The K-theory of Triangulated Categories

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Summary. The purpose of this survey is to explain the open problems in the K–theory of triangulated categories. The survey is intended to be very easy for non-experts to read; I gave it to a couple of fourth-year undergraduates, who had little trouble with it. Perhaps the hardest part is the first section, which discusses the history of the subject. It is hard to give a brief historical account without assuming prior knowledge. The students are advised to skip directly to Section 2.

1 Historical Survey

The fact that the groups K_0 and K_1 are related to derived categories is so obvious that it was observed right at the beginnings of the subject. We remind the reader.

Let $\mathcal{A}$ be a small abelian (or exact) category. Let $D^b(\mathcal{A})$ be its bounded derived category.[1] The category $D^b(\mathcal{A})$ is a triangulated category. What we will now do is define, for every triangulated category $\mathcal{T}$, an abelian group $K_0(\mathcal{T})$. This definition has the virtue that there is a natural isomorphism $K_0(\mathcal{A}) = K_0\big(D^b(\mathcal{A})\big)$. By $K_0(\mathcal{A})$ we understand the usual Grothendieck group of the exact category $\mathcal{A}$, while $K_0\big(D^b(\mathcal{A})\big)$ is as follows:

Definition 1. *Let $\mathcal{T}$ be a small triangulated category. Consider the abelian group freely generated by the isomorphism classes $[X]$ of objects $X \in \mathcal{T}$. The group $K_0(\mathcal{T})$ is obtained by dividing by the relations generated by all expressions $[X] - [Y] + [Z]$, where there exists a distinguished triangle*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

* This manuscript contains the expanded notes of a talk I gave in Paris, at the working seminar run by Georges Maltsiniotis and Bernhard Keller. I would like to thank Keller and Maltsiniotis for inviting me to speak, and Jussieu for its hospitality during my visit.

[1] For an abelian category $\mathcal{A}$, the definition of $D^b(\mathcal{A})$ is classical. See Verdier [93], or Hartshorne [41, Chapter I]. When $\mathcal{A}$ is only an exact category there was some confusion about how to define $D^b(\mathcal{A})$; see [64].

The relation between $K_1(\mathcal{A})$ and $D^b(\mathcal{A})$ is not so simple. For example it was not known, until quite recently, how to give a definition of $K_1(\mathcal{A})$ which builds on $D^b(\mathcal{A})$. But the fact that K_1 is related (more loosely) to derived categories was known. This goes back to Whitehead's work on determinants of automorphisms of chain complexes and simple homotopy type.

Practically as soon as higher K–theory was defined, its relation with derived category was implicit. One of the first theorems in Quillen's foundational paper on the subject is the resolution theorem [77, Theorem 3 and Corollary 1 of §4]. The theorem says approximately the following:

Theorem 1. (Modified version of Quillen's theorem) *Let $i : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful, exact embedding of the exact category $\mathcal{A}$ into the exact category $\mathcal{B}$. Assume that the induced map of bounded derived categories $D^b(i) : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$ is an equivalence. Then the induced map in Quillen's K–theory $K(i) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ is a homotopy equivalence.*

The reader is referred to Quillen's original paper, or to Theorem 4 of this article, for Quillen's precise formulation (which does not explicitly mention derived categories).

To make K–theory into a useful tool, it is important to understand how $K(\mathcal{A})$ changes with $\mathcal{A}$. Let $f : \mathcal{B} \longrightarrow \mathcal{C}$ be an exact functor of exact categories. It induces a continuous map $K(f) : K(\mathcal{B}) \longrightarrow K(\mathcal{C})$. The homotopy fiber of this map is a spectrum, and it turns out to be very useful to describe it in some computable way, for example as $K(\mathcal{A})$ for some $\mathcal{A}$. The first theorem of this sort was Quillen's localisation theorem [77, Theorem 5 of §5]. Quillen's theorem was very powerful, with many important consequences, for example in algebraic geometry. But, while on the subject of the algebro-geometric applications, it should be noted that to apply the theorem effectively one had to restrict to smooth varieties, or varieties with very mild singularities. Important work followed, trying to generalise this to singular varieties. The reader is referred to Levine [58, 57] and Weibel [96, 97]. The definitive treatment did not come until Thomason [89], and for his work Thomason needed a more powerful foundational basis. It turns out that the homotopy fiber of the map $K(f)$ above can be expressed as $K(\mathcal{A})$, but only if we are willing to understand by this the Waldhausen K–theory of a suitable Waldhausen category $\mathcal{A}$. In other words, to obtain a sufficiently powerful general theorem one needed the domain of the K–theory functor to be expanded. Progress depended on K–theory being defined in greater generality.

Walhausen's work [95] provided a far more general setting for studying K–theory. To every Waldhausen model category $\mathcal{C}$ one attaches a K–theory spectrum $K(\mathcal{C})$. There is a brief discussion of Waldhausen model categories, and of their relation with triangulated categories, in Section 3. For our purposes the important observation is that, once again, there is a clear relation with triangulated categories. To each Waldhausen category $\mathcal{C}$ one can associate a triangulated category $\mathrm{ho}(\mathcal{C})$. Waldhausen's approximation theorem says that, under some technical hypotheses,

Theorem 2. (Waldhausen Approximation Theorem, without the technical hypotheses) *Suppose $i : \mathcal{C} \longrightarrow \mathcal{D}$ is an exact functor of Waldhausen model categories. Suppose the induced map of triangulated categories*

$$\mathrm{ho}(i) : \mathrm{ho}(\mathcal{C}) \longrightarrow \mathrm{ho}(\mathcal{D})$$

is an equivalence. Then the K–theory map $K(i) : K(\mathcal{C}) \longrightarrow K(\mathcal{D})$ is a homotopy equivalence.

All of this suggests very naturally that K–theory and triangulated categories ought to be related. We still do not understand the relation, and this survey is mainly about the many open problems in the field.

But while we are still on the history of the problem, let me discuss the work that has been done. In the light of Waldhausen's approximation theorem, it is natural to ask whether Waldhausen's K–theory depends only on triangulated categories. Given a Waldhausen category $\mathcal{C}$, Waldhausen defined a spectrum $K(\mathcal{C})$. Does this spectrum only depend on $\mathrm{ho}(\mathcal{C})$? If so, is the dependence functorial? I believe the question was first asked in Thomason [89].

The answer turns out to be No. In a paper by myself [65] I produce an example of a pair of Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, and a triangulated functor $f : \mathrm{ho}(\mathcal{C}) \longrightarrow \mathrm{ho}(\mathcal{D})$ which cannot possibly induce a map in Waldhausen K–theory. More recently Schlichting [85] produces a pair of Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, with $\mathrm{ho}(\mathcal{C}) \simeq \mathrm{ho}(\mathcal{D})$ but $K(\mathcal{C}) \not\simeq K(\mathcal{D})$.

This establishes that Waldhausen's K–theory $K(\mathcal{C})$ depends on more than just $\mathrm{ho}(\mathcal{C})$. But it still leaves unresolved the question of whether we can recover Quillen's K–theory of an abelian (or exact?) category $\mathcal{A}$ from the triangulated category $D^b(\mathcal{A})$. This question has interested people since the 1980's. Kapranov tells me that they held a seminar about it in Moscow at the time. There were several counterexamples produced. The reader can see some of them in Hinich and Schechtman [42, 43] and Vaknin [90, 91]. By the mid 1980's, the consensus was that it could not be done.

Then in the late 1980's and early 1990's I proved a theorem, establishing the unexpected. For abelian categories, all of Quillen's higher K–theory may be recovered directly from the derived category. In the first half of this survey I state carefully the results I proved, and in the second half I explain the many open problems that remain.

Still in the historical survey, I should mention that Matthias Künzer also worked on this. He produced a construction and several very interesting conjectures. Unfortunately none of this ever appeared in print. His constructions were actually quite similar to mine. The key difference was that his constructions did not come with coherent differentials (these will de described in detail in Definition 7). For what it may be worth, let me quote Thomason who said that the key input in my work was the coherent differentials.

Also deserving mention is the fascinating work of the school around Maltsiniotis. Their work begins with something intermediate between the Waldhausen category $\mathcal{C}$ and its triangulated category $\mathrm{ho}(\mathcal{C})$. Starting from the derivator associated to $\mathcal{C}$, one can define a K–theory by modifying Waldhausen's construction in a straight-forward way. It is interesting to study this, and the reader can find an excellent account in

`http://www.math.jussieu.fr/~maltsin/Gtder.html`

2 Introduction

The aim of this manuscript is to explain just how little we know about the K–theory of triangulated categories. There are many fascinating open problems in the subject. I am going to try to make the point that a bright young mathematician, with plenty of imagination, could make impressive progress in the field. What we now know is enough to establish that the field is interesting. But the most basic, immediate questions that beg to be answered are completely open.

The best way to explain how little we know is to tell you all of it. Therefore we begin with a fairly careful account of all the existing theorems in the field.

Unfortunately, this requires us to be a little technical. It forces us to introduce five simplicial sets and four maps connecting them. Let $\mathcal{T}$ be a triangulated category with a bounded t–structure. Let $\mathcal{A}$ be the heart. Suppose $\mathcal{T}$ has at least one Waldhausen model. The first half of the manuscript produces five simplicial sets and four maps

$$S_*(\mathcal{A}) \xrightarrow{\alpha} {}^wS_*(\mathcal{T}) \xrightarrow{\beta} S_*({}^d\mathcal{T}) \xrightarrow{\gamma} S_*({}^v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b\mathcal{A}).$$

The only simplicial set the reader might already be familiar with is $S_*(\mathcal{A})$, the Waldhausen S_*–construction applied to the abelian category $\mathcal{A}$.

The main theorem is Theorem 3. It tells us

(i) The composite $\delta\gamma\beta\alpha$ induces a homotopy equivalence.
(ii) The map α induces a homotopy equivalence.
(iii) The simplicial set $S_*({}^v\mathcal{T})$ has a homotopy type which depends only on $\mathcal{A}$. That is, $S_*({}^v\mathcal{T}) \cong S_*\big({}^vD^b(\mathcal{A})\big)$.

Perhaps part (i) of this is the most striking. Each of the simplicial sets ${}^wS_*(\mathcal{T})$, $S_*({}^d\mathcal{T})$ and $S_*({}^v\mathcal{T})$ defines a K–theory for our triangulated category $\mathcal{T}$. We have three candidates for what the right definition might be. By Theorem 3(i), all of them contain the Quillen K–theory of $\mathcal{A}$ as a retract. Any half-way sensible definition of the K–theory of derived categories contains Quillen's K–theory. Passing to the derived category most certainly does not lose K–theoretic information.

I have tried to organise the material so that the introductory part, the part where we define the four simplicial maps α, β, γ and δ, is short. I tried to condense this part of the manuscript without sacrificing the accuracy. It is helpful to have the exact statements of the theorems we now know. It helps delineate the extent of our ignorance.

After setting up the simplicial machinery and stating Theorem 3, we very briefly explain how it can be used to draw very strong conclusions about K–theory. This part is very brief. As I have already said, we focus mostly on the shortcomings of the theory, as it now stands. This allows us to highlight the many open problems.

In this entire document we will consider only small categories. The abelian categories, triangulated categories and Walhausen model categories will all be small categories.

3 Waldhausen Model Categories and Triangulated Categories

In this survey we assume some familiarity with triangulated categories. It also helps to know a little bit about their models. This modest introductory section will attempt to provide the very minimum, bare essentials. Instead of developing the axiomatic formalism, we will give the key examples of interest.

Example 1. Let $\mathcal{A}$ be an abelian category. The category $C(\mathcal{A})$ is the category of chain complexes in $\mathcal{A}$. The objects are the chain complexes

$$\cdots \xrightarrow{\partial} x_{i-1} \xrightarrow{\partial} x_i \xrightarrow{\partial} x_{i+1} \xrightarrow{\partial} \cdots$$

where $\partial\partial = 0$. The morphisms are the chain maps; that is the commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & x_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & x_i \arrow[r] \arrow[d, "f_i"] & x_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & y_{i-1} \arrow[r] & y_i \arrow[r] & y_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

So far, we have defined a category.

It is customary to consider $C(\mathcal{A})$ as a Waldhausen category. This means endowing it with a great deal of extra structure. First of all, we consider three subcategories $cC(\mathcal{A})$, $fC(\mathcal{A})$ and $wC(\mathcal{A})$. These subcategories all have the same objects, namely all the objects of $C(\mathcal{A})$. It is the morphisms that are restricted. The restrictions are

(i) A morphism in $cC(\mathcal{A})$, also called a *cofibration in* $C(\mathcal{A})$, is a chain map of chain complexes so that, for every $i \in \mathbb{Z}$, the map $f_i : x_i \longrightarrow y_i$ is a split monomorphism. (The splittings are not assumed to be chain maps).

(ii) A morphism in $fC(\mathcal{A})$, also called a *fibration in* $C(\mathcal{A})$, is a chain map of chain complexes so that, for every $i \in \mathbb{Z}$, the map $f_i : x_i \longrightarrow y_i$ is a split epimorphism. (Once again, the splittings are not assumed to be chain maps).

(iii) A morphism in $wC(\mathcal{A})$, also called a *weak equivalence in* $C(\mathcal{A})$, is a chain map of chain complexes inducing an isomorphism in homology.

One also assumes that there is a functor, called the *cylinder functor,* taking a morphism in $C(\mathcal{A})$ to an object, called the *mapping cylinder.* Let me not remind the reader of the detail of this construction. In Example 2 we will see the related construction of the *mapping cone,* which is more relevant for us. An important consequence of the existence of mapping cylinders (or mapping cones) is that the category $C(\mathcal{A})$ has an authomorphism, called the suspension functor, and denoted $\Sigma : C(\mathcal{A}) \longrightarrow C(\mathcal{A})$. It takes the complex

$$\cdots \xrightarrow{\partial} x_{i-1} \xrightarrow{\partial} x_i \xrightarrow{\partial} x_{i+1} \xrightarrow{\partial} \cdots$$

to the complex

$$\cdots \xrightarrow{-\partial} x_i \xrightarrow{-\partial} x_{i+1} \xrightarrow{-\partial} x_{i+2} \xrightarrow{-\partial} \cdots$$

In other words, Σ shifts the degrees by one, and changes the sign of the differential ∂.

Remark 1. The data above, that is the three subcategories $cC(\mathcal{A})$, $fC(\mathcal{A})$ and $wC(\mathcal{A})$ and the cylinder functor, satisfy a long list of compatibility conditions. We omit all of them. The interested reader can find a much more thorough treatment in Chapter 1 of Thomason's [89]. Thomason calls the categories satisfying this long list of properties *biWaldhausen complicial categories.* In this paper we will call them *Waldhausen model categories,* or just *Waldhausen categories* for brevity. The experts, please note: what we call Waldhausen model categories is exactly the same as Thomason's biWaldhausen complicial categories. This allows us to freely quote results from [89].

Example 2. Suppose we start with a Waldhausen model category, like $C(\mathcal{A})$. We can form a category, often denoted ho$C(\mathcal{A})$. It is called the *homotopy category of* $C(\mathcal{A})$*,* and is obtained from $C(\mathcal{A})$ by formally inverting the weak equivalences. In the case of the Waldhausen category $C(\mathcal{A})$, the category ho$C(\mathcal{A})$ is usually called the *derived category of* $\mathcal{A}$*,* and denoted $D(\mathcal{A})$. The suspension functor descends to an automorphism of ho$C(\mathcal{A}) = D(\mathcal{A})$. The category $D(\mathcal{A})$ is a *triangulated category;* it satisfies a very short list of axioms. Basically, the only construction one has is the mapping cone. Suppose we are given two chain complexes X and Y, and a map of chain complexes $f : X \longrightarrow Y$. That is, we are given a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & x_{i-1} \arrow[r, "\partial"] \arrow[d, "f_{i-1}"] & x_i \arrow[r, "\partial"] \arrow[d, "f_i"] & x_{i+1} \arrow[r, "\partial"] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r, "\partial"'] & y_{i-1} \arrow[r, "\partial"'] & y_i \arrow[r, "\partial"'] & y_{i+1} \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

We can form the mapping cone, which is a chain complex

$$\cdots \longrightarrow x_i \oplus y_{i-1} \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_i & \partial \end{pmatrix}} x_{i+1} \oplus y_i \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_{i+1} & \partial \end{pmatrix}} x_{i+2} \oplus y_{i+1} \longrightarrow \cdots$$

It turns out that this mapping cone, which we will denote $\mathrm{Cone}(f)$, is well-defined in the category $\mathrm{ho}C(\mathcal{A}) = D(\mathcal{A})$. One can look at the maps

$$X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Cone}(f).$$

Of course, there is nothing to stop us from iterating this process. We can continue to

$$X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Cone}(f) \xrightarrow{h} \mathrm{Cone}(g) \xrightarrow{i} \mathrm{Cone}(h) \xrightarrow{j} \cdots$$

Contrary to what we might expect, this process soon begins to iterate. There is a natural commutative square in $D(\mathcal{A})$, where the vertical maps are isomorphisms

$$\begin{tikzcd} \mathrm{Cone}(g) \arrow[r, "i"] \arrow[d, "\wr"] & \mathrm{Cone}(h) \arrow[d, "\wr"] \\ \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

That is, up to suspension and sign, the diagram is periodic with period 3. We call any diagram

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

isomorphic to

$$X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Cone}(f) \xrightarrow{h} \mathrm{Cone}(g)$$

a *distinguished triangle* in $D(\mathcal{A})$. There is a very short list of axioms which distinguished triangles satisfy, and that is all the structure there is in $D(\mathcal{A})$. The axiomatic treatment may be found, for example, in Verdier's thesis [93], in Hartshorne [41, Chapter 1], or in the recent book [73].

Remark 2. It is quite possible for a single triangulated category $\mathcal{T}$ to have many different Waldhausen models. For instance, there are many known examples of abelian categories $\mathcal{A}$ and $\mathcal{B}$, with $D(\mathcal{A}) = D(\mathcal{B})$.[2] The models $C(\mathcal{A})$ and $C(\mathcal{B})$ are quite different, non-isomorphic Waldhausen categories. The passage from $C(\mathcal{A})$ to $\mathrm{ho}C(\mathcal{A}) = D(\mathcal{A})$ loses a great deal of information. What we will try to explain is that higher K–theory is not among the information which is lost.

[2] The first example may have been the one in Beilinson's 1978 article [11]. By now, a quarter of a century later, we know a wealth of other examples. A very brief discussion is included in an appendix; see Section 17.

4 Virtual Triangles

We need to remind the reader briefly of some of the results in Vaknin's [92]. In any triangulated category $\mathcal{T}$, Vaknin defined a hierarchy of triangles. When we use the word *triangle* without an adjective, we mean a diagram

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

so that $vu = wv = \{\Sigma u\}w = 0$. Vaknin defines classes of triangles

$$\text{splitting} \quad \subset \quad \text{distinguished} \quad \subset \quad \text{exact} \quad \subset \quad \text{virtual.}$$

The definitions are as follows.

