A MOVE ON DIAGRAMS THAT GENERATES S-EQUIVALENCE OF KNOTS

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Abstract: Two knots in three-space are S-equivalent if they are indistinguishable by Seifert matrices. We show that S-equivalence is generated by the doubled-delta move on knot diagrams. It follows as a corollary that a knot has trivial Alexander polynomial if and only if it can be undone by doubled-delta moves.

We consider tame, oriented knots in oriented S^3, with equivalence being ambient isotopy. A Seifert surface for such a knot is an oriented surface whose boundary is the given knot, and whose orientation induces the given orientation on the knot. An oriented surface S in S^3 has a linking form $\langle *, * \rangle$ on the homology $H_1(S)$, where $\langle x, y \rangle$ is defined to be the linking number of the cycle x with the cycle y slightly pushed off S in a direction determined by the orientation of S. Given a knot K, choose a Seifert surface S for K and a basis for $H_1(S)$. Then the linking form is represented by an integer matrix M, which is called a Seifert matrix for K. Two knots are called S-equivalent if they have a common Seifert matrix (which is the same as saying that they have a common Seifert form). We sketch a proof at the end of the paper that this is an equivalence relation. The reader who wishes may ignore this proof and take S-equivalence to be the smallest equivalence relation that includes any pair of knots which have a common Seifert matrix. Two knots K and K' are then S-equivalent if and only if there exists a sequence $K = K_1, K_2, \ldots K_m = K'$, such that for all $1 \leq i < m$, K_i and K_{i+1} have a common Seifert matrix. It makes no difference in the proof of Theorem A which definition we use, since in either case what we need to show is that two knots share a common Seifert matrix if and only if they are equivalent by certain diagram moves which we will call doubled-delta moves.

The usual way to define S-equivalence is to define it first for matrices, and then to define it for knots by saying that knots with S-equivalent matrices are S-equivalent. See Gordon [3] and Kawauchi [5] for the standard definition, for further references, and for more detail on the following statements. S-equivalence of matrices was first introduced by Trotter in [13] under the name h-equivalence. Murasugi [10] and Rice [11] applied it to matrices obtained from knot diagrams. None of the abelian invariants, such as the Alexander polynomials, homology of cyclic and branched covers, or signatures, can distinguish between S-equivalent knots. It was shown by Levine [7] that in higher dimensions simple knots are characterized by S-equivalence. Two knots are S-equivalent if and only if their (integral) Blanchfield pairings are isometric. This follows from work of Levine [7] and Kearton [6], and was also proved by Trotter [14] from a purely algebraic point of view.

A knot may be given by a regular projection in the usual way, with equivalence of diagrams given by the Reidemeister moves. For more details on knots, diagrams, and Seifert surfaces and matrices, see Rolfsen [12]. or Kawauchi [5]. We consider now the delta move and the *doubled-delta move*, shown in Figure 1.

† Research supported in part by the Naval Academy Research Council

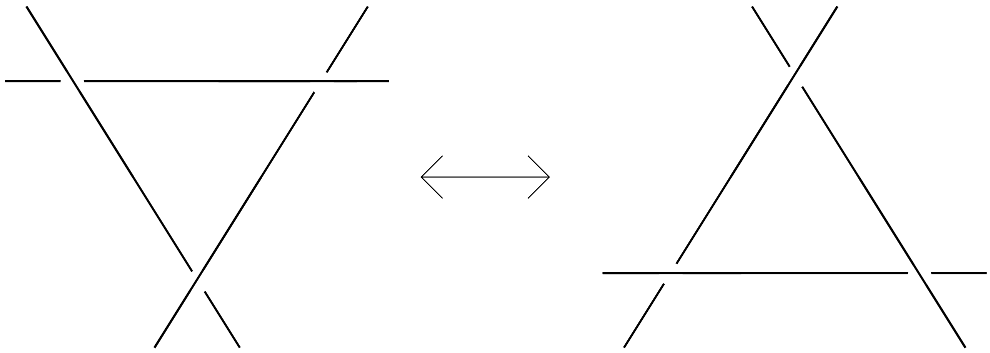

Figure 1a. The delta move.

