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Dedekind sums and the signature of $f(x, y) + z^N$

András Némethi

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§1. Introduction

Almost twenty years ago A. Durfee conjectured [4] that the signature $\sigma(g)$ of the Milnor fiber associated with an isolated complete intersection singularity $g : (\mathbf{C}^{k+2}, 0) \to (\mathbf{C}^k, 0)$ is negative. On the other hand, J. Wahl found a smoothing of a (noncomplete intersection) singularity with positive signature [25]. But mathematicians working in singularity theory firmly believe in Durfee's conjecture (at least) in the case of hypersurfaces (see, for example, the list of open problems in [30]).

Actually, in the same paper [4], Durfee conjectured a much-much stronger inequality, namely that the geometric genus p_g and the Milnor number μ satisfy

$$p_g \leq \mu/6. \tag{$*$}$$

Using some examples, e.g. the germ $x^a + y^a + z^a$, it is easy to verify that the coefficient $1/6$ in this inequality is the optimal one (cf. (4.5) and the end of (4.10)).

For quasi-homogeneous hypersurface singularities, Y. Xu and S. S.-T. Yau verified the inequality $(*)$ [27], [28] (even a stronger version, see below). In this paper we will prove the inequality $(*)$ for singularities of type $f(x, y) + z^N$, where f is an irreducible plane curve singularity (Theorem 5.1), and also a sharper inequality for Brieskorn singularities. Moreover, in section 5, we verify that for any reducible curve singularity $f = \prod_{i=1}^r f_i$ (f_i irreducible) one has

$$-3\sigma(f + z^N) \geq (N-1)\left(\sum_i \mu(f_i) + 3r - 3\right) \geq (N-1)(m_0(f) + 2r - 3) \geq 0,$$

where $m_0(f)$ denotes the multiplicity of f. In particular, $\sigma(f+z^N)<0$ for any f.

As a first step, we find a formula for the signature of $f+z^N$ (f arbitrary) in terms of *generalized Dedekind sums* associated with the multiplicities of the irreducible exceptional divisors of the embedding resolution of f (Theorem 2.3). The proof involves results about plane curve and suspension singularities: $\sigma(f+z^N)$ is computed via the *eta-invariant* of f ([9], [10]), the eta-invariant is computed from the *spectral pairs* (or *equivariant Hodge numbers*) of f and the number N (as in [9]), and finally, the set of spectral pairs of f can be computed from the embedded resolution graph of f (see [22] and [14]).

This formula (2.3) generalizes Proposition (2.5) of W. Neumann and J. Wahl [17], where the signature of $g=f(x,y)+z^N$ is computed in the case when the link of g is an integer homology sphere (equivalently, f is irreducible, and N is relatively prime with the Newton pairs of f, cf. [loc. cit.]). (Unfortunately, the method of [17] cannot be used in our general situation; for this reason we developed the new approach.)

Theorem 2.3, applied for the particular case of Brieskorn singularities $x^a+y^b+z^c$ (via Brieskorn formula of the signature [3]), gives (see 4.6) the number of lattice points in the open tetrahedron $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ in terms of Dedekind sums. This provides a new proof of this famous lattice point problem, which was solved by Mordell [8] in the case when (a,b,c) are pairwise relative prime numbers, and recently by Pommersheim [18] in the general case. (For the importance of this problem in the theory of toric varieties, see [18].)

The general result of section 2, applied for irreducible germs f, gives in section 3 nice inductive formulas (see 3.2). In particular, it is proven that the signature of $f+z^N$ is a sum of signatures of Brieskorn type singularities (see 3.3). (If N is generic, then we recover the result (2.5) of [17], mentioned above.) This provides the inductive step in the proof of Durfee's conjecture. On the other hand, we emphasize that the inequality $(*)$ for a Brieskorn singularity does not imply the general case: we need stronger inequalities for the Brieskorn case.

For quasi-homogeneous singularities $g:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ (in particular, for Brieskorn singularities as well), Y. Xu and S. S.-T. Yau [27], [28] proved the inequality $6p_g\le\mu-m_0(g)+1$ (which is stronger than $(*)$, but not sufficiently strong for our inductive step). Our sharp inequality is presented in Theorem 4.1. Its proof is technical, it uses identities and inequalities about Dedekind sums (e.g. the reciprocity law). These facts are separated in the Appendix.

We mention (as [27], [28] already suggest) that the difficult part in the proof of Durfee's conjecture is not the computation of the signature in terms of Dedekind sums, but in finding good estimates for the Dedekind sums.

In the last years, different inequalities of type $p_g\le c\cdot\mu$ are proven (where c is some constant) for particular classes of singularities (see, e.g. [24], [26]). Notice that our class is rather general with respect to these classes.

Recently a preprint of Ashikaga [1] appeared in which he proves that $-3\sigma(f+$

z^N) has at least the order of $m_0(f)^2 \cdot N$. Our approach in this article is more conceptual, shorter and gives stronger results (actually the optimal one) in the case when f is irreducible. The case f reducible is discussed in [12]. We mention that the negativity of the signature of $f + z^N$ (f arbitrary) is proved in [13] by a short, rather elementary argument.