(i) A *splitting triangle* is a direct sum of three triangles

$$A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \Sigma A$$

$$0 \longrightarrow B \xrightarrow{1} B \longrightarrow 0$$

$$\Sigma^{-1}C \longrightarrow 0 \longrightarrow C \xrightarrow{1} C$$

(ii) A *distinguished triangle* is part of the structure that comes for free, just because $\mathcal{T}$ is a triangulated category.

(iii) A triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

is *exact* if there exist maps u', v' and w' so that the following three triangles

$$A \xrightarrow{u'} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

$$A \xrightarrow{u} B \xrightarrow{v'} C \xrightarrow{w} \Sigma A$$

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w'} \Sigma A$$

are all distinguished.

(iv) A triangle T is *virtual* if there exists a splitting triangle S so that $S \oplus T$ is exact.

The important facts for us to observe here are

Lemma 1. *All distinguished triangles are virtual.*

Lemma 2. *Homological functors take virtual triangles to long exact sequences.*

Proof. Lemma 1 may be found in Vaknin's [92, Remark 1.4]. For Lemma 2, see [92, Definition 1.6 and Theorem 1.11].

5 Categories with Squares

The input we will need to define K–theory is a *category with squares.* In Section 7 we will see how, starting with a category with squares, one can define a K–theory. This section prepares the background. We will see the definition of a category with squares, and also the key examples of interest.

Definition 2. *An additive category* $\mathcal{T}$ *will be called a* category with squares *provided*

(i) $\mathcal{T}$ *has an automorphism* $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$.
(ii) $\mathcal{T}$ *comes with a collection of* special squares

$$\begin{tikzcd} C \arrow[r] & D \arrow[dl, bend left=60, "(1)"] \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

This means that the square

$$\begin{tikzcd} C \arrow[r] & D \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

is commutative in $\mathcal{T}$*, and there is a map* $D \longrightarrow \Sigma A$*, which we denote by the curly arrow*

$$\begin{tikzcd} & D \arrow[dl, bend left=60, "(1)"] \\ A & \end{tikzcd}$$

The (1) *in the label of the arrow is to remind us that the map is of degree* 1*, that is a map* $D \longrightarrow \Sigma A$.

Definition 3. *Given two categories with squares, a* special functor

$$F : \mathcal{S} \longrightarrow \mathcal{T}$$

is an additive functor such that

(i) *There is a natural isomorphism* $\Sigma F \cong F\Sigma$.
(ii) *The functor* F *takes special squares in* $\mathcal{S}$ *to special squares in* $\mathcal{T}$.

The next definition is a convenient tool in the discussion of the examples.

Definition 4. *Let $\mathcal{T}$ be an additive category with an automorphism $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$. Suppose we are given a square*

$$\begin{tikzcd}
C \arrow[r, "\delta"] & D \arrow[dl, bend left=90, "\mu"] \\
A \arrow[u, "\beta"] \arrow[r, "\alpha"] & B \arrow[u, "\gamma"']
\end{tikzcd}$$

The fold of this square *will be the the sequence*

$$A \xrightarrow{\begin{pmatrix}\alpha \\ -\beta\end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix}\gamma & \delta\end{pmatrix}} D \xrightarrow{\quad\mu\quad} \Sigma A.$$

Example 3. Let $\mathcal{T}$ be a triangulated category. Then $\mathcal{T}$ is an additive category, and it comes with an automorphism $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$. A square is defined to be special if and only if its fold is a distinguished triangle in $\mathcal{T}$. When we think of the triangulated category $\mathcal{T}$ as being the category with squares defined above, then we will denote it as ${}^d\mathcal{T}$.

Example 4. Given a triangulated category $\mathcal{T}$, we wish to consider yet another possible structure one can give it, as a category with squares. The suspension functor $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ is the same as in ${}^d\mathcal{T}$. But there are more special squares. In the category which we will call ${}^v\mathcal{T}$, a square will be special if and only if its fold is a *virtual* triangle, in the sense of Vaknin [92] (see also Section 4).

Example 5. Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Gr}^b\mathcal{A}$ be the category of bounded, graded objects in $\mathcal{A}$. We remind the reader. A *graded object* of $\mathcal{A}$ is a sequence of objects $\{a_i \mid i \in \mathbb{Z}, a_i \in \mathcal{A}\}$. The sequence $\{a_i\}$ is *bounded* if $a_i = 0$ except for finitely many $i \in \mathbb{Z}$.

We define the functor $\Sigma : \mathrm{Gr}^b\mathcal{A} \longrightarrow \mathrm{Gr}^b\mathcal{A}$ to be the shift. That is,

$$\Sigma\{a_i\} = \{b_i\}$$

with $b_i = a_{i+1}$. A square in $\mathrm{Gr}^b\mathcal{A}$ is defined to be *special* if the fold

$$A \xrightarrow{\begin{pmatrix}\alpha \\ -\beta\end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix}\gamma & \delta\end{pmatrix}} D \xrightarrow{\quad\mu\quad} \Sigma A$$

gives a long exact sequence in $\mathcal{A}$. That is, the fold gives us a sequence

$$\cdots \longrightarrow D_{i-1} \longrightarrow A_i \longrightarrow B_i \oplus C_i \longrightarrow D_i \longrightarrow A_{i+1} \longrightarrow \cdots$$

and we require that this sequence be exact everywhere.

Example 6. In Definition 3, a *special functor* $\mathcal{S} \longrightarrow \mathcal{T}$ was defined to be an additive functor taking special squares in $\mathcal{S}$ to special squares in $\mathcal{T}$. Let $\mathcal{T}$ be a triangulated category. Lemma 1 tells us that the identity functor $1 : \mathcal{T} \longrightarrow \mathcal{T}$ gives a special functor $\gamma : {}^d\mathcal{T} \longrightarrow {}^v\mathcal{T}$. Any special square in ${}^d\mathcal{T}$ is automatically a special square in ${}^v\mathcal{T}$.

Let $H : \mathcal{T} \longrightarrow \mathcal{A}$ be a homological functor from the triangulated category $\mathcal{T}$ to the abelian category $\mathcal{A}$. Suppose H is bounded. That is, for each $t \in \mathcal{T}$ there exists $N \in \mathbb{N}$ with $H(\Sigma^i t) = 0$ unless $-N < i < N$. By Lemma 2, H takes virtual triangles in $\mathcal{T}$ to long exact sequences in $\mathcal{A}$. The functor taking $t \in \mathcal{T}$ to the graded object $\{H(\Sigma^i t) \mid i \in \mathbb{Z}\}$ is a special functor

$$\delta : {}^v\mathcal{T} \longrightarrow \mathrm{Gr}^b\mathcal{A}$$

of categories with squares. Summarising, we have produced special functors

$${}^d\mathcal{T} \xrightarrow{\ \gamma\ } {}^v\mathcal{T} \xrightarrow{\ \delta\ } \mathrm{Gr}^b\mathcal{A}.$$

In some very simple cases, for example if $\mathcal{T} = D^b(k)$ is the derived category of a field k and H is ordinary homology, the maps γ and δ are equivalences of categories with squares.

6 Regions

In Section 5 we learned what is meant by a category with squares. We learned the definition, and the three examples we will refer to in this article. In the current section we will study regions $\mathcal{R} \subset \mathbb{Z}\times\mathbb{Z}$, and then in Section 7 we put it all together. The K–theory of a category with squares $\mathcal{T}$ is defined from the simplicial set of certain functors from regions $\mathcal{R} \subset \mathbb{Z}\times\mathbb{Z}$ to the category with squares $\mathcal{T}$.

Let us agree first that, from this point on, $\mathbb{Z}$ will be understood to be a category. The objects are the integers, and

$$\mathrm{Hom}(i,j) = \begin{cases} \emptyset & \text{if } i > j \\ 1 & \text{if } i \le j \end{cases}$$

That is, $\mathrm{Hom}(i,j)$ is either empty or has one element. It is non-empty exactly when $i \le j$. There is only one possible composition law.

Definition 5. *A* region *will mean a full subcategory* $\mathcal{R} \subset \mathbb{Z}\times\mathbb{Z}$.

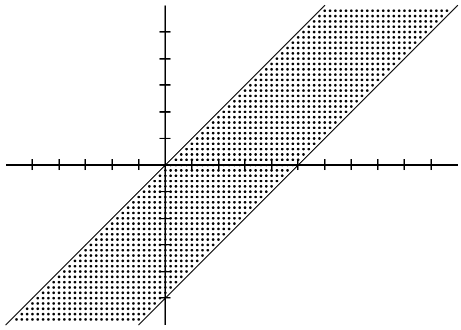
Definition 6. *Let* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *be two regions. A* morphism of regions $\mathcal{R}_1 \longrightarrow \mathcal{R}_2$ *is a functor* $F : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$*, so that there exist two functors* $f_1 : \mathbb{Z} \longrightarrow \mathbb{Z}$*,* $f_2 : \mathbb{Z} \longrightarrow \mathbb{Z}$ *and a commutative square*

$$\begin{tikzcd} \mathcal{R}_1 \arrow[r, "F"] \arrow[d, hook] & \mathcal{R}_2 \arrow[d, hook] \\ \mathbb{Z}\times\mathbb{Z} \arrow[r, "f_1\times f_2"] & \mathbb{Z}\times\mathbb{Z} \end{tikzcd}$$

Remark 3. In this article, the regions we most care about are

$$\mathcal{R}_n \quad = \quad \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid 0 \le x-y \le n+1\}\,.$$

We consider them when $n \geq 0$. The picture is



Part of the reason we care about the $\mathcal{R}_n$'s is the following.

Remark 4. Recall the category $\varDelta$ of finite ordered sets. The objects are $\mathbf{n} = \{0,1,\ldots,n\}$. The morphisms are the order preserving maps. I assert that there is a functor θ from $\varDelta$ to the category of regions in $\mathbb{Z}\times\mathbb{Z}$. We define the functor θ as follows.

(i) On objects: For an object $\mathbf{n} \in \varDelta$, put $\theta(\mathbf{n}) = \mathcal{R}_n$, as in Remark 3.

(ii) On morphisms: Suppose we are given a morphism $\varphi : \mathbf{m} \longrightarrow \mathbf{n}$ in $\varDelta$. We define $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ as follows. Any integer in $\mathbb{Z}$ can be expressed, uniquely, as $a(m+1)+b$, with $0 \le b \le m$. Put

$$f(a(m+1)+b) = a(n+1) + \varphi(b).$$

Then f is an order-preserving map $\mathbb{Z} \longrightarrow \mathbb{Z}$ (that is, a functor when we view $\mathbb{Z}$ as a category). The reader can show that

$$f\times f : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}$$

takes $\mathcal{R}_m \subset \mathbb{Z}\times\mathbb{Z}$ into $\mathcal{R}_n \subset \mathbb{Z}\times\mathbb{Z}$. We define $\theta(\varphi)$ to be the map $\mathcal{R}_m \longrightarrow \mathcal{R}_n$ induced by $f\times f : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}$.

It is useful to note that $\theta(\varphi)$ takes the boundary of the region $\theta(\mathbf{m}) = \mathcal{R}_m$ to the boundary of the region $\theta(\mathbf{n}) = \mathcal{R}_n$. More explicitly, the boundary point $(y,y) \in \mathcal{R}_m$ gets mapped to the boundary point $\big(f(y)\,,\,f(y)\big) \in \mathcal{R}_n$. The boundary point $(y+m+1\,,\,y) \in \mathcal{R}_m$ gets mapped to the boundary point $\big(f(y)+n+1\,,\,f(y)\big) \in \mathcal{R}_n$.

7 The Simplicial Set

Now we know what we mean by

(i) Regions in $\mathbb{Z} \times \mathbb{Z}$.
(ii) Categories with square.

It is time to put it together and define K–theory. The key ingredient is

Definition 7. *Let $\mathcal{T}$ be a category with squares. Let $\mathcal{R}$ be a region in $\mathbb{Z} \times \mathbb{Z}$. An* augmented diagram for the pair $(\mathcal{R}, \mathcal{T})$ *is defined to be*

(i) *A functor $F : \mathcal{R} \longrightarrow \mathcal{T}$.*
(ii) *Suppose we are given four integers $i \leq i'$ and $j \leq j'$. These four integers define a commutative square in $\mathbb{Z} \times \mathbb{Z}$, namely*

$$\begin{tikzcd}
(i,j') \arrow[r] & (i',j') \\
(i,j) \arrow[u] \arrow[r] & (i',j) \arrow[u]
\end{tikzcd}$$

If this square happens to be contained in the region $\mathcal{R}$, then the functor F, of part (i) *above, takes it to a commutative square in $\mathcal{T}$*

$$\begin{tikzcd}
F(i,j') \arrow[r] & F(i',j') \\
F(i,j) \arrow[u] \arrow[r] & F(i',j) \arrow[u]
\end{tikzcd}$$

We require that all such squares extend to special squares in $\mathcal{T}$. That is, we must be given a map

$$\delta_{i,j}^{i',j'} : F(i',j') \longrightarrow \Sigma F(i,j)$$

yielding a special square.

(iii) *The maps $\delta_{i,j}^{i',j'}$ should be compatible, in the following sense. Suppose we are given two squares in $\mathbb{Z} \times \mathbb{Z}$, one inside the other. That is, we have integers $I \leq i \leq i' \leq I'$ and $J \leq j \leq j' \leq J'$, giving in $\mathbb{Z} \times \mathbb{Z}$ the commutative diagram*

$$\begin{tikzcd}
(I,J') \arrow[r] & (i,J') \arrow[r] & (i',J') \arrow[r] & (I',J') \\
(I,j') \arrow[u] \arrow[r] & (i,j') \arrow[u] \arrow[r] & (i',j') \arrow[u] \arrow[r] & (I',j') \arrow[u] \\
(I,j) \arrow[u] \arrow[r] & (i,j) \arrow[u] \arrow[r] & (i',j) \arrow[u] \arrow[r] & (I',j) \arrow[u] \\
(I,J) \arrow[u] \arrow[r] & (i,J) \arrow[u] \arrow[r] & (i',J) \arrow[u] \arrow[r] & (I',J) \arrow[u]
\end{tikzcd}$$

Suppose the small, middle square and the outside, large square both lie entirely in $\mathcal{R}$. That is, we have two squares in $\mathcal{R}$, one contained in the other

$$\begin{tikzcd} (i,j') \arrow[r] & (i',j') & & (I,J') \arrow[r] & (I',J') \\ (i,j) \arrow[r] \arrow[u] & (i',j) \arrow[u] & & (I,J) \arrow[r] \arrow[u] & (I',J) \arrow[u] \end{tikzcd}$$

Part (ii) *above gives us two maps*

$$\delta_{i,j}^{i',j'} : F(i',j') \longrightarrow \Sigma F(i,j)$$

$$\delta_{I,J}^{I',J'} : F(I',J') \longrightarrow \Sigma F(I,J)$$

The compatibility requirement is that $\delta_{i,j}^{i',j'}$ should be the composite

$$F(i',j') \xrightarrow{F(\alpha)} F(I',J') \xrightarrow{\delta_{I,J}^{I',J'}} \Sigma F(I,J) \xrightarrow{\Sigma F(\beta)} \Sigma F(i,j)$$

where $\beta : (I,J) \longrightarrow (i,j)$ and $\alpha : (i',j') \longrightarrow (I',J')$ are the unique maps in $\mathbb{Z} \times \mathbb{Z}$.

Remark 5. The definition of augmented diagrams is clearly functorial in the pairs $\mathcal{R}, \mathcal{T}$. Given a morphism of regions $f : \mathcal{R} \longrightarrow \mathcal{R}'$ and a special functor of categories with squares $g : \mathcal{T} \longrightarrow \mathcal{T}'$, then composition induces a natural map

$$\left\{ \begin{array}{c} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R}',\mathcal{T}) \end{array} \right\} \xrightarrow{\quad (f,g) \quad} \left\{ \begin{array}{c} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R},\mathcal{T}') \end{array} \right\} .$$

This says that there is a functor

$$\left\{ \begin{array}{c} \text{Regions} \\ \mathcal{R} \subset \mathbb{Z} \times \mathbb{Z} \end{array} \right\}^{\mathrm{op}} \times \left\{ \begin{array}{c} \text{Categories} \\ \text{with squares} \end{array} \right\} \xrightarrow{\quad \Phi \quad} \{\text{Sets}\}$$

which takes the pair $(\mathcal{R},\mathcal{T}) \in \{\text{Regions}\} \times \{\text{Categories with squares}\}$ to

$$\Phi(\mathcal{R},\mathcal{T}) \quad = \quad \left\{ \begin{array}{c} \text{Augmented diagrams} \\ \text{for the pair } (\mathcal{R},\mathcal{T}) \end{array} \right\} .$$

This functor is contravariant in the region $\mathcal{R}$, covariant in $\mathcal{T}$ (the category with squares).

Now, finally, we come to our simplicial set.

Definition 8. *Remark 4 provides us with a functor*

$$\theta : \Delta \longrightarrow \{\text{regions in } \mathbb{Z}\times\mathbb{Z}\}.$$

Remark 5 gives a functor

$$\Phi : \left\{\begin{matrix}\text{Regions}\\ \mathcal{R}\subset\mathbb{Z}\times\mathbb{Z}\end{matrix}\right\}^{\mathrm{op}} \times \left\{\begin{matrix}\text{Categories}\\ \text{with squares}\end{matrix}\right\} \longrightarrow \{\text{Sets}\}\ .$$

Let $\mathcal{T}$ be a category with squares. Then the functor taking $(-)\in\Delta$ to

$$\Phi\big(\theta(-),\mathcal{T}\big)$$

is a functor $\Delta^{\mathrm{op}}\longrightarrow\{\text{Sets}\}$. We wish to consider a simplicial subset

$$S_*(\mathcal{T}) \quad\subset\quad \Phi\big(\theta(-),\mathcal{T}\big).$$

The elements of $S_n(\mathcal{T})$ form a subset of

$$\Phi\big(\theta(\mathbf{n}),\mathcal{T}\big)=\Phi(\mathcal{R}_n,\mathcal{T}).$$

The set $\Phi(\mathcal{R}_n,\mathcal{T})$ consists of all augmented diagrams for the pair $(\mathcal{R}_n,\mathcal{T})$. The subset $S_n(\mathcal{T})$ are the augmented diagrams which vanish on the boundary. Recall: An augmented diagram gives, among other things, a functor $F:\mathcal{R}_n\longrightarrow\mathcal{T}$. The augmented diagram belongs to $S_n(\mathcal{T})$ if

$$F(y,y)=0=F(y+n+1\,,\,y).$$

Remark 6. At the end of Remark 4 we noted that, if $\varphi:\mathbf{m}\longrightarrow\mathbf{n}$ is any morphism in Δ, then $\theta(\varphi)$ takes points on the boundary of the region $\mathcal{R}_m=\theta(\mathbf{m})$ to boundary points of $\mathcal{R}_n=\theta(\mathbf{n})$. Augmented diagrams which vanish on the boundary of $\mathcal{R}_n$ therefore go to augmented diagrams vanishing on the boundary of $\mathcal{R}_m$, and hence $S_*(\mathcal{T})$ really is a simplicial subset of $\Phi\big(\theta(-),\mathcal{T}\big)$.

