The semi–ring structure and the spectral pairs of sesqui–linear forms

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1 Introduction

The basic motivation of this paper is related to the theory of complex hypersurface singularities.

Let $\mathcal{S}_n$ ($n \geq 1$) be the set of analytic germs $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ with isolated singularity at the origin. Let $\mathbf{N}[\mathcal{S}_n]$ be the free semi–group generated by $\mathcal{S}_n$ and let the graded semi–group $\mathbf{N}[\mathcal{S}] = \oplus_{n\geq 1}\mathbf{N}[\mathcal{S}_n]$ be their direct sum. $\mathbf{N}[\mathcal{S}]$ has a graded semi–ring structure generated by the following multiplicative law: if $f \in \mathcal{S}_n$ and $g \in \mathcal{S}_m$, then $f \oplus g \in \mathcal{S}_{n+m}$ is defined by $(f \oplus g)(x, y) = f(x) + g(y)$, where $x \in (\mathbf{C}^n, 0)$ and $y \in (\mathbf{C}^m, 0)$.

An important analytic invariant of $f \in \mathcal{S}_n$ is the collection $Spp(f)$ of its spectral pairs. $Spp(f)$ lies in the free semi–group $(\mathbf{N}[\mathbf{Q}\times\mathbf{N}], +)$ generated by $\mathbf{Q} \times \mathbf{N}$ [8, 9, 7]. On $(\mathbf{N}[\mathbf{Q}\times\mathbf{N}], +)$ there is a compatible semi–ring structure defined by the (multiplicative) law: $(\alpha, w) * (\beta, \omega) = (\alpha + \beta + 1, w + \omega + 1)$.

By [7], the map $Spp : (\mathbf{N}[\mathcal{S}], +, \oplus) \to (\mathbf{N}[\mathbf{Q} \times \mathbf{N}], +, *)$ is a semi–ring morphism, in particular $Spp(f \oplus g) = Spp(f) * Spp(g)$.

The most important topological invariant of $f \in \mathcal{S}_n$ is its Seifert form V_f. It is well–known that V_f is non–degenerate and it is equivalent to the variation map of f.

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Let $(\mathcal{F}, +, \otimes_s)$ be the semi–ring of the non–degenerate, graded sesqui–linear forms (i.e. the collection of forms $G = (G_n)_{n\geq 1}$, where each G_n is non–degenerate). The additive law is $(G+H)_n = G_n \oplus H_n$, and the multiplicative one is $(G \otimes_s H)_k = \oplus_{n+m=k}(-1)^{nm} G_n \otimes H_m$. By a result of Sakamoto [6] (or Deligne, see [1]), if $f \in \mathcal{S}_n$ and $g \in \mathcal{S}_m$, then $V_{f\oplus g} = (-1)^{nm} V_f \otimes V_g$. In particular $f \mapsto V_f \otimes 1_{\mathbf{C}}$ induces a semi–ring morphism $V : \mathbf{N}[\mathcal{S}] \to (\mathcal{F}, +, \otimes_s)$.

Consider the natural projection $pr_{mod-2} : (\mathbf{N}[\mathbf{Q}\times\mathbf{N}], +, *) \to (\mathbf{N}[(\mathbf{Q}/2\mathbf{Z})\times \mathbf{N}], +, *)$ induced by the natural map $\mathbf{Q} \to \mathbf{Q}/2\mathbf{Z}$. In [4] it is proved that the information contained in the complex (equivalently, in the real) Seifert form is equivalent to the information contained in $pr_{mod-2}(Spp(f))$. In fact, a map $Spp_{mod-2} : \mathcal{F} \to \mathbf{N}[(\mathbf{Q}/2\mathbf{Z}) \times \mathbf{N}]$ is constructed, such that the following diagram is commutative:

$$\begin{array}{ccc}
\mathbf{N}[\mathcal{S}] & \stackrel{Spp}{\longrightarrow} & \mathbf{N}[\mathbf{Q}\times\mathbf{N}] \\
\downarrow V & & \downarrow pr_{mod-2} \\
\mathcal{F} & \stackrel{Spp_{mod-2}}{\longrightarrow} & \mathbf{N}[(\mathbf{Q}/2\mathbf{Z})\times\mathbf{N}]
\end{array}$$

In this note, we study the semi–ring structure of $\mathcal{F}$ and we prove that Spp_{mod-2} is a morphism of semi–rings. This seems to be a purely algebraic problem, and it can be formulated independently from the singularity theory.