We will use the following notations: μ_0 is the dimension of the kernel of the Milnor lattice L of a singularity, μ_+ (resp. μ_-) is the dimension of a maximal subspace where L is positive (negative) definite. The generalized Dedekind sum (cf. [19], [29]) $s(b, c; a)$ is defined by

$$s(b,c;a) = \sum_{k=1}^{a-1} \left(\left(\frac{kb}{a} \right) \right) \left(\left(\frac{kc}{a} \right) \right),$$

where $((x))$ is defined via the fractional part $\{x\}$ as

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \notin \mathbf{Z} \\ 0 & \text{otherwise.} \end{cases}$$

§2. σ_N in terms of the embedded resolution graph of f

2.1. Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a germ of an analytic function which defines an isolated singularity at the origin. We consider an embedded resolution $\phi : (\mathcal{Y}, D) \to (\mathbf{C}^2, f^{-1}(0))$ of $(f^{-1}(0), 0) \subset (\mathbf{C}^2, 0)$ (here $D = \phi^{-1}(f^{-1}(0))$). Let $E = \phi^{-1}(0)$ be the exceptional divisor and $E = \cup_{w \in \mathcal{W}} E_w$ its decomposition in irreducible divisors. If $f = \prod_{a \in \mathcal{A}} f_a$ is the irreducible decomposition of f, then $D = E \cup \cup_{a \in \mathcal{A}} S_a$, where S_a is the strict transform of $f_a^{-1}(0)$. Let G_f be the resolution graph of f, i.e. its vertices $\mathcal{V} = \mathcal{W} \coprod \mathcal{A}$ consist of the nonarrowhead vertices $\mathcal{W}$ (corresponding to the irreducible exceptional divisors), and arrowhead vertices $\mathcal{A}$ (corresponding to the strict transform divisors of D). We will assume that no irreducible exceptional divisor has an autointersection and $\mathcal{W} \neq 0$. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have an intersection point, then (v_1, v_2) $(= (v_2, v_1))$ is an edge of G_f. The set of edges is denoted by $\mathcal{E}$.

For any $w \in \mathcal{W}$, we denote by $\mathcal{V}_w$ the set of vertices $v \in \mathcal{V}$ adjacent to w. Set $\delta_w = \#\mathcal{V}_w$ for any $w \in \mathcal{W}$. If $\delta_w > 2$, then $w \in \mathcal{W}$ is called "rupture point". The set of rupture points is denoted by $\mathcal{R}$.

The graph G_f is decorated by the self-intersection (or Euler-) numbers $e_w := E_w \cdot E_w$ for any $w \in \mathcal{W}$.

For any $v \in \mathcal{V}$, let m_v be the multiplicity of $f \circ \phi$ along the irreducible divisor corresponding to v. In particular, for any $a \in \mathcal{A}$ one has $m_a = 1$. The multiplicities satisfy the following relations. For any $w \in \mathcal{W}$ one has

$$e_w m_w + \sum_{v \in \mathcal{V}_m} m_v = 0. \tag{2.2}$$

It is convenient to use the following notations:

(a) for any $w \in \mathcal{W}$, we define $M_w := gcd(m_w, m_{v_1}, \dots, m_{v_t})$, where $\mathcal{V}_w = \{v_1, \dots, v_t\}$; and

(b) for any $e = (v_1, v_2) \in \mathcal{E}$, we define $m_e := gcd(m_{v_1}, m_{v_2})$.

With these notations one has:

2.3. Theorem. *Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be an isolated plane curve singularity as above. Then the signature $\sigma_N(f)$ of the Milnor fiber of the suspension $f(x,y) + z^N$ is*

$$\sigma_N(f) = \eta(f; N) - N \cdot \eta(f; 1), \quad \textit{where}$$

$$\eta(f;K) = \#\mathcal{A} - 1 + \sum_{e \in \mathcal{E}} \big((K, m_e) - 1\big) - \sum_{w \in \mathcal{W}} \big((K, M_w) - 1\big) +$$
$$+4 \cdot \sum_{w \in \mathcal{R}} \sum_{v \in \mathcal{V}_w} \sum_{k=1}^{m_w} \bigg(\bigg(\frac{km_v}{m_w}\bigg)\bigg) \cdot \bigg(\bigg(\frac{kK}{m_w}\bigg)\bigg).$$

Notice also that $K \mapsto \eta(f;K)$ is a periodic function. The last term is a sum of generalized Dedekind sums.

Proof. Let F be the Milnor fiber of f and $(H; b, h, V)$ the variation structure associated with f (see [9], [10], [11]). This means that $H = H_1(F, \mathbf{C})$; $b : H \to H^*$ corresponds to the intersection form $\langle , \rangle : H \otimes H \to \mathbf{C}$ via $b(x)(y) = \langle x, y \rangle$; $h : H \to H$ is the monodromy operator and $V : H^* \to H$ is the variation map (here $H_1(F, \partial F)$ is identified with the dual space H^* via the perfect pair $H_1(F) \otimes H_1(F, \partial F) \to \mathbf{C}$).

For any natural number K we define $V(K) := (I + h + \cdots + h^{K-1}) \circ V$. Then the system $(H; b, h^K, V(K))$ has a spectral decomposition $\oplus_\chi (H_\chi; b_\chi, (h^K)_\chi, V(K)_\chi)$ with respect to the automorphism h^K; i.e. H_χ is the χ-generalized eigenspace of h^K, and the spectral decomposition is compatible with the extra-structure $(b, V(K))$ (see [9]).