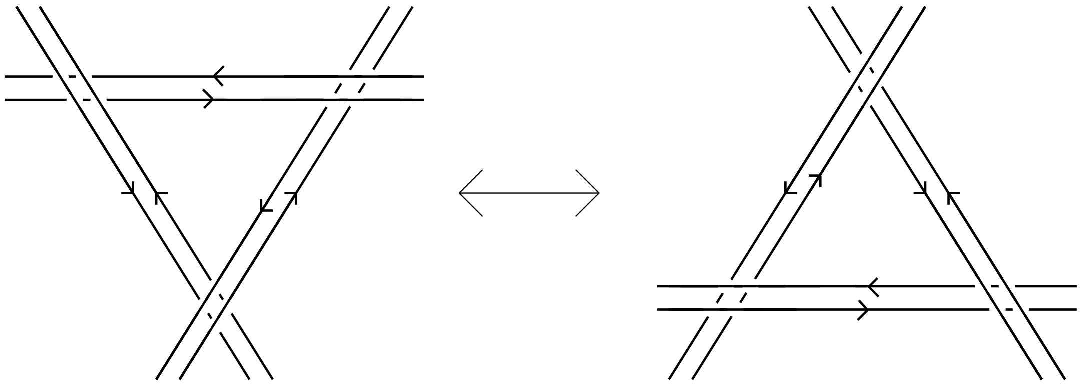

Figure 1b. The doubled-delta move.

The orientations shown on the strands of the doubled-delta move matter only in that the arrows on each pair of parallel strands are oriented oppositely. Any two such choices of orientation give essentially the same move, since cancelling half-twists may be added to pairs of strands just outside the disk where the move takes place.

Theorem A. Two knots K and K' are S-equivalent if and only if they are equivalent by a sequence of doubled-delta moves.

Any knot with a trivial Alexander polynomial has a trivial Alexander module and therefore a trivial Blanchfield pairing. Therefore, any two knots with trivial Alexander polynomial are S-equivalent. (Note, however, that it is not true in general that knots with the same Alexander polynomial are S-equivalent.) Thus we have

Corollary B. A knot may be undone by doubled-delta moves if and only if it has trivial Alexander polynomial.

Since the doubled-delta move takes place inside a disk, S-equivalence and S-triviality may be considered to be local properties in some sense.

Theorem A and Corollary B are reminiscent of the result of Kauffman [4], which states that two knots have the same Arf invariant if and only if they are equivalent by a sequence of band-pass moves as in Figure 2. The idea is that a band-pass move can be used to undo the knotting and linking of the bands in a Seifert surface, with the Arf invariant (which is always either 0 or 1) being the only obstruction to completely trivializing the surface and thus the knot. The doubled-delta move is used to undo the knotting and linking of the bands of a Seifert surface in a similar way. However, since the delta move preserves linking numbers, the doubled-delta move cannot change the Seifert form of a surface or of a knot.

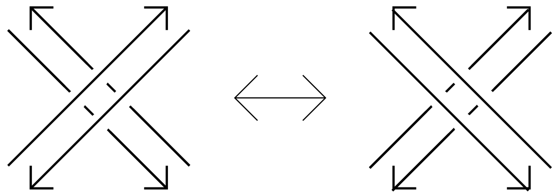

Figure 2. The band-pass move.

Matveev [8] and Murakami and Nakanishi [9] have shown that two links are equivalent by delta moves if and only if they have the same pairwise linking numbers. We shall need to generalize this result to string links. Although the proofs of [8] and [9] appear to generalize, we shall give a different proof.

Theorem C: Two string links are equivalent by a sequence of delta moves if and only if they have the same pairwise linking numbers.