Remark 7. It is clear that $S_*(\mathcal{T})$ is functorial in $\mathcal{T}$. Given a special functor of categories with squares $\mathcal{S}\longrightarrow\mathcal{T}$, then composition induces a map

$$S_*(\mathcal{S})\longrightarrow S_*(\mathcal{T}).$$

In Example 6 we saw that, given a triangulated category $\mathcal{T}$, an abelian category $\mathcal{A}$ and a bounded homological functor $H:\mathcal{T}\longrightarrow\mathcal{A}$, there are special functors of categories with squares

$${}^d\mathcal{T}\xrightarrow{\ \gamma\ }{}^v\mathcal{T}\xrightarrow{\ \delta\ }\mathrm{Gr}^b\mathcal{A}.$$

We conclude that there are simplicial maps of simplicial sets

$$S_*({}^d\mathcal{T})\xrightarrow{\ \gamma\ }S_*({}^v\mathcal{T})\xrightarrow{\ \delta\ }S_*(\mathrm{Gr}^b\mathcal{A}).$$

Note that, in an abuse of notation, the letter γ stands for both the map ${}^d\mathcal{T}\longrightarrow{}^v\mathcal{T}$ and for the map it induces on the simplicial sets, and similarly for the letter δ.

Definition 9. *For a category with squares $\mathfrak{T}$, its K–theory $K(\mathfrak{T})$ is defined to be the loop space of the geometric realisation of the simplicial set $S_*(\mathfrak{T})$. In symbols:*

$$K(\mathfrak{T}) = \Omega|S_*(\mathfrak{T})|$$

Remark 8. Taking loop spaces of the geometric realisation of the maps in Remark 7, we deduce continuous maps of spaces

$$K({}^d\mathfrak{T}) \xrightarrow{\ \gamma\ } K({}^v\mathfrak{T}) \xrightarrow{\ \delta\ } K(\mathrm{Gr}^b\mathcal{A}).$$

8 What It All Means

Until now our treatment has been very abstract. We have constructed certain simplicial sets and simplicial maps. It might be helpful to work out explicitly what are the low-dimensional simplices. The definition says

$$S_n(\mathfrak{T}) = \left\{ \begin{array}{c|c} \text{Augmented diagrams} & \text{The functor } F : \mathcal{R}_n \longrightarrow \mathfrak{T}\text{, given} \\ & \text{as part of the data of the} \\ \text{for the pair } (\mathcal{R}_n, \mathfrak{T}) & \text{augmented diagram, satisfies} \\ & F(y,y) = 0 = F(y+n+1\,,\,y) \end{array} \right\}$$

Let us now work this out, in low dimensions, for the category with squares ${}^d\mathfrak{T}$.

Case 1. $S_0({}^d\mathfrak{T})$ is easy to compute. The region $\mathcal{R}_0$ is the region $0 \le x - y \le 1$, and all the points are boundary points. That is, for every $(x, y) \in \mathcal{R}_0$ we have that $x - y$ is either 0 or 1. There is only one element in $S_0(\mathfrak{T})$. It is the diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] & {} \\
 & 0 \arrow[r] & 0 \arrow[u] & \\
{} \arrow[r] & 0 \arrow[u] & &
\end{tikzcd}$$

Case 2. Slightly more interesting is $S_1({}^d\mathfrak{T})$. The region $\mathcal{R}_1$ is $0 \le x - y \le 2$, and the boundary consists of the points where $x - y$ is 0 or 2. A simplex is therefore a diagram

$$
\begin{tikzcd}
 & & 0 & \\
 & 0 \arrow[r] & x_{n+1} \arrow[u] \arrow[r] & 0 \\
 0 \arrow[r] & x_n \arrow[u] \arrow[r] & 0 \arrow[u] & \\
 x_{n-1} \arrow[u] \arrow[r] & 0 \arrow[u] & & \\
 0 \arrow[u] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & x_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

In this diagram, each square

$$
\begin{tikzcd}
0 \arrow[r] & x_{n+1} \\
x_n \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}
$$

is a special square. It comes with a map $\delta_n : x_{n+1} \longrightarrow \Sigma x_n$. In the case of the category with squares ${}^d\mathfrak{T}$, the fact that the square is special means that we have a distinguished triangle

$$
x_n \longrightarrow 0 \longrightarrow x_{n+1} \xrightarrow{\delta_n} \Sigma x_n .
$$

In other words, the map $\delta_n : x_{n+1} \longrightarrow \Sigma x_n$ must be an isomorphism. The diagram defining the simplex is canonically isomorphic to

$$
\begin{tikzcd}
 & & & 0 & \\
 & & 0 \arrow[r] & \Sigma x_0 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[r] & x_0 \arrow[u] \arrow[r] & 0 \arrow[u] & \\
 0 \arrow[r] & \Sigma^{-1} x_0 \arrow[u] \arrow[r] & 0 \arrow[u] & & \\
 & 0 \arrow[u] & & &
\end{tikzcd}
$$

Up to canonical isomorphism, the simplices in $S_1({}^d\mathfrak{T})$ are just the objects of $\mathfrak{T}$.

Case 3. Next we consider $S_2({}^d\mathcal{T})$. The region $\mathcal{R}_2$ is $0 \leq x - y \leq 3$, and the boundary consists of the points where $x-y$ is 0 or 3. A 2-simplex is a diagram

$$\begin{tikzcd}
 & & & 0 & & \\
 & & 0 \arrow[r] & y' \arrow[r] \arrow[u] & z' \arrow[r] & 0 \\
 & 0 \arrow[r] & z \arrow[r] \arrow[u] & x' \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & x \arrow[r] \arrow[u] & y \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
 & & 0 \arrow[u] & & &
\end{tikzcd}$$

The special squares

$$\begin{tikzcd}
0 \arrow[r] & x' \\
x \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & y' \\
y \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & z' \\
z \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

have differentials

$$\delta_x : x' \longrightarrow \Sigma x, \qquad \delta_y : y' \longrightarrow \Sigma y, \qquad \delta_z : z' \longrightarrow \Sigma z.$$

As in Case 2 above, these differentials must be isomorphisms. The diagram as a whole is therefore canonically isomorphic to

$$\begin{tikzcd}
 & & & 0 & & \\
 & & 0 \arrow[r] & \Sigma y \arrow[r, "v'"] \arrow[u] & \Sigma z \arrow[r] & 0 \\
 & 0 \arrow[r] & z \arrow[r, "w"] \arrow[u] & \Sigma x \arrow[r] \arrow[u, "u'"'] & 0 \arrow[u] & \\
0 \arrow[r] & x \arrow[r, "u"] \arrow[u] & y \arrow[r] \arrow[u, "v"'] & 0 \arrow[u] & & \\
 & & 0 \arrow[u] & & &
\end{tikzcd}$$

The isomorphism is such that, in the special squares

$$\begin{tikzcd} 0 \arrow[r] & \Sigma x \\ x \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \Sigma y \\ y \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \Sigma z \\ z \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

the differentials are all identity maps. Next we will use the fact that the differentials are coherent, to compute the maps in the diagram.

Consider the following little bit of the larger diagram above

$$\begin{tikzcd} & 0 \arrow[r] & \Sigma y \\ 0 \arrow[r] & z \arrow[u] \arrow[r, "w"] & \Sigma x \arrow[u, "u'"'] \\ x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

There are three squares in this bit, namely

$$\begin{tikzcd} 0 \arrow[r] & \Sigma x \\ x \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \Sigma y \\ y \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, "w"] & \Sigma x \\ y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

These are three special squares, with compatible differentials. The differentials of the first two squares are

$$\delta_x = 1 : \Sigma x \longrightarrow \Sigma x, \qquad\qquad \delta_y = 1 : \Sigma y \longrightarrow \Sigma y.$$

The compatibility says that the differential of the third square

$$\begin{tikzcd} z \arrow[r, "w"] & \Sigma x \\ y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

can be computed as either of the composites

$$\Sigma x \xrightarrow{\delta_x} \Sigma x \xrightarrow{\Sigma u} \Sigma y$$

$$\Sigma x \xrightarrow{u'} \Sigma y \xrightarrow{\delta_y} \Sigma y$$

We conclude that $u' = \Sigma u$. The diagram

$$\begin{tikzcd}
0 \arrow[r] & \Sigma y \arrow[r, "v'"] & \Sigma z \\
z \arrow[u] \arrow[r, "w"] & \Sigma x \arrow[u, "u'"'] \arrow[r] & 0 \arrow[u] \\
y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] &
\end{tikzcd}$$

permits us to compute that $v' = \Sigma v$, and so on. The simplex becomes

$$\begin{tikzcd}
 & & & 0 & & \\
 & & 0 \arrow[r] & \Sigma y \arrow[u] \arrow[r, "\Sigma v"] & \Sigma z \arrow[r] & 0 \\
 & 0 \arrow[r] & z \arrow[u] \arrow[r, "w"] & \Sigma x \arrow[u, "\Sigma u"'] \arrow[r] & 0 \arrow[u] & \\
0 \arrow[r] & x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] & & \\
 & & 0 \arrow[u] & & &
\end{tikzcd}$$

In this diagram there are many special squares. So far, we have focused mainly on the special squares of the form

$$\begin{tikzcd}
0 \arrow[r] & \Sigma A \\
A \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

where the differential $\Sigma A \longrightarrow \Sigma A$ is the identity. But there are other special squares. For example

$$\begin{tikzcd}
0 \arrow[r] & z \\
x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"']
\end{tikzcd}$$

The differential of this special square may be computed from the fact that, in the diagram

$$\begin{tikzcd}
0 \arrow[r] & z \arrow[r, "w"] & \Sigma x \\
x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

the larger special square

$$\begin{tikzcd} 0 \arrow[r] & \Sigma x \\ x \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

has for its differential the map $1 : \Sigma x \longrightarrow \Sigma x$. Compatibility tells us that the differential of

$$\begin{tikzcd} 0 \arrow[r] & z \\ x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \end{tikzcd}$$

must be $w : z \longrightarrow \Sigma x$. But in Example 3 we defined special squares in ${}^d\mathfrak{T}$ to be squares

$$\begin{tikzcd} C \arrow[r, "\delta"] & D \arrow[dl, bend left, "\mu"] \\ A \arrow[u, "\beta"] \arrow[r, "\alpha"] & B \arrow[u, "\gamma"'] \end{tikzcd}$$

for which the sequence

$$A \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} D \xrightarrow{\mu} \Sigma A$$

is a distinguished triangle. In our case, this becomes a distinguished triangle

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x.$$

One of the miracles here is that the signs take care of themselves. The special square

$$\begin{tikzcd} z \arrow[r, "w"] & \Sigma x \\ y \arrow[u, "v"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

has a differential, which is easily computed to be $\Sigma u : \Sigma x \longrightarrow \Sigma y$. This gives a distinguished triangle

$$y \xrightarrow{-v} z \xrightarrow{w} \Sigma x \xrightarrow{\Sigma u} \Sigma y.$$

The fact that the morphism $v : y \longrightarrow z$ in the square is vertical automatically takes care of the sign.

We conclude that the only real restriction on the diagram

$$\begin{tikzcd}
 & & & 0 & & \\
 & & 0 \arrow[r] & \Sigma y \arrow[u] \arrow[r, "\Sigma v"] & \Sigma z \arrow[r] & 0 \\
 & 0 \arrow[r] & z \arrow[u] \arrow[r, "w"] & \Sigma x \arrow[u, "\Sigma u"'] \arrow[r] & 0 \arrow[u] & \\
0 \arrow[r] & x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \arrow[r] & 0 \arrow[u] & & \\
 & & 0 \arrow[u] & & &
\end{tikzcd}$$

is the fact that

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$$

is a distinguished triangle. The other special squares give distinguished triangles which are just rotations of the above. In conclusion: Any element in $S_2({}^d\mathcal{T})$ is canonically isomorphic to a diagram which arises as above from a distinguished triangle

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x.$$

There are three face maps

$$S_2({}^d\mathcal{T}) \Rrightarrow S_1({}^d\mathcal{T}).$$

In the above, we identified the elements of $S_2({}^d\mathcal{T})$ with distinguished triangles in $\mathcal{T}$. In Case 2, we identified the elements of $S_1({}^d\mathcal{T})$ with the objects of $\mathcal{T}$. The face maps above take the distinguished triangle

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$$

to z, y and x, respectively.

Remark 9. We now have an explicit identification of the elements of $S_2({}^d\mathcal{T})$ and $S_1({}^d\mathcal{T})$, and of the three face maps

$$S_2({}^d\mathcal{T}) \Rrightarrow S_1({}^d\mathcal{T}).$$

Using this, one can compute the first homology group of the space $|S_*({}^d\mathcal{T})|$. Since $|S_*({}^d\mathcal{T})|$ is an H–space, we have

$$H_1|S_*({}^d\mathcal{T})| \;=\; \pi_1|S_*({}^d\mathcal{T})| \;=\; \pi_0 K({}^d\mathcal{T}).$$

An explicit computation easily shows this to be the usual Grothendieck group of Definition 1.

Remark 10. In Case 2 we saw that the diagram for a 1–simplex has objects which repeat (up to suspension). In Case 3 we saw that the morphisms in a 2–simplex also repeat, again up to suspension. Take an element $x \in S_n({}^d\mathcal{T})$, with $n \geq 2$. Then x is a diagram in $\mathcal{T}$. The objects of this diagram are all objects of 1–dimensional faces of x, and the morphisms are all composites of morphisms in 2–dimensional faces of x. From Cases 2 and 3 we conclude that the entire diagram is periodic.

More explicitly, a fundamental region for the diagram $x \in S_n({}^d\mathcal{T})$ is given by

$$\mathcal{D}_n \quad = \quad \left\{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; \begin{array}{c} 0 \leq a \leq n \\ 0 \leq b \leq n \\ 0 \leq a-b \end{array} \right\}.$$

Thus, a 1–simplex is completely determined by the diagram

$$\begin{tikzcd} & 0 \\ 0 \arrow[r] & x \arrow[u] \end{tikzcd}$$

and a 2–simplex is determined by

$$\begin{tikzcd} & & 0 \\ & 0 \arrow[r] & z \arrow[u] \\ 0 \arrow[r] & x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \end{tikzcd}$$

If the reader is worried that the map $w : z \longrightarrow \Sigma x$ does not seem to appear, the point is simple. It is the differential of the special square

$$\begin{tikzcd} 0 \arrow[r] & z \\ x \arrow[u] \arrow[r, "u"] & y \arrow[u, "v"'] \end{tikzcd}$$

What is being asserted is the following. The region $\mathcal{R}_n$ contains the region $\mathcal{D}_n$. An element $x \in S_n({}^d\mathcal{T})$ is an augmented diagram for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$. It restricts to an augmented diagram for the pair $(\mathcal{D}_n, {}^d\mathcal{T})$. The assertion is that the smaller diagram determines, up to canonical isomorphism, the larger one.

Case 4. Next we wish to study the elements of $S_3(\mathcal{T})$. By Remark 10, the simplex is determined by its restriction to $\mathcal{D}_3 \subset \mathcal{R}_3$. We have a diagram

$$
\begin{tikzcd}
 & & & 0 \\
 & & 0 \arrow[r] & z \arrow[u] \\
 & 0 \arrow[r] & x \arrow[r] \arrow[u] & y \arrow[u] \\
0 \arrow[r] & u \arrow[r] \arrow[u] & v \arrow[r] \arrow[u] & w \arrow[u]
\end{tikzcd}
$$

A simplex in $S_3(\mathfrak{T})$ is obtained from this by periodicity, up to suspension. The simplex will look like

$$
\begin{tikzcd}
 & & & & 0 & & & \\
 & & & 0 \arrow[r] & \Sigma w \arrow[r] \arrow[u] & \Sigma y \arrow[r] & \Sigma z \arrow[r] & 0 \\
 & & 0 \arrow[r] & z \arrow[r] \arrow[u] & \Sigma v \arrow[r] \arrow[u] & \Sigma x \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[r] & x \arrow[r] \arrow[u] & y \arrow[r] \arrow[u] & \Sigma u \arrow[r] \arrow[u] & 0 \arrow[u] & & \\
0 \arrow[r] & u \arrow[r] \arrow[u] & v \arrow[r] \arrow[u] & w \arrow[r] \arrow[u] & 0 \arrow[u] & & & \\
 & & & 0 \arrow[u] & & & &
\end{tikzcd}
$$

What does it all mean?

We have two composable morphisms

$$u \longrightarrow v \longrightarrow w.$$

The special squares

$$
\begin{tikzcd}
0 \arrow[r] & x \\
u \arrow[r] \arrow[u] & v \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & y \\
u \arrow[r] \arrow[u] & w \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & z \\
v \arrow[r] \arrow[u] & w \arrow[u]
\end{tikzcd}
$$

give three distinguished triangles

$$u \longrightarrow v \longrightarrow x \longrightarrow \Sigma u$$

$$u \longrightarrow w \longrightarrow y \longrightarrow \Sigma u$$

$$v \longrightarrow w \longrightarrow z \longrightarrow \Sigma v$$

and the special square

$$\begin{tikzcd} 0 \arrow[r] & z \\ x \arrow[u] \arrow[r] & y \arrow[u] \end{tikzcd}$$

tells us that the mapping cones x, y and z of the maps $u \longrightarrow v$, $u \longrightarrow w$ and $v \longrightarrow w$ fit in a distinguished triangle

$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x.$$

This should hopefully look familiar. What we have here is an octahedron, with its four distinguished triangles and four commutative triangles.

Remark 11. Our octahedron is somewhat special. We have special squares

$$\begin{tikzcd} x \arrow[r] & y \\ v \arrow[u] \arrow[r] & w \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r] & \Sigma v \\ y \arrow[u] \arrow[r] & \Sigma u \arrow[u] \end{tikzcd}$$

These come with differentials, and fold to give distinguished triangles. Thus a 3–simplex in the simplicial set $S_*({}^d\mathcal{T})$ is more than just an octahedron. It is an octahedron where the two commutative squares are special.

I observed the existence of such octahedra in [66, Remark 5.5]. This existence may be viewed as a refinement of the old octahedral lemma.

