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The Derived Category of an Exact Category

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INTRODUCTION

There is some confusion in the literature regarding the derived category of an exact category $\mathscr{E}$. Thomason gives a satisfactory treatment of the bounded derived category in [T, 1.11.6 (see also Appendix A)]. The result is that provided all weakly split epimorphisms in $\mathscr{E}$ are admissible, the bounded derived category may be defined as usual. Although Thomason does not say it, the categories $D^+(\mathscr{E})$, $D^-(\mathscr{E})$ may also be defined in the same way.

The definition of the unbounded derived category is more difficult, and is very poorly treated in the literature. For instance, in [BBD] this derived category is only defined provided every morphism in $\mathscr{E}$ admits a kernel. (See [BBD, 1.1.4]; this is very unsatisfactory since very few exact categories satisfy the condition.)

In this article we will show that $D(\mathscr{E})$ may be defined whenever $\mathscr{E}$ is saturated ("Karoubian" in Thomason's terminology). An exact category is saturated if every idempotent splits; i.e., $\mathscr{E}$ contains all direct summands of its objects.

For the purpose of comparing with Thomason's result, the bounded derived category is defined for more $\mathscr{E}$'s. All that is required to define $D^b(\mathscr{E})$ is that whenever an idempotent e: $A \to A$ factors as $A \xrightarrow{f} B \xrightarrow{g} A$ with $f \circ g = 1_B$, then e is split. Thomason calls such idempotents *weakly split*, and we will honor his notation. We will show here that these constructions are in some sense best possible (Remark 1.8 for $D(\mathscr{E})$, Remarks 1.9 and 1.10 for $D^+(\mathscr{E})$, $D^-(\mathscr{E})$, $D^b(\mathscr{E})$).

There are two reasons why I wrote this note. One is to correct the misconceptions in the literature. But, more importantly, the proof of the key result, Lemma 1.2, depends on an important characterization of épaisse subcategories due to Rickard, and this article is intended to highlight Rickard's criterion: a full triangulated subcategory $\mathscr{S}$ of a triangulated

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gory $\mathscr{T}$ is épaisse if and only if every $\mathscr{T}$-direct summand of an object
$^{\circ}$ is in $\mathscr{S}$.

ickard's criterion, once stated, is really trivial to prove. For complete-
we include a proof, which is slightly simpler than the original one in
:ard's paper. But there are other proofs, and the reader can amuse
self constructing some. The point of the criterion is not its proof but the
that it is the right characterization of épaisse subcategories.

1. The Construction

et $\mathscr{E}$ be an exact category. Let $K(\mathscr{E})$ be the homotopy category of
n complexes of objects of $\mathscr{E}$. Let $A(\mathscr{E})$ be the full subcategory of $K(\mathscr{E})$
sisting of acyclic complexes: a complex

$$X^n \longrightarrow X^{n+1} \longrightarrow X^{n+2}$$

cyclic if each map $X^n \to X^{n+1}$ decomposes in $\mathscr{E}$ as $X^n \xrightarrow{e_n} D^n \xrightarrow{m_n} X^{n+1}$
re m_n is an admissible mono and e_n is an admissible epi; furthermore,
$\xrightarrow{m_n} X^n \xrightarrow{e_{n+1}} D^{n+1}$ must be an exact sequence.

EMMA 1.1. *The category $A(\mathscr{E})$ is triangulated.*

'roof. This is the trivial lemma. Suppose we are given two complexes
and $Y^{\cdot}$ in $A(\mathscr{E})$ and a chain map $X^{\cdot} \to Y^{\cdot}$. We need to show that the
pping cone is also in $A(\mathscr{E})$.

uppose therefore that the morphism of chain complexes $X^{\cdot} \to Y^{\cdot}$ is
n by the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & X^n & \xrightarrow{d_n} & X^{n+1} & \xrightarrow{d_{n+1}} & X^{n+2} & \longrightarrow \\
& \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_{n+2}} & \\
\longrightarrow & Y^n & \xrightarrow{d'_n} & Y^{n+1} & \xrightarrow{d'_{n+1}} & Y^{n+2} & \longrightarrow
\end{array}.$$

Ve need to check that the complex

$$\longrightarrow X^{n+1} \oplus Y^n \xrightarrow{\begin{pmatrix} -d_{n+1} & 0 \\ f_{n+1} & d'_n \end{pmatrix}} X^{n+2} \oplus Y^{n+1} \longrightarrow$$

n $A(\mathscr{E})$. Choose an exact embedding of $\mathscr{E}$ in an abelian category as a
category closed under extensions; the main point is that

$$\ker \begin{pmatrix} -d_{n+1} & 0 \\ f_{n+1} & d'_n \end{pmatrix}$$

is an extension of $\ker\{d_{n+1})$ by $\ker\{d'_n)$, and hence is in $\mathscr{E}$. The fact that the sequence

$$\ker\begin{pmatrix} -d_{n+1} & 0 \\ f_{n+1} & d'_n \end{pmatrix} \longrightarrow X^{n+1} \oplus Y^n \longrightarrow \ker\begin{pmatrix} -d_{n+2} & 0 \\ f_{n+2} & d'_{n+1} \end{pmatrix}$$