Proof of Theorem A: First we need to show that a doubled-delta move does not change the S-equivalence class of a knot. Consider a knot diagram with the left-hand side of a doubled-delta move inside a planar disk D. Temporarily cut the bands of the doubled-delta move—replace the left-hand side of Figure 1a with Figure 3. Apply Seifert's algorithm to the resulting link to obtain an oriented surface, and then add the bands back in to obtain a Seifert surface for the original knot. For this surface, it is clear that the doubled-delta move may be applied without changing the linking form.

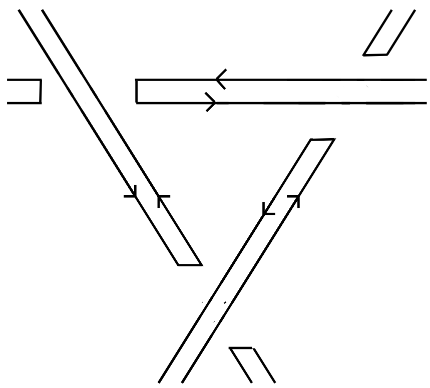

Figure 3.

Now let K_1 be a knot with Seifert surface S_1 of genus n and associated Seifert matrix M, and let K_2 be a knot with Seifert surface S_2 and Seifert matrix M. Let F_n be the abstract surface with genus n and with one boundary component, specifically realized as a disk with bands as shown in Figure 4. An ordered basis for $H_1(F_n)$ is given by

$\beta = ([a_1], [a_2], \dots [a_{2n}])$. We take F_n and the a_i to be oriented such that $\langle [a_1], [a_2] \rangle = 1$ and $\langle [a_2], [a_1] \rangle = 0$ The matrix M represents the linking form of S_1 with respect to some basis B_1 of $H_1(S_1)$. Let $\phi_1 : F_n \to S_1$ be an orientation-preserving homeomorphism. Then $\phi_1(\beta) = (\phi_1([a_1], \phi_1([a_2]), \dots \phi_1([a_{2n}])$ is also a basis for $H_1(S_1)$, and so there exists an invertible integer matrix A_1 such that $N_1 = A_1 M A_1^T$ represents the linking form of S_1 with respect to the basis $\phi_1(\beta)$. (If A is a matrix, we denote its transpose by A^T and the inverse of A^T by A^{-T}.) Define B_2, ϕ_2, A_2, and N_2 the same way for the surface S_2.

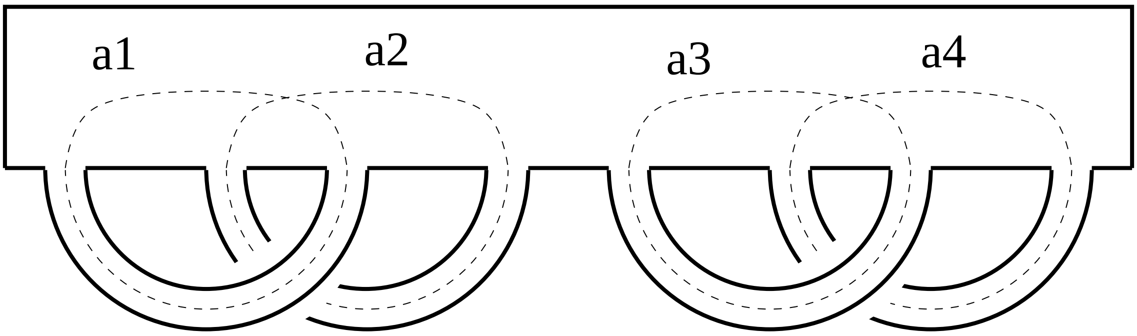

Figure 4. The standard surface F_2.