Remark 12. It is perhaps worth explaining this point even further. In Remark 10 we observed that a simplex in $S_n({}^d\mathcal{T})$ is determined by its restriction to the region $\mathcal{D}_n \subset \mathcal{R}_n$. But it is only right to warn the reader that not every augmented diagram for the pair $(\mathcal{D}_n, {}^d\mathcal{T})$, vanishing on the top diagonal, extends to a simplex in $S_n({}^d\mathcal{T})$. If the extension exists then it is unique up to canonical isomorphism; but there is no guarantee of existence. For clarity, let us illustrate this when $n = 3$.

An augmented diagram for the pair $(\mathcal{D}_3, {}^d\mathcal{T})$, vanishing on the top diagonal, is a diagram

$$\begin{tikzcd}
 & & & 0 \\
 & & 0 \arrow[r] & z \arrow[u] \\
 & 0 \arrow[r] & x \arrow[r] \arrow[u] & y \arrow[u] \\
0 \arrow[r] & u \arrow[r] \arrow[u] & v \arrow[r] \arrow[u] & w \arrow[u]
\end{tikzcd}$$

together with compatible differentials, and where we have five special squares. By the periodicity of Remark 10, we can extend this to a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[r] & \Sigma w \\
 & & 0 \arrow[r] & z \arrow[r] \arrow[u] & \Sigma v \arrow[u] \\
 & 0 \arrow[r] & x \arrow[r] \arrow[u] & y \arrow[r] \arrow[u] & \Sigma u \arrow[u] \\
0 \arrow[r] & u \arrow[r] \arrow[u] & v \arrow[r] \arrow[u] & w \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The periodicity provides us all the maps and differentials we might care to have. The problem is that nothing guarantees that

$$\begin{tikzcd}
z \arrow[r] & \Sigma v \\
y \arrow[r] \arrow[u] & \Sigma u \arrow[u]
\end{tikzcd}$$

should be a special square. In general it will not be.

It turns out that, for the categories with squares ${}^v\mathfrak{T}$ and $\mathrm{Gr}^b\mathcal{A}$, this problem disappears; every augmented diagram for the pair $(\mathcal{D}_n, {}^v\mathfrak{T})$ (resp. $(\mathcal{D}_n, \mathrm{Gr}^b\mathcal{A})$), vanishing on the top diagonal, extends to a simplex in $S_n({}^v\mathfrak{T})$ (resp. $S_n(\mathrm{Gr}^b\mathcal{A})$). The point is that in the diagram above we have

$$\begin{tikzcd}
z \arrow[r] & \Sigma v \\
y \arrow[r] \arrow[u] & \Sigma u \arrow[u] \\
w \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The squares

$$\begin{tikzcd} z \arrow[r] & \Sigma v \\ w \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r] & \Sigma u \\ w \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

are both special, being the rotations of given virtual triangles (resp. long exact sequences). The 2-out-of-3 property holds, implying that the square

$$\begin{tikzcd} z \arrow[r] & \Sigma v \\ y \arrow[u] \arrow[r] & \Sigma u \arrow[u] \end{tikzcd}$$

is also special. For ${}^v\mathcal{T}$ the 2-out-of-3 property is proved in [92, Section 2.4]. For $\mathrm{Gr}^b\mathcal{A}$ the proof may be found in [70, Lemma 4.3].

Remark 13. The elements of $S_n(\mathcal{T})$ can be thought of as refinements of the higher octahedra. Let $x \in S_n(\mathcal{T})$ be a simplex. It is an augmented diagram for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$. In $\mathcal{R}_n \subset \mathbb{Z} \times \mathbb{Z}$, consider the intersection with $\mathbb{Z} \times \{0\}$. It is the set $\{(i,0) \mid 0 \le i \le n+1\}$. On the region $\mathcal{R}_n \cap \big\{\mathbb{Z} \times \{0\}\big\}$, the restriction of $x \in S_n(\mathcal{T})$ is just

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n \longrightarrow 0.$$

This gives us $(n-1)$ composable morphisms. The restriction of x to the fundamental region $\mathcal{D}_n \subset \mathcal{R}_n$ of Remark 10 is a diagram which contains all the mapping cones on the maps $x_i \longrightarrow x_j$. And the simplex keeps track of the relations among these. Note that the simplex remembers more data than the higher octahedra of [8, Remarque 1.1.14]. We already observed this in the case of 3–simplices. Somehow the coherent differentials and all the special squares tell us of the existence of many distinguished triangles.

9 Waldhausen Models and the Existence of Large Simplices

Let $\mathcal{T}$ be a category with squares. In Section 7 we defined a simplicial set $S_*(\mathcal{T})$. In Section 8 we analysed the low-dimensional simplices of $S_*({}^d\mathcal{T})$, where ${}^d\mathcal{T}$ is the category with squares obtained from a triangulated category $\mathcal{T}$ as in Example 3. The analysis of Section 8 tells us that the 1–simplices correspond to objects, the 2–simplices correspond to distinguished triangles, and the 3–simplices correspond to special octahedra. The refined octahedral axiom guarantees the existence of a great many 3–simplices. For $n \ge 4$, the n–simplices are complicated diagrams, and it is not clear if any non-degenerate examples exist. It is therefore of some interest to see how a Waldhausen model can be used to construct simplices.

Let $\mathcal{A}$ be an abelian category, $C(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. As in Section 3, our Waldhausen categories will all be assumed to be full subcategories of $C(\mathcal{A})$. We begin with a definition

Definition 10. *A commutative square in* $C(\mathcal{A})$

$$\begin{tikzcd} b' \arrow[r, "\delta"] & c \\ a \arrow[u, "\beta"] \arrow[r, "\alpha"'] & b \arrow[u, "\gamma"'] \end{tikzcd}$$

is called bicartesian *if the sequence*

$$0 \longrightarrow a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} c \longrightarrow 0$$

is a short exact sequence of chain complexes.

Remark 14. Suppose we have a bicartesian square in $C(\mathcal{A})$ as in Definition 10. The fact that the composite

$$a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} c$$

vanishes gives us a natural map from the mapping cone of $\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$ to c. This map must be a homology isomorphism. It therefore becomes invertible in $\mathrm{ho}C(\mathcal{A}) = D(\mathcal{A})$. Unless confusion is likely to arise (that is, if there are several possibilities for α, β, γ and δ), we will omit them entirely in the notation. The map will be written

$$\mathrm{Cone}(a \longrightarrow b \oplus b') \longrightarrow c.$$

The key lemma is

Lemma 3. *Let $C(\mathcal{A})$ be a Waldhausen category. Let*

$$\begin{tikzcd} b' \arrow[r] & c \\ a \arrow[u] \arrow[r] & b \arrow[u] \end{tikzcd}$$

be a bicartesian square in $C(\mathcal{A})$. There exists a canonical choice for a differential $\partial : c \longrightarrow \Sigma a$ rendering the diagram into a special square in

${}^d\mathrm{ho}C(\mathcal{A}) = {}^dD(\mathcal{A})$. *Furthermore, this choice of differentials is coherent. That is, given a digram in* $C(\mathcal{A})$ *where all the squares are bicartesian*

$$\begin{tikzcd}
d''' \arrow[r] & e'' \arrow[r] & f' \arrow[r] & g \\
c'' \arrow[r] \arrow[u] & d'' \arrow[r] \arrow[u] & e' \arrow[r] \arrow[u] & f \arrow[u] \\
b' \arrow[r] \arrow[u] & c' \arrow[r] \arrow[u] & d' \arrow[r] \arrow[u] & e \arrow[u] \\
a \arrow[r] \arrow[u] & b \arrow[r] \arrow[u] & c \arrow[r] \arrow[u] & d \arrow[u]
\end{tikzcd}$$

we deduce two bicartesian squares, one contained in the other

$$\begin{tikzcd}
d'' \arrow[r] & e' \\
c' \arrow[r] \arrow[u] & d' \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
d''' \arrow[r] & g \\
a \arrow[r] \arrow[u] & d \arrow[u]
\end{tikzcd}$$

The above tells us that there are canonical choices for two differentials

$$\delta_1 : e' \longrightarrow \Sigma c'$$

$$\delta_2 : g \longrightarrow \Sigma a.$$

The compatibility requirement, which we assert is automatic for the canonical choices of differentials, is that δ_1 *should be the composite*

$$e' \longrightarrow g \xrightarrow{\delta_2} \Sigma a \longrightarrow \Sigma c'.$$

Proof. Let

$$\begin{tikzcd}
b' \arrow[r] & c \\
a \arrow[r, "\alpha"] \arrow[u, "\beta"] & b \arrow[u]
\end{tikzcd}$$

be a bicartesian square in $C(\mathcal{A})$. Let X be the mapping cone on the map

$$a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b'.$$

We have maps

$$\Sigma a \xleftarrow{f} X \xrightarrow{g} c.$$

By Remark 14, the map $g : X \longrightarrow c$ is invertible in $D(\mathcal{A})$. The canonical choice for the differential is fg^{-1}. The compatibility of these differentials comes from the commutative diagram

$$\begin{tikzcd}
\Sigma a \arrow[d] & \mathrm{Cone}(a \longrightarrow d \oplus d''') \arrow[l] \arrow[r] \arrow[d] & g \arrow[d, equal] \\
\Sigma c' \arrow[d, equal] & \mathrm{Cone}(c' \longrightarrow e \oplus e'') \arrow[l] \arrow[r] & g \\
\Sigma c' & \mathrm{Cone}(c' \longrightarrow d' \oplus d'') \arrow[l] \arrow[r] \arrow[u] & e' \arrow[u]
\end{tikzcd}$$

Corollary 1. *Let $\mathcal{R} \subset \mathbb{Z} \times \mathbb{Z}$ be a region. Assume that $\mathcal{R}$ is* convex. *Suppose we have a functor $F : \mathcal{R} \longrightarrow C(\mathcal{A})$. Any time we have four integers $i \leq i'$ and $j \leq j'$, these four integers define a commutative square in $\mathbb{Z} \times \mathbb{Z}$, namely*

$$\begin{tikzcd}
(i,j') \arrow[r] & (i',j') \\
(i,j) \arrow[r] \arrow[u] & (i',j) \arrow[u]
\end{tikzcd}$$

Suppose that, whenever the square above happens to be contained in the region $\mathcal{R}$, the functor F takes it to a bicartesian square in $C(\mathcal{A})$

$$\begin{tikzcd}
F(i,j') \arrow[r] & F(i',j') \\
F(i,j) \arrow[r] \arrow[u] & F(i',j) \arrow[u]
\end{tikzcd}$$

Then there is a canonical way to associate to the functor F an augmented diagram for the pair $\big(\mathcal{R}, {}^dD(\mathcal{A})\big)$.

Proof. We certainly have a functor

$$\mathcal{R} \xrightarrow{F} C(\mathcal{A}) \longrightarrow D(\mathcal{A}).$$

For any square lying in $\mathcal{R}$, the bicartesian square in $C(\mathcal{A})$

$$\begin{tikzcd}
F(i,j') \arrow[r] & F(i',j') \\
F(i,j) \arrow[r] \arrow[u] & F(i',j) \arrow[u]
\end{tikzcd}$$

permits us, using Lemma 3, to make the canonical choice of differential $F(i',j') \longrightarrow \Sigma F(i,j)$. It only remains to check that the choices are coherent.

Suppose therefore that we have a diagram in $\mathbb{Z} \times \mathbb{Z}$

$$\begin{tikzcd}
(I,J') \arrow[r] & (i,J') \arrow[r] & (i',J') \arrow[r] & (I',J') \\
(I,j') \arrow[r] \arrow[u] & (i,j') \arrow[r] \arrow[u] & (i',j') \arrow[r] \arrow[u] & (I',j') \arrow[u] \\
(I,j) \arrow[r] \arrow[u] & (i,j) \arrow[r] \arrow[u] & (i',j) \arrow[r] \arrow[u] & (I',j) \arrow[u] \\
(I,J) \arrow[r] \arrow[u] & (i,J) \arrow[r] \arrow[u] & (i',J) \arrow[r] \arrow[u] & (I',J) \arrow[u]
\end{tikzcd}$$

If the large square

$$\begin{tikzcd}
(I,J') \arrow[r] & (I',J') \\
(I,J) \arrow[r] \arrow[u] & (I',J) \arrow[u]
\end{tikzcd}$$

lies in the region $\mathcal{R}$, then the convexity of $\mathcal{R}$ tells us that so does the entire diagram. We can therefore apply F to it, obtaining a diagram of bicartesian squares in $C(\mathcal{A})$

$$\begin{tikzcd}
F(I,J') \arrow[r] & F(i,J') \arrow[r] & F(i',J') \arrow[r] & F(I',J') \\
F(I,j') \arrow[r] \arrow[u] & F(i,j') \arrow[r] \arrow[u] & F(i',j') \arrow[r] \arrow[u] & F(I',j') \arrow[u] \\
F(I,j) \arrow[r] \arrow[u] & F(i,j) \arrow[r] \arrow[u] & F(i',j) \arrow[r] \arrow[u] & F(I',j) \arrow[u] \\
F(I,J) \arrow[r] \arrow[u] & F(i,J) \arrow[r] \arrow[u] & F(i',J) \arrow[r] \arrow[u] & F(I',J) \arrow[u]
\end{tikzcd}$$

Lemma 3 therefore applies, and tells us that the two special squares

$$\begin{tikzcd}
F(i,j') \arrow[r] & F(i',j') \\
F(i,j) \arrow[r] \arrow[u] & F(i',j) \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(I,J') \arrow[r] & F(I',J') \\
F(I,J) \arrow[r] \arrow[u] & F(I',J) \arrow[u]
\end{tikzcd}$$

have compatible differentials.

Remark 15. It is clear that proof of Corollary 1 uses less than the full strength of the convexity hypothesis. The corollary remains true for some non-convex regions. In this article, the main region of interest in $\mathcal{R}_n = \{(x,y) \mid 0 \leq x - y \leq n+1\}$, and $\mathcal{R}_n$ is clearly convex. Hence we do not take the trouble to give the strongest version of the corollary.

Remark 16. Now we want to use Corollary 1 to construct simplices in $S_n({}^d\mathcal{T})$. As in Remark 13, we begin with the restriction of a putative simplex to $\mathcal{R}_n \cap \big\{\mathbb{Z} \times \{0\}\big\}$. In other words, we have sequence of composable maps in $\mathcal{T}$

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n \longrightarrow 0.$$

and we want to show that this sequence may be extended to a simplex.

Let $\mathcal{C}$ be any Waldhausen model for $\mathcal{T}$. The first observation is that we may choose a lifting of this sequence of composable maps to $\mathcal{C}$. We will define, by descending induction on i, a sequence of morphisms in $\mathcal{C}$

$$y_i \longrightarrow y_{i+1} \longrightarrow \cdots \longrightarrow y_{n-1} \longrightarrow y_n$$

isomorphic in $\mathcal{T}$ to the sequence

$$x_i \longrightarrow x_{i+1} \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n.$$

Choose y_n to be any object of $\mathcal{C}$ isomorphic to x_n; this defines the sequence for $i = n$. Suppose the sequence has been defined for i. The morphism $x_{i-1} \longrightarrow x_i \simeq y_i$ is a map in $\mathcal{T}$, and by the calculus of fractions in biWaldhausen complicial categories (which we call Waldhausen model categories), it may be represented as $\alpha\beta^{-1}$, with α and β as below and β a weak equivalence

$$x_{i-1} \xleftarrow{\ \beta\ } y_{i-1} \xrightarrow{\ \alpha\ } y_i.$$

The map α can be used to extend our sequence to

$$y_{i-1} \longrightarrow y_i \longrightarrow y_{i+1} \longrightarrow \cdots \longrightarrow y_{n-1} \longrightarrow y_n.$$

This completes the induction. Replacing the x_i by y_i, we now assume our sequence lies in $\mathcal{C}$.

Now we need to construct the simplex. Choose in $\mathcal{C}$ a cofibration $x_1 \longrightarrow y_1^1$, with y_1^1 contractible. (For example, y_1^1 could be the mapping cone on $1 : x_1 \longrightarrow x_1$). Pushing out allows us to obtain a diagram of bicartesian squares

$$\begin{tikzcd}
 & y_1^1 \arrow[r] & y_1^2 \arrow[r] & \cdots \arrow[r] & y_1^{n-1} \arrow[r] & y_1^n \arrow[r] & y_1^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & x_1 \arrow[r] \arrow[u] & x_2 \arrow[r] \arrow[u] & \cdots \arrow[r] & x_{n-1} \arrow[r] \arrow[u] & x_n \arrow[r] \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Choosing a cofibration $y_1^2 \rightarrowtail y_2^2$, with y_2^2 contractible, we can continue to

$$\begin{tikzcd}
 & & y_2^2 \arrow[r] & \cdots \arrow[r] & y_2^n \arrow[r] & y_2^{n+1} \arrow[r] & y_2^{n+2} \arrow[r] & 0 \\
 & y_1^1 \arrow[r] & y_1^2 \arrow[r] \arrow[u] & \cdots \arrow[r] & y_1^n \arrow[r] \arrow[u] & y_1^{n+1} \arrow[r] \arrow[u] & 0 \arrow[u] & \\
0 \arrow[r] & x_1 \arrow[r] \arrow[u] & x_2 \arrow[r] \arrow[u] & \cdots \arrow[r] & x_n \arrow[r] \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Clearly, we can iterate this process, obtaining a commutative diagram where each square is bicartesian. We can also continue this diagram in the negative direction. Suppose y_{-1}^n is contractible, and suppose we have a fibration $y_{-1}^n \twoheadrightarrow x_n$. We can pull back to obtain

$$\begin{tikzcd}
 & 0 \arrow[r] & x_1 \arrow[r] & x_2 \arrow[r] & \cdots \arrow[r] & x_{n-1} \arrow[r] & x_n \arrow[r] & 0 \\
0 \arrow[r] & y_{-1}^0 \arrow[r] \arrow[u] & y_{-1}^1 \arrow[r] \arrow[u] & y_{-1}^2 \arrow[r] \arrow[u] & \cdots \arrow[r] & y_{-1}^{n-1} \arrow[r] \arrow[u] & y_{-1}^n \arrow[u] &
\end{tikzcd}$$

By iterating this construction in both the negative and positive direction, we obtain a functor from the region $\mathcal{R}_n \subset \mathbb{Z} \times \mathbb{Z}$ to $\mathcal{C}$. In the category $\mathrm{ho}\mathcal{C} = \mathcal{T}$, the object y_i^i and y_{-i}^{n+1-i} are isomorphic to zero. Consider the composite functor

$$\mathcal{R}_n \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}.$$

It vanishes at the boundary of the region $\mathcal{R}_n$. Corollary 1 then tells us that we have a simplex in $S_n({}^d\mathcal{T})$.