For any χ we define

$$\eta(f;K)_\chi = \begin{cases} (1 - 2c) \cdot \text{signature}(ib_\chi) & \text{if } \chi = e^{2\pi i c},\ 0 < c < 1; \\ -\text{signature}\big[\big(1 + (h^K)_\chi^{-1}\big) V(K)_\chi\big] & \text{if } \chi = 1. \end{cases}$$

Now, for any $K > 0$, the eta-invariant of f is defined by [9], [10]:

$$\eta(f;K) = \sum_\chi \eta(f;K)_\chi.$$

In [9], [10] it is proved that $\sigma_N(f) = \eta(f;N) - N \cdot \eta(f;1)$.

The eta-invariant $\eta(f;K)$ can be computed from the *spectral pairs* of f. First we recall that the set of spectral pairs $Spp(f) \in \mathbf{Z}[\mathbf{Q} \times \mathbf{Z}]$ of f codifies the equivariant

Hodge numbers $\{h_\lambda^{pq}\}$ of the mixed Hodge structure of the vanishing cohomology of f via the relation [23]

$$Spp(f) = \sum_{(\alpha,\omega)} h^{1+[-\alpha],\omega+s_\alpha-1-[-\alpha]}_{\exp(-2\pi i\alpha)} \cdot (\alpha,\omega), \tag{2.4}$$

where $s_\alpha = 0$ if $\alpha \notin \mathbf{Z}$, and $= 1$ otherwise (cf. 5.20 [9]).

The set of *spectral* (or *characteristic*) numbers is defined ([23]) by

$$Sp(f) = \sum \alpha \in \mathbf{Z}[\mathbf{Q}] \text{ (the sum over the spectral pairs } (\alpha,\omega)).$$

Then we set (cf. 5.20. [9]):

$$\Sigma p_{\lambda,\pm}(f) = \#\big\{c|\ c \text{ is a spectral number with } e^{-2\pi ic} = \lambda, \text{ and } (-1)^{[c]} = \pm 1\big\}.$$

Then the general result (5.21) [9] in the plane singularity case reads

$$\eta(f;K) = \sum_{\substack{\lambda^K=1\\ \lambda\neq 1}} h_\lambda^{11} - \sum_{\substack{\lambda^K\neq 1 \text{ or } \lambda=1\\ \lambda=e^{-2\pi ic};\ 0\le c<1}} \big(1-2\{Kc\}\big)\big(\Sigma p_{\lambda,-}(f) - \Sigma p_{\lambda,+}(f)\big). \tag{2.5}$$

In particular, $\eta(f;K)$ can be computed explicitly from the spectral pairs. On the other hand, the set of spectral pairs can be computed in terms of the embedded resolution graph G_f (see [22] and [14]).

The formula (6.5) in [14] (via the transformation (2.11) [14]) reads

$$\begin{aligned} Spp(f) = (\#\mathcal{A}-1)(0,1) &+ \sum_{e\in\mathcal{E}} \sum_{0<k<m_e} \left(\left(\frac{k}{m_e},0\right) + \left(-\frac{k}{m_e},2\right)\right) \\ &- \sum_{w\in\mathcal{W}} \sum_{\substack{0<k<m_w\\ R_w^k=0}} \left(\left(\frac{k}{m_w},0\right) + \left(-\frac{k}{m_w},2\right)\right) \\ &+ \sum_{w\in\mathcal{W}} \sum_{\substack{0<k<m_w\\ R_w^k\neq 0}} (R_w^k-1)\left(\left(1-\frac{k}{m_w},1\right) + \left(-1+\frac{k}{m_w},1\right)\right), \end{aligned}$$

where $R_w^k := \sum_{v\in\mathcal{V}_m}\{k\cdot m_v/m_w\}$. Notice that $\Sigma p_{1,-} = 0$, $\Sigma p_{1,+} = h_1^{11} = \#\mathcal{A}-1$. In order to compute $\sum_{\lambda^K=1,\lambda\neq 1} h_\lambda^{11}$ in (2.4), we need spectral pairs with $[-\alpha]=0$, $s_\alpha = 0$, $\omega = 2$ and $\alpha K \in \mathbf{Z}$. Hence, by the above formula:

$$\sum_{\lambda^K=1,\ \lambda\neq 1} h_\lambda^{11} = \sum_e \sum_{\substack{0<k<m_e\\ m_e|Kk}} 1 - \sum_w \sum_{\substack{0<k<m_w\\ R_w^k=0,\ m_w|Kk}} 1 \tag{2.6}$$

which is exactly $\sum_e((K,m_e)-1)-\sum_w((K,M_w)-1)$. In the sum $\Sigma := \Sigma_{\lambda^K\neq 1}(1-2\{Kc\})(\Sigma p_{\lambda,-}-\Sigma p_{\lambda,+})$, a set of numbers of type $\sum_k(k/m,\omega)$ has no contribution, therefore

$$\begin{aligned}\Sigma &= \sum_w\sum_k R^k_w\left(\left(1-2\left\{K\left(1-\frac{k}{m_w}\right)\right\}\right)-\left(1-2\left\{K\frac{k}{m_w}\right\}\right)\right)\\ &= 4\sum_w\sum_k R^k_w\left(\left(\frac{kK}{m_w}\right)\right).\end{aligned}$$

In this sum the nonrupture points have trivial contribution. Indeed, by (2.2), if $\delta_w\leq 2$, then $\sum_{v\in\mathcal{V}_w}((km_v/m_w))=0$ for any k.