In the study of isolated singularities it is convenient to study the variation map (Seifert form) V_f together with the intersection form b_f and the monodromy operator h_f (even if b_f and h_f are determined by V_f). Similarly, in the pure algebraic study of the sesqui–linear forms V, we preffer to work with a triplet (V, b, h) (which satisfies some axioms, inspired from the properties of (V_f, b_f, h_f)). We call these triplets variation structures. In [4] there is proved that any variation structure (with non–degenerate V) is a direct sum of some indecomposable structures, and the indecomposable structures are listed. Using this, in this paper, we determine the (topological) semi–ring structure of $\mathcal{F}$.

(For more information about the above diagram, and its relation with other invariants of the singularities, see [4].)

2 ε-hermitian variation structures

If U is a finite dimensional vector space then U^* is its dual $Hom(U, \mathbf{C})$. We have the natural isomorphism $\theta : U \to U^{**}$ given by $\theta(u)(\varphi) = \varphi(u)$. It is convenient to write $\varepsilon = \pm 1$ in the form $\varepsilon = (-1)^n$.

2.1. **Definition**. An ε–hermitian variation structure (abbreviated by HVS) over $\mathbf{C}$ is a system $(U; b, h, V)$, where

a.) U is a finite dimensional $\mathbf{C}$–vector space,

b.) $b : U \to U^*$ is an ε–symmetric $\mathbf{C}$–linear endomorphism, i.e. $\overline{b^* \circ \theta} = \varepsilon b$.

c.) h is b–orthogonal automorphism of U, i.e. $\overline{h^*} \circ b \circ h = b$.

d.) $V : U^* \to U$ is a $\mathbf{C}$–linear endomorphism, with

i) $\overline{\theta^{-1} \circ V^*} = -\varepsilon V \circ \overline{h}^*$,

ii) $V \circ b = h - I$.

2.2. We have two immediate properties: $b \circ V = \overline{h}^{*,-1} - I$, and $h \circ V \circ \overline{h}^* = V$.

2.3. **Definition.** The HVS $(U; b, h, V)$ is called *nondegenerate* (resp. *simple*) if b (resp. V) is an isomorphism.

2.4. **Remarks.**

a.) If b is an isomorphism then $V = (h - I)b^{-1}$ and the HVS $(U; b, h, V)$ is completely determined by the *isometric structure* $(U; b, h)$ (i.e. triplets with axioms a-b-c and with non–degenerate form b). For their classification, see the papers of Milnor [3] and Neumann [5].

b.) If V is an isomorphism, then $h = -\varepsilon V \overline{V}^{*,-1}$ and $b = -V^{-1} - \varepsilon \overline{V}^{*,-1}$. In particular, the classification of *simple* HVS-s is equivalent to the classification of $\mathbf{C}$–linear isomorphisms $V : U^* \to U$ or to the classification of sesqui–linear forms on finite dimensional vector spaces.

If we would like to emphasize ε then the ε–HVS determined by V is denoted by ${}_\varepsilon\mathcal{V}$.

c.) If two real nondegenerate bilinear forms are isomorphic over $\mathbf{C}$, then they are isomorphic over $\mathbf{R}$. In particular, the study of *real* simple variation structures is equivalent to the study of the complex ones.