Remark 17. We have shown how to construct elements of $S_n({}^d\mathcal{T})$ starting from diagrams of bicartesian squares in a Waldhausen model. An element $s \in S_n({}^d\mathcal{T})$ is called *Waldhausen liftable* is there exists some Waldhausen model $\mathcal{C}$ for $\mathcal{T}$, a diagram y of bicartesian squares in $\mathcal{C}$, and an ismorphism of augmented $(\mathcal{R}_n, {}^d\mathcal{T})$ diagrams $y \cong s$.

Definition 11. *The simplicial subset ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ is defined to be the simplicial set of all Waldhausen liftable simplices.*

Remark 18. Note that the simplicial subset ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ does not depend on a choice of model. A simplex is liftable if there exists some model $\mathcal{C}$ for $\mathcal{T}$, and a lifting to $\mathcal{C}$. The model $\mathcal{C}$ is not specified in advance.

Remark 19. If we let β be the inclusion map ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$, then what we have so far are four simplicial maps

$$ {}^wS_*(\mathcal{T}) \xrightarrow{\beta} S_*({}^d\mathcal{T}) \xrightarrow{\gamma} S_*({}^v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b\mathcal{A}). $$

Next we define the fifth and last map.

Remark 20. For the remainder of this section, we will assume that the reader is familiar with t–structures in triangulated categories. For an excellent account, the reader is referred to Chapter 1 of Beilinson, Bernstein and Deligne's [8]. In this section, we will use the following facts. Given a triangulated category $\mathcal{T}$ with a t–structure, there is a full subcategory $\mathcal{A} \subset \mathcal{T}$, called the *heart* of $\mathcal{T}$. It satisfies

(i) $\mathcal{A}$ is an abelian category.
(ii) Given a monomorphism $f : a \longrightarrow b$ in $\mathcal{A}$, there is a canonically unique way to complete it to a distinguished triangle

$$ a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a $$

The object c lies in $\mathcal{A} \subset \mathcal{T}$, and

$$ 0 \longrightarrow a \xrightarrow{f} b \xrightarrow{g} c \longrightarrow 0 $$

is a short exact sequence in $\mathcal{A}$ (this already makes the cokernel map $g : b \longrightarrow c$ unique up to canonical isomorphism). What is being asserted is that, given $g : b \longrightarrow c$, the map $h : c \longrightarrow \Sigma a$ is unique. See [8, Corollaire 1.1.10(ii)].
(iii) There is a canonical way to define a homological functor $H : \mathcal{T} \longrightarrow \mathcal{A}$.
(iv) The t–structure is called *bounded* if H is a bounded homological functor (see Example 6 for the definition of bounded homological functors), and if

$$ \{\forall i \in \mathbb{Z}, H(\Sigma^i x) = 0\} \quad \Longrightarrow \quad x = 0. $$

Example 7. Let $\mathcal{T} = D(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. There is a t–structure on $\mathcal{T} = D(\mathcal{A})$, called the *standard t–structure.* The heart of $\mathcal{T}$ is $\mathcal{A} \subset D(\mathcal{A})$, where $\mathcal{A}$ is embedded in $D(\mathcal{A})$ as the complexes which vanish in all degrees but zero. The homological functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ of part (iii) is just the functor taking a chain complex $X \in D(\mathcal{A})$ to $H^0(X)$. This t–structure is not bounded on $\mathcal{T} = D(\mathcal{A})$. Define a full subcategory $D^b(\mathcal{A}) \subset D(\mathcal{A})$ by

$$ \mathrm{Ob}\big(D^b(\mathcal{A})\big) \quad = \quad \left\{ X \in D(\mathcal{A}) \;\middle|\; \begin{array}{c} H^n(X) = 0 \text{ for all but} \\ \text{finitely many } n \in \mathbb{Z} \end{array} \right\}. $$

Then $D^b(\mathcal{A})$ is a triangulated subcategory of $D(\mathcal{A})$. The standard t–structure on $\mathcal{T} = D(\mathcal{A})$ restricts to a standard t–structure on $D^b(\mathcal{A}) \subset D(\mathcal{A})$. The heart is still $\mathcal{A}$, and the t–structure on $D^b(\mathcal{A})$ is bounded, as in (iv) above.

Lemma 4. *Suppose $\mathcal{T}$ is a triangulated category with a t–structure, and let $\mathcal{A}$ be the heart. Suppose $\mathcal{T}$ has at least one Waldhausen model. Then there is a simplicial map*

$$\alpha : S_*(\mathcal{A}) \longrightarrow {}^wS_*(\mathcal{T}).$$

Here, by $S_*(\mathcal{A})$ we mean the Waldhausen S_*–construction on the abelian category $\mathcal{A}$.

Proof. An element in Waldhausen's $S_n(\mathcal{A})$ is a string of $(n-1)$ composable monomorphisms in $\mathcal{A}$, together with a (canonically unique) choice of the quotients. That is, maps

$$0 \longrightarrow x_1 \rightarrowtail x_2 \rightarrowtail \cdots \rightarrowtail x_{n-1} \rightarrowtail x_n$$

together with a choice of the quotients x_j/x_i for all $i<j$. Choose any Waldhausen model $\mathcal{C}$ for $\mathcal{T}$. In Remark 16 we saw that the sequence

$$0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n \longrightarrow 0$$

can be extended to a simplex in $S_*({}^d\mathcal{T})$, with a Waldhausen lifting to $\mathcal{C}$. That is, it can be extended to a simplex in ${}^wS_*(\mathcal{T})$.

But now the restriction of this simplex to the region $\mathcal{D}_n \subset \mathcal{R}_n$ gives us nothing other than the sequence of monomorphisms

$$0 \longrightarrow x_1 \rightarrowtail x_2 \rightarrowtail \cdots \rightarrowtail x_{n-1} \rightarrowtail x_n$$

together with a choice of the quotients x_j/x_i. This choice of the quotients must be canonically isomorphic to the choice that comes from the simplex in $S_n(\mathcal{A})$. Remark 20(ii) tells us that even the differentials are canonically unique. But by Remark 10 the simplex is determined by its restriction to $\mathcal{D}_n \subset \mathcal{R}_n$. [The careful reader, mindful of Remark 12, will recall that not all augmented diagrams on $\mathcal{D}_n$ extend to $\mathcal{R}_n$. But here we know that the extension exists, and the uniqueness always holds].

10 The Main Theorems

Up until now, we have produced is a string of definitions. Let $\mathcal{T}$ be a triangulated category. Assume $\mathcal{T}$ has at least one Waldhausen model. Assume it has a bounded t–structure, with heart $\mathcal{A}$. We have constructed simplicial maps

$$S_*(\mathcal{A}) \xrightarrow{\alpha} {}^wS_*(\mathcal{T}) \xrightarrow{\beta} S_*({}^d\mathcal{T}) \xrightarrow{\gamma} S_*({}^v\mathcal{T}) \xrightarrow{\delta} S_*(\mathrm{Gr}^b\mathcal{A}).$$

Consider the loop spaces of the geometric realisations of these maps. Write them as

$$K(\mathcal{A}) \xrightarrow{\alpha} K({}^w\mathcal{T}) \xrightarrow{\beta} K({}^d\mathcal{T}) \xrightarrow{\gamma} K({}^v\mathcal{T}) \xrightarrow{\delta} K(\mathrm{Gr}^b\mathcal{A}).$$

The main theorem is

Theorem 3. *With the notation as above, we have*

(i) *The composite* $\delta\gamma\beta\alpha : K(\mathcal{A}) \longrightarrow K(\mathrm{Gr}^b\mathcal{A})$ *is a homotopy equivalence.*
(ii) *The map* $\alpha : K(\mathcal{A}) \longrightarrow K({}^w\mathcal{T})$ *is a homotopy equivalence.*
(iii) *The space* $K({}^v\mathcal{T})$ *has a homotopy type which depends only on* $\mathcal{A}$. *That is,* $K({}^v\mathcal{T}) \cong K\big({}^vD^b(\mathcal{A})\big)$.

Proof. The proofs of these statements are, at least at the moment, long and very difficult. The proof of (i) may be found in [70] and [71], or in [66] and [67]. For the proof of (ii), see [72], or [68] and [69]. The detailed proof of (iii) does not yet exist in print. The idea is that it follows by a slight modification of the proof of (ii).

In the sections which follow, I will try to highlight the problems which naturally arise. The aim of this survey is to explain why the theorems we now know, that is Theorem 3(i), (ii) and (iii), are deeply unsatisfying and cry for improvement.

Before we launch into an exhaustive treatment of the defects in what we know, in this section I will give a brief discussion of the positive. Here are some remarkable consequences of the theorems.

Remark 21. From Theorem 3(i) we know that the spaces $K({}^w\mathcal{T})$, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$ all contain Quillen's K–theory $K(\mathcal{A})$ as a retract. Far from losing all information about higher K–theory, the passage to the derived category has, if anything, added more information.

Remark 22. From Theorem 3(ii) we conclude the following. Suppose $\mathcal{T}$ is a triangulated category with at least one Waldhausen model. Suppose it admits two bounded t–structures, with hearts $\mathcal{A}$ and $\mathcal{B}$. Then the Quillen K–theories of $\mathcal{A}$ and $\mathcal{B}$ agree. In symbols, we have

$$K(\mathcal{A}) \cong K(\mathcal{B}).$$

After all both are isomorphic, by Theorem 3(ii), with $K({}^w\mathcal{T})$.

This was unknown even for the "standard t–structures" of Example 7. In other words, a special case of the above is where we have two abelian categories $\mathcal{A}$ and $\mathcal{B}$, with $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$. Put $\mathcal{T} = D^b(\mathcal{A}) = D^b(\mathcal{B})$. Then $\mathcal{T}$ certainly has at least one Waldhausen model, namely $C^b(\mathcal{A})$. It has two bounded t–structures, namely the standard one on $D^b(\mathcal{A})$ and the standard one on $D^b(\mathcal{B})$. The hearts of these two t–structures are $\mathcal{A}$ and $\mathcal{B}$ respectively. We conclude that $K(\mathcal{A}) \cong K(\mathcal{B})$.

Remark 23. In comparing the consequences of Theorem 3 with what was known earlier, it is helpful to recall some of the work of Waldhausen.

To each Waldhausen category $\mathcal{C}$, Waldhausen associates a K–theory. Let us call it $WK(\mathcal{C})$, for the Waldhausen K–theory of $\mathcal{C}$. Suppose we are given an exact functor of Waldhausen categories $\alpha : \mathcal{C} \longrightarrow \mathcal{D}$. Suppose that

$$\mathrm{ho}(\alpha) : \mathrm{ho}\mathcal{C} \longrightarrow \mathrm{ho}\mathcal{D}$$

is an equivalence of triangulated categories. From Waldhausen's Approximation Theorem, it is possible to deduce fairly easily that

$$WK(\alpha) : WK(\mathcal{C}) \longrightarrow WK(\mathcal{D})$$

is a homotopy equivalence. For details see Thomason [89, Theorem 1.9.8]. It follows that, given any zigzag of exact functors of Waldhausen categories

$$\begin{tikzcd}[column sep=small] & \mathcal{C}_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & \arrow[dl, "\alpha_3"] & \cdots & \arrow[dr, "\alpha_{n-2}"'] & & \mathcal{C}_{n-1} \arrow[dl, "\alpha_{n-1}"'] \arrow[dr, "\alpha_n"] & \\ \mathcal{C}_0 & & \mathcal{C}_2 & & & & \mathcal{C}_{n-2} & & \mathcal{C}_n \end{tikzcd}$$

if each $\mathrm{ho}(\alpha_i)$ is an equivalence of triangulated categories, then $WK(\mathcal{C}_0) \cong WK(\mathcal{C}_n)$.

Example 8. For example, let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and assume that the categories $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$ are equivalent. Assume further that the equivalence can be lifted to models. This means there is a zigzag of exact functors of Waldhausen models from $C^b(\mathcal{A})$ to $C^b(\mathcal{B})$, as in Remark 23. Then it follows that $K(\mathcal{A}) \cong K(\mathcal{B})$. Already in this baby application there is an advantage to Theorem 3 over the older results. The advantage is that, in applying Theorem 3, there is no need to assume the equivalence $D^b(\mathcal{A}) \cong D^b(\mathcal{B})$ can be lifted to models.

While on the subject of comparing Theorem 3 with the earlier results, let us mention a question raised by Thomason. Thomason asked the following: Does there exist a pair of Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$, with

$$\mathrm{ho}\mathcal{C} \cong \mathrm{ho}\mathcal{D} \qquad \text{but} \qquad WK(\mathcal{C}) \not\cong WK(\mathcal{D})?$$

By Remark 23, a pair of the sort Thomason asked for could not possibly be compared by a zigzag of maps of models as above. Not quite so obvious is the fact that, if no such pair exists, then the "standard t–structure" case of my theorem above becomes a consequence of Waldhausen's work.

We now know that such a pair exists. The result may be found in Schlichting [85]. In this very precise sense, my result cannot be deduced from Waldhausen's.

Remark 24. Quillen defined a K–theory space $K(\mathcal{A})$ for any abelian category $\mathcal{A}$. If we define $K'(\mathcal{A}) = K\big({}^vD^b(\mathcal{A})\big)$, we have a functor such that

(i) By Theorem 3(i), there is a natural split inclusion $K(\mathcal{A}) \longrightarrow K'(\mathcal{A})$.
(ii) By Theorem 3(iii), if $\mathcal{A}$ is the heart of a bounded t–structure, on a triangulated category $\mathcal{T}$ with at least one Waldhausen model, then

$$K'(\mathcal{A}) \cong K({}^v\mathcal{T}).$$

No information is lost if we replace $K(\mathcal{A})$ by $K'(\mathcal{A})$, and for all we know $K'(\mathcal{A})$ might be better.

11 Computational Problems

It is time to turn to the problems in the subject, which are very many. Let us begin with what ought to be the easiest. We should be able to compute the various maps, at least for low dimensions.

Theorem 3(i) tells us that the spaces $K({}^w\mathcal{T})$, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$ all contain Quillen's K–theory $K(\mathcal{A})$ as a retract. It is easy to see that in K_0, this is an isomorphism

$$K_0(\mathcal{A}) = K_0({}^w\mathcal{T}) = K_0({}^d\mathcal{T}) = K_0({}^v\mathcal{T}).$$

Very embarrasingly, this is all we know. The first question would be

Problem 1. Is it true that

$$K_1(\mathcal{A}) = K_1({}^w\mathcal{T}) = K_1({}^d\mathcal{T}) = K_1({}^v\mathcal{T})?$$

If not, can one say anything about the other direct summands?

It goes without saying that the same problem is entirely open for K_i, for any $i > 1$. I stated Problem 1 as a problem about K_1 for two reasons.

(i) In order to show how embarrassingly little we know.
(ii) Because very recently, as a result of Vaknin [90, 91], we actually have a half-way usable description of K_1 of a triangulated category $\mathcal{T}$.

Remark 25. One way to compute K_1 is from the definition we gave. The K–theory spaces are the loop spaces of the simplicial sets ${}^wS_*(\mathcal{T})$, $S_*({}^d\mathcal{T})$ and $S_*({}^v\mathcal{T})$ respectively. This means the groups K_1 are the second homotopy groups

$$\pi_2|{}^wS_*(\mathcal{T})|, \quad \pi_2|S_*({}^d\mathcal{T})|, \quad \pi_2|S_*({}^v\mathcal{T})|$$

I do not consider this a computationally-friendly description. The comment (ii) above reminds the reader that, from the recent work by Vaknin [90, 91], we have a much more useful description. It is for this reason that the problem might be more manageable in K_1.

So far we have looked only at hearts of t–structures, which are always abelian categories. One special case is $\mathcal{T} = D^b(\mathcal{A})$, with the standard t–structure as in Example 7. We know that there are maps in K–theory

$$K(\mathcal{A}) \xrightarrow{\alpha} K\big({}^wD^b(\mathcal{A})\big) \xrightarrow{\beta} K\big({}^dD^b(\mathcal{A})\big) \xrightarrow{\gamma} K\big({}^vD^b(\mathcal{A})\big),$$

and that the map $\gamma\beta\alpha : K(\mathcal{A}) \longrightarrow K\big({}^vD^b(\mathcal{A})\big)$ is a monomorphism (it is even split injective). It is natural to wonder what happens if we replace the abelian category $\mathcal{A}$ by an exact category $\mathcal{E}$. There is a sensible way to define the derived category $D^b(\mathcal{E})$ for any exact category $\mathcal{E}$. The category $D^b(\mathcal{E})$ is definitely a triangulated category. This construction may be found in [64].

The general formalism, valid for any triangulated category, specialises in the case of $D^b(\mathcal{E})$ to give maps

$$K\big({}^wD^b(\mathcal{E})\big) \xrightarrow{\beta} K\big({}^dD^b(\mathcal{E})\big) \xrightarrow{\gamma} K\big({}^vD^b(\mathcal{E})\big).$$

Not quite so immediate, but nevertheless true, is that there is also a continuous map $\alpha : K(\mathcal{E}) \longrightarrow K\big({}^wD^b(\mathcal{E})\big)$. From Vaknin's direct computations [90], we have

Proposition 1. *For certain choices of the exact category $\mathcal{E}$, the induced map $K_1(\alpha) : K_1(\mathcal{E}) \longrightarrow K_1\big({}^wD^b(\mathcal{E})\big)$ has a non-trivial kernel.*

Note that this is quite unlike what happens when $\mathcal{E}$ is abelian; in the abelian case we know that $K(\mathcal{E})$ is a retract of each of $K\big({}^wD^b(\mathcal{E})\big)$, $K\big({}^dD^b(\mathcal{E})\big)$ and $K\big({}^vD^b(\mathcal{E})\big)$. This leads to:

Problem 2. For an exact category $\mathcal{E}$, compute the maps

$$K(\mathcal{E}) \xrightarrow{\alpha} K\big({}^wD^b(\mathcal{E})\big) \xrightarrow{\beta} K\big({}^dD^b(\mathcal{E})\big) \xrightarrow{\gamma} K\big({}^vD^b(\mathcal{E})\big).$$

Even an explicit computational understanding of what happens in K_1 would be a vast improvement over what we now know.

12 Functoriality Problems

Starting with any triangulated category $\mathcal{T}$, we have defined three possible candidates for its K–theory. They are the spaces $K({}^w\mathcal{T})$, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. Of the three, $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$ are functors in $\mathcal{T}$. Given a triangulated functor of triangulated categories $f : \mathcal{S} \longrightarrow \mathcal{T}$, there are natural induced maps

$$K({}^df) : K({}^d\mathcal{S}) \longrightarrow K({}^d\mathcal{T}) \qquad \text{and} \qquad K({}^vf) : K({}^v\mathcal{S}) \longrightarrow K({}^v\mathcal{T}).$$

Remark 26. The simplicial sets $S_*({}^d\mathcal{T})$ and $S_*({}^v\mathcal{T})$ have a very nice addition defined on them, allowing us to construct an infinite loop structure on $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. From now on, we will view these as spectra.