Notice that $\sum_k((\frac{Kk}{m_w}))=0$, so the result follows. □

2.7. Remark. If f is irreducible ($\#\mathcal{A}=1$), or if the monodromy of f has finite order, then

$$\sum_{e\in\mathcal{E}}\big((N,m_e)-1\big)-\sum_{w\in\mathcal{W}}\big((N,M_w)-1\big)=0.$$

Indeed, the above expression in the proof of (2.3) is identified with $\sum_{\lambda^N=1,\lambda\neq 1}h^{11}_\lambda$ (cf. 2.6). But for $\lambda\neq 1$, the Hodge number h^{11}_λ is exactly the number of Jordan blocks of the monodromy operator of f with eigenvalue λ and size two. If f is irreducible then this number is zero by a result of Lê [6]. This follows also from [15], [16] (cf. also [5]).

§3. $\sigma_N(f)$ for irreducible germs f

It is well-known that some invariants of plane curve singularities behave additively with respect to the splicing of their embedded resolution graph (or with respect to their rupture points). We ask the following natural question: is the signature $\sigma_N(f)$ of $f(x,y)+z^N$ additive in this sense? Reformulating this: is $\sigma_N(f)$ a sum of signatures of some Brieskorn type singularities? Actually, the isometric structure or the Seifert form of f is "additive" only if f is irreducible (cf. [5] (15.3)); therefore we can expect a nice answer of this type only in this case.

Consider an irreducible germ f with Newton pairs $(p_i,q_i)_{i=1}^s$ (cf. [5] page 49). Define also the integers $\{a_i\}_{i=1}^s$ by

$$a_1=q_1,\quad\text{and}\quad a_{i+1}=q_{i+1}+p_{i+1}p_ia_i\quad\text{if}\quad i\geq 1. \tag{3.1}$$

We will denote by $\sigma(a,b,c)$ (respective by $\mu(a,b,c)$) the signature (respective the Milnor number $(a-1)(b-1)(c-1)$) of the Brieskorn singularity $g(x,y,z)=x^a+y^b+z^c$.

3.2. Theorem. *Assume that f is irreducible with Newton pairs $(p_i, q_i)_{i=1}^s$. If $d_i = (N, p_{i+1} \dots p_s)$ for $1 \le i < s$ and $d_s = 1$, then*

$$\sigma_N(f) = \sum_{i=1}^{s} d_i \cdot \sigma(a_i, p_i, N/d_i).$$

Proof. By (2.3) and (2.7)

$$\eta(f; N) = 4 \cdot \sum_{w \in \mathcal{R}} \sum_{v \in \mathcal{V}_w} s(m_v, N; m_w).$$

The multiplicities $\{m_w\}_{w\in\mathcal{R}}$ and $\{m_v\}_{v\in\mathcal{V}_w}$, can be computed easily using the Eisenbud-Neumann splicing graph of f [5] (page 51). The "rupture points" $\{w_1, \dots, w_s\}$ of this graph correspond to the rupture points of G_f (for the correspondence, see [loc. cit. chap. V]); and the multiplicity m_{w_i} is the multiplicity of a generic fiber of the i^{th} Seifert component. By [loc. cit. chap. III]: $m_{w_i} = a_i p_i p_{i+1} \dots p_s$. Since f is irreducible, each rupture point has exactly three adjacent vertices in G_f. Their multiplicities $\{m_v\}$ can be computed as follows (see [16] or [21] p. 127). Since $(a_i, p_i) = 1$, we can consider two integers u_i and v_i such that $a_i u_i + p_i v_i = 1$. Then the multiplicities $\{m_v\}_{v\in\mathcal{V}_{w_i}}$ (modulo m_{w_i}) are

$$\big(- u_i a_i p_{i+1} \dots p_s,\ -v_i p_i p_{i+1} \dots p_s,\ p_{i+1} \dots p_s \big).$$

On the other hand, the multiplicity of the unique rupture point of $G(x^{a_i} + y^{p_i})$ is $a_i p_i$, and (by the same argument) the multiplicities of the adjacent vertices are $(-u_i a_i, -v_i p_i, 1)$ (cf. the proof of (4.1) too). Therefore, by (A.1) $\eta(f; N) = \sum_{i=1}^s d_i \cdot \eta(x^{a_i} + y^{p_i}; N/d_i)$. Now (2.3) applied for f and $x^{a_i} + y^{p_i}$ gives the result. $\square$

3.3. Corollary. *Fix an irreducible singularity f with Newton pairs $(p_i, q_i)_{i=1}^s$ as above. For $1 \le l \le s$ let $f_{(l)} : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be an irreducible singularity with Newton pairs $(p_i, q_i)_{i=1}^l$. Then for any $1 < l \le s$ one has*

$$\sigma_N(f_{(l)}) = \sigma(a_l, p_l, N) + (N, p_l) \cdot \sigma_{N'}(f_{(l-1)}),$$

where $N' = N/(N, p_l)$.