For $i = 1, 2$, we have $N_i - N_i^T = X_n$, where $X_1 = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$ and X_n is built up with n copies of X_1 down the diagonal and 0 everywhere else. Since $N_1 = A_1 A_2^{-1} N_2 A_2^{-T} A_1^T$, we get $X_n = A_1 A_2^{-1} X_n A_2^{-T} A_1^{-1}$, so $A_1 A_2^{-1} = C = [C_{i,j}]$ is a symplectic matrix. The mapping class group of a surface with 0 or 1 boundary components acts on H_1 of the surface, and if a basis is chosen for H_1 with the same intersection properties as the a_i curves on F_n, then the matrices that represent this action are symplectic matrices. Moreover, this map from the mapping class group to the symplectic group is well-known to be surjective, and therefore there exists a homeomorphism $g : S_2 \to S_2$ such that $[g(\phi_2(a_i))] = \sum_{j=1}^{2n} C_{i,j} [\phi_2(a_j)]$ for all $1 \le i \le 2n$, and N_1 represents the linking form of S_2 with respect to the ordered basis $([g(\phi_2(a_1))], [(\phi_2(a_2))], \dots [g(\phi_2(a_{2n}))])$. Now we may use ϕ_1 and $g \circ \phi_2$ to put S_1 and S_2, respectively, into a disk and band form as in Figure 5. The only difference now between S_1 and S_2 is in the (framed) string links L_1 and L_2, and the pairwise linking numbers of these two string links are identical to each other, both being given by the matrix N_1. Hence by Theorem C there exists a sequence of delta moves taking L_1 to L_2, and therefore a sequence of doubled-delta moves taking K_1 to K_2.

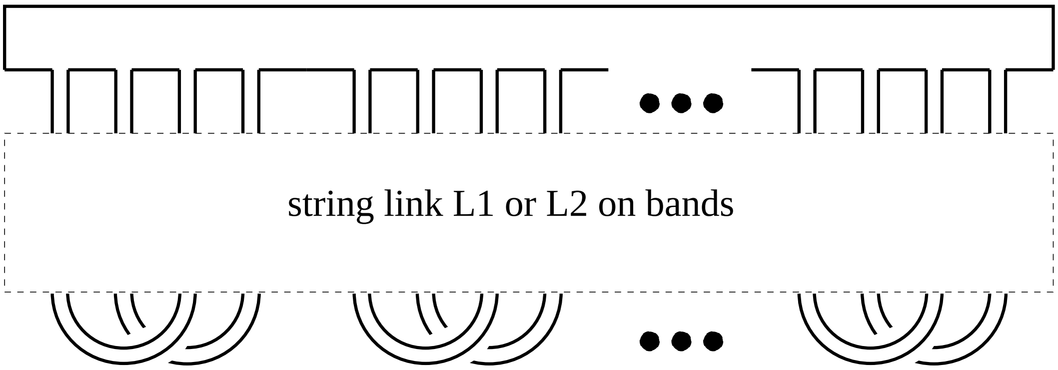

Figure 5. The standard form for K_1 and K_2.

Note that although Theorem C doesn't address the framing issue, it is clear that a delta move doesn't change the framing on any strand of a string link, and a doubled-delta move doesn't change the self-linking of any of the bands in the surfaces S_1 and S_2. Moreover, each band in L_1 must have the same framing as the corresponding band in L_2, because the framings in both diagrams are given by the diagonal entries of N_1. $\square$

Proof of Theorem C: The "only if" part is easy to check, so it is left to show that two string links with matching linking numbers are connected by a sequence of delta moves.