Theorem 3 tells us little about $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. All we know is that, if $\mathcal{T}$ had a bounded t–structure with heart $\mathcal{A}$, then $K(\mathcal{A})$ is a retract of both $K({}^d\mathcal{T})$ and $K({}^v\mathcal{T})$. The good theorem is about $K({}^w\mathcal{T})$. Suppose $\mathcal{T}$ has at least one Waldhausen model. Theorem 3(ii) tells us that $K(\mathcal{A}) = K({}^w\mathcal{T})$. This suggests that we define the K–theory of the triangulated category $\mathcal{T}$ to be $K({}^w\mathcal{T})$, and forget about the other options. Let me now point to all the faults of $K({}^w\mathcal{T})$. First we should remind the reader of the definition of $K({}^w\mathcal{T})$.

Given a triangulated category $\mathcal{T}$, there is a simplicial set $S_*({}^d\mathcal{T})$. The set $S_n({}^d\mathcal{T})$ has for its elements the augmented diagrams for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$, which vanish on the boundary of the region $\mathcal{R}_n$. In Section 9, we defined what it means for an element of $S_n({}^d\mathcal{T})$ to have a Waldhausen lifting (see Remark 17). The simplicial subset ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ is defined to be the simplicial subset of all Waldhausen liftable simplices. The K–theory $K({}^w\mathcal{T})$ is the loop space of the geometric realisation of ${}^wS_*(\mathcal{T})$.

Remark 27. There is no obvious H–space structure on ${}^wS_*(\mathcal{T})$. Suppose we are given two n–simplices. Both are augmented diagrams for the pair $(\mathcal{R}_n, {}^d\mathcal{T})$. Each diagram has a lifting to some Waldhausen model. Suppose the first diagram lifts to a model $\mathcal{C}_1$ and the second lifts to a model $\mathcal{C}_2$. For all we know, the direct sum may not have a lifting to any model.

Since ${}^wS_*(\mathcal{T})$ is not obviously an H–space, it most certainly is not obviously an infinite loop space. Let us now be careful about what Theorem 3(ii) tells us. If $\mathcal{A}$ is the heart of a bounded t–structure on $\mathcal{T}$, the theorem asserts that $K(\mathcal{A}) \cong K({}^w\mathcal{T})$. This is only a homotopy equivalence of spaces. It is not an H–map of H–spaces, and most certainly not an infinite loop map of infinite loop spaces. In Remark 22 we observed that, if $\mathcal{A}$ and $\mathcal{B}$ are two hearts of two bounded t–structures on a single triangulated category $\mathcal{T}$, then $K(\mathcal{A}) \cong K(\mathcal{B})$. Both $K(\mathcal{A})$ and $K(\mathcal{B})$ are naturally infinite loop spaces, but the above isomorphism is only as spaces. It is not an infinite loop map.

Remark 28. Unlike the many open problems I am in the process of outlining, this problem is settled. Suppose we are in the situation above. That is, $\mathcal{T}$ is a triangulated category with at least one Waldhausen model, and $\mathcal{A}$ and $\mathcal{B}$ are two hearts of two bounded t–structures on $\mathcal{T}$. Then $K(\mathcal{A}) \cong K(\mathcal{B})$, even as infinite loop spaces. The proof is to introduce yet another simplicial set, which we can denote ${}^+S_*(\mathcal{T})$. We define ${}^+S_*(\mathcal{T})$ to be a subset of $S_*({}^d\mathcal{T})$. A simplex in $S_*({}^d\mathcal{T})$ belongs to ${}^+S_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ if it can be written as a direct sum of simplices, each with a Waldhausen lifting. In other words, we obtain ${}^+S_*(\mathcal{T})$ as the closure of ${}^wS_*(\mathcal{T}) \subset S_*({}^d\mathcal{T})$ under direct sums. Define $K({}^+\mathcal{T})$ to be the loop space of the geometric realisation of ${}^+S_*(\mathcal{T})$.

It is now easy to see that $K({}^+\mathcal{T})$ is an infinite loop space. It turns out that the proof of Theorem 3(ii) works well for $K({}^+\mathcal{T})$. We conclude that the map $K(\mathcal{A}) \longrightarrow K({}^+\mathcal{T})$ is a homotopy equivalence. Since it is an infinite loop map of infinite loop spaces, the problem posed by Remark 27 is solved.

There is something quite unappetising about the nature of the proof outlined in Remark 28. Surely we do not want to have to introduce a new simplicial set, and a new definition for the K–theory of the triangulated category $\mathcal{T}$, every time we wish to prove a new theorem. This method of proof by modification of the simplicial set is the best we know; presumably there is a good choice of the simplicial set, rendering such trickery unnecessary.

Remark 29. The most serious problem with $K({}^w\mathcal{T})$ is that it is not a functor of $\mathcal{T}$. Let $f : \mathcal{S} \longrightarrow \mathcal{T}$ be a triangulated functor of triangulated categories. I do not know how to construct an induced map $K({}^w\mathcal{S}) \longrightarrow K({}^w\mathcal{T})$. The same problem also holds for the simplicial sets of Remark 28. Starting with a triangulated functor $f : \mathcal{S} \longrightarrow \mathcal{T}$, I do not know how to construct an induced map $K({}^{+}\mathcal{S}) \longrightarrow K({}^{+}\mathcal{T})$.

Problem 3. Find a simplicial set $K({}^{?}\mathcal{T})$, which is a functor of $\mathcal{T}$ and for which the strong statement of Theorem 3(ii) holds.

It is quite possible that $K({}^{?}\mathcal{T})$ is already on the list of possibilities we have considered, and that the problem is that we do not yet know how to prove enough about it.

13 Localisation

In order to turn the K–theory of triangulated categories into a powerful tool, one would need to have some theorems about the way $K({}^{?}\mathcal{T})$ changes as $\mathcal{T}$ varies. Note that I have left it vague which particular K–theory one should consider. At this point our ignorance is so profound that we should do the unprejudiced thing and consider all the possibilities. When we know more, we will presumably know which of the simplicial sets can safely be forgotten.

There is one obvious conjecture. Suppose $\mathcal{S}$ is a triangulated category, and suppose that $\mathcal{R} \subset \mathcal{S}$ is a thick subcategory. This means that $\mathcal{R}$ is a full, triangulated subcategory of $\mathcal{S}$, and that if $y \in \mathcal{R}$ decomposes as $y = x \oplus x'$ in the category $\mathcal{S}$, then both x and x' lie in $\mathcal{R}$. That is, $\mathcal{R}$ is closed under the formation in $\mathcal{S}$ of direct summands of its objects. Verdier thesis [93] taught us how to form the quotient category $\mathcal{T} = \mathcal{S}/\mathcal{R}$. We have triangulated functors of triangulated categories

$$\mathcal{R} \longrightarrow \mathcal{S} \longrightarrow \mathcal{T},$$

and the composite $\mathcal{R} \longrightarrow \mathcal{T}$ is naturally isomorphic to the zero map.

Problem 4. Find a suitable K–theory of triangulated categories $K({}^{?}-)$ so that

(i) $K({}^{?}\mathcal{T})$ is a functor of the triangulated category $\mathcal{T}$.
(ii) By (i) we know that the functor $K({}^{?}-)$ yields continuous maps

$$K({}^{?}\mathcal{R}) \longrightarrow K({}^{?}\mathcal{S}) \longrightarrow K({}^{?}\mathcal{T}).$$

The composite $K({}^{?}\mathcal{R}) \longrightarrow K({}^{?}\mathcal{T})$ must be the null map, and there is a natural map from $K({}^{?}\mathcal{R})$ to the homotopy fiber of $K({}^{?}\mathcal{S}) \longrightarrow K({}^{?}\mathcal{T})$. We want this map to be a homotopy equivalence.

Remark 30. The natural candidates for the functor $K({}^{?}-)$ are $K({}^{d}-)$ and $K({}^{v}-)$; what makes them natural is that we know they are functors. Unless we have a functor, the question makes no sense. Without a functor, the maps $\mathcal{R} \longrightarrow \mathcal{S} \longrightarrow \mathcal{T}$ will not, in general, induce maps in K–theory, and it would be meaningless to ask whether the induced sequence is a homotopy fibration. For $K({}^d-)$ and $K({}^v-)$, the problem is concrete enough. We are asking whether one or both of the sequences

$$K({}^d\mathcal{R}) \longrightarrow K({}^d\mathcal{S}) \longrightarrow K({}^d\mathcal{T})$$

$$K({}^v\mathcal{R}) \longrightarrow K({}^v\mathcal{S}) \longrightarrow K({}^v\mathcal{T})$$

is a homotopy fibration.

I spent a long time working on this problem. It goes without saying that I do not know the answer; if I did, I would not keep it secret.

Remark 31. Suppose we succeed in finding a K–theory $K({}^?-)$ of triangulated categories, so that

(i) As in Problem 4, when $\mathcal{T} = \mathcal{S}/\mathcal{R}$ we have a homotopy fibration

$$K({}^?\mathcal{R}) \longrightarrow K({}^?\mathcal{S}) \longrightarrow K({}^?\mathcal{T}).$$

(ii) If $\mathcal{A}$ is the heart of a bounded t–structure on $\mathcal{T}$, then there is a natural isomorphism

$$K(\mathcal{A}) \longrightarrow K({}^?\mathcal{T}).$$

Then Quillen's localisation theorem [77, Theorem 5 of §5] follows easily. Given abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ with $\mathcal{C} = \mathcal{B}/\mathcal{A}$, we have triangulated categories $D^b(\mathcal{C}) = \dfrac{D^b(\mathcal{B})}{D^b_{\mathcal{A}}(\mathcal{B})}$ where $D^b_{\mathcal{A}}(\mathcal{B})$ is the category of bounded chain complexes in $\mathcal{B}$, whose cohomology lies in $\mathcal{A} \subset \mathcal{B}$. Applying (i) and (ii) above to these triangulated categories with the obvious t–structures, Quillen's localisation theorem is immediate.

14 Bounded δ–Functors

Theorem 3(i) is very intriguing. We remind the reader. In this article we constructed maps

$$K(\mathcal{A}) \xrightarrow{\alpha} K({}^w\mathcal{T}) \xrightarrow{\beta} K({}^d\mathcal{T}) \xrightarrow{\gamma} K({}^v\mathcal{T}) \xrightarrow{\delta} K(\mathrm{Gr}^b\mathcal{A}).$$

Theorem 3(i) asserts that the composite $\delta\gamma\beta\alpha : K(\mathcal{A}) \longrightarrow K(\mathrm{Gr}^b\mathcal{A})$ is a homotopy equivalence. What is quite surprising is that this composite is independent of the triangulated category $\mathcal{T}$.

For any abelian category $\mathcal{A}$, Example 5 constructs for us a category with squares $\mathrm{Gr}^b\mathcal{A}$, and we formally have a simplicial set $S_*(\mathrm{Gr}^b\mathcal{A})$. The space $K(\mathrm{Gr}^b\mathcal{A})$ is the loop space of the geometric realisation of $S_*(\mathrm{Gr}^b\mathcal{A})$. Quillen's

K–theory $K(\mathcal{A})$ is the loop space of the geometric realisation of Waldhausen's simplicial set $S_*(\mathcal{A})$. The maps α, β, γ and δ are all the loops on the geometric realisations of explicit simplicial maps. It is not difficult to compute the composite; it amounts to remembering the definitions of the maps α, β, γ and δ. We leave the details to the reader; let us only state the conclusion. In the next paragraphs, we tell the reader what the map $\delta\gamma\beta\alpha$ does to a simplex in Waldhausen's $S_*(\mathcal{A})$.

Suppose $s \in S_n(\mathcal{A})$ is an n–simplex in Waldhausen's simplicial set $S_*(\mathcal{A})$. The simplex s is a sequence of monomorphisms in $\mathcal{A}$

$$0 \longrightarrow x_1 \rightarrowtail x_2 \rightarrowtail \cdots \rightarrowtail x_{n-1} \rightarrowtail x_n$$

together with choices for the cokernels y_i^j of each monomorphism $x_i \longrightarrow x_j$. Recall the region $\mathcal{D}_n \subset \mathcal{R}_n$ of Remark 10. The simplex $s \in S_n(\mathcal{A})$ is a functor

$$\mathcal{D}_n \longrightarrow \mathcal{A} \subset \mathrm{Gr}^b\mathcal{A}.$$

To make it into an augmented diagram for the pair $(\mathcal{D}_n, \mathrm{Gr}^b\mathcal{A})$ we only need to choose the coherent differentials; we choose them all to be zero. The region $\mathcal{D}_n \subset \mathcal{R}_n$ is a fundamental domain for augmented diagrams on $\mathcal{R}_n$. Any augmented diagram on $\mathcal{R}_n$ is uniquely determined by its restriction to $\mathcal{D}_n$. Furthermore, by the last paragraph of Remark 12, for the category with squares $\mathrm{Gr}^b\mathcal{A})$ there is no extension problem; our augmented diagram on $\mathcal{D}_n$ extends (uniquely) to an augmented diagram on $\mathcal{R}_n$. The simplicial map $\delta\gamma\beta\alpha : S_*(\mathcal{A}) \longrightarrow S_*(\mathrm{Gr}^b\mathcal{A})$ takes $s \in S_n(\mathcal{A})$ to this augmented diagram for the pair $(\mathcal{R}_n, \mathrm{Gr}^b\mathcal{A})$.

The next step is to generalise this to arbitrary δ–functors. We should begin by reminding the reader what a δ–functor is. I will only give a sketch here; much more detail may be found in Grothendieck [35]. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A δ–functor $f^* : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor taking short exact sequences in $\mathcal{A}$ to long exact sequences in $\mathcal{B}$. More precisely

Definition 12. *A δ–functor $f^* : \mathcal{A} \longrightarrow \mathcal{B}$ is*

(i) *For each integer $i \in \mathbb{Z}$, an additive functor $f^i : \mathcal{A} \longrightarrow \mathcal{B}$.*
(ii) *For each integer $i \in \mathbb{Z}$ and each short exact sequence in $\mathcal{A}$*

$$0 \longrightarrow a' \longrightarrow a \longrightarrow a'' \longrightarrow 0,$$

a map $\partial : f^i(a'') \longrightarrow f^{i+1}(a')$.
(iii) *The maps ∂ are natural in the short exact sequences. Given an integer $i \in \mathbb{Z}$ and a map of short exact sequences in $\mathcal{A}$*

$$\begin{tikzcd} 0 \arrow[r] & a' \arrow[r] \arrow[d, "\alpha'"] & a \arrow[r] \arrow[d, "\alpha"] & a'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\ 0 \arrow[r] & b' \arrow[r] & b \arrow[r] & b'' \arrow[r] & 0 \end{tikzcd}$$

there is a commutative square

$$\begin{tikzcd} f^i(a'') \arrow[r, "\partial"] \arrow[d, "f^i(\alpha'')"'] & f^{i+1}(a') \arrow[d, "f^{i+1}(\alpha')"] \\ f^i(b'') \arrow[r, "\partial"'] & f^{i+1}(b') \end{tikzcd}$$

(iv) *Every short exact sequence in* $\mathcal{A}$

$$0 \longrightarrow a' \longrightarrow a \longrightarrow a'' \longrightarrow 0$$

goes to a long exact sequence in $\mathcal{B}$

$$\cdots \longrightarrow f^{i-1}(a'') \xrightarrow{\partial} f^i(a') \longrightarrow f^i(a) \longrightarrow f^i(a'') \xrightarrow{\partial} f^{i+1}(a') \longrightarrow \cdots$$

A δ*–functor* $f^* : \mathcal{A} \longrightarrow \mathcal{B}$ *is called* bounded *if, for every object* $a \in \mathcal{A}$, $f^i(a)$ *vanishes for all but finitely many* $i \in \mathbb{Z}$.

Now that we have recalled the definition, we make the observation

Lemma 5. *Let* $f^* : \mathcal{A} \longrightarrow \mathcal{B}$ *be a bounded* δ*–functor. Define a functor, which by abuse of notation we will write as*

$$f^* : \mathcal{A} \longrightarrow \mathrm{Gr}^b(\mathcal{B}).$$

It is the functor taking $a \in \mathcal{A}$ *to the sequence* $\{f^i(a) \mid i \in \mathbb{Z}\}$*. Given a bicartesian square in* $\mathcal{A}$

$$\begin{tikzcd} b' \arrow[r, "\delta"] & c \\ a \arrow[u, "\beta"] \arrow[r, "\alpha"'] & b \arrow[u, "\gamma"'] \end{tikzcd}$$

we assert that the functor $f^* : \mathcal{A} \longrightarrow \mathrm{Gr}^b(\mathcal{B})$ *takes it to a special square in* $\mathrm{Gr}^b(\mathcal{B})$*. Furthermore, if we are given a diagram of bicartesian squares in* $\mathcal{A}$

$$\begin{tikzcd} d''' \arrow[r] & e'' \arrow[r] & f' \arrow[r] & g \\ c'' \arrow[u] \arrow[r] & d'' \arrow[u] \arrow[r] & e' \arrow[u] \arrow[r] & f \arrow[u] \\ b' \arrow[u] \arrow[r] & c' \arrow[u] \arrow[r] & d' \arrow[u] \arrow[r] & e \arrow[u] \\ a \arrow[u] \arrow[r] & b \arrow[u] \arrow[r] & c \arrow[u] \arrow[r] & d \arrow[u] \end{tikzcd}$$

we deduce two bicartesian squares, one contained in the other

$$\begin{tikzcd} d'' \arrow[r] & e' \\ c' \arrow[u] \arrow[r] & d' \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} d''' \arrow[r] & g \\ a \arrow[u] \arrow[r] & d \arrow[u] \end{tikzcd}$$

The functor f^* takes these to two special squares, with differentials

$$\partial_1 : f^*(e') \longrightarrow \Sigma f^*(c')$$

$$\partial_2 : f^*(g) \longrightarrow \Sigma f^*(a).$$

These differentials are compatible; that is, ∂_1 is the composite

$$f^*(e') \longrightarrow f^*(g) \xrightarrow{\partial_2} \Sigma f^*(a) \longrightarrow \Sigma f^*(c').$$

Proof. The commutative square

$$\begin{tikzcd} b' \arrow[r, "\delta"] & c \\ a \arrow[u, "\beta"] \arrow[r, "\alpha"'] & b \arrow[u, "\gamma"'] \end{tikzcd}$$

is bicartesian, and by Definition 10 this means that

$$0 \longrightarrow a \xrightarrow{\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}} b \oplus b' \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} c \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$. But then the δ functor f^* gives us a map $\partial : f^*(c) \longrightarrow \Sigma f^*(a)$, so that

$$f^*(a) \xrightarrow{\begin{pmatrix} f^*(\alpha) \\ -f^*(\beta) \end{pmatrix}} f^*(b) \oplus f^*(b') \xrightarrow{\begin{pmatrix} f^*(\gamma) & f^*(\delta) \end{pmatrix}} f^*(c) \xrightarrow{\partial} \Sigma f^*(a)$$

is a long exact sequence. In other words, the differential $\partial : f^*(c) \longrightarrow \Sigma f^*(a)$ together with the commutative square

$$\begin{tikzcd} f^*(b') \arrow[r, "f^*(\delta)"] & f^*(c) \\ f^*(a) \arrow[u, "f^*(\beta)"] \arrow[r, "f^*(\alpha)"'] & f^*(b) \arrow[u, "f^*(\gamma)"'] \end{tikzcd}$$

give us a special square in $\mathrm{Gr}^b(\mathcal{B})$. It remains to establish the coherence of the differentials.