Proof. Use (3.2) and the identity $(N, ab) = (N, a) \cdot (N/(N, a), b)$. $\square$

3.4. Remark. (3.3) can be compared with the following relation. For simplicity, we denote $\mu_0(f + z^N)$ by $\mu_{0,N}(f)$. Then one has

$$\mu_{0,N}(f_{(l)}) = \mu_0(a_l, p_l, N) + (N, p_l) \cdot \mu_{0,N'}(f_{(l-1)}),$$

where $N' = N/(N, p_l)$ as above.

In order to see this, notice that $\mu_{0,N}(g)$ is $\#\{\lambda$ eigenvalue of the monodromy of g; $\lambda^N=1$, $\lambda\neq 1\}$ (counted with multiplicities). Now, the above relation follows from the following formula of the characteristic polynomials of the monodromy operators:

$$P_{f_{(l)}}(t)=P_{x^{a_l}+y^{p_l}}(t)\cdot P_{f_{(l-1)}}(t^{p_l}).$$

3.5. Remark. Compare (3.3) and (3.4) with the corresponding inductive formula of the Milnor numbers. (Below $\mu_N(f_{(l)})$ denotes the Milnor number of $f_{(l)}+z^N$)

$$\mu_N(f_{(l)})=\mu(a_l,p_l,N)+p_l\cdot\mu_N(f_{(l-1)}).$$

Notice that μ_N preserves the corresponding information from $f_{(l-1)}$ with "higher weight" (p_l versus (N,p_l)).

§4. Inequalities for Brieskorn singularities

It is well-known that the signature $\sigma(a,b,c)$ of Brieskorn singularities is negative. Therefore, by (3.2), for any irreducible singularity f, the signature of $f+z^N$ is negative. In the next section we will prove a stronger (and in some sense, optimal) inequality: $-3\sigma(f+z^N)\geq\mu(f+z^N)+3\mu_0(f+z^N)$. A short analysis of the inductive formulas (3.3), (3.4) and (3.5) shows that the same inequality for Brieskorn singularities does not imply the general one. The needed, stronger inequality, which is valid for Brieskorn singularities is given in the next theorem.

4.1. Theorem. *Assume that $a\leq b\leq c$. Then*

$$-3\sigma(a,b,c)\geq\mu(a,b,c)\cdot\frac{a+1}{a}+(c-1)\cdot\frac{a-1}{a}.$$

In particular, for any $2\leq a\leq c$ and $2\leq b$ one has

$$-3\sigma(a,b,c)\geq\mu(a,b,c)+(a-1)\cdot(c-1)b/a.$$

Proof. We will apply (2.3) for the germ $f(x,y)=x^a+y^b$ and $N=c$. We will use the notations $(a,b)=d$, $a=dp$, $b=dq$. Obviously $q\geq p\geq 1$ and $N\geq dq$.

The germ f has exactly d irreducible components. There is only one rupture point r with multiplicity $m_r=dpq$.

If $p>1$ then in the minimal resolution (or splicing graph) $\delta_r=d+2$. If $q>1$ and $p=1$ (respectively $p=q=1$) then in the minimal graph $\delta_r=d+1$ (respectively $\delta_r=d$). But also in these two special cases we can assume $\delta_r=d+2$, if we work with nonminimal graphs (i.e. in the splicing graph we introduce edges with weights equal to one). (Actually, if $p=q=1$, then $\sigma(a,a,c)$ is given in (4.5), and the inequality follows easily from the properties of the Dedekind sums listed in the Appendix.)

Since (2.7) is valid, by (2.3),

$$\eta(f,N) = d - 1 + 4 \sum_{v \in \mathcal{V}_r} s(m_v, N; dpq).$$

The multiplicities $\{m_v\}$ ($v \in \mathcal{V}_r$) can be computed as follows ([16] or [21] p. 127, cf. also the proof of (3.2)). Set $\alpha_q \cdot p + \alpha_p \cdot q = 1$. Then the $d+2$ multiplicities, modulo m_r, are $(-\alpha_p dq, -\alpha_q dp, 1, \ldots, 1)$. Therefore $[\eta(f;N) - (d-1)]/4 = s(-\alpha_p dq, N; dpq) + s(-\alpha_q dp, N; dpq) + d \cdot s(1, N; dpq)$. Using the properties of the Dedekind sums: $s(\alpha_q dq, N; dpq) = (dq, N) \cdot s(\alpha_p, N/(dq, N); p) = (dq, N) \cdot s(1, \alpha_p^{-1} N/(dq, N); p) = (dq, N) \cdot s(1, qN/(dq, N); p)$, where α_p^{-1} is the inverse of α_p modulo p (cf. A.1). By similar computation of the second term, (and again by (A.1)) one has

$$\begin{aligned} \frac{1}{4}\eta(f;N) = \frac{d-1}{4} - (dq,N) s\left(\frac{qN}{(dq,N)}, 1; p\right) - (dp,N) s\left(\frac{pN}{(dp,N)}, 1; q\right) + \\ + d \cdot s\left(\frac{N}{(dpq,N)}, 1; \frac{dpq}{(dpq,N)}\right). \end{aligned} \tag{4.2}$$