Let P_n be the group of pure braids, those elements of the braid group B_n which induce the identity permutation on the endpoints of the strands. For details and presentations of P_n and B_n, see Birman [2]. For any $1 \leq i < j \leq n$, the map $P_n \to \mathbb{Z}$ which measures the linking number between the ith and jth strands in a pure braid is a group homomorphism. In fact, these are all the abelianizing homomorphisms of P_n, so that $p \in P_n$ has all its linking numbers 0 if and only if $p \in P_n'$, the commutator subgroup of P_n. Three other facts are also easy to check. First, any commutator of the form

$$(D) \qquad p_{i,j} p_{j,k} p_{i,j}^{-1} p_{j,k}^{-1} \in P_n$$

may be undone by a delta move, where $p_{i,j}$ is the standard pure braid generator which links the ith and the jth strands. Second, adding the commutator (D) to a presentation of P_n abelianizes the three-strand subgroup of P_n generated by $p_{i,j}, p_{j,k}, p_{i,k}$. Third, abelianizing all such three-strand subgroups of P_n abelianizes P_n. Thus if a pure braid p has all its linking numbers 0 then it can be undone by delta moves.

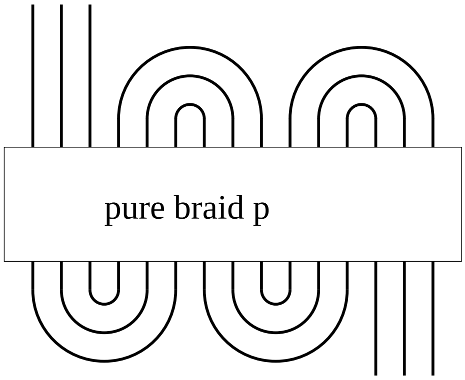

Figure 6.

Now let L be an n-strand string link with all pairwise linking numbers equal to 0. For some $k > 0$, L is represented by a diagram as in Figure 6, where p is a pure braid on kn strands. Let us label the strands of p with a double index (i, a), indicating the ath braid strand of the ith string link strand. For example, the 4th braid strand in Figure 6 (in the usual sense) will be labeled $(3, 2)$, and the 5th will be labeled $(2, 2)$. Let $p_{(i,a)(j,b)}$ be the standard braid generator, as above, linking the (i, a) strand with the (j, b) strand. We have $p_{(i,a)(j,b)} = p_{(j,b)(i,a)}$, and for notational convenience we set $p_{(i,a)(i,a)} = 1$ for all i, a. Let $\mathrm{lk}((i, a), (j, b))$ be the linking number between the (i, a) strand and the (j, b) strand. We measure this linking number with respect to the braid orientation on the strands, which coincides with the string link orientation exactly when the second index of the strand is odd. The linking number 0 condition on L becomes

$$(E) \qquad \sum_{a=1}^{k}\sum_{b=1}^{k}(-1)^{a+b}\mathrm{lk}((i, a), (j, b)) = 0$$

for all $1 \leq i \neq j \leq n$. Let b be even, $1 \leq a, b \leq k$, and $1 \leq i, j \leq n$. Then we may replace p with $(p_{(i,a)(j,b)}p_{(i,a)(j,b+1)})^{\pm 1}p$ without affecting L. Similarly, if b is odd, then we may replace p by $p(p_{(i,a)(j,b)}p_{(i,a)(j,b+1)})^{\pm 1}$ without affecting L. The effect of multiplication by $(p_{(i,a)(j,b)}p_{(i,a),(j,b+1)})^{\pm 1}$ on the linking numbers of p is the same in either case. If $i = j$ and $a = b$ or $a = b + 1$, then $\mathrm{lk}((i, b)(i, b + 1))$ goes up by one or down by one. Otherwise, $\mathrm{lk}((i, a), (j, b))$ and $\mathrm{lk}((i, a), (j, b+1))$ both either go up by one or go down by one. Repeated multiplications by $(p_{(i,a)(j,b)}p_{(i,a),(j,b+1)})^{\pm 1}$ for various appropriate values of i, j, a, b thus suffice to make $\mathrm{lk}((i, a), (b, j)) = 0$ for all $a > 1$ and $b > 1$, and condition (E) then forces all the remaining linking numbers to be 0 as well.

