $\partial N$ is diffeomorphic to $\partial N'$, and $L$ diffeomorphic to the product of either with an interval. This shows that we can effectively identify $N$ and $N'$ (they are diffeomorphic, with a diffeomorphism diffeotopic in $N$ to the identity), so the presentation of $N'$ leads to the required one of $N$.

COROLLARY. *Any automorphism of $H_n(N)$ respecting the invariants can be realised by a diffeomorphism of $N$.*

The statement that the invariants of a presentation were a complete set means precisely that, given an isomorphism of such a set, there exists a corresponding diffeomorphism.

## Relations between the invariants

Thus in fact $c_{ij} = e_i \cdot e_j$ $(i \neq j)$ and $\alpha(e_i)$ already give a complete set of diffeomorphism invariants for $N$. But $e_i \cdot e_i$, and the values of $\alpha$ on other elements, are also diffeomorphism invariants, so must be functions of the above. Hence, there are relations between our invariants. We now determine all of these.

We first recall a little homotopy theory: there is a canonical homomorphism $J\colon \pi_r(SO_n) \to \pi_{n+r}(S^n)$ and the Hopf invariant $H\colon \pi_{2n-1}(S^n) \to Z$. We also use a section of the homotopy sequence of the fibering $S^n = SO_{n+1}/SO_n$;

$$(4) \qquad \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SO_n) \xrightarrow{S} \pi_{n-1}(SO_{n+1}) \cong \pi_{n-1}(SO_{2n}) .$$

Let $\iota_n \in \pi_n(S^n)$ be the class of the identity map; then $J\partial\iota_n$ is the Whitehead product $[\iota_n, \iota_n]$, whose Hopf invariant is $\pm 2$ ($n$ even). We suppose orientations chosen so that $HJ\partial\iota_n = 2$ ($n$ even).

LEMMA 2. *We have*

$$(5) \qquad x^2 = HJ\alpha(x) ,$$

$$(6) \qquad \alpha(x + y) = \alpha(x) + \alpha(y) + xy(\partial\iota_n) .$$

PROOF. We can identify $HJ\colon \pi_{n-1}(SO_n) \to Z$ with the homomorphism $\pi_{n-1}(SO_n) \to \pi_{n-1}(S^{n-1})$ induced by projection. To find $x^2$ we take a sphere, representing $x$, and a tubular neighborhood, defined by $\alpha(x)$, and consider the intersection of the sphere with a neighboring sphere in general position. But this intersection is simply the obstruction to finding a cross-section of the associated $S^{n-1}$-bundle. By the remark above, this is $HJ\alpha(x)$.

For the second part we may join two imbedded spheres representing $x$ and $y$ by a tube in $N$, and note that we then have an immersed sphere, representing $x + y$, with self-intersection number $xy$, and with normal bundle defined by $\alpha(x) + \alpha(y)$. We now propose to show that, for each removal of a self-intersection, we must add $\partial\iota_n$ to the normal bundle. The result will then follow.

First of all, we remark that it is a fairly simple exercise in homotopy theory to prove that if there is no element of Hopf invariant 1 in $\pi_{2k-1}(S^k)$, i.e., if $k \neq 1, 3, 7$, then the homomorphism $\pi_k(SO_k) \to \pi_k(SO_{k+2})$ is onto, and so $\pi_k(SO_{k+2}/SO_k) \to \pi_{k-1}(SO_k)$ is a monomorphism; hence, by stability, so is the map $\theta\colon \pi_k(V_{2k,k}) \to \pi_{k-1}(SO_k)$.

Moreover, the maps $\pi_k(S^k) \to \pi_k(SO_{k+2}/SO_k) \to \pi_k(V_{2k,k})$ are onto. Hence the image of $\theta$ is the cyclic subgroup generated by $\partial\iota_n$.

It follows by a result of Smale [16] that two immersions of $S^n$ in $E^{2n}$ are regularly homotopic if and only if a certain obstruction in $\pi_k(V_{2k,k})$ vanishes. Using the above, we now see that this is true (for $n \neq 1, 3, 7$) if and only if they have the same normal bundles, and that the characteristic classes of any such bundles are multiples of $\partial\iota_n$. Also it has been proved by Kervaire [9] that such an immersion is regularly homotopic to an imbedding if and only if its Smale invariant $x$ vanishes, i.e., it has trivial normal bundle.

Now we refer to a paper of Whitney [25]. It is known that his methods are valid if euclidean $2n$-space is replaced by a simply connected manifold. First, he defines the self-intersection number of an immersed manifold $M^n$ in $E^{2n}$; this being an integer if $n$ is even, and an integer mod 2 if $n$ is odd. It is proved that if this number is zero, the immersion is regularly homotopic to an imbedding. Whitney also gives a method of locally introducing a self-intersection, which adds $\pm 1$ to this number.

Comparing these results with the above, we deduce the following: up to sign, the intersection number of an immersion of $S^n$ in $E^{2n}$ is $\lambda$ if and only if the normal bundle is given by the characteristic class $\lambda\partial\iota_n$. It follows that the local introduction of a single self-intersection for $S^n$ adds (or maybe subtracts) $\partial\iota_n$ to (from) its normal bundle. But since this is an entirely local property, it holds for an immersion of $S^n$ in any $2n$-manifold. Then, following Whitney, pairs of opposite self-intersections can be removed without changing the regular homotopy class, and thus the normal bundle.

Now if $n = 3, 7$, relation (6) becomes trivial. If $n$ is any other odd number (1 excluded), $\partial\iota_n$ has order 2, and the relation is proved. If $n$ is even, our argument has established the relation up to the sign of $\partial\iota_n$; and the fact that this sign is correct now follows by applying $HJ$ to each side, using (5).

In the case when $n$ is even, we can give a much simpler proof. We first suspend, and show that $S\alpha$ is a homomorphism of $H$ to $\pi_{n-1}(SO_{n+1})$: it is sufficient (by stability) to prove that $S^n\alpha\colon H \to \pi_{n-1}(SO_{2n})$ is additive. But since the tangent bundle of a sphere representing $x$ is stably trivial, $S^n\alpha(x)$ defines the restriction to such a sphere of the tangent bundle of $N$. But this is certainly additive. Now by the exact sequence (4), writing $\beta = \alpha(x + y) - \alpha(x) - \alpha(y)$, since $S\beta = 0$, $\beta$ lies in the image of $\partial$, so for some integer $\lambda$, $\beta = \lambda(\partial\iota_n)$. Now apply $HJ$, in the case when $n$ is even:

$$(x + y)^2 - x^2 - y^2 = 2\lambda\ . \tag{7}$$

Thus $\lambda = xy$, and the result follows.

The group $H$, with $n$-symmetric product $H \otimes H \to Z$ and map $\alpha\colon H \to \pi_{n-1}(SO_n)$ satisfying the formulae (5), (6), we call a pre-$n$-space. Then the classification theorem may be compactly stated, using Theorem 1:

*Elements of $\mathcal{H}(n)$ $(n \geqq 3)$ stand in $(1-1)$ correspondence with pre-$n$-spaces.*

The advantage of this over Lemma 1 is of course in its coordinate-free, functorial form. As we have seen, it is essentially the same statement.

## Almost closed $2n$-manifolds

We have digressed somewhat from our main theme in considering $N$ any element of $\mathcal{H}(n)$. We now want to restrict the boundary of $N$ to be a sphere. Certainly, since $n \geqq 3$, the boundary of $N$ is simply connected; in fact, one sees at once that its homology is non-zero only in dimensions $(n-1)$ and $n$. Thus it is easy to give conditions for it to be a homotopy sphere.

Let $H^*$ be the integral dual of $H$. It is a free abelian group with rank $r$ and basis $e_1^*, \cdots, e_r^*$, with a Kronecker product defined by $\langle e_i, e_j^* \rangle = \delta_{ij}$. The product in $H$ defines a map $\pi$: $H \to H^*$ by

$$\langle y, \pi(x) \rangle = yx \;, \tag{8}$$

since $H^* = \operatorname{Hom}(H, Z)$.