In the case $N = 1$ we can apply the reciprocity law (A.2), once for the first two terms and then for the third one (notice that $s(dpq, 1; 1) = 0$). We obtain

$$\eta(f;1) = \frac{d^2p^2q^2 - p^2 - q^2 + 1}{3pq}. \tag{4.3}$$

Using the relation $\sigma_N(f) = \eta(f;N) - N\eta(f;1)$, we have to verify that

$$3\eta(f;N) \leq 3N\eta(f;1) - (dp-1)(dq-1)(N-1)\frac{dp+1}{dp} - (N-1)\frac{dp-1}{dp}.$$

The right hand side $R(N,d)$ of this inequality (via (4.3)) is

$$R(N,d) = N\left[dp - \frac{p}{q} + \frac{1}{pq} - 1\right] + d^2pq - dp - \frac{q}{p} + 1.$$

First notice that we can assume that $(N,d) = 1$. Indeed, with the notation $(N,d) = u$ consider the plane curve singularity $f' = x^{a/u} + y^{b/u}$ and $N = uN'$, $d = ud'$. In the next lines, we show that the inequality applied for f' and N' implies the inequality for f and N (i.e. $3\eta(f';N') \leq R(N',d')$ implies $3\eta(f,N) \leq R(N,d)$). For this, notice that $3\eta(f;N) = -3 + u(3\eta(f';N') + 3)$; hence we have to show that $3\eta(f';N') \leq R(N',d')$ implies

$$R(uN', ud') + 3 - u \cdot (3\eta(f';N') + 3) \geq 0.$$

The left hand side of this last inequality (via the definition of R) has the form $Q(u) = Au^2 + Bu + C$ with $A \geq 0$, $2A \geq C$. Moreover, $3\eta(f'; N') \leq R(N', d')$ is equivalent with $Q(1) \geq 0$. Now, $Q(1) = A + B + C \geq 0$ and $2A \geq C$ imply $Q(u) \geq Q(1)$ for any $u \geq 2$. In this discussion, if $a = u$, then f' defines a smooth germ, but in this case $Q(1) = 0$ hence $Q(u) \geq 0$, so the wanted inequality is proven and we can stop. If $a > u$, then f' is still a singular germ and $Q(u) \geq Q(1)$ shows that it is enough to verify (4.1) for f' and N' only.

So, in the sequel we will assume $(d, N) = 1$ (and $q \geq p \geq 1$, $d \geq 1$, $N \geq dq$ and $dp \geq 2$).

Using (A.3) one has $3\eta(f; N) \leq S$, where

$$S = -1 + (q, N)\left(p - 3 + \frac{2}{p}\right) + (p, N)\left(q - 3 + \frac{2}{q}\right) + \frac{d^2pq}{(p, N)(q, N)}.$$

Consider the function $S(x, y) = -1 + xA + yB + C/(xy)$, for $1 \leq x \leq q$, $1 \leq y \leq p$ (with $C > 0$). Since S is convex its maximum value is $\max\{S(1, 1), S(1, p), S(q, 1), S(q, p)\}$. If $C \geq pB$ and $C \geq qA$, then $S(1, 1) \geq S(1, p)$ and $S(1, 1) \geq S(q, 1)$. In our case $A = p - 3 + 2/p$, $B = q - 3 + 2/q$ and $C = d^2pq$, hence $S \leq \max\{S(1, 1), S(q, p)\}$. Notice that $R(N, d)$ is an increasing function in N, so it is enough to verify $\max\{S(1, 1), S(q, p)\} \leq R(dq, d)$.

The inequality $S(q, p) \leq R(dq, d)$ is always true. Indeed, write the inequality in the form $Q(d) = Ad^2 + Bd + C \geq 0$, and verify that $2A + B \geq 0$. Hence $Q(d) \geq Q(1)$. Now, $Q(1) \geq 0$ if $p \geq 2$. If $p = 1$, then $d \geq 2$ (because $a = dp \geq 2$), and $Q(d) \geq Q(2)$ (because $2A + B \geq 0$), and $Q(2) \geq 0$ is easy.

The inequality $S(1, 1) \leq R(dq, d)$ is true unless $d = 1$ and $p = 2$. The proof is similar as above; $Q(2) \geq 0$ always, and $Q(1) \geq 0$ if $p \geq 3$. (It is helpful to notice that the expressions $Q(1)$ and $Q(2)$ are increasing functions in q, so it is enough to consider the case $q = p$.)

If $d = 1$ and $p = 2$, then $S(1, 1) \leq R(2q, 1)$, hence in this case also $S \leq R(2q, d)$ is valid.

Therefore, the only case which is not covered by the above discussion is $d = 1$, $p = 2$ and $q \leq N \leq 2q$. In this case the needed inequality $3\eta(f; N) \leq R(N, d)$ reads as (cf. (4.2) and (A1)):

$$-12(2, N) \cdot s\left(\frac{2N}{(2, N)}, 1; q\right) + 12 \cdot s(N, 1; 2q) \leq R(N, 1). \tag{4.4}$$

For the first Dedekind sum we will again use (A.3). Since $2q \geq N$, for the second Dedekind sum, we obtain a better estimate if we transform it (by the reciprocity law (A.2)) in a Dedekind sum with denominator N, and we use (A.3) for this latter one:

$$\begin{aligned}12s(N,1;2q) &= -12s(2q,1;N) - 3 + \frac{4q^2+N^2+(N,2q)^2}{2qN}\\ &\le -12s\left(\frac{2q}{(2,N)},1;\frac{N}{(2,N)}\right) - 3 + \frac{4q^2+N^2+(2qN)}{2qN}\\ &\le \frac{N}{(2,N)} + \frac{2(2,N)}{N} + \frac{2q}{N} + \frac{N}{2q} - 5.\end{aligned}$$