2.5. **Examples.**

1. If $\mathcal{V}_i = (U_i; b_i, h_i, V_i)$ $(i = 1, 2)$ are variation structures, then $\mathcal{V}_1 \oplus \mathcal{V}_2 = (U_1 \oplus U_2; b_1 \oplus b_2, h_1 \oplus h_2, V_1 \oplus V_2)$ is their direct sum in the category of HVS-s. If $\mathcal{V} = (U; b, h, V)$ then $-\mathcal{V}$ denotes $(U; -b, h, -V)$ with the same ε.

Assume that $\varepsilon_1, \varepsilon_2$ and ε are fixed. If $\mathcal{V}_i$, $(i = 1, 2)$ are *simple* ε_i–hermitian variation structures, then the tensor product $V_1 \otimes V_2$ defines a new simple ε–

structure. The corresponding automorphisms are related by ${}^{\otimes}h = -\varepsilon\varepsilon_1\varepsilon_2 h_1 \otimes h_2$. In this paper always $\varepsilon\varepsilon_1\varepsilon_2 = -1$, i.e. $h = h_1 \otimes h_2$.

The conjugate of $\mathcal{V} = (U; b, h, V)$ is $\overline{\mathcal{V}} = (U; \overline{b}, \overline{h}, \overline{V})$.

2. In the next examples J_k denotes the $k \times k$–Jordan block with eigenvalue $= 1$.

Consider $\lambda \in \mathbf{C}^* - S^1$. ($S^1$ denotes the unit circle.) The ε–HVS $\mathcal{V}^{2k}(\lambda)$ is defined by:

$$\mathcal{V}^{2k}_{\lambda} = (\mathbf{C}^{2k}; \begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix}, \begin{pmatrix} \lambda J_k & 0 \\ 0 & \frac{1}{\bar\lambda} J_k^{*,-1} \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon(\lambda J_k - I) \\ \frac{1}{\bar\lambda} J_k^{*,-1} - I & 0 \end{pmatrix}).$$

Note that $\mathcal{V}^{2k}_{\lambda} \approx \mathcal{V}^{2k}_{1/\bar\lambda} \approx -\mathcal{V}^{2k}_{\lambda}$.

3. We are looking for *nondegenerate* forms b such that $\bar b^* = \varepsilon b$ and $J_k^* b J_k = b$. It is immediate that $b_{ij} = 0$ if $i + j \leq k$ and $b_{k+1-i,i} = (-1)^{i+1} b_{k,1}$. By [3] b is determined by $b_{k,1}$. Since b is nondegenerate $b_{k,1} \neq 0$, so we can assume that $b_{k,1} = \omega \in S^1$. By the symmetry of b one has $\overline{\omega} = \varepsilon(-1)^{k-1}\omega$. This equation has two solutions. In conclusion, there are exactly two nondegenerate forms $b = b^k_{\pm}$ (up to isomorphism) with $\overline{b}^* = \varepsilon b$ and $J_k^* b J_k = b$. Their representatives are chosen so that $(b^k_{\pm})_{k,1} = \pm i^{-n^2-k+1}$.

Let $\lambda \in S^1$. If $h = \lambda J_k$, then by the above argument, there are exactly two *nondegenerate* ε–HVS-s (up to isomorphism):

$$\mathcal{V}^k_{\lambda}(\pm 1) = (\mathbf{C}^k; b^k_{\pm}, \lambda J_k, (\lambda J_k - I)(b^k_{\pm})^{-1})$$

where $\omega = (b^k_{\pm})_{k,1} = \pm i^{-n^2-k+1}$.

If $\lambda \neq 1$, then by (d-ii) any HVS with $h = \lambda J_k$ is nondegenerate. If $h = J_k$, then there are some degenerate structures, too.