LEMMA 3. *There are natural isomorphisms*

$$H_{n-1}(\partial N) \cong \operatorname{Coker} \pi, \qquad H_n(\partial N) \cong \operatorname{Ker} \pi \;.$$

PROOF. $\partial N$ is constructed from $S^{2n-1}$ and $f$ (given in (1)) by first removing the interiors of the $r$ copies of $S^{n-1} \times D^n$ and then pasting in $r$ copies of $D^n \times S^{n-1}$. Write $X = S^{2n-1} - r(S^{n-1} \times D^n) = \partial N - r(D^n \times S^{n-1})$. Then $H_{n-2}(X) = 0$, $H_{n-1}(X) \cong H^*$, for the homology class of an $(n-1)$-cycle is determined by its linking numbers with the $S_i^{n-1}$, which may be arbitrary, $H_n(X) = 0$. To obtain $\partial N$, we paste in $r$ copies of $D^n$ (which is all that affects the middle dimensions). Thus we have chain groups, zero in dimensions $n-2$, $n+1$, $C_{n-1} = H^*$, $C_n \cong H$, and the boundary homomorphism is given by the attaching maps. This shows that it must be $\pi$. The result then follows.

COROLLARY. *$\partial N$ is a homotopy sphere if and only if $\pi$ is an isomorphism.*

A pre-$n$-space $H$, whose corresponding map $\pi$ is an isomorphism, is called an $n$-space. Equivalent conditions are that $\pi$ be onto, and that the bilinear form be unimodular.

By results of Smale [18], a homotopy sphere $T^{2n-1}$, with $n \geqq 3$, is homeomorphic, and even combinatorially equivalent, to the standard sphere $S^{2n-1}$. Hence it defines an element of the group $\Gamma_{2n-1}$ [22], [15]. Thus $N$ is derived from a closed manifold $M$ by removing the interior of a disc if and only if $\partial N$ is a homotopy sphere, and defines the zero element of $\Gamma_{2n-1}$.

The reader will note, however, that even if $\partial N$ is diffeomorphic to $S^{2n-1}$, we must choose some diffeomorphism to form $M$ from $N$, and in general, different diffeomorphisms lead to different $M$. However, if two diffeomorphisms belong to the same orbit under the operation of the subgroup of diffeomorphisms of $S^{2n-1}$ which extend to $D^{2n}$, we clearly get a diffeomorphic $M$, hence the indeterminacy lies in the quotient group $\Gamma_{2n}$. Two such manifolds $M$ are related by the fact that one is the connected sum of the other with a manifold homeomorphic (in fact, combinatorially equivalent) to $S^{2n}$, which defines the element of $\Gamma_{2n}$.

In this paper, we shall not investigate the problem of different manifolds $M$ corresponding to the same invariants (or $N$). We merely consider uniqueness mod $\Gamma_{2n}$, and it is with this understanding that we obtain a complete diffeomorphism classification.

We now investigate the boundary of $N$ in the case when this is a homotopy sphere. First, however, let us again consolidate our gains.

A manifold $N^{2n}$ is called almost closed if its boundary is an element of $\Gamma_{2n-1}$ (i.e., is combinatorially equivalent to $S^{2n-1}$). Then we have shown:

*Almost closed $(n-1)$-connected $2n$-manifolds stand in $(1-1)$ correspondence with $n$-spaces $H$.*

(That almost closed $(n-1)$-connected $2n$-manifolds are handlebodies follows from Smale [19, 1.2]. We recall that all manifolds in this paper are oriented.)

Thus each $n$-space $H$ determines uniquely an element of $\Gamma_{2n-1}$. The next stage is to compute which element. This is facilitated by Lemma 4 which follows.

We define the direct sum $H_1 \oplus H_2$ of two $n$-spaces in the obvious manner: products and $\alpha$ are to be reckoned coordinate-wise and added: $(x_1, x_2)(y_1, y_2) = x_1y_1 + x_2y_2$ and $\alpha(x, y) = \alpha(x) + \alpha(y)$. It is immediate that this sum satisfies (5), (6), and defines an $n$-space. The sum of two $2n$-manifolds with boundary is defined in Smale [17]: the manifolds are to be glued together along a $(2n-1)$-disc imbedded in the boundary of each, with opposite orientations.

LEMMA 4. *Forming the sum of two almost closed $(n-1)$-connected $2n$-manifolds corresponds to taking the direct sum of the corresponding $n$-spaces.*

PROOF. This is completely trivial. The inclusions of the summands in the sum lead to an injective representation of $H$ as a direct sum. Two homology classes in a summand have the same (geometrical and therefore algebraic) intersection in that summand or in the sum, and two in different summands have zero intersection. Likewise the normal bundle of an imbedded sphere is unaltered on proceeding to the sum. The result now follows using bilinearity of intersections and using (6).

COROLLARY 1. *Taking the direct sum of two $n$-spaces corresponds to adding the corresponding elements of $\Gamma_{2n-1}$.*

We may put this in a more algebraic form, using the Grothendieck group of $n$-spaces. First take the free abelian group on all (isomorphism classes of) $n$-spaces. Then consider the subgroup generated by relations $\{H_1 \oplus H_2\} - \{H_1\} - \{H_2\}$. The quotient group is the Grothendieck group $\mathcal{G}_n$. The above corollary can now be re-phrased as

COROLLARY 2. *The map from n-spaces to* $\Gamma_{2n-1}$ *defines a homomorphism*

$$\mathfrak{o}\colon \mathcal{G}_n \to \Gamma_{2n-1} . \tag{9}$$

We shall now investigate the homomorphism $\mathfrak{o}$. It is first necessary to compute $\mathcal{G}_n$, and for this we need to know what the groups $\pi_{n-1}(SO_n)$ actually are, so we now give these groups, and use our knowledge of them.

### $n$-Spaces for odd $n$

The groups $\pi_{n-1}(SO_n)$ have been calculated by Kervaire [7], using results of Bott. We now quote them. We must distinguish no less than 7 cases (and recall $n \geqq 3$).

*n even.* In each case, $S \oplus HJ\colon \pi_{n-1}(SO_n) \to \pi_{n-1}(SO) \oplus Z$ is a monomorphism, with image of index 2.

| | | | |
|---|---|---|---|
| (1) | $n \equiv 0\,(4),\ n > 8$ . | $\pi_{n-1}(SO) \cong Z$ , | $HJ(\alpha)$ is even. |
| (2) | $n = 4, 8$ . | $\pi_{n-1}(SO) \cong Z$ , | $S(\alpha) + HJ(\alpha)$ is even. |
| (3) | $n \equiv 2\,(8)$ . | $\pi_{n-1}(SO) \cong Z_2$ , | $HJ(\alpha)$ is even. |
| (4) | $n \equiv 6\,(8)$ . | $\pi_{n-1}(SO) \cong 0$ , | $HJ(\alpha)$ is even. |

*n odd.* In this case, $HJ(\alpha) \equiv 0$.

(5) $\quad n \equiv 1\,(8)$ . $\quad \pi_{n-1}(SO_n) \cong 2Z_2$ ,

$S$ projects onto the first component; let $\varphi$ denote projection on the second. $S(\partial\iota_n) = 0$, $\varphi(\partial\iota_n) = 1$, and $J(\partial\iota_n) \neq 0$. $\varphi$ is not well defined; we could equally well use $\varphi' = S + \varphi$.

(6) $\quad n \equiv 3, 5, 7\,(8),\ n \neq 3, 7$ . $\quad \pi_{n-1}(SO_n) \cong Z_2$ .

Let $\varphi$ be the isomorphism. $\varphi(\partial\iota_n) = 1$, and $J$ is a monomorphism.

(7) $\quad n = 3, 7$ . $\quad \pi_{n-1}(SO_n) = 0$ .

Now in the case when $n$ is odd, the product on the $n$-space $H$ is skew-symmetric; in these cases, we are able to give a complete classification.

*Case* 7. In this case, an $n$-space is a free abelian group with a unimodular skew-symmetric form. Then by a well-known result, the rank is even, say $r = 2s$, and we may choose a base in $H$ (a symplectic base) $e_i, e'_i$ $(1 \leqq i \leqq s)$ so that $e_i e'_i = -e'_i e_i = 1$ $(1 \leqq i \leqq s)$, and all other products of basic elements are zero. Thus the only invariant is $s$, which can be any natural number.

Recall that, by the usual proof, any indivisible element of $H$ can be taken as $e_1$.

*Case* 6. An $n$-space now is a free abelian group $H$ with a unimodular skew form, and a map $\varphi$: $H \to Z_2$ with $\varphi(x + y) = \varphi(x) + \varphi(y) + xy$. Again the rank is even. We may take a symplectic base, and it is known [27] that the only invariant apart from $s$ is the Arf invariant

$$\Phi = \sum_{i=1}^{s} \varphi(e_i)\varphi(e'_i) \qquad (\text{mod } 2) . \tag{10}$$

Since this is less well-known in this context, we shall give a proof.