Now, we can consider the two cases $(2,N)=1$ and $(2,N)=2$, and (4.4) follows in both cases by an elementary computation. □

4.5. Example.

(a) If $a=b$, then by the proof of (4.1) one has

$$\sigma(a,a,c) = a - 1 + 4a\cdot s(c,1;a) - c(a^2-1)/3.$$

In particular, if $a=b=c$, then $-3\sigma(a,a,a)=(a-1)(a^2+a-3)$. Then (4.1) reads as $(a-1)(a^2+a-3)\ge a(a-1)^2$.

(b) If $a=2$ and $2\le b\le c$ then $-6\sigma(a,b,c)\ge(c-1)(3b-2)$, in particular, $\sigma(a,b,c)\le-\mu(a,b,c)/2$.

Actually, for any germ $g=f(x,y)+z^2$, Tomari [24] proved the inequality $\sigma\le-\mu/2$. Notice that our coefficient $(a+1)/3a$ of μ in (4.1) generalizes Tomari's coefficient $1/2$ (case $a=2$).

4.6. Remark. The invariant $\mu_0(x^a+y^b+z^c)$ can be computed as follows (see, for example, [7]):

$$\mu_0 = \frac{(a,b)(a,c)(b,c)}{(a,b,c)} - (a,b) - (a,c) - (b,c) + 2. \tag{4.7}$$

Using the identities $\mu=\mu_0+\mu_++\mu_-$ and $\sigma=\mu_+-\mu_-$ and Brieskorn's formula about the signature [3], the computation of $\sigma(a,b,c)$ is equivalent to the computation of the lattice points in the open tetrahedron with vertices $(0,0,0),(a,0,0),(0,b,0),(0,0,c)$ (which is exactly $\mu_+/2$). Via these relations, our theorem (2.3) applied for Brieskorn singularities (more precisely (4.2) and (4.3)) gives the number of these lattice points in terms of Dedekind sums. In the case of pairwise relative prime numbers, this was computed by Mordell [8], and in general case recently by Pommersheim [18].

4.8. By Durfee's formula [4], the geometric genus p_g of an isolated singularity $f:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ satisfies $2\cdot p_g=\mu_o+\mu_+$.

4.9. Corollary. *Assume $a \le b \le c$. Then the geometric genus of $x^a + y^b + z^c$ satisfies*

$$p_g \le \frac{1}{6}\left(1 - \frac{1}{2a}\right)\mu + \frac{1}{4}\mu_0 - \frac{(c-1)(a-1)}{12a}.$$

Proof. Use (4.1) and the relation: $4p_g = \mu + \sigma + \mu_0$. □

The next result proves Durfee's conjecture in the case of Brieskorn singularities.

4.10. Theorem. *For a singularity $x^a + y^b + z^c$ the following holds:*

$$p_g \le \frac{1}{6}\mu.$$

Proof. The verification of the following fact is elementary but tedious. We leave it to the reader (use (4.7)).

Fact. *If $a \le b \le c$ and $(a, b, c) \notin \{(a, a, a), (a, a, 2a), (a, 2a, 2a)\}$, then*

$$\mu_0 \le \frac{(c-1)(a-1)b}{3a}.$$

This inequality together with (4.9) implies the result, except for the three cases which appear in the fact. In these exceptional cases σ and μ_0 can be computed by (4.5) and (4.7). The values of p_g are

$$a(a-1)(a-2)/6, \quad a(a-1)(a-2)/3, \quad a(a-1)(4a-5)/6,$$

hence the inequality follows. □

§5. Inequalities for singularities of type $f(x, y) + z^N$

We start with the following theorem.

5.1. Theorem. *Assume that $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is irreducible. Then the invariants p_g, σ, μ and μ_0 of $f + z^N$ satisfy*

$$p_g \le \frac{1}{6}\mu \quad \textit{or equivalently} \quad -3\sigma \ge \mu + 3\mu_0.$$

Proof. Consider the numerical invariants $(p_i, q_i)_{i=1}^s$, $(a_i)_{i=1}^s$ as in §3, and construct the germs $f_{(l)}$ as in (3.3). We will use induction over l. The case $l = 1$ follows from (4.10). For the inductive step, we need two facts.

5.2. Lemma.

a) *For any* $l \geq 2$ *one has* $a_l > 1 + 3p_l + p_l \cdot \mu(f_{(l-1)})$.
b) $\mu_0(a_l, p_l, N) \leq N - 1$.

Proof. a) Use induction and (3.1). The inductive step is $a_{l+1} \geq 1+p_{l+1}[(p_l-1)(a_l-1)+p_l+a_l-1] \geq 1+p_{l+1}[(p_l-1)(a_l-1)+4p_l+p_l\mu(f_{(l-1)})] = 1+p_{l+1}\mu(f_{(l)})+4p_lp_{l+1}$. Here we used $\mu(f_{(l)}) = p_l\mu(f_{l-1}) + (p_l - 1)(a_l - 1)$.