Assume therefore that we are given a diagram of bicartesian squares in $\mathcal{A}$

$$\begin{tikzcd}
d''' \arrow[r] & e'' \arrow[r] & f' \arrow[r] & g \\
c'' \arrow[r] \arrow[u] & d'' \arrow[r] \arrow[u] & e' \arrow[r] \arrow[u] & f \arrow[u] \\
b' \arrow[r] \arrow[u] & c' \arrow[r] \arrow[u] & d' \arrow[r] \arrow[u] & e \arrow[u] \\
a \arrow[r] \arrow[u] & b \arrow[r] \arrow[u] & c \arrow[r] \arrow[u] & d \arrow[u]
\end{tikzcd}$$

We deduce maps of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & a \arrow[r] \arrow[d] & d \oplus d''' \arrow[r] \arrow[d] & g \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c' \arrow[r] \arrow[d, equal] & e \oplus e'' \arrow[r] & g \arrow[r] & 0 \\
0 \arrow[r] & c' \arrow[r] & d' \oplus d'' \arrow[r] \arrow[u] & e' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The fact that f^* is a δ–functor gives us commutative squares

$$\begin{tikzcd}
f^*(g) \arrow[r, "\partial_2"] \arrow[d, equal] & \Sigma f^*(a) \arrow[d] \\
f^*(g) \arrow[r, "\partial_3"] & \Sigma f^*(c') \arrow[d, equal] \\
f^*(e') \arrow[r, "\partial_1"] \arrow[u] & \Sigma f^*(c')
\end{tikzcd}$$

from which the coherence for the differentials immediately follows.

Proposition 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Then any δ–functor $f^* : \mathcal{A} \longrightarrow \mathcal{B}$ induces a simplicial map of simplicial sets*

$$S_*(\mathcal{A}) \longrightarrow S_*(\mathrm{Gr}^b\mathcal{B}).$$

Proof. As at the beginning of this section, a simplex $s \in S_n(\mathcal{A})$ is a sequence of monomorphisms in $\mathcal{A}$

$$0 \longrightarrow x_1 \rightarrowtail x_2 \rightarrowtail \cdots \rightarrowtail x_{n-1} \rightarrowtail x_n$$

together with choices for the cokernels y_i^j of each monomorphism $x_i \longrightarrow x_j$. Applying f^* to the diagram as in Lemma 5, we deduce an augmented diagram for the pair $(\mathcal{D}_n, \mathrm{Gr}^b\mathcal{B})$. The region $\mathcal{D}_n \subset \mathcal{R}_n$ is a fundamental domain for $\mathcal{R}_n$ by Remark 10, and there is no extension problem by the last paragraph of Remark 12. Hence the diagram extends uniquely to an augmented diagram for the pair $(\mathcal{R}_n, \mathrm{Gr}^b\mathcal{B})$, vanishing on the boundary. That is, we have a simplex in $S_n(\mathrm{Gr}^b\mathcal{B})$. This defines the simplicial map.

Remark 32. The identity functor $1 : \mathcal{A} \longrightarrow \mathcal{A}$ can always be viewed as a δ–functor. That is, we define a δ–functor $i^* : \mathcal{A} \longrightarrow \mathcal{A}$ by putting $i^0 = 1$, and $i^j = 0$ if $j \neq 0$. The differential ∂ is zero for every short exact sequence in $\mathcal{A}$.

In terms of Proposition 2, the computation at the beginning of the section says that the map $\delta\gamma\beta\alpha : S_*(\mathcal{A}) \longrightarrow S_*(\mathrm{Gr}^b\mathcal{A})$ is nothing other than the map induced by the the trivial δ–functor i^*. That is,

$$\delta\gamma\beta\alpha = i^* : S_*(\mathcal{A}) \longrightarrow S_*(\mathrm{Gr}^b\mathcal{A}).$$

Theorem 3(i) asserts that i^* induces a homotopy equivalence.

Given any δ–functor $f^* : \mathcal{A} \longrightarrow \mathcal{B}$, we can now define an induced map $K(f^*) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$. Consider the diagram

$$\begin{tikzcd} S_*(\mathcal{A}) \arrow[dr, "f^*"] & & S_*(\mathcal{B}) \arrow[dl, "i^*"'] \\ & S_*(\mathrm{Gr}^b\mathcal{B}) & \end{tikzcd}$$

If we pass to loop spaces of geometric realisations, we have a diagram

$$\begin{tikzcd} K(\mathcal{A}) \arrow[dr, "K(f^*)"] & & K(\mathcal{B}) \arrow[dl, "K(i^*)"'] \\ & K(\mathrm{Gr}^b\mathcal{B}) & \end{tikzcd}$$

and the map $K(i^*)$ is a homotopy equivalence. The map induced by f^* is simply

$$K(i^*)^{-1}K(f^*) : K(\mathcal{A}) \longrightarrow K(\mathcal{B}).$$

What I find so puzzling about this theorem is

Problem 5. What happens to the composite of two δ–functors? Suppose we have three abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and two δ–functors

$$\mathcal{A} \xrightarrow{f^*} \mathcal{B} \xrightarrow{g^*} \mathcal{C}.$$

The above tells us how to construct maps in K–theory

$$K(\mathcal{A}) \xrightarrow{K(f^*)} K(\mathcal{B}) \xrightarrow{K(g^*)} K(\mathcal{C}).$$

There is a composite map $K(g^*)K(f^*) : K(\mathcal{A}) \longrightarrow K(\mathcal{C})$. What is the homological algebra data inducing it?

Presumably the composite map $K(g^*)K(f^*)$ must be induced by the composite g^*f^*. But what is the composite of two δ–functors? A δ–functor is a strange beast, taking short exact sequences in $\mathcal{A}$ to long exact sequences in $\mathcal{B}$. What does the composite of two such things do? Does it take short exact sequences in $\mathcal{A}$ to spectral sequences in $\mathcal{B}$? If so, how?

It would already be interesting if someone could formulate a plausible conjecture for Problem 5.

15 Devissage

There are two theorems about the K–theory of abelian categories which are formally very similar. They are Quillen's resolution theorem [77, Theorem 3 and Corollary 1 of §4] and Quillen's devissage theorem [77, Theorem 4 of §5]. Let me remind the reader.

Theorem 4. *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful, exact embedding of exact categories. If either* (i) *or* (ii) *below holds, then the induced map*

$$K(f) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$$

is a homotopy equivalence. It remains to tell the reader what are the hypotheses (i) *and* (ii).

(i) Resolution: *Whenever we have an exact sequence*

$$0 \longrightarrow b' \longrightarrow b \longrightarrow b'' \longrightarrow 0$$

in $\mathcal{B}$, then

$$\{b, b'' \in \mathcal{A}\} \Longrightarrow b' \in \mathcal{A} \qquad \text{and} \qquad \{b', b'' \in \mathcal{A}\} \Longrightarrow b \in \mathcal{A}$$

Furthermore, every object $y \in \mathcal{B}$ admits a resolution

$$0 \longrightarrow x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_1 \longrightarrow x_0 \longrightarrow y \longrightarrow 0,$$

with all the x_i's in $\mathcal{A}$.

(ii) Devissage: *The categories $\mathcal{A}$ and $\mathcal{B}$ are both abelian. Furthermore, every object $y \in \mathcal{B}$ admits a filtration*

$$0 = x_n \subset x_{n-1} \subset \cdots \subset x_1 \subset x_0 = y,$$

with all the intermediate quotients x_i/x_{i+1} in $\mathcal{A}$.

We can wonder what these theorems mean in the K–theory of triangulated categories. For resolution, the following well-known lemma is suggestive.

Lemma 6. *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful, exact embedding of exact categories. Suppose the resolution hypothesis holds. Then the natural map*

$$D^b(f) : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$$

is an equivalence of categories.

Remark 33. In most of this article I have avoided all mention of exact categories, focusing instead on the special case of abelian categories. This is mostly because we know much more about the K–theory of derived categories of abelian categories. For the resolution theorem, it would be a mistake to try to state it only for abelian categories. The reason is simple. If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a fully faithful, exact embedding of abelian categories, and if $D^b(f) : D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$ is an equivalence of categories, then one can easily show that f must be an equivalence of categories. For an embedding of abelian categories $f : \mathcal{A} \longrightarrow \mathcal{B}$, Quillen's resolution theorem is content-free.

As we mentioned in Remark 33 our main result, Theorem 3, is about abelian categories. Therefore Quillen's resolution theorem does not formally follow. But morally we have been learning that K–theory depends only on the derived category. In the light of Lemma 6, Quillen's resolution theorem is hardly surprising.

The devissage theorem, by contrast, has always been very puzzling. Since the statement is so similar to the resolution theorem, one has to wonder whether the two have a common generalisation. Let me try to propose one. In both cases, the theorem asserts that an inclusion $\mathcal{A} \subset \mathcal{B}$ induces a homotopy equivalence in K–theory. Let us, for simplicity, look at resolutions and filtrations of length 1. Conditions (i) and (ii), of the resolution and devissage theorems in the special case of length 1 resolutions and filtrations, are

(i) *Resolution:* Every object $y \in \mathcal{B}$ admits an exact sequence

$$0 \longrightarrow x \longrightarrow x' \longrightarrow y \longrightarrow 0$$

with x, x' in $\mathcal{A}$.

(ii) *Devissage:* Every object $y \in \mathcal{B}$ admits an exact sequence

$$0 \longrightarrow x \longrightarrow y \longrightarrow x' \longrightarrow 0$$

with x, x' in $\mathcal{A}$.

The point I want to make is that, in the derived category, these become indistinguishable. In other words, if the inclusion $\mathcal{A} \subset \mathcal{B}$ satisfies the hypothesis of devissage, then the natural map

$$D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B})$$

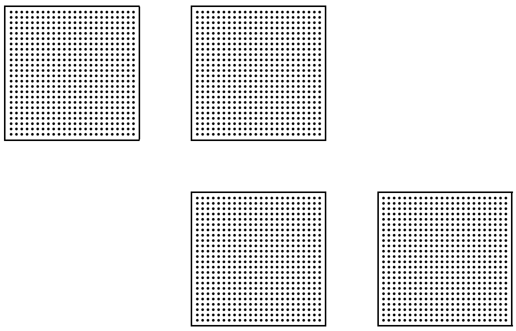
should satisfy a something analogous to the hypothesis of resolution. And morally resolution is the statement that K–theory of $\mathcal{A}$ is really a functor of $D^b(\mathcal{A})$.

This leads one to expect that there should be some construction, which we will call the derived category of a triangulated category. In fact, categories ought to be infinitely differentiable. Given a category $\mathcal{T}$, it should be possible to define its derived category $D^b(\mathcal{T})$, and this category should have a K–theory isomorphic to the K–theory of $\mathcal{T}$. Devissage is presumably the statement that the K–theory of an abelian category depends only on the derived category of its derived category.

Since this problem is so ill-posed, let me not try to say much more. The major thrust of the results in Theorem 3 is that K–theory is an invariant that captures relatively little of the homological structure we have been using. Perhaps the clearest evidence for this is the fact that even a δ–functor is enough to induce a map in higher K–theory; see Section 14. So perhaps the problem I am trying to pose in this section is: Find the right homological algebra gadget, which comes closer to being completely detected by K–theory.

16 About the Proofs

There are several ideas that come into the proofs which, as I have already said, are long and very difficult. One way to explain the strategy is the following. To define the K–theory of a triangulated category, we looked at the cosimplicial region $\mathcal{R}_n$ of Section 6. It turns out that there are many other cosimplicial regions. For example, we can look at regions in $\mathbb{Z} \times \mathbb{Z}$ which look like



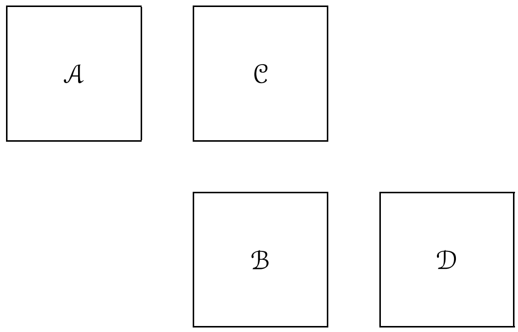
It turns out to be very easy to make this into a cosimplicial region. That is, there is a straight-forward way of finding a functor

$$\Theta : \Delta \longrightarrow \{\text{Regions in } \mathbb{Z} \times \mathbb{Z}\}$$

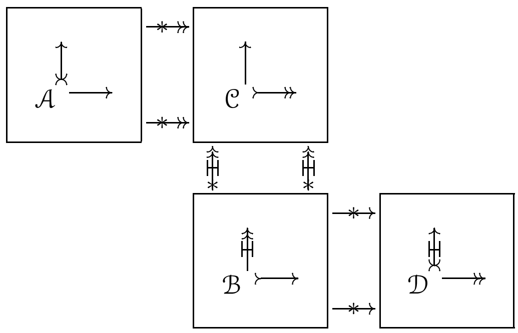
which takes an object $\mathbf{n} \in \Delta$ to a region with the indicated shape. Let $\mathcal{T}$ be a category with squares. As in Section 7, we can take the functor sending $\mathbf{n} \in \Delta$ to augmented diagrams for the pair $(\Theta(\mathbf{n}), \mathcal{T})$. This is a functor $\Delta^{\mathrm{op}} \longrightarrow$

{Sets}, that is a simplicial set. The idea is to study many such simplicial sets, for many choices of cosimplicial regions.

In fact, we can produce many variants. Our region is the disjoint union of four subregions, which I have drawn well separated from each other. One way to produce variants is by imposing different restrictions on each subregion. If we have four subcategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{T}$, we can look at the simplicial subset



This just means that the augmented diagram takes the indicated subregions to the prescribed subcategories. We can also place restrictions on the horizontal and vertical morphisms in each subregion, and on the morphisms connecting the subregions:



The reader will notice that the papers containing the proofs have many such simplicial sets and simplicial maps among them. At some level, the proofs amount to a combinatorial manipulation of the many possible simplicial sets that arise this way. Each of the main steps in the proofs shows that two regions, with all the adornment indicated above, give rise to homotopy equivalent simplicial sets.

This raises the obvious problem:

Problem 6. Are there more conceptual, less combinatorial proofs? Is it possible to give easier proofs of the main theorems?

For what it is worth, let me quote what Thomason had to say about this. When he came to believe that I really had a proof of the theorems I was claiming, his comment was: "There *has* to be a better proof." What I have tried to explain in this manuscript is that, before looking for the optimal proof, perhaps we should search for improved theorems. Thomason was undoubtedly right about the existence of a better proof. All I wish to add to Thomason's remark is: "There *has* to be a better theorem".

17 Appendix: Examples of $D(\mathcal{A}) = D(\mathcal{B})$

In this appendix we outline the many examples now known, of pairs of abelian categories $\mathcal{A}$ and $\mathcal{B}$ with $D(\mathcal{A}) = D(\mathcal{B})$. Let me thank Bernhard Keller and Idun Reiten for much help with this appendix. However, all responsibility for mistakes rests with me.

The overwhelming majority of known examples fall into three types.

(i) Both $\mathcal{A}$ and $\mathcal{B}$ are categories of modules, for different rings R and S.
(ii) $\mathcal{A}$ is a category of modules over some ring R, and $\mathcal{B}$ is the category of (quasi)-coherent sheaves on some projective variety (or a noncommutative analog of a projective variety).
(iii) Both $\mathcal{A}$ and $\mathcal{B}$ are categories of (quasi)-coherent sheaves, on some projective varieties X and Y.

The first example was probably Beilinson's 1978 article [11]. Beilinson produces three abelian categories with $D(\mathcal{A}) = D(\mathcal{B}) = D(\mathcal{C})$. In the example, $\mathcal{A}$ is the category of coherent sheaves on $\mathbb{P}^n$ (the n–dimensional projective space). For $\mathcal{B}$ and $\mathcal{C}$ Beilinson produced two rings R and S, and $\mathcal{B}$ and $\mathcal{C}$ are the categories of finite modules over R and S, respectively. Since the module categories for the rings R and S are not equivalent, Beilinson's example is simultaneously of types (i) and (ii).

The first example of type (iii) seems to be in Mukai's 1981 article [62]. In Mukai's example, $\mathcal{A}$ and $\mathcal{B}$ are the categories of coherent sheaves on an abelian variety X and on its dual $\widehat{X}$, respectively.

Both Beilinson's and Mukai's example have been infinitely generalised and extended since. Let us first discuss type (iii). Let $\mathcal{A}$ and $\mathcal{B}$ be the abelian categories of coherent sheaves on smooth, projective varieties X and Y. Orlov's paper [74] gives a characterisation of all the equivalences $D(\mathcal{A}) = D(\mathcal{B})$. Bondal and Orlov [15] show that if the canonical bundle on X is ample or its negative is ample, then $D(\mathcal{A}) = D(\mathcal{B})$ implies $X = Y$. Kawamata shows [49] that if X is of general type or if the Kodaira dimension of $-K_X$ is the dimension of X, then $D(\mathcal{A}) = D(\mathcal{B})$ implies that X and Y are birational. Non-birational examples (where the Kodaira dimension is restricted by the above) may be found first of all in Mukai's original papers [62, 63], but more recently also in Bridgeland [20, 21], Bridgeland and Maciocia [24, 25], Orlov [74, 75] and Polishchuk [76]. But in some sense the case where X and Y are birational is most interesting, since it seems to be closely related to the minimal models program. It is conjectured that whenever X and Y are related by a sequence of flops then the derived categories should be the same. The first paper to prove such a theorem, for certain smooth flops, was Bondal and Orlov [17]. A particularly nice treatment for general smooth 3–fold flops, in terms of a certain moduli problem, may be found in Bridgeland [22]. For 3–fold flops with only terminal Gorenstein singularities this was done by Chen [30], and for flops with only quotient singularities by Kawamata [50]. One of the problems with more general singularities is that it is not quite clear what the precise statement should be. That is, just exactly which derived category of sheaves is right. The reader can find a brief discussion of the conjectured relationship between derived categories and birational geometry in Reid [78, §3.6]. Another algebro-geometric example of $D(\mathcal{A}) = D(\mathcal{B})$ comes from the McKay correspondence in Bridgeland, King and Reid [23]. It is slightly different from the above in that $\mathcal{A}$ is not just the abelian category of sheaves on some variety, but rather the category of sheaves with some compatible group action. The reader can find a much more thorough survey of all the algebro-geometric examples in Bondal and Orlov [16].