Note that $\varphi$ is determined by the $\varphi(e_i)$ and $\varphi(e'_i)$. In fact $H$ is a direct sum of $s$ $n$-spaces, those spanned by $(e_i, e'_i)$ for $1 \leqq i \leqq s$. For each of these, $\varphi$ is determined by $\varphi(e_i)$, $\varphi(e'_i)$. Write $W_{ab}$ for the $n$-space spanned by $e$, $e'$ with $ee' = -e'e = 1$, $\varphi(e) = a$, $\varphi(e') = b$. Then $W_{01} \cong W_{10} \cong W_{00}$, for, first change coordinates by $e \to e'$, $e' \to -e$, and then write $f = e + e'$, $f' = e'$. Thus the only two indecomposable $n$-spaces are $W_{00}$ and $W_{11}$, which we note are distinguished by $\Phi$. That $\Phi$ is the only invariant follows, since $W_{11} \oplus W_{11} \cong W_{10} \oplus W_{01}$ by the change of variables

$$f_1 = e_1, \qquad f'_1 = e'_1 - e'_2, \qquad f_2 = e_1 + e_2, \qquad f'_2 = e'_2 .$$

Finally, $\Phi$ is an invariant: this may be checked directly, or since $\varphi$ is really a function on $H/2H$, which has $2^{2s}$ elements, and if $\Phi = 1$, $\varphi = 1$ on $2^{2s-1} + 2^{s-1}$ of them; if $\Phi = 0$, $\varphi = 1$ on $2^{2s-1} - 2^{s-1}$ of them, as can easily be checked.

Thus here enumeration is given by $s$, $\Phi$ (mod 2), where $s \geqq 0$; if $s = 0$, $\Phi = 0$.

*Case* 5. $S\alpha$ is a homomorphism $H \to Z_2$, so as $\pi : H \cong H^* = \text{Hom}(H, Z)$, and $\text{Hom}(H, Z_2) \cong \text{Hom}(H, Z) \otimes Z_2 \cong H \otimes Z_2$, determines an element $\chi$ of $H$ (determined mod $2H$) with $S\alpha(x) = \chi x$ (mod 2) for all $x \in H$. There are now two cases. If $\chi = 0$ (mod 2), reduction proceeds exactly as in *Case* 6. If $\chi \neq 0$ (mod 2), we may suppose that $\chi$ is indivisible; then it may be taken as the first element of a symplectic base. As in *Case* 6, the $n$-space splits into the direct sum of irreducible $n$-spaces with $s = 1$; those other than the first may now be dealt with as before. The space with $\chi = e$ and $\varphi(e) = a$, $\varphi(e') = b$ we will refer to as $X_{ab}$. However, now, $a = \varphi(\chi)$ is a genuine invariant mod 2. We have $X_{00} \cong X_{01}$ by $e' \to e + e'$, but $X_{00}$, $X_{10}$, $X_{11}$ are distinct. The first is distinguished by $\varphi(\chi)$, the latter two by $\Phi$. We may now assemble these results. Call the $n$-space of *type* 0 if $\chi \neq 0$ (mod 2); of *type* I, if $\chi = 0$.

LEMMA 5. *For $n$ odd, we can classify $n$-spaces as follows.*

*In Case* 5, *the non-negative integer $s$, the type $T$, $\Phi$, and $\varphi(\chi)$ form a* *complete set of invariants which are related by:*

*If* $s = 0$ *then* $T = \text{I}$, $\Phi = \varphi(\chi) = 0$.

*If* $T = \text{I}$ *then* $\varphi(\chi) = 0$.

*If* $s = 1$, $T = 0$ *and* $\varphi(\chi) = 0$, *then* $\Phi = 0$.

*In Case* 6, $s$, $\Phi$ *form a complete set of invariants, and* $\Phi = 0$ *if* $s = 0$.

*In Case* 7, $s$ *is a complete set of invariants.*

*Note*. In *Case* 5, instead of $\varphi$, we might equally well use $\varphi' = \varphi + S$. The new invariants will then be related to the old by

$$\varphi'(\chi) = \varphi(\chi), \qquad \Phi' = \Phi + \varphi(\chi) ,$$

as is easy to verify.

From these enumerations it is easy to compute the Grothendieck groups $\mathcal{G}_n$ ($n$ odd). The only one of the invariants above which is not additive under direct sums is the type $T$ in *Case* 5, which is multiplicative. This has no effect on $\mathcal{G}_n$. Thus we have:

*Case* 7. $\mathcal{G}_n \cong Z$ by taking $s$.

*Case* 6. $\mathcal{G}_n \cong Z \oplus Z_2$ by $s$, $\Phi$.

*Case* 5. $\mathcal{G}_n \cong Z \oplus Z_2 \oplus Z_2$ by $s$, $\Phi$, $\varphi(\chi)$.

## Grothendieck groups of $n$-spaces, $n$ even

In the case when $n$ is even, we can give no such simple classification. This would involve a classification of quadratic forms, a task not yet achieved. But even if a form were given, it would not be easy to classify the corresponding $n$-spaces. Hence in these cases, we shall ignore the classification problem, and proceed directly to determine the Grothendieck groups.

We need some information on unimodular quadratic forms. In the first place, these are classified as of type II or I depending upon whether or not $x^2$ is always even. (Note: there are two definitions of integral quadratic forms; strictly speaking, we refer to symmetric bilinear forms.) Note that in *Cases* 1, 3, 4 we deal solely with quadratic forms of type II. We may now quote the standard theorems. For their elucidation and further references see Milnor [13].

PROPOSITION 3. *The rank, index* (i.e. *signature*) *and type form a complete system of invariants for the genus of an unimodular quadratic form. These satisfy just four conditions:* $|\tau| \leqq r$; $\tau \equiv r \pmod 2$; *for type* II, $\tau \equiv 0 \pmod 8$; *and for type* I, $r > 0$.

This is given force by the next

PROPOSITION 4. *Two indefinite unimodular forms are equivalent if and only if they have the same genus.*

LEMMA 6. *The Grothendieck group $\mathcal{G}_{\text{II}}$ for forms of type* II *is isomorphic to $2Z$. An isomorphism is defined by the map $((r-\tau)/2, \tau/8)$. The Grothendieck group $\mathcal{G}_{\text{I}}$ for all forms is also isomorphic to $2Z$, under the map $((r-\tau)/2, \tau)$.*

PROOF. Take two forms of the same genus, and add to each an indefinite form of type II, such as $U$ below. By the last proposition, they become equivalent. Hence they define the same element of the Grothendieck group. It now follows that $r$ and $\tau$ are the only valid invariants (in the first case all forms have type II; and in the second, type is not preserved on addition); we know the additive relations between them, so the result follows.

*Note.* It has been pointed out to me by J. Milnor that all these lemmas up to the determination of the Grothendieck groups can be proved using only his lemma:

PROPOSITION. *An indefinite unimodular quadratic form over the integers represents zero.*

Since the proof of this, however, entails most of the classification of forms over the rationals, and as we shall need the more detailed results later, we may as well use them here.

*Case* 4 is now settled. The invariant in that case consisted precisely of a quadratic form of type II. In fact we now note in general that, since $S \oplus HJ$ is a monomorphism and $HJ\alpha(x) = x^2$, the invariant $\alpha$ can be replaced by the invariant $S\alpha$: $H \to \pi_{n-1}(SO)$ which, as before, determines by duality an element $\chi$ of $H \otimes \pi_{n-1}(SO)$ such that for all $x \in H$, $S\alpha(x) = \chi x$. We may now re-phrase the $n$-space structure of $H$ ($n$ even) as follows:

(1) We have a quadratic form of type II on $H$, and an element $\chi$ of $H$.

(2) We have a quadratic form, and an element $\chi$ of $H$, such that for all $x \in H$, $\chi x \equiv x^2 \pmod 2$.

(3) We have a quadratic form of type II on $H$, and $\chi \in H \otimes Z_2$.

(4) We have a quadratic form of type II on $H$.