4. Suppose that $k \geq 2$ and $h = J_k$ but b is degenerate. Then $\mathbf{ker} b = \mathbf{ker}(h - I)$ by (d-ii). Similarly as above, any degenerated form b with $\mathbf{ker} b = \mathbf{ker}(J_k - I)$ and $\overline{b}^* = \varepsilon b$ and $\overline{h}^* b h = b$ has the properties $b_{i,j} = 0$ if $i + j \leq k + 1$, and $b_{k+2-i,i} = (-1)^i b_{k,2}$. Therefore $b_{k,2} \neq 0$ and we can assume that $b_{k,2} = \omega \in S^1$. By symmetry, $\overline{\omega} = (-1)^{n+k}\omega$. Similarly as in the Milnor argument, b is completely determined by $b_{k,2}$. So, we have exactly two solutions $\tilde b^k_{\pm}$ with $(\tilde b^k_{\pm})_{k,2} = \pm(-1)^{n+1} i^{-(n+1)^2-k+1}$. Moreover, V is completely determined by h and b (up to isomorphism). In particular, there are exactly two degenerate structures with $h = J_k$ and $k \geq 2$:

$$\tilde{\mathcal{V}}^k_1(\pm 1) = (\mathbf{C}^k; \tilde b^k_{\pm}, J_k, \tilde V^k_{\pm}),$$

where $(\tilde{b}^k_\pm)_{k,2} = \pm i^{-n^2-k+2}$. In fact:

$$b = \tilde{b}^k_\pm = \begin{pmatrix} 0 & 0 \\ 0 & b^{k-1}_\pm \end{pmatrix}.$$

Note that the structure can also be recognized from $((\tilde{V}^k_\pm)^{-1})_{k,1} = \pm i^{-n^2-k+2}$; (use the identity $b = V^{-1}(h-I)$).

By computation we get that $\tilde{V}^k_\pm$ is an isomorphism. In particular, the variation structures $\mathcal{V}^k_\lambda(\pm 1)$, where $\lambda \in S^1 - \{1\}$ resp. $k \geq 1$ and $\tilde{\mathcal{V}}^k_1(\pm 1)$ where $k \geq 2$, are simple, and they are determined by the corresponding isometric structures $(\mathbf{C}^k; b, h)$.

5. Suppose that $U = \mathbf{C}$ and $h = 1_{\mathbf{C}}$. Then there are exactly five HVS-s:

$$\mathcal{V}^1_1(\pm 1) = (\mathbf{C}; \pm i^{-n^2}, 1_{\mathbf{C}}, 0);$$

$$\tilde{\mathcal{V}}^1_1(\pm 1) = (\mathbf{C}; 0, 1_{\mathbf{C}}, \pm i^{n^2-1});$$

and

$$\mathcal{T} = (\mathbf{C}; 0, 1_{\mathbf{C}}, 0).$$

Note that in $\tilde{\mathcal{V}}^1_1(\pm 1)$ the variation structure is *not* determined by its underlying (degenerate) isometric structure.

6. In order to unify the notations of simple structures, we introduce: $\mathcal{W}^k_\lambda(\pm 1) = \mathcal{V}^k_\lambda(\pm 1)$ if $\lambda \in S^1 - \{1\}$, and $= \tilde{\mathcal{V}}^k_1(\pm 1)$ if $\lambda = 1$. Set $s = 1$ if $\lambda = 1$ and $= 0$ otherwise. Then: $\overline{\mathcal{W}}^k_\lambda(\pm 1) = \mathcal{W}^k_{\bar{\lambda}}(\pm(-1)^{-n^2-k+1+s})$.

2.6. In [4] it is proved the following

Theorem. *A simple ε–hermitian variation structure is uniquely expressible as a sum of indecomposable ones up to order of summands and isomorphism. The indecomposable ones are:*

$$\mathcal{W}^k_\lambda(\pm 1) \quad \textit{where } k \geq 1;\ \lambda \in S^1;\ \textit{and}$$

$$\mathcal{V}^{2k}_\lambda \quad \textit{where } k \geq 1;\ 0 < |\lambda| < 1.$$

This theorem gives a classification of comlex sesquilinear forms (with respect to complex conjugation) over finite dimensional $\mathbf{C}$–vector spaces (cf. 2.6.b).