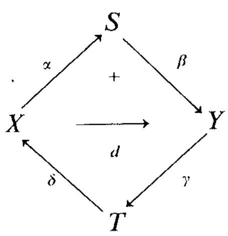
is exact can also be checked after the embedding in the abelian category, where it is obvious. ∎

LEMMA 1.2. *The category* $A(\mathscr{E}) \subset K(\mathscr{E})$ *is an épaisse subcategory, provided* $\mathscr{E}$ *is saturated.*

To prove this, we use a criterion of Rickard's; for convenience, we include a proof.

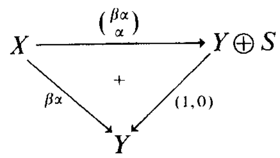
CRITERION 1.3 (Rickard's Criterion [R, Proposition 1.4]). *A full, triangulated subcategory* $\mathscr{S}$ *of a triangulated category* $\mathscr{T}$ *is épaisse if and only if every direct summand of an object of* $\mathscr{S}$ *is in* $\mathscr{S}$.

Proof. It is trivial to check that an épaisse subcategory satisfies Rickard's criterion. We check the converse. Let us be given the diagram in $\mathscr{T}$



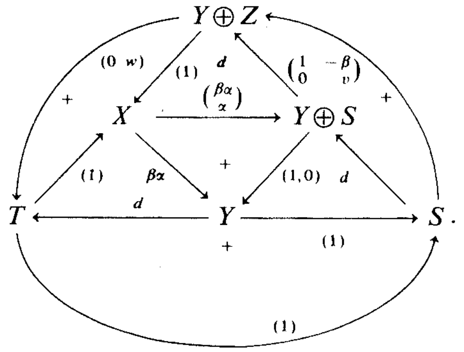
and suppose S and T are objects of $\mathscr{S}$. We need to prove that so are X and Y.

Then consider the commutative triangle



(∗)

Let $X \xrightarrow{\alpha} S \xrightarrow{v} Z \xrightarrow{w} \sum X$ be a triangle. Then we can complete (∗) to an octahedron



follows that there is a triangle $S \to Y \oplus Z \to T \to \sum S$.

ut S and T are objects of $\mathscr{S}$ and $\mathscr{S}$ is assumed triangulated; it follows $Y \oplus Z$ is an object of $\mathscr{S}$. Therefore $\mathscr{S}$, being closed under direct mands, must contain Y, and it easily follows that X is also an object $\mathscr{S}$. ∎

emark 1.4. This proof is here mostly for completeness, but it should ointed out that it is "better" than Rickards proof in that the octahedral m is only used once. Rickards proof involved the use of two octahedra.

roof of Lemma 1.2. We need to prove $A(\mathscr{E})$ épaisse. By Rickard's rion, it suffices to prove that if $Z^{\cdot}$ is an object of $A(\mathscr{E})$, and $\simeq X^{\cdot} \oplus Y^{\cdot}$, where $X^{\cdot}$, $Y^{\cdot}$, and the isomorphism are in $K(\mathscr{E})$, then $X^{\cdot}$ and re actually acyclic.

o give an isomorphism $X^{\cdot} \oplus Y^{\cdot} \simeq Z^{\cdot}$ in $K(\mathscr{E})$ is to give two chain maps

$$X^{\cdot} \oplus Y^{\cdot} \xrightarrow{\varphi} Z^{\cdot}$$

$$Z^{\cdot} \xrightarrow{\psi} X^{\cdot} \oplus Y^{\cdot}$$

that $\psi \circ \varphi$ is homotopic to 1. Therefore $1 - \psi \circ \varphi$ is null-homotopic; it easily be checked that any null-homotopic map $f\colon A^{\cdot} \to B^{\cdot}$ factors ugh the contractible complex $C(A^{\cdot} \to^{1} A^{\cdot})$, the mapping cone of the tity map $1\colon A^{\cdot} \to A^{\cdot}$. The reader will easily check that $C(A^{\cdot} \to A^{\cdot})$ is an lic complex.

our particular case we deduce that $1 - \psi \circ \varphi$ factors as

$$X \oplus Y \xrightarrow{\alpha} C(X \oplus Y \xrightarrow{1} X \oplus Y) \xrightarrow{\beta} X \oplus Y.$$

s we have

$$X \oplus Y \xrightarrow{\binom{\varphi}{\alpha}} Z \oplus C(X \oplus Y \xrightarrow{1} X \oplus Y) \xrightarrow{(\psi\ \beta)} X \oplus Y$$

and the composite $(\psi\ \beta)\binom{\varphi}{\alpha} = 1_{X\oplus Y}$. Therefore if we replace Z by $Z' = Z \oplus C(X\oplus Y \xrightarrow{1} X \oplus Y)$, then Z' is acyclic, and there are maps

$$X\oplus Y \xrightarrow{\varphi'} Z' \xrightarrow{\psi'} X\oplus Y$$

such that $\psi' \circ \varphi' = 1$ (equality being genuine, not only up to homotopy).