First consider *Case* 1. Clearly $\chi^2$ is invariant, and since the form is of type II, we may write $N = \frac{1}{2}\chi^2$. Consider the auxiliary $n$-space $U$ spanned by $e_1$ and $e_2$ with $e_1^2 = e_2^2 = 0$, $e_1 e_2 = e_2 e_1 = 1$. If $\chi = \lambda e_1 + \mu e_2$ we denote this space by $U(\lambda, \mu)$.

LEMMA 7. *$H \oplus U(0, 1) \cong H_0 \oplus U(N, 1)$ where $H_0$ denotes the $n$-space derived from $H$ by setting $\chi = 0$.*

PROOF. Let $x_1, \cdots, x_r$ be a basis for $H$. Set $x_i' = x_i - (x_i\chi)e_1$ in $H \oplus U(0, 1)$: then $x_i'\chi = 0$ for $1 \leq i \leq r$. Also $x_i'$ is orthogonal to $e_1$ (though not, in general, to $e_2$), and $x_i' x_j' = x_i x_j$. Write $H'$ for the subspace spanned by the $x_i'$: then $x_i \to x_i'$ induces an isomorphism $\gamma$: $H \to H'$ preserving products:

$$x_i'e_2 = -x_i\chi = -x_i'\gamma(\chi)\ .$$

Set $e_2' = e_2 + \gamma(\chi)$. Then $H'$ is orthogonal to $e_1$, $e_2'$, so we have another splitting as a direct sum. $e_1^2 = 0$, $e_1e_2' = e_1e_2$ (since $e_1$ is orthogonal to $H'$) and

$$\begin{aligned} e_2'^2 &= e_2^2 + 2e_2\gamma(\chi) + (\gamma(\chi))^2 \\ &= 0 - 2\chi\cdot\chi + \chi^2 = -2N\,; \end{aligned}$$

so writing $e_2'' = e_2' + Ne_1$, we have $e_1^2 = 0$, $e_1e_2'' = 1$, $e_2''^2 = 0$. Finally, the '$\chi$' for the direct sum is $\chi + e_2$, and since $\gamma(\chi) = \chi - \chi^2e_1 = \chi - 2Ne_1$, this is $\gamma(\chi) + 2Ne_1 + e_2 = 2Ne_1 + e_2' = Ne_1 + e_2''$. The result is proved.

It follows from this that two $n$-spaces with the same quadratic form and value of $N$ determine the same element of the Grothendieck group. Since $N$ is invariant, our problem is solved. $((r-\tau)/2, \tau/8, N)$ determines an isomorphism of $\mathcal{G}_n$ onto $Z \oplus Z \oplus Z$, since taking $\chi = 0$, we know, by Lemma 5, we get an isomorphism onto the first two components, and $U(N, 1) \to (1, 0, N)$.

In *Case* 3, we utilise the above to see that $r$, $\tau$ and $N$ are the only possible invariants, but now, since $\chi$ is only defined modulo 2, $U(N, 1) = U(N', 1)$ for $N \equiv N'$ (mod 2). But $N$ (mod 2) is invariant, for if $\chi' = \chi + 2x$, we have

$$2N' = \chi'^2 = \chi^2 + 4\chi x + 4x^2 \equiv \chi^2 = 2N \qquad (\text{mod } 4)$$

so that $N' \equiv N$ (mod 2).

*Case* 4 has already been dealt with. It remains only to consider *Case* 2. Our forms are no longer of type II, but we still have invariants $((r-\tau)/2, \tau, \chi^2)$. We have the following examples of $n$-spaces: spanned by the single element $e$, with $e^2 = \pm 1$, and $\chi = e$. We call them $T$, $T'$. Given an $n$-space, we may add to it copies of these spaces and reduce $\chi^2$ to zero. We may thus suppose $\chi^2 = 0$. We note that $\chi^2 - \tau$ is unaltered by this process.

Similarly, by adding (if necessary) one copy each of $T$, $T'$, we may suppose also that $\chi$ is indivisible. Then we can find $\psi'$, with $\chi\psi' = 1$. Thus $\psi'^2 \equiv \chi\psi' = 1$ (mod 2) is odd. Set $\psi = \psi' - \frac{1}{2}(\psi'^2 - 1)\chi$. Then $\chi^2 = 0$, $\chi\psi = 1$, $\psi^2 = 1$. The space $S$ spanned by $\chi$ and $\psi$ is unimodular, so a direct summand. Write $H = S \oplus H'$. Then in $H'$, $\chi = 0$, so $r$, $\tau$ are the only invariants remaining for consideration. Thus $r$, $\tau$, $\chi^2$ is a complete set of invariants for $\mathcal{G}_n$ in this case also.

Now consider $H' : \chi = 0$. For any $x \in H'$, $x^2 \equiv \chi x = 0$ (mod 2). Thus for $H'$, $\tau$ is divisible by 8, hence so is $\chi^2 - \tau$. The same is valid for $H$, since $\chi^2$, $\tau$ both vanish for $S$. But we remarked above that the step from an arbitrary $n$-space to $H$ left $\chi^2 - \tau$ unaltered; hence this is always divisible by 8.

We have now determined all the Grothendieck groups $\mathcal{G}_n$, and summarise them for further convenience. As well as stating invariants, we give examples of spaces to show that all values of the invariants are attained. We first introduce symbols for certain matrices:

$$T = (1)\,, \quad T' = (-1)\,, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

THEOREM 2. *For each $n$, $\mathcal{G}_n \cong Z \oplus \pi_{n-1}(SO_n)$. These isomorphisms are defined as follows:*

(1) $\left(\frac{r-\tau}{2}, \frac{\tau}{8}, \frac{\chi^2}{2}\right) \quad : \mathcal{G}_n \cong Z \oplus Z \oplus Z\,;$

(2) $\left(\frac{r-\tau}{2}, \tau, \frac{\chi^2-\tau}{8}\right) : \mathcal{G}_n \cong Z \oplus Z \oplus Z\,;$

(3) $\left(\frac{r-\tau}{2}, \frac{\tau}{8}, \frac{\chi^2}{2}\right) \quad : \mathcal{G}_n \cong Z \oplus Z \oplus Z_2\,;$

(4) $\left(\frac{r-\tau}{2}, \frac{\tau}{8}\right) \quad : \mathcal{G}_n \cong Z \oplus Z\,;$

(5) $\left(\frac{r}{2}, \Phi, \varphi(\chi)\right) \quad : \mathcal{G}_n \cong Z \oplus Z_2 \oplus Z_2\,;$

(6) $\left(\frac{r}{2}, \Phi\right) \quad : \mathcal{G}_n \cong Z \oplus Z_2\,;$

(7) $\left(\frac{r}{2}\right) \quad : \mathcal{G}_n \cong Z\,.$

*Values of the invariants are realised as follows*:

(1) $U(0, 0) \to (1, 0, 0)\,, \quad V(\chi = 0) \to (0, 1, 0)\,, \quad U(1, 1) \to (1, 0, 1)\,;$

(2) $U(0,0)\to(1,0,0)$ , $T(\chi=e)\to(0,1,0)$ , $T(\chi=3e)\to(0,1,1)$ ;

(3) $U(0,0)\to(1,0,0)$ , $V(\chi=0)\to(0,1,0)$ , $U(1,1)\to(1,0,1)$ ;

(4) $U\to(1,0)$ , $V\to(0,1)$ ;

$$W(\chi=0,\varphi(e_1)=\varphi(e_2)=0)\to(1,0,0)\ ,$$
(5)
$$W(\chi=0,\varphi(e_1)=\varphi(e_2)=1)\to(1,1,0)\ ,$$
$$W(\chi=e_1,\varphi(e_1)=1,\varphi(e_2)=0)\to(1,0,1)\ ;$$

(6) $W(\varphi(e_1)=\varphi(e_2)=0)\to(1,0)$ , $W(\varphi(e_1)=\varphi(e_2)=1)\to(1,1)$ ;

(7) $$W\to 1\ .$$

*Thus there are no relations between the invariants as stated.*

**Obstruction to closing $N$**

Each $n$-space determines an almost closed manifold $N$. Taking the boundary of $N$ defines a homomorphism $\mathfrak{o}$: $\mathcal{G}_n\to\Gamma_{2n-1}$ by Lemma 4, Corollary 2. We must now consider this homomorphism. We note that if $N$ actually corresponds to a closed manifold $M$, $\mathfrak{o}$ must vanish on $N$. Hence it is desirable to have some examples of closed, $(n-1)$-connected $2n$-manifolds. The first obvious example is the product manifold $S^n\times S^n$. We can at once verify which element of $\mathcal{G}_n$ corresponds to this. In each case, the first invariant of the element of $\mathcal{G}_n$ is 1; the others (if any) zero. Our invariants were chosen partly with this in mind. Thus the first component of $\mathcal{G}_n$ is irrelevant in computing the obstruction. We can enunciate this simple and general result:

THEOREM 3. *The obstruction to closing an almost closed $(n-1)$-connected $N$ depends only on $\tau$, $\chi^2$ ($n$ even) and $\Phi$, $\varphi(\chi)$ ($n$ odd).*

However, it is difficult to make any further progress, especially in the case of the elements of order 2. We first refer to work on almost parallelisable closed manifolds by Milnor and Kervaire [12]. A $2n$-manifold is almost parallelisable if the restriction of its tangent bundle to the $(2n-1)$-skeleton is trivial. For $(n-1)$-connected manifolds, we have the obstruction $\chi$ to triviality on the $n$-skeleton. Further obstructions lie in zero groups. Hence, for us, $\chi=0$ is equivalent to almost parallelisability. Then if $n$ is even and $\chi=0$, the index $\tau$ must be divisible by a certain integer $I(M_0^{2n})=8I_l$ say, where $n=2l$.

PROPOSITION 5. *If $n=2l$, $\chi=0$, the obstruction is* $(\tau/8)(\text{mod } I_l)$.
In fact we already know that in this case the obstruction is of this form. But in [12], some facts are given about $I_l$.

PROPOSITION 6.

(11) $$I_l=2^{2l-4}(2^{2l-1}-1)B_l j_l a_l/l\ ,$$

*where* $B_l$ *is the* $l^{\text{th}}$ *Bernoulli number,* $j_l$ *the order of the image of* $J$: $\pi_{4l-1}(SO_N) \to \pi_{4l+N-1}(S^N)$ *for* $N$ *large, and* $a_l = 2$ *for* $l$ *odd,* 1 *for* $l$ *even.*

It can then be deduced, for example (cf., [12]),

$$2^{2l-2}(2^{2l-1} - 1)a_l \mid I_l \,, \tag{12}$$

e.g., $2^2 \cdot 7 \mid I_2$, $2^6 \cdot 127 \mid I_4$, $2^5 \cdot 31 \mid I_3$. In fact it is known [11] that $I_2 = 2^2 \cdot 7$, $I_3 = 2^5 \cdot 31$, $I_4 = 2^6 \cdot 127$. However, higher values of $I_l$ are not yet known.

Likewise, although for almost parallelisable manifolds with $n$ odd, the obstruction must be $\Phi$ or zero; it is not known, except in one case, whether it is $\Phi$ or zero.

Although, up to this point, we have considered $n$ arbitrary ($n \geqq 3$), we now solve the problem only for $n \leqq 8$ (this has the advantage of taking in the special Cases 2, 7; and these are, in fact, easier than the general case).

$n = 3, 7$, we have *Case* 7, and there is no obstruction.

$n = 5$, we have *Case* 6, and the obstruction was computed by Kervaire [10]: for a closed manifold, $\Phi = 0$.

$n = 6$, we have *Case* 4, and the obstruction depends only on $\tau$: we found above that it is $(\tau/8)(\text{mod } 2^5 \cdot 31)$.

$n = 4, 8$, we have *Case* 2. In these cases we are able to construct further closed manifolds: the projective planes over the quaternions and Cayley numbers. These have index 1, and the Pontrjagin class $p$ (and hence $\chi$) is known. We find that $\chi^2 = 1$. Hence both $\to (0, 1, 0)$. Thus the obstruction in these cases depends only on $(\chi^2 - \tau)/8$. But in the case $\chi = 0$, the obstruction was determined above: it is $(\tau/8)(\text{mod } I_l)$. Hence the obstruction is $(\chi^2 - \tau)/8(\text{mod } 28)$ or $(\text{mod } 2^6 \cdot 127)$ in the two cases. Now we have this obstruction, it is not without interest to compute, and find that in each case we obtain the Todd genus $\hat{A}$: for $n = 4$, $\hat{A} = (1/28)\{(\chi^2 - \tau)/8\}$, and for $n = 8$, $\hat{A} = 2^{-6} \cdot (1/127)\{(\chi^2 - \tau)/8\}$. (Cf. Eells and Kuiper [4].)

THEOREM 4. *For the range* $3 \leqq n \leqq 8$, *the obstruction is as follows: for* $n = 3$ *or* 7, *it is zero; for* $n = 5$, *it is* $\Phi$; *for* $n = 6$, *it is* $(\tau/8) \text{ mod } 2^5 \cdot 31$; *for* $n = 4$ (*respectively* 8), *it is* $(\chi^2 - \tau)/8 \text{ mod } 28$ (*resp.* $2^6 \cdot 127$).

Thus in these cases we can in principle enumerate all closed $(n - 1)$-connected $2n$-manifolds.

If we agree to ignore the groups $Z_2$, then, we find the following. For $n$ odd, there is nothing left. For $n \equiv 2 \pmod 4$ the single obstruction is $(\tau/8)(\text{mod } I_l)$. *Case* 2 is completely dealt with above; this leaves *Case* 1. We know again that if $\chi^2 = 0$, the obstruction is $(\tau/8)(\text{mod } I_l)$: thus the main question remaining for consideration is: what values of $\chi^2$ are possible? Unfortunately, the author has been unable to settle this question. However, we may restate it in line with current notation. Set $n = 4m$.

We have, by Kervaire [8], the formula

$$p_m = \pm a_m(2m-1)!\chi , \tag{13}$$

and by Hirzebruch [6], a formula $\tau = L(p_m^2, p_{2m})$ for the index in terms of Pontrjagin numbers. Hence linear combinations of $\chi^2$, $\tau$ can be equally well regarded as linear combinations of $p_m^2$, $p_{2m}$. Then, certainly, for a closed manifold, $p_{2m}$ is integral, which gives one condition, and the Todd genus $\hat{A}(p_m^2, p_{2m})$ is integral [1], which yields another.

We will now compute these relations explicitly.

For any genus as defined by Hirzebruch [6] from the formal power series $Q(z)$ we have, in the case when $p_m$ and $p_{2m}$ are the only non-vanishing Pontrjagin classes,

$$K(M) = s_{2m}p_{2m} + \frac{1}{2}(s_m^2 - s_{2m})p_m^2 ,$$

where the coefficients $s_m$ are given by

$$R(z) \equiv Q(z)\frac{d}{dz}\left(z/Q(z)\right) = 1 + \sum_1^\infty(-1)^j s_j z^j .$$

We shall apply this in two cases: for the $L$-genus we have

$$Q(z) = \sqrt{z}/\tanh(\sqrt{z}) = 1 + \sum_1^\infty(-1)^{k-1}2^{2k}B_k z^k/(2k)! ,$$

so

$$R(z) = \frac{1}{2}\left(1 + 2\sqrt{z}/\mathrm{sh}\,(2\sqrt{z})\right) = 1 + \sum_1^\infty(-1)^k 2^{2k}(2^{2k-1}-1)B_k z^k/(2k)!$$

and

$$s_j = 2^{2j}(2^{2j-1}-1)B_j/(2j)! .$$

For the $\hat{A}$-genus of Borel and Hirzebruch [2], we have

$$Q(z) = \frac{1}{2}\sqrt{z}\Big/\mathrm{sh}\left(\frac{1}{2}\sqrt{z}\right) = 1 + \sum_1^\infty(-1)^k 2^{1-2k}(2^{2k-1}-1)B_k z^k/(2k)! ,$$

so

$$R(z) = \frac{1}{2}\left(1 + \frac{1}{2}\sqrt{z}\Big/\tanh\left(\frac{1}{2}\sqrt{z}\right)\right) = 1 + \sum_1^\infty(-1)^{k-1}B_k z^k/(2(2k)!),$$

and

$$s_j = -\frac{1}{2}B_j/(2j)! .$$

These formulae give explicity $L(M) = \tau$ and $\hat{A}(M)$ as linear combinations of $p_m^2$, $p_{2m}$. Solving these equations in terms of $\chi^2$ and $\tau$, where $p_m^2 = a_m^2((2m-1)!)^2\chi^2$, we find

$$(14)_m \qquad p_{2m} = \frac{a_m^2}{2}((2m-1)!)^2\left\{1 - \frac{(2^{2m-1}-1)^2}{2^{4m-1}-1}\binom{4m}{2m}\frac{B_m^2}{B_{2m}}\right\}\chi^2 + \frac{(4m)!}{2^{4m}(2^{4m-1}-1)B_{2m}}\tau ,$$

$$(15)_m \qquad \hat{A}_{2m} = \left\{\frac{(2^{2m}-1)a_m B_m}{2m}\right\}^2 \frac{\chi^2}{8(2^{4m-1}-1)} - \frac{\tau}{2^{4m+1}(2^{4m-1}-1)} .$$