2.7. If we do not want to relate this presentation and classification to the singularity and Hodge theory, then the sign–convention can be simplified. For the sign–motivation, see [4].

3 The multiplicative structure

3.1. If we want to study only the sesqui-linear forms, then the sign of ε is irrelevant. But in some cases, we want to see the invariants of the associated ε–symmetric form b, too. From this reason, it is convenient to work in a $\mathbf{Z}_2$–graded theory; the gradation given by ε. On the other hand, we have a substructure given by the automorphism h. The nilpotent element $\log J_k$ gives a filtration of the ambient space of $\mathcal{W}_\lambda^k(\pm 1)$. In fact, the action of $\log J_k$ can be extended to an irreducible representation of $sl_2(\mathbf{C})$. In the classical theory of the representations of $sl_2(\mathbf{C})$–modules this weight filtration is centered at zero. Inspired again by the Hodge theory, we want to work with arbitrary centers. This gives an $\mathbf{N}$–graded theory. Since in the theory of singularities, the symmetry of the bilinear form and the center of the weight filtration are related and are coordinated by the dimension, we will use (only) an $\mathbf{N}$–gradation. (Our results can be extended to a $\mathbf{Z}$–graded version in a trivial way.)
3.2. **Definition.** A graded hermitian variation structure over $\mathbf{C}$ (abbreviated by $GHVS$) is a system of $\mathbf{N}$–graded objects $\mathcal{V} = \oplus_{n\geq 0}\mathcal{V}_{n+1}$ (finite direct sum), such that $\mathcal{V}_{n+1}$ is a $(-1)^n$–hermition variation structure. $\mathcal{V}$ is called simple if for any $n \geq 0$ $\mathcal{V}_{n+1}$ is either simple or zero.

The direct sum and the tenzor product (twisted with a sign) are defined by:

$$(\mathcal{V}\oplus\mathcal{W})_{n+1} = \mathcal{V}_{n+1}\oplus\mathcal{W}_{n+1};$$
$$(\mathcal{V}\otimes_s\mathcal{W})_{k+1} = \bigoplus_{n+m+1=k}(-1)^{(n+1)(m+1)}\mathcal{V}_{n+1}\otimes\mathcal{W}_{m+1}.$$

(Here, by the above definition, $\mathcal{V}_{n+1}\otimes\mathcal{W}_{m+1}$ is $(-1)^{n+m+1}$–symmetric.)
3.3. The additive structure of GHVS-s is completely determined by theorem 2.6. As a byproduct of the multiplication structure, we must reobtain the behaviour of the tensor product $J_k\otimes J_l$. In terms of $sl_2(\mathbf{C})$–representations, this problem is equivalent to the splitting of the tensor product of two irreducible representation in irreducible ones, i.e. in the Clebsch–Gordan series (see for example [2]). Now, each $h = \lambda J_k$ ($\lambda \in S^1$) belongs to exactly two HVS, but, as we will see, the parameter space of the eigenvalues and this $\mathbf{Z}_2$–sign fit together in a moduli space which is a $\mathbf{Z}_2$–covering of the parameter space of the eigenvalues. Therefore, the multiplicative structure is coordinated by two objects: the Clebsch–Gordan series and this moduli space.

3.4. A computation (or an argument using the signature) gives for $\xi \in S^1$: $\lim_{\lambda\to\xi;\lambda\notin S^1} \mathcal{V}_\lambda^{2k} = \mathcal{V}_\xi^k(+1) \oplus \mathcal{V}_\xi^k(-1)$; (cf. 3.8). This, and also the next discussions, motivate the following definition: for $\xi \in S^1$ set $\mathcal{V}_\xi^{2k} = \mathcal{V}_\xi^k(+1) \oplus \mathcal{V}_\xi^k(-1)$.