It follows that $Z' \cong X\oplus Y\oplus W$ as a chain complex; degree by degree φ' and ψ' must determine a splitting of Z compatible with the differential. Therefore, after embedding in a suitable abelian category, the kernel

$$\ker[(Z')^n \to (Z')^{n+1}]$$

must split as a direct sum

$$\ker(X^n \to X^{n+1}) \oplus \ker(Y^n \to Y^{n+1}) \oplus \ker(W^n \to W^{n+1}),$$

and because $\mathscr{E}$ is saturated, the fact that $\ker[(Z')^n \to (Z')^{n+1}]$ is an object in $\mathscr{E}$ (which follows from the acyclicity of Z') implies that so are $\ker(X^n \to X^{n+1})$, $\ker(Y^n \to Y^{n+1})$, and $\ker(W^n \to W^{n+1})$. The exactness of

$$\ker(X^n \to X^{n+1}) \to X^n \to \ker(X^{n+1} \to X^{n+2})$$

is also obvious; hence the proof is complete. ∎

Construction 1.5. The derived category $D(\mathscr{E})$ is the quotient of $K(\mathscr{E})$ by the épaisse subcategory $A(\mathscr{E})$.

Remark 1.6. If $\mathscr{E}$ is not saturated, one can still define $D(\mathscr{E})$ to be the quotient of $K(\mathscr{E})$ by the épaisse closure of $A(\mathscr{E})$.

Remark 1.7. It follows trivially from Rickard's criterion that the épaisse closure of a full triangulated subcategory $\mathscr{S}$ of a triangulated category $\mathscr{T}$ is the full subcategory consisting of $\mathscr{T}$-direct summands of objects of $\mathscr{S}$.

Remark 1.8. If $\mathscr{E}$ is not saturated, then $A(\mathscr{E})$ is not épaisse in $K(\mathscr{E})$. Let A be an object of $\mathscr{E}$, $e \in \mathrm{End}(A)$ a non-split idempotent. Then the chain complex $(A, e)^{\cdot}$ given by

$$\xrightarrow{1-e} A \xrightarrow{e} A \xrightarrow{1-e} A \xrightarrow{e} A \xrightarrow{1-e}$$

has the property that $(A, e)^{\cdot} \oplus \Sigma (A, e)^{\cdot}$ is acyclic. In fact, the reader will easily verify that if X is the complex

$$\cdots \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} A \xrightarrow{0} \cdots,$$

then $(A, e)^{\cdot} \oplus \Sigma (A, e)^{\cdot} \cong C(X \xrightarrow{1} X)$. Thus $(A, e)^{\cdot}$ is a direct summand of an acyclic complex, but $(A, e)^{\cdot}$ itself is not acyclic; $\ker(e)$ is not an object of $\mathscr{E}$.

Remark 1.9. Thomason essentially proves that for the bounded omotopy category, $A^b(\mathscr{E})$ is épaisse provided only that weakly split pimorphisms are split (see Remark 1.10). We will show that Thomason's iterion is also best possible.

Suppose $\mathscr{E}$ is an exact category. Suppose $f\colon A \to B$ and $g\colon B \to A$ are iorphisms in $\mathscr{E}$ such that $fg\colon B \to B$ is the identity. Suppose $gf\colon A \to A$ is ot split. We will show that $A^b(\mathscr{E})$ is not épaisse.

Consider the chain complex $X^{\boldsymbol{\cdot}}$ given by

$$^{\boldsymbol{\cdot}} = \cdots \longrightarrow 0 \longrightarrow B \longrightarrow A \longrightarrow A \longrightarrow B \longrightarrow 0 \longrightarrow \cdots.$$

assert that $X^{\boldsymbol{\cdot}} \oplus \sum X^{\boldsymbol{\cdot}}$ is acyclic; for $X^{\boldsymbol{\cdot}} \oplus \sum X^{\boldsymbol{\cdot}}$ is the complex

$$B \xrightarrow{\begin{pmatrix} 0 \\ g \end{pmatrix}} B \oplus A \xrightarrow{\begin{pmatrix} -g & 0 \\ 0 & 1-gf \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} -1+gf & 0 \\ 0 & f \end{pmatrix}} A \oplus B \xrightarrow{(-f\ \ 0)} B$$