This last formula is surprisingly simple, almost all the terms having cancelled out, and is the simplest relation between any 3 of the 4 invariants. We have already noted that for $m = 1, 2$ it reduces to

$$(15)_1 \qquad \hat{A}_2 = 2^{-5}(\chi^2 - \tau)/7$$

and

$$(15)_2 \qquad \hat{A}_4 = 2^{-9}(\chi^2 - \tau)/127 ,$$

and in these cases the integrality of $\hat{A}$ guarantees the vanishing of the obstruction. This, however, is not true in general, for in the very next case we have

$$(15)_3 \qquad \hat{A}_6 = 2^{-13}(2^8\chi^2 - \tau)/2047 ,$$

but

$$(14)_3 \qquad p_6 = \frac{3^6\cdot5^3\cdot7^2\cdot11\cdot13\tau - 2^8\cdot3^2\cdot5^2\cdot82,573\chi^2}{2\cdot2047\cdot691} ,$$

and neither condition of divisibility implies the other, though the divisibility conditions mod 2047 are the same.

Let us in fact examine these conditions more closely. By a theorem of Euler [28],

$$e_m = 2^{2m-2}a_m(2^{2m}-1)B_m/m$$

is an integer. Then the expression for $\hat{A}$ may be written

$$(15) \qquad 2^{4m+1}(2^{4m-1}-1)\hat{A}_{2m} = e_m^2\chi^2 - \tau$$

so this clearly represents a divisibility condition by precisely $2^{4m-2}(2^{4m-1}-1)$, as in the special cases already listed. The other condition is more complicated. Write $q_{2m} = p_{2m} - \frac{1}{2}p_m^2$. The conditions that $p_{2m}$ and $q_{2m}$ be integral are clearly equivalent.

Setting $B_m = N_m/D_m$, as a fraction in its lowest terms, we assert that it follows, from the fact that $\hat{A}_{2m}$ is an integer, that $N_{2m}q_{2m}$ is also. In fact, we can show more. Write $m = K_m L_m$, where the prime factors of $K_m$ divide $D_m$, and those of $L_m$ do not. Then von Staudt's second theorem [28] states that $N_m = L_m R_m$, with $R_m$ an integer. Then we claim that, if $\hat{A}_{2m}$, $\chi^2$ are integers, so is $R_{2m} q_{2m}$. For the formula for $\hat{A}_{2m}$ in terms of $p_{2m}$, $p_m^2$ leads to

$$2\hat{A}_{2m} = \left(\frac{a_m B_m}{4m}\right)^2 \chi^2 - \frac{B_{2m}}{(4m)!} q_{2m} ,$$

and so to

$$R_{2m} q_{2m} = \frac{2D_{2m}}{C_{2m}} (4m)! \left\{ \left(\frac{a_m B_m}{4m}\right)^2 \chi^2 - 2\hat{A}_{2m} \right\} .$$

Here, the coefficient of $\hat{A}_{2m}$ is integral since $C_{2m}$ is, by definition, a factor of $2m$, and we may verify that the same holds for $\chi^2$, since $16m^2 D_m C_{2m}$ is a factor of $(4m)!$ for $1 < m$, and we have studied above the case $m = 1$.

Thus the further divisibility condition involves $R_{2m}$. Nothing is known about this number, except that it is prime to $D_{2m}$ (and so odd), and tends to have rather large prime factors. In every case known to the author, $R_{2m}$ is prime to $2m(2^{4m-1} - 1)$, and we get a divisibility condition independent of the one deduced from $\hat{A}$, and one may conjecture that this holds in general. If this were so, we note that we have no divisibility condition applying to $\chi^2$ alone.

Although it is certainly necessary that both $\hat{A}_{2m}$ and $q_{2m}$ be integral for the obstructions to vanish, it is by no means clear that this is sufficient (cf. (11)); and, for the present, this seems the most unsatisfactory part of the whole theory. It would be most desirable at this stage if one could find constructions giving examples of closed, differential, $(n-1)$-connected $2n$-manifolds, but for the present this must stand as an unsolved problem.

## Homotopy and combinatorial classifications

In this section we continue our policy of integrating as far as possible all known facts on $(n-1)$-connected $2n$-manifolds in a single framework, and express homotopy type and combinatorial invariants in terms of our main invariant, the $n$-space.

Notice that if $N$ is an almost closed $(n-1)$-connected $2n$-manifold, then its boundary is combinatorially equivalent to a simplex boundary. Hence we can construct from $N$ a closed combinatorial manifold $M$ in a natural, unique way. We shall consider the set of such manifolds.

First consider homotopy type classification. Since we have simply connected polyhedra, we can use the homology decomposition of Hilton and

Eckmann [3]. $H_n(M)$ is given by the integer $r$ as $rZ$; there is then a single $k'$-invariant $\beta \in \pi_{2n-1}(M^n)$, where $M^n$ is the $n$-skeleton. This is a deformation retract of $N$, and is a bouquet of $r$ copies of $S^n$. Define maps $e_i$: $S^n \to M^n$ corresponding to the generators of $\pi_n(M^n) \cong H$.

Now $\pi_{2n-1}(M^n)$ is known, since $M^n$ is a bouquet, and in fact

(16) $$\pi_{2n-1}(M^n) \cong r\pi_{2n-1}(S^n) \oplus \binom{r}{2}\pi_{2n-1}(S^{2n-1}) .$$

Write

(17) $$\beta = \sum_i e_i \circ \beta_i + \sum_{i<j} \gamma_{ij}[e_i, e_j] .$$

Then $\beta$ is the invariant; or, $\{\beta_i, \gamma_{ij}\}$ is the set of invariants. Now the $\gamma_{ij}$ define the cup product in $M$. If $e_1^*$ is the base of $H^n(M)$ dual to the $e_i$, we have

(18) $$e_i^* \cdot e_j^* = \gamma_{ij}(\text{if } i < j) \quad = (-1)^n \gamma_{ij}(\text{if } i > j) \quad = H(\beta_i)(\text{if } i = j) .$$

Define $\gamma_{ij}$ for all $i, j$ as $e_i^* \cdot e_j^*$. Then $(\gamma_{ij})$ is up to sign the inverse of $(c_{ij})$, for write $f_i = M \cap e_i^*$, and $e_i = M \cap g_i$. Then

$$\langle f_i, e_j^* \rangle = \langle M, e_i^* \cup e_j^* \rangle = \gamma_{ij} ,$$

so $f_i = \sum \gamma_{ij} e_j$, and $f_i \cdot e_k = \sum \gamma_{ij} c_{jk}$. But

$$f_i \cdot e_k = (-1)^n e_k \cap M \cap e_i^* = (-1)^n e_k \cap e_i^* = (-1)^n \delta_{ik} .$$

We now interpret $\beta_i$. In the case $r = 1$, a result of Milnor [14] states that $e_i^* \circ \beta = J\alpha(e_i)$, where $e_i^*$ is the map $M^n \to S^n$ inducing the cohomology element $e_i^*$. The proper generalisation of this is clearly:

(19) $$g_i \circ \beta = J\alpha(e_i)$$

where $g_i \in \pi^n(M^n)$ corresponds to $g_i \in H^n(M^n)$, so $g_i \circ e_j = c_{ij}$.

From these interpretations of $\gamma_{ij}$ and $\beta$ we deduce the following

LEMMA 8. *Given an $n$-space, a complete set of homotopy type invariants for the corresponding $M$ is given by the bilinear form $H \otimes H \to H$ and by $J\alpha$: $H \to \pi_{2n-1}(S^n)$.*

Now in *Cases* 5 and 6, $\partial \iota_n$ is an element of order 2 in $\pi_{n-1}(SO_n)$. It is well-known that $J(\partial \iota_n) = [\iota_n, \iota_n]$, and that this is non-zero. Thus $\varphi$ (in *Case* 6), and $\varphi$ or $\varphi'$ (in *Case* 5) (we cannot tell which, but if the result does not hold for both, define $\varphi$ as the one for which it does) is a homotopy type invariant. Thus in *Cases* 4, 6, 7, homotopy type classification coincides with diffeomorphism classification (modulo $\Gamma_{2n}$). These cases are given by $n \equiv 3, 5, 6, 7 \pmod 8$; i.e., by $\pi_{n-1}(SO) = 0$.