Finally, if L and L' are two string links with the same pairwise linking numbers, let $p \in P_n$ have the same pairwise linking numbers as L and L'. Clearly L and L' are equivalent by delta moves if and only if $p^{-1}L$ and $p^{-1}L'$ are equivalent by delta moves, but these last two string links have all pairwise linking numbers 0, and so are both equivalent to the unlink by delta moves. $\square$

Sketch of proof that S-equivalence is an equivalence relation: The reflexive and symmetric properties are obvious, so we only need to consider transitivity. If M and M' are Seifert matrices, then we may say that $M' > M$ if M' can be obtained from M by a sequence of unimodular congruences, column enlargements, and row enlargements. See for example Kawauchi [5]. These matrix operations correspond to changing the basis of H_1 of the Seifert surface, and to adding a tube to a Seifert surface. It is well-known that if $<>$ is the equivalence relation generated by $<$, and M and M' are two Seifert matrices of the same knot, then $M <> M'$. For a recent elementary proof that two Seifert surfaces of a knot are tube-equivalent, see Bar-Natan, Fulman, and Kauffman [1].

We make two claims: First, that if M_1 and M_2 are Seifert matrices and $M_1 <> M_2$, then there exists a Seifert matrix M_3 such that $M_3 > M_1$ and $M_3 > M_2$. Second, if M_1 is a Seifert matrix for a knot K, and $M_2 > M_1$, then M_2 is also a Seifert matrix for K. Both of these are verified with elementary matrix and tube operations.

Now suppose that knots K_1 and K_2 share a common matrix M_{12} and that knots K_2 and K_3 share a common matrix M_{23}. Then by the first claim there exists a Seifert matrix M_2 for K_2 such that $M_2 > M_{12}$ and $M_2 > M_{23}$, and by the second claim M_2 is a Seifert matrix for both K_1 and K_3. $\square$

References.

[1] D. Bar-Natan, J. Fulman, and L.H. Kauffman. *An elementary proof that all spanning surfaces of a link are tube-equivalent.* Journal of Knot Theory and its Ramifications 7 (1998), 873–879.

[2] J.S. Birman. "Braids, Links and Mapping Class Groups." Annals of Mathematics Studies 82. Princeton University Press, 1975.

[3] C.McA. Gordon. *Some aspects of classical knot theory*, "Knot Theory" (ed. J. C. Hausmann), 1–60, Lecture Notes in Mathematics 685, Springer–Verlag, 1978.

[4] L.H. Kauffman. "Formal knot theory." Mathematical Notes 30. Princeton University Press, 1983.

[5] A. Kawauchi. "A Survey of Knot Theory." Birkhauser, 1996.

[6] C. Kearton. *Blanchfield duality and simple knots.* Transactions of the American Mathematical Society 202 (1975), 141–160.

[7] J. Levine. *An Algebraic Classification of some Knots of Codimension 2.* Commentarii Math. Helv., 15 (1970) 185–198.

[8] S.V. Matveev. *Generalized surgeries of three-dimensional manifolds and representations of homology spheres.* Matematicheskie Zametki 42 (1987), no. 2, 268–278. (English translation in Mathematical Notes 42 (1987) 651–656.)

[9] Murakami and Nakanishi. *On a certain move generating link homology.* Mathematische Annalen 284 (1989), 75–89.

[10] K. Murasugi. *On a Certain Numerical Invariant of Link Types*, Transactions of the American Mathematical Society 117 (1965) 387–422.

[11] P.M. Rice. *Equivalence of Alexander Matrices*, Mathematische Annalen 193 (1971) 65–75.

[12] D. Rolfsen. "Knots and Links" Mathematics Lecture Series 7. Publish or Perish, Wilmington, DE, 1976.

[13] H.F. Trotter. *Homology of Group Systems with applications to Knot Theory.* Annals of Mathematics 76 (1962) 464–498.

[14] H.F. Trotter. *On S-Equivalence of Seifert Matrices*, Inventiones Mathematicae 20 (1973) 173–207.

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