Now the following chain map expresses an isomorphism of $X^{\boldsymbol{\cdot}} \oplus \sum X$ ith an obviously acyclic complex

$$\begin{array}{ccccccccc}
B & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & B \oplus A & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} & A \oplus A & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & A \oplus B & \xrightarrow{(0\ \ 1)} & B \\
\downarrow 1 & & \downarrow \begin{pmatrix} 0 & -f \\ g & 1-gf \end{pmatrix} & & \downarrow \begin{pmatrix} -1+gf & gf \\ gf & 1-gf \end{pmatrix} & & \downarrow \begin{pmatrix} 1-gf & -g \\ f & 0 \end{pmatrix} & & \downarrow 1 \\
B & \xrightarrow[\begin{pmatrix} 0 \\ g \end{pmatrix}]{} & B \oplus A & \xrightarrow[\begin{pmatrix} -g & 0 \\ 0 & 1-gf \end{pmatrix}]{} & A \oplus A & \xrightarrow[\begin{pmatrix} -1+gf & 0 \\ 0 & f \end{pmatrix}]{} & A \oplus B & \xrightarrow[(-f,\ 0)]{} & B.
\end{array}$$

We leave it to the reader to check the commutativity of the diagram and verify that the vertical maps are isomorphisms. The top row is clearly cyclic. Thus $X^{\boldsymbol{\cdot}}$ is a direct summand of an acyclic complex, but it is not self acyclic; after all, by hypothesis $A \xrightarrow{f} B$ is not an admissible epi.

Remark 1.10. The condition that weakly split epis split is self-dual. homason proves that a category $\mathscr{E}$ satisfying the hypothesis admits a fully ithful embedding into an abelian category $\mathscr{A}$ such that $\mathscr{E} \subset \mathscr{A}$ is

(1) closed under extensions;

(2) the exact functor $\mathscr{E} \to \mathscr{A}$ reflects exact sequences;

(3) every morphism in $\mathscr{E}$ which maps to an epimorphism in $\mathscr{A}$ is an dmissible epi in $\mathscr{E}$.

By the self-duality of the condition, it follows that there is an embedding $\to \mathscr{B}$ for some other abelian category, where (1), (2), and the *dual* of (3) re satisfied.

A complex in $K^-(\mathscr{E})$ is acyclic if and only if its image in $K^-(\mathscr{A})$ is cyclic. A complex in $K^+(\mathscr{E})$ is acyclic if and only if its image in $K^+(\mathscr{B})$ acyclic. It trivially follows that $A^+(\mathscr{E})$, $A^-(\mathscr{E})$, and $A^b(\mathscr{E})$ are épaisse (this is essentially Thomason's proof; the only reason I repeated it here is because Thomason fails to mention the obvious extension of his argument to $K^+(\mathscr{E})$, $K^-(\mathscr{E})$).

DEFINITION 1.11. It is natural, in view of 1.9 and 1.10, to define an exact category to be *semi-saturated* if every weakly split epimorphism is an admissible epi.

Remark 1.12. Using Remark 1.6, we may define the derived category of any exact category. The following assertions are left to the reader.

1.12.1. Every exact category $\mathscr{E}$ has a semi-saturation; there exists a category $\mathscr{E}^{ss}$ such that any functor $\mathscr{E} \to \mathscr{F}$, where $\mathscr{F}$ is semi-saturated, factors uniquely through $\mathscr{E}^{ss}$.

1.12.2. Every exact category $\mathscr{E}$ has a saturation; there exists an exact catagory $\mathscr{E}^s$ such that any functor $\mathscr{E} \to \mathscr{F}$, where $\mathscr{F}$ is saturated, factors uniquely through $\mathscr{E}^s$. (This is well known; see Karoubi [K].)

1.12.3. The natural inclusions $D^b(\mathscr{E}) \to D^b(\mathscr{E}^{ss})$, $D^-(\mathscr{E}) \to D^-(\mathscr{E}^{ss})$, $D^+(\mathscr{E}) \to D^b(\mathscr{E}^{ss})$ are equivalences of categories.

1.12.4. The natural inclusions $D^-(\mathscr{E}) \to D^-(\mathscr{E}^s)$, $D^+(\mathscr{E}) \to D^+(\mathscr{E}^s)$, $D(\mathscr{E}) \to D(\mathscr{E}^s)$ are equivalences (in particular, $D^-(\mathscr{E}^{ss}) \to D^-(\mathscr{E}^s)$, and $D^+(\mathscr{E}^{ss}) \to D^+(\mathscr{E}^s)$ are equivalences).

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