THEOREM 5. *If $\pi_{n-1}(SO) = 0$, $n \geqq 3$, and $M_1$, $M_2$ are differential* $(n-1)$-

*connected $2n$-manifolds of the same homotopy type, then for some manifold $T$ homeomorphic (and so combinatorially equivalent) to $S^{2n}$, $M_1$ is diffeomorphic to $M_2 \# T$. If $n = 3, 6$, $M_1$ is diffeomorphic to $M_2$.*

The last clause in the theorem follows since $\Gamma_6 \cong \Gamma_{12} \cong 0$.

Now we consider the combinatorial classification of these manifolds. Of course, a homotopy type invariant is *a fortiori* a combinatorial invariant, so, in particular, are the bilinear form, and the map $\varphi$ (when $n$ is odd). Also, in the case when $n = 4m$, the Pontrjagin class $p_m(M)$ is defined, and combinatorially invariant with rational coefficients, by Thom [21]. Since $M$ is torsion free, this shows (by (13)) that in this case, $\chi$ is a combinatorial invariant. Hence in *Cases* 1, 2, 4, 6, 7 the combinatorial classification coincides with the differential. In *Cases* 3, 5 we do not know whether or not $\chi$ is a combinatorial invariant. If we could prove $J$ a monomorphism in the stable range when $n \equiv 1, 2 \pmod 8$, $\chi$ would be an invariant of homotopy type, and the classifications again coincide; by Kervaire [7], this holds for $n = 9, 10$, and it is also known for $n = 17, 18$.

We observe that in *Cases* 1 and 2 the combinatorial classification differs widely from that according to homotopy type; in fact for any manifold with $H \neq 0$, we can find infinitely many inequivalent manifolds all with the same homotopy type (by varying $\chi$), and, being simply connected, the same simple homotopy type [24]. Note in particular that they will in general have different Pontrjagin numbers. The simplest example is with $n = 4$, $r = 1$, $H$ generated by $e$ with $e^2 = 1$, and $\chi = e$ or $97e$ (both yielding closed differential manifolds). Nothing, however, appears to be known about the homeomorphism classification of such manifolds.

## Applications

We now apply our results to various problems that seem of interest. Our conclusions are mostly not new qualitatively; but it is of interest to fit these questions in our framework, and see how theorems follow from our general classification, without need of special investigations.

*Problem* 1. *Existence of differential structures on manifolds.* We have noted that a closed combinatorial manifold corresponds to any $n$-space, but that there is an obstruction to the existence of corresponding closed differential manifolds. This point needs no further discussion. However, in some cases we can even find homotopy type obstructions. This is clear, for example, if the index is restricted to certain values, as in Proposition 5. Now consider the case $n = 4$: the obstruction is $(\chi^2 - \tau)/8 \pmod{28}$. Now $J\chi$ is a homotopy type invariant, i.e., $\chi$ is invariant mod 24. Then $\chi^2$ is invariant mod 48, and so $(\chi^2 - \tau)/8$ mod 6. Thus $(\chi^2 - \tau)/8$ mod 2 is a homotopy type obstruction to the existence of differential structure. Similarly for $n = 8$, we have $(\chi^2 - \tau)/8 \bmod 4$. For $n = 5$ we have $\Phi$ which, as noted above, is also a homotopy type invariant. Thus in all these cases there are closed combinatorial manifolds not of the homotopy type of any closed differential manifold. These results are due to Kervaire [10] and Milnor.

*Problem* 2. *Additive decompositions of manifolds*. The connected sum of two closed differential manifolds is defined in Milnor [15]. Given such a sum, $M = M_1 \# M_2$, we call this an (additive) decomposition of $M$. It is clear that $M$ is $(n - 1)$-connected if and only if both $M_1$ and $M_2$ are. We first consider the problem:

*Problem* 2A. When do any two additive decompositions of $M$ admit isomorphic refinements? (The unique factorisation problem.) For this problem, the non-uniqueness of $M$ corresponding to $\Gamma_{2n}$ is unimportant, and it is sufficient to examine the corresponding $n$-spaces. If $n$ is even, unique factorisation breaks down completely. This follows from the corresponding statement for quadratic forms of type II. In fact, choose an $n$-space with $r = \tau \neq 0$, $\chi = 0$, and zero obstruction (we omit *Case* 2, which may be similarly discussed, and even more explicitly); such exists by Proposition 5 if and only if $\tau$ is divisible by $8I_{n/2}$ — we take $\tau = 8I_{n/2}$. Then the corresponding manifold $M_1$ is indecomposable. By changing orientation we obtain an indecomposable manifold $M_2$ with $\tau = -8I_{n/2}$. But for the connected sum $M_1 \# M_2$, the index is zero, and the form indefinite, hence, using Proposition 4, it decomposes as a direct sum of $r$ forms given by the matrix $U$. We deduce

$$(20) \qquad M_1 \# M_2 = \#^r(S^n \times S^n) \# T ,$$

where $T$ is an element of $\Gamma_{2n}$ (a unit). Here each element is indecomposable. The discussion is similar, but simpler, in the combinatorial case.

If $n$ is odd, on the other hand, unique decomposition nearly holds. For, as shown above, each $n$-space splits as a sum of spaces of rank 2, and we must then compare these pairwise. We find the following. In *Case* 7, unique decomposition holds (Smale [19]). In *Case* 6, it fails for combinatorial manifolds; it holds for closed differential manifolds if and only if $\Phi$ is zero for all these. In *Case* 5, it never holds; in the combinatorial case, this again follows by using $\Phi$. In the differential case, even if $\Phi$, $\varphi(\chi) = 0$ in every case, we still have one invariant: the type, and non-uniqueness again follows. Specifically, let $W_1$, $W_2$ be the $n$-spaces with matrix $W$, $\varphi(e_1) = \varphi(e_2) = 0$ for each, and $\chi = 0$, $\chi = e_1$ respectively. Then the obstruction certainly vanishes for each, so they correspond to closed manifolds. But now $W_1 \neq W_2$, both are indecomposable, and we have

$W_2 \oplus W_2 = W_1 \oplus W_2$, by our classification. (Lemma 5.)

*Problem* 2B. Which manifolds $M$ are indecomposable? (Problem of primes.) This problem is of interest, since if we can describe all indecomposable manifolds, we know that all the rest are connected sums of these known ones.

Let us first consider the problem for $n$-spaces. Then we see at once that any decomposition of the bilinear form determines one of the $n$-space ($\alpha$ then splits automatically). For $n$ odd, then any indecomposable space must have matrix $W$. For $n$ even, any indefinite form is decomposable unless it has matrix $U$; but the structure of definite forms is complicated, and we know of no general theorems of any use for this problem. Applying these results to the combinatorial case, we have

LEMMA 9. *A combinatorial manifold of the type considered is decomposable unless*

(a) *it has rank* 2 *and matrix* $U$ *or* $W$, *or*

(b) $n$ *is even and the quadratic form definite and indecomposable. A differential manifold* (*of this type*) *with* $n$ *odd is decomposable unless it has rank* 2.

PROOF. For the last part it suffices to note that the decomposition explicitly produced in our classification of $n$-spaces above has the property that, if one or both of $\Phi$, $\varphi(\chi)$ vanish for the total, the same holds for each decomposed part; in fact, $\chi$ was non-zero only in one summand, and we gave an explicit method to make $\Phi$ zero.

When $n$ is even, we are in general unable to solve the problem posed above. This is partly on account of our lack of knowledge of the obstruction. However, if we restrict ourselves to the almost parallelisable case ($\chi = 0$) it is possible to make some progress, and in *Case* 4 this is no restriction. Even so, all we are able to prove is: An almost parallelisable manifold, with indefinite quadratic form, is decomposable unless it has rank 2. For in this case, the $n$-space decomposes, with $U$ as one summand. Since $\chi = 0$ throughout, the obstruction vanishes for $U$; since it vanishes for the total, it does also for the other summand. Hence the manifold decomposes, with $(S^n \times S^n)$ as one summand.