For b) use $(a_l, p_l) = 1$ in (4.7). □

Using these, we have the following inequalities:

$$\begin{aligned} -3\sigma(a_l, p_l, N) &\overset{(4.1)}{\geq} \mu(a_l, p_l, N) + (p_l - 1)N(a_l - 1)/p_l \\ &\overset{(5.2)}{\geq} \mu(a_l, p_l, N) + (p_l - 1)N\mu(f_{(l-1)}) + 3\mu_0(a_l, p_l, N). \end{aligned}$$

Above, we used the second inequality from (4.1) ($a = p_l$, $b = N$ and $c = a_l$). Now, the verification of the inductive step (in the proof of 5.1) is easy (use (3.3), (3.4) and (3.5)). □

5.3. When f is reducible, we will need the following inequality. Let $g : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ be an isolated singularity with Milnor lattice L_g. If g_t is a deformation of g such that $g_0 = g$ and g_t, for $t \neq 0$ small, has k singular points with Milnor lattices $L_1, \ldots, L_k$ then there is an embedding $\oplus_{i=1}^k L_i \hookrightarrow L_g$. If c is the codimension of this embedding, then $\sigma(g) \leq c + \sum_i \sigma(L_i)$.

5.4. Now assume that $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ defines a reducible isolated singularity. Let $f = \prod_{i=1}^r f_i$ be its decomposition in irreducible factors. Set $f_t = \prod_i f_i \circ T_i(t)$, where $T_i(t) = \mathrm{Id} + tA_i$ is a generic affine transformation with $T_i(0) = \mathrm{Id}$. The degeneration $f_t + z^N$ of $f + z^N$ gives an embedding

$$\bigoplus_{i=1}^r L(f_i + z^N) \oplus \left(\sum_{i<j} m_0(f_i, f_j) \right) \cdot L(x^2 + y^2 + z^N) \hookrightarrow L(f + z^N).$$

(Above $m_0(,)$ denotes the intersection multiplicity at the origin.) Since

$$\mu(f) = \sum_i \mu(f_i) + 2\sum_{i<j} m_0(f_i, f_j) - r + 1,$$

the codimension c of the embedding is $(N - 1)(\sum_{i<j} m_0(f_i, f_j) - r + 1)$. Therefore, (5.3) gives:

5.5. Theorem. *Assume that* $f = \prod_{i=1}^{r}$. *Then*

$$\sigma(f + z^N) \leq \sum_{i=1}^{r} \sigma(f_i + z^N) + (1 - r)(N - 1).$$

The inequality $-3\sigma \geq \mu$ for $f_i + z^N$ (cf. 5.1) implies:

5.6. Corollary. *For an arbitrary isolated plane curve singularity* f *one has*

$$-3\sigma(f + z^N) \geq (N - 1)\left(\sum_{i=1}^{r} \mu(f_i) + 3r - 3\right).$$

Notice that $\mu(f_i) \geq (m_0(f_i) - 1)^2 \geq m_0(f_i) - 1$, hence the inequality from the introduction follows.

§6. Appendix

For definitions, properties about (generalized) Dedekind sums, see [20], [19], [29]. Using the identity

$$\sum_{t=0}^{d-1} \left(\left(\frac{x+t}{d} \cdot n\right)\right) = (n, d) \cdot \left(\left(\frac{n}{(n,d)} \cdot x\right)\right),$$

it is not difficult to prove that

$$s(b, c; a) = (a, b, c) \cdot s\left(\frac{b}{(a,b)}, \frac{c}{(a,b,c)}; \frac{a}{(a,b)}\right); \tag{A.1}$$

(Write $(a, b) = d$, $a' = a/d$, $x = r/a'$ and $k = r + a't$ with $0 \leq r \leq a' - 1$ and $0 \leq t \leq d - 1$.)

Even more generally, if k divides a and b, $a' = a/k$, $b' = b/k$, $c' = c/(c, k)$, then

$$s(b, c; a) = (c, k) \cdot s(b', c'; a').$$

If $(a, b) = 1$, then the sum $s(b, 1; a)$ is the classical Dedekind sum $s(b, a)$. The famous *reciprocity law of Dedekind* asserts that for any two numbers a and b (if they are not relative prime numbers, use (A.1)):

$$s(b, 1; a) + s(a, 1; b) = -\frac{1}{4} + \frac{a^2 + b^2 + (a, b)^2}{12ab}. \tag{A.2}$$

Notice that $s(b, c; 1) = 0$, hence by (A.2) $s(1, 1; a) = (a-1)(a-2)/(12a)$. Moreover:

Lemma (see, e.g. [2], page 208)**.** *Assume that* $a > 0$ *and* $(a, b) = 1$. *Then*

$$|s(b, 1; a)| \leq s(1, 1; a) = \frac{(a-1)(a-2)}{12a}. \tag{A.3}$$

Proof. By the Cauchy-Schwartz inequality: $|s(b, 1; a)| = |\sum((bk/a))((k/a))| \leq (\sum((bk/a))^2 \sum((k/a))^2)^{1/2} = s(1, 1; a)$. $\square$

Actually, using (A.1), the above inequality is valid even if $(a, b) \neq 1$.

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A. Némethi
The Ohio State University
Dept. of Mathematics
231 West 18th Avenue
Columbus, OH 43210
USA
e-mail: nemethi@math.ohio-state.edu