Next we mention more examples of type (ii). That is, $\mathcal{A}$ is a module category and $\mathcal{B}$ is a category of sheaves on some projective variety, and $D(\mathcal{A}) = D(\mathcal{B})$. The general case, of how such equivalences come about, was studied by Dagmar Baer [7] and by Alexei Bondal [14]. Baer applied it to coherent sheaves on weighted projective lines. Then the algebras R are *Ringel's canonical algebras*. Bondal studied braid group actions on the collection of exceptional sequences in $D(\mathcal{B})$. Kapranov generalised Beilinson's example to other homogenous spaces; see [45, 46, 47, 48]. In the realm of non–commutative algebraic geometry, see LeBruyn's [56] work on Weyl algebras, which was extended by Berest-Wilson [9] (note the appendix by Michel Van den Bergh). Kapranov-Vasserot's McKay equivalence [44] is also almost of type (ii).

The richest collection of known examples are the ones of type (i). It is probably fair to say that the subject began with Happel's Habilitationsschrift [36, 37]. Happel observes that, if (R, T, S) is a tilting triple (that is, R and S are rings and T is an $R-S$–bimodule satisfying certain conditions), then there is an equivalence of categories $D(\mathcal{A}) = D(\mathcal{B})$. Here $\mathcal{A}$ and $\mathcal{B}$ are, respectively, the categories of R– and of S–modules.

Remark 34. We should make a historical note here. Tilting triples predate Happel's work. One of Happel's key contributions was to observe that they naturally give rise to equivalences of the form $D(\mathcal{A}) = D(\mathcal{B})$. For historical completeness we note

– Important precursors of tilting triples may be found in Gelfand-Ponomarev [33, 34], Bernstein-Gelfand-Ponomarev [10], Auslander-Platzeck-Reiten [6] and Marmaridis [60]. One should note that Street independently developed similar ideas in his (unpublished) 1968 PhD thesis. See also his article [88].
– The people (before Happel) who gave tilting theory its modern form: Brenner-Butler [19], who first proved the 'tilting theorem', Happel-Ringel

[40], who improved the theorem and defined tilted algebras, Bongartz [18], who streamlined the theory, and Miyashita [61], who generalized it to tilting modules of projective dimension > 1.

Remark 35. Happel found that the existence of a tilting module was sufficient to give an equivalence $D(\mathcal{A}) = D(\mathcal{B})$. A necessary and sufficient condition appeared soon after in Rickard's work [80].

Remark 36. For a concise introduction to tilting theory and its link with derived equivalences the reader is referred to Keller [53]. There is also Chapter XII in Gabriel-Roiter's book [31], the lecture notes edited by König-Zimmermann [54] and Assem's introduction [3].

There is a long list of applications of tilting theory (that is, of examples of rings R and S with $D(R) = D(S)$). If R is a hereditary algebra, the reader is referred to Happel-Rickard-Schofield [39] for a general theorem about the possible S's. For certain specific R's (precisely, for R the algebra of a quiver of Dynkin type) there is a complete classifications of all possible S's. For type A, this is in Keller-Vossieck [51] and Assem-Happel [4]. For type D see Keller [52]. Type $\tilde{A}$ may be found in Assem-Skowronski [5], while types B and C are in Assem [2].

More examples of algebras R, for which all S's with $D(R) = D(S)$ have been classified, are the Brauer tree algebras treated by Rickard [79], the representation-finite selfinjective algebras of Asashiba [1], the discrete algebras introduced by Dieter Vossieck [94], Brüstle's derived tame tree algebras in [29] (the main theorem was independently obtained by Geiss [32]), or Bocian-Holm-Skowronski's weakly symmetric algebras of Euclidean type [12] (the preprint is available at Thorsten Holm's homepage).

Another large source of examples comes from Broué's abelian defect group conjecture. Let me state the conjecture:

Conjecture 1. Let p be a prime, let O be a complete discrete valuation ring of characteristic zero with residue field k of characteristic p. Suppose that O and k are large enough.

Let G be a finite group, let R be a block algebra of the group algebra OG that has an abelian defect group D, and let S be the Brauer correspondent of R. We remind the reader that S is a block algebra of $ON_G(D)$, the group algebra of the normalizer of D in G. In any case R and S are rings. Their module categories will be $\mathcal{A}$ and $\mathcal{B}$.

Then Broué conjectured, in his 1990 paper [27], that there is an equivalence $D(\mathcal{A}) = D(\mathcal{B})$.

Remark 37. It might be helpful to give the reader a special case, which is already very interesting. Suppose k is an algebraically closed field of characteristic $p > 0$. Let G be a finite group, P a p–Sylow subgroup of G. Assume P is abelian. Let $N_G(P)$ be the normaliser of P in G. Let R and S be the principal blocks of kG and $kN_G(P)$, respectively. It follows from Conjecture 1 that the derived categories of R and S are equivalent.

For more on the conjecture see Broué [27, 26, 28], König–Zimmermann [54] and Rickard [81, 82]. For us the relevance is that the cases where the conjecture has been verified give equivalences $D(\mathcal{A}) = D(\mathcal{B})$. In the cases where $\mathcal{A}$ and $\mathcal{B}$ are not equivalent (and there are many of these), this gives examples of type (i).

A list of three of the large classes of known examples so far is:

1. All blocks with cyclic defect groups. See Rickard [79], Linckelmann [59] and Rouquier [83, 84].
2. All blocks of symmetric groups with abelian defect groups of order at most p^5. (Preprint by Chuang and Kessar).
3. The non-principal block with full defect of $SL_2(p^2)$ in characteristic p. The defect group is $C_p \times C_p$. (Preprint by Holloway).

A much more complete and up-to-date list may be found on Jeremy Rickard's home page, at

`http://www.maths.bris.ac.uk/~majcr/adgc/which.html`

Remark 38. It is perhaps worth noting that the original evidence, which led Broué to formulate his conjecture, was obtained by counting characters. In other words, the evidence was mostly K_0 computations.

We should say a little bit about examples not of the three types (i), (ii) and (iii). The first to find a technique to produce such examples were Happel, Reiten and Smalø [38]. For a different approach see Schneiders [86]. A discussion of both approaches, the relation between them and improvements to the theorem may be found in Bondal and van den Bergh [13, Section 5.4 and Appendix B].

In all of the above I have said nothing about the uniqueness of an equivalence $D(\mathcal{A}) \simeq D(\mathcal{B})$. Any such equivalence is unique up to an automorphism of $D(\mathcal{A})$. If X is a Calabi–Yau manifold and $\mathcal{A}$ the category of coherent sheaves on it, then $D(\mathcal{A})$ is expected to have a large automorphism group, and this is expected to be related to the mirror partner of X. The reader can find more about this in Kontsevich [55] or Seidel and Thomas [87]. There has been some beautiful work on this, but our survey must end at some point.

References

1. Hideto Asashiba, *The derived equivalence classification of representation-finite selfinjective algebras*, J. Algebra **214** (1999), no. 1, 182–221.
2. Ibrahim Assem, *Iterated tilted algebras of types B_n and C_n*, J. Algebra **84** (1983), no. 2, 361–390.

3. ———, *Tilting theory—an introduction*, Topics in algebra, Part 1 (Warsaw, 1988), Banach Center Publ., vol. 26, PWN, Warsaw, 1990, pp. 127–180.
4. Ibrahim Assem and Dieter Happel, *Generalized tilted algebras of type A_n*, Comm. Algebra **9** (1981), no. 20, 2101–2125.
5. Ibrahim Assem and Andrzej Skowroński, *Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$*, Math. Z. **195** (1987), no. 2, 269–290.
6. Maurice Auslander, María Inés Platzeck, and Idun Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc. **250** (1979), 1–46.
7. Dagmar Baer, *Tilting sheaves in representation theory of algebras*, Manuscripta Math. **60** (1988), no. 3, 323–347.
8. A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Soc. Math. France, Paris, 1982, pp. 5–171.
9. Yuri Berest and George Wilson, *Ideal classes of the Weyl algebra and noncommutative projective geometry*, Int. Math. Res. Not. (2002), no. 26, 1347–1396, With an appendix by Michel Van den Bergh.
10. I. N. Bernšteĭ n, I. M. Gel′fand, and V. A. Ponomarev, *Coxeter functors, and Gabriel's theorem*, Uspehi Mat. Nauk **28** (1973), no. 2(170), 19–33.
11. A. A. Beĭ linson, *Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69.
12. R. Bocian, T. Holm, and A. Skowroński, *Derived equivalence classification of weakly symmetric algebras of euclidean type*, preprint, 2003, 32 pages.
13. A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.
14. A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 25–44.
15. Alexei Bondal and Dmitri Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), no. 3, 327–344.
16. ———, *Derived categories of coherent sheaves*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 47–56.
17. ———, *Semiorthogonal decompositions for algebraic varieties*, math.AG/9506012.
18. Klaus Bongartz, *Tilted algebras*, Representations of algebras (Puebla, 1980), Lecture Notes in Math., vol. 903, Springer, Berlin, 1981, pp. 26–38.
19. Sheila Brenner and M. C. R. Butler, *Generalizations of the Bernstein-Gel′fand-Ponomarev reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 103–169.
20. Tom Bridgeland, *Fourier-Mukai transforms for elliptic surfaces*, J. Reine Angew. Math. **498** (1998), 115–133.
21. ———, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. **31** (1999), no. 1, 25–34.
22. ———, *Flops and derived categories*, Invent. Math. **147** (2002), no. 3, 613–632.
23. Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554 (electronic).

24. Tom Bridgeland and Antony Maciocia, *Complex surfaces with equivalent derived categories*, Math. Z. **236** (2001), no. 4, 677–697.
25. ———, *Fourier-Mukai transforms for $K3$ and elliptic fibrations*, J. Algebraic Geom. **11** (2002), no. 4, 629–657.
26. M. Broué, *Rickard equivalences and block theory*, Groups '93 Galway/St. Andrews, Vol. 1 (Galway, 1993), London Math. Soc. Lecture Note Ser., vol. 211, Cambridge Univ. Press, Cambridge, 1995, pp. 58–79.
27. Michel Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque (1990), no. 181-182, 61–92.
28. ———, *Equivalences of blocks of group algebras*, Finite-dimensional algebras and related topics (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, pp. 1–26.
29. Thomas Brüstle, *Derived-tame tree algebras*, Compositio Math. **129** (2001), no. 3, 301–323.
30. Jiun-Cheng Chen, *Flops and equivalences of derived categories for threefolds with only terminal Gorenstein singularities*, J. Differential Geom. **61** (2002), no. 2, 227–261.
31. P. Gabriel and A. V. Roĭ ter, *Representations of finite-dimensional algebras*, Algebra, VIII, Encyclopaedia Math. Sci., vol. 73, Springer, Berlin, 1992, With a chapter by B. Keller, pp. 1–177.
32. Ch. Geiß, *Derived tame algebras and Euler-forms*, Math. Z. **239** (2002), no. 4, 829–862, With an appendix by the author and B. Keller.
33. I. M. Gel′fand and V. A. Ponomarev, *Quadruples of subspaces of finite-dimensional vector space*, Dokl. Akad. Nauk SSSR **197** (1971), 762–765.
34. ———, *Problems of linear algebra and classification of quadruples of subspaces in a finite-dimensional vector space*, Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), North-Holland, Amsterdam, 1972, pp. 163–237. Colloq. Math. Soc. János Bolyai, 5.
35. Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.
36. Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339–389.
37. ———, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.
38. Dieter Happel, Idun Reiten, and Sverre O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575, viii+ 88.
39. Dieter Happel, Jeremy Rickard, and Aidan Schofield, *Piecewise hereditary algebras*, Bull. London Math. Soc. **20** (1988), no. 1, 23–28.
40. Dieter Happel and Claus Michael Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443.
41. Robin Hartshorne, *Residues and duality*, Springer-Verlag, Berlin, 1966.
42. V. A. Hinich and V. V. Schechtman, *Geometry of a category of complexes and algebraic K-theory*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 2, 83–84.
43. ———, *Geometry of a category of complexes and algebraic K-theory*, Duke Math. J. **52** (1985), no. 2, 399–430.
44. M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. **316** (2000), no. 3, 565–576.
45. M. M. Kapranov, *The derived category of coherent sheaves on Grassmann varieties*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 78–79.

46. ______, *Derived category of coherent sheaves on Grassmann manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 1, 192–202.
47. ______, *Derived category of coherent bundles on a quadric*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 2, 67.
48. ______, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), no. 3, 479–508.
49. Yujiro Kawamata, *D-equivalence and K-equivalence*, J. Differential Geom. **61** (2002), no. 1, 147–171.
50. ______, *Francia's flip and derived categories*, Algebraic geometry, de Gruyter, Berlin, 2002, pp. 197–215.
51. B. Keller and D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253, Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987).
52. Bernhard Keller, *Algèbres héréditaires par morceaux de type* $\mathbf{D}_n$, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 7, 483–486.
53. ______, *Basculement et homologie cyclique*, Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), Sémin. Congr., vol. 2, Soc. Math. France, Paris, 1997, pp. 13–33.
54. Steffen König and Alexander Zimmermann, *Derived equivalences for group rings*, Lecture Notes in Mathematics, vol. 1685, Springer-Verlag, Berlin, 1998, With contributions by Bernhard Keller, Markus Linckelmann, Jeremy Rickard and Raphaël Rouquier.
55. Maxim Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.
56. Lieven Le Bruyn, *Moduli spaces for right ideals of the Weyl algebra*, J. Algebra **172** (1995), no. 1, 32–48.
57. Marc Levine, *Erratum to: "Localization on singular varieties"*, Invent. Math. **93** (1988), no. 3, 715–716.
58. ______, *Localization on singular varieties*, Invent. Math. **91** (1988), no. 3, 423–464.
59. Markus Linckelmann, *Derived equivalence for cyclic blocks over a P-adic ring*, Math. Z. **207** (1991), no. 2, 293–304.
60. Nikolaos Marmaridis, *Reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 382–395.
61. Yoichi Miyashita, *Tilting modules of finite projective dimension*, Math. Z. **193** (1986), no. 1, 113–146.
62. Shigeru Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.
63. ______, *On the moduli space of bundles on $K3$ surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., vol. 11, Tata Inst. Fund. Res., Bombay, 1987, pp. 341–413.
64. Amnon Neeman, *The derived category of an exact category*, J. Algebra **135** (1990), no. 2, 388–394.
65. ______, *Stable homotopy as a triangulated functor*, Invent. Math. **109** (1992), no. 1, 17–40.
66. ______, *K-theory for triangulated categories. $I(A)$. Homological functors*, Asian J. Math. **1** (1997), no. 2, 330–417.

67. ______, *K-theory for triangulated categories. I(B). Homological functors*, Asian J. Math. **1** (1997), no. 3, 435–529.
68. ______, *K-theory for triangulated categories. III(A). The theorem of the heart*, Asian J. Math. **2** (1998), no. 3, 495–589.
69. ______, *K-theory for triangulated categories. III(B). The theorem of the heart*, Asian J. Math. **3** (1999), no. 3, 557–608.
70. ______, *K-theory for triangulated categories $3\frac{1}{2}$. A. A detailed proof of the theorem of homological functors*, K-Theory **20** (2000), no. 2, 97–174, Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part II.
71. ______, *K-theory for triangulated categories $3\frac{1}{2}$. B. A detailed proof of the theorem of homological functors*, K-Theory **20** (2000), no. 3, 243–298, Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part III.
72. ______, *K-theory for triangulated categories $3\frac{3}{4}$: a direct proof of the theorem of the heart*, K-Theory **22** (2001), no. 1-2, 1–144.
73. ______, *Triangulated categories*, Princeton University Press, Princeton, NJ, 2001.
74. D. O. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381, Algebraic geometry, 7.
75. ______, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, Izv. Ross. Akad. Nauk Ser. Mat. **66** (2002), no. 3, 131–158.
76. A. Polishchuk, *Symplectic biextensions and a generalization of the Fourier-Mukai transform*, Math. Res. Lett. **3** (1996), no. 6, 813–828.
77. Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.
78. Miles Reid, *Update on 3-folds*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 513–524.
79. Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.
80. ______, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.
81. ______, *The abelian defect group conjecture*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), no. Extra Vol. II, 1998, pp. 121–128 (electronic).
82. ______, *The abelian defect group conjecture: some recent progress*, Algebra—representation theory (Constanta, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 28, Kluwer Acad. Publ., Dordrecht, 2001, pp. 273–299.
83. Raphaël Rouquier, *From stable equivalences to Rickard equivalences for blocks with cyclic defect*, Groups '93 Galway/St. Andrews, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 212, Cambridge Univ. Press, Cambridge, 1995, pp. 512–523.
84. ______, *The derived category of blocks with cyclic defect groups*, Derived equivalences for group rings, Lecture Notes in Math., vol. 1685, Springer, Berlin, 1998, pp. 199–220.
85. Marco Schlichting, *A note on K-theory and triangulated categories*, Invent. Math. **150** (2002), no. 1, 111–116.

86. Jean-Pierre Schneiders, *Quasi-abelian categories and sheaves*, Mém. Soc. Math. Fr. (N.S.) (1999), no. 76, vi+134.
87. Paul Seidel and Richard Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108.
88. Ross Street, *Homotopy classification by diagrams of interlocking sequences*, Math. Colloq. Univ. Cape Town **13** (1984), 83–120.
89. R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.
90. Avishay Vaknin, *Determinants in triangulated categories*, K-Theory **24** (2001), no. 1, 57–68.
91. ______, *K_1 for triangulated categories*, K-Theory **24** (2001), no. 1, 1–56.
92. ______, *Virtual triangles*, K-Theory **22** (2001), no. 1-2, 161–198.
93. Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.
94. Dieter Vossieck, *The algebras with discrete derived category*, J. Algebra **243** (2001), no. 1, 168–176.
95. Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Springer, Berlin, 1985, pp. 318–419.
96. Charles A. Weibel, *Negative K-theory of varieties with isolated singularities*, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), vol. 34, 1984, pp. 331–342.
97. ______, *A Brown-Gersten spectral sequence for the K-theory of varieties with isolated singularities*, Adv. in Math. **73** (1989), no. 2, 192–203.

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