While considering indecomposable manifolds it seems appropriate to give geometrically one obvious example that the reader may have expected us to mention earlier; namely, that of $n$-sphere bundles over $S^n$. These are classified by $\pi_{n-1}(SO_{n+1})$, onto which $S$ maps $\pi_{n-1}(SO_n)$ by (4). Picking a reduction to $SO_n$, given by $\alpha \in \pi_{n-1}(SO_n)$, determines a cross-section $e_1$, the characteristic element of whose normal bundle is $\alpha$. Another imbedded sphere is a fibre $e_2$ which of course has trivial normal bundle.

Our invariants are then

$$e_1^2 = HJ(\alpha)\ ,\quad e_1e_2 = (-1)^n e_2e_1 = 1\ ,\quad e_2^2 = 0\ ,\quad \alpha(e_1) = \alpha\ ,\quad \alpha(e_2) = 0\ .$$

If we are not in *Case* 2, we can pick the reduction so that $HJ(\alpha) = 0$. The reader will now notice that the element determines the same element of $\mathcal{G}_n$ as does $(S^n \times S^n)$, so throws no further light on the obstruction map $\mathfrak{o}$. If $n$ is odd, the only non-trivial case is *Case* 5, where the bundles with $S(\alpha) = 0$ (the product) resp. $\neq 0$ correspond to the spaces $W_1$, $W_2$ above. If $n$ is even, a simple computation shows that all $n$-spaces with $r = 2$, $\chi^2 = \tau = 0$, are given by $U$ with $\chi = de_1$ ($d$ even in *Case* 2) or, in *Case* 2, $S$ with $\chi = de_1$ ($d$ odd), and we now see that all these are given by bundles of this type.

*Problem* 2C. Does $M_1 \# M_2 = M_1 \# M_3$ imply $M_2 = M_3$? (Unique subtraction problem.) This problem is related to 2A, but is a more refined question. If we regard the connected sum (which is clearly commutative and associative) as turning our set of manifolds into a monoid, with an equivalence relation induced by the operation of the maximal subgroup $\Gamma_{2n}$ consisting of all invertible elements of the monoid [15], [18], then Problem 2A asks whether the monoid is free, Problem 2B asks for a minimal set of generators, and Problem 2C asks whether subtraction is possible in the monoid, so that its canonical map into a group is an imbedding.

If $n$ is even, the result is false. For choose two inequivalent quadratic forms of the same genus (necessarily definite), of type II and with given index $\tau$ (divisible by 8, and greater than 8); this is possible by a result of Eichler [26]. Set $\chi = 0$; the forms are supposed of type II, so this is legitimate even in *Case* 2. Let them define manifolds $M_1$, $M_2$. Then $M_1$, $M_2$ have not the same homotopy type. But $M_1 \# (S^n \times S^n) = M_2 \# (S^n \times S^n)$, since both have the same invariants.

(Note that though we only prove isomorphism modulo $\Gamma_{2n}$, this gives an adequate counter-example.)

Even in the case $n = 2$ the result is known to be false, for if $\bar{P}_2(C)$ denotes the complex projective plane with orientation reversed, then

$$P_2(C) \# P_2(C) \# \bar{P}_2(C) = P_2(C) \# (S^2 \times S^2)\ , \tag{21}$$

as has been shown by Hirzebruch, but $P_2(C) \# \bar{P}_2(C)$ has different homotopy type (and even genus) from $S^2 \times S^2$. Replacing in this example $n = 2$ by 4 resp. 8, and $C$ by quaternions resp. Cayley numbers, (21) continues to hold, for we can produce an explicit isomorphism of the corresponding $n$-spaces. In fact, if $e_1$, $e_2$, $e_3$ is a base of the homology cooresponding to the left-hand decomposition, set $f_1 = \chi = e_1 + e_2 + e_3$, $f_2 = e_1 + e_3$, $f_3 = e_2 + e_3$, and we obtain the decomposition on the right of (21). It is also of interest to note that if $n = 1$, the answer is *yes* if and only if we restrict attention to oriented manifolds (which form a free monoid).

For $n$ odd, in Cases 6 and 7, the conjecture holds for our class of manifolds, since the only invariants, $r$ and $\Phi$, are additive. But in *Case* 5, the counter-example given to Problem 2A serves in this case also, in the differential case, and in the combinatorial case whenever $\chi$ is a combinatorial invariant, e.g., for $n = 9$.

Finally we remark that although this problem is superficially similar to that of Whitehead (Question 3 of [24]), the latter is much more difficult, and our results have no relevance to it. Whitehead's problem is posed for bounded manifolds, and is: Does $M_1 + M_2 = M_1 + M_3$ imply that $M_2$, $M_3$ have the same simple homotopy type? But our bounded manifolds $N$ had the homotopy types of bouquets of spheres, so the conjecture is trivially true for them.

*Problem* 3. *Multiplicative decompositions of manifolds*. The only case which we contemplate is that in which each of the factors is a homotopy sphere. The invariants of such a product are easy to compute. The intersection matrix is $U$ or $W$ according as $n$ is even or odd (since it is homotopy invariant); and in the latter case, for the same reason, $\varphi(e_1) = \varphi(e_2) = 0$. Let the product be $T_1 \times T_2$, and let the characteristic elements of the tangent boundles of $T_1$, $T_2$ be $\alpha_1, \alpha_2 \in \pi_{n-1}(SO_{n+1})$. It is known that they are zero except perhaps in Cases 3, 5 and are annihilated by the $J$-homomorphism; it is probable that they must be zero. Then a straightforward computation with bundles gives $S\alpha(e_i) = \alpha_i$ $(i = 1, 2)$, and so $\chi = \alpha_2 e_1 + \alpha_1 e_2$.

Thus in all *Cases* except 3, 5 the product has the same invariants as $S^n \times S^n$. This could have been predicted, since in these cases the combinatorial classification agrees with the differential and (for $n \geqq 5$), each of $T_1$, $T_2$ must be combinatorially equivalent to $S^n$ (Smale [18], [31]). Note that if there exist $T_i$ with $\alpha_i$ not vanishing, we find that in *Case* 3, $\frac{1}{2}\chi^2$, and in *Case* 5, $\varphi(\chi)$ is non-zero, thus we gain more information about the obstructions. Of course, there is no converse inference. If just one $\alpha_i$ is non-zero, we have the same invariants as for a certain $n$-sphere bundle over $S^n$, by a remark above.

Let us now consider the case $\alpha_1 = \alpha_2 = 0$. We have proved

LEMMA 10. *Let $T_1$, $T_2$ be two homotopy $n$-spheres which are $\pi$-manifolds. Then for some $T \in \Gamma_{2n}$, we have $T_1 \times T_2 = (S^n \times S^n) \# T$.*

This is really precise if $\Gamma_{2n} = 0$; however, for $n \geqq 3$, this is only known to occur for $n = 3, 6$.

Actually, for $n = 6$, $T_i$ is diffeomorphic to $S^6$, so this result is trivial. But we can easily find a non-trivial case. For example, $\Gamma_7 \cong Z_{28}$, $\Gamma_{14} \cong Z_2$.

Hence all the products $T_1 \times T_2$ ($T_i \in \Gamma_7$) fall into one of two diffeomorphism classes, so at least one of these admits many differentially distinct multiplicative decompositions. In fact, we are able to decide which class. This was pointed out to me by J. Milnor. First we note that all the manifolds under discussion are $\pi$-manifolds. Moreover $T_1$ bounds a $\pi$-manifold, hence so does $T_1 \times T_2$, and of course $S^n \times S^n$ does. Hence $T$ bounds a $\pi$-manifold, and this shows that $T$ defines the zero element of $\Gamma_{14}$.

COROLLARY 1. *Let $T_1$, $T_2$ be differential manifolds homeomorphic to $S^7$. Then $T_1 \times T_2$ is diffeomorphic to $S^7 \times S^7$.*

COROLLARY 2. *Let $T_1$, $T_2$ be two homotopy 3-spheres. Then $T_1 \times T_2$ is diffeomorphic to $S^3 \times S^3$.*

This result is of interest with relation to the Poincaré conjecture. Of course, the power of the result derives from Smale's theorem that a compact contractible 6-manifold with simply connected boundary is diffeomorphic to a disc [17].

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