3.5. **Theorem.** *The multiplication table, given on homogeneous elements, is:*

a)

$$\mathcal{V}_\lambda^{2k} \otimes_s \mathcal{V}_\eta^{2l} = \bigoplus_{t=1}^{\min(k,l)} \mathcal{V}_{\lambda\eta}^{2(k+l+1-2t)} \oplus \bigoplus_{t=1}^{\min(k,l)} \mathcal{V}_{\lambda/\bar\eta}^{2(k+l+1-2t)}$$

where $\lambda \notin S^1$ and $\eta \notin S^1$;

b)

$$\mathcal{V}_\lambda^{2k} \otimes_s \mathcal{W}_\xi^l(\pm1) = \bigoplus_{t=1}^{\min(k,l)} \mathcal{V}_{\lambda\xi}^{2(k+l+1-2t)}$$

where $\lambda \notin S^1$ and $\xi \in S^1$;

c)

$$(\mathcal{W}_\lambda^k(u))_{n+1} \otimes_s (\mathcal{W}_\eta^l(v))_{m+1} = \bigoplus_{t=1}^{\min(k,l)} \mathcal{W}_{\lambda\eta}^{k+l+1-2t}(uv(-1)^{t+1}s(\lambda,\eta))$$

where $\lambda = e^{-2\pi i\alpha}$; $\eta = e^{-2\pi i\beta}$; $u = \pm1$; $v = \pm1$ and $s(\lambda,\eta) = (-1)^{[\alpha+\beta]-[\alpha]-[\beta]}$.

Proof. The first two parts follow from the classification theorem (2.6) and the Clebsch–Gordan theorem. (As a remark, note that $\mathcal{V}_\lambda^{2k} \otimes \cdot$ is not one-to-one.)

We prove c) in several steps. For the left hand side of 3.5.c we use the notation $\mathcal{V}^\otimes = (b^\otimes, h^\otimes, V^\otimes)$.

Step 1: The case $\lambda = 1$, $\eta \neq 1$ and $l = 1$.

a.) If $k = 1$, then $V^\otimes = (-1)^{(n+1)(m+1)} u i^{n^2-1} \cdot v(\eta-1) i^{m^2} = uv(\eta-1) i^{(n+m+1)^2}$. This corresponds to $\mathcal{V}_\eta^1(uv)_{n+m+1}$.

b.) If $k \geq 2$, then $V^\otimes = (-1)^{(n+1)(m+1)}(\eta-1) v i^{m^2} \tilde{V}_u^k$. Therefore, $b^\otimes = (V^\otimes)^{-1}(\eta J_k - I) = (V^\otimes)^{-1}[\eta(J_k - I) + (\eta-1)I] = (-1)^{(n+1)(m+1)} v i^{-m^2}[(\tilde{V}_u^k)^{-1} + \frac{\eta}{\eta-1}\tilde{b}_u^k]$. Hence $(b^\otimes)_{k,1} = (-1)^{(n+1)(m+1)} v i^{-m^2} (\tilde{V}_u^k)^{-1}_{k,1} = (-1)^{(n+1)(m+1)} v i^{-m^2} u i^{-n^2-k+2} = uvi^{-(n+m+1)^2-k+1}$.

Step 2: The case $\lambda = \eta = 1$.

a) The case $l = 1$ follows from $((V^\otimes)^{-1})_{k,1} = (-1)^{(n+1)(m+1)} (\tilde{V}_u^k)^{-1}_{k,1} \cdot (\tilde{V}_v^1)^{-1}_{1,1} = (-1)^{(n+1)(m+1)} u i^{-n^2-k+2} v i^{-m^2-1+2} = uvi^{-(n+m+1)^2-k+2}$.

b) If $l = 2$, then by Clebsch–Gordan theorem, we know that $\mathcal{V}^\otimes = \tilde{\mathcal{V}}_1^{k+1}(x) \oplus$

$\tilde{\mathcal{V}}_1^{k-1}(y)$. If $\{e_1,\dots,e_k\}$ resp. $\{f_1,f_2\}$ are the standard bases such that $\log J_k e_i = e_{i-1}$ $(i=2,\dots,k)$ and $\log J_2 f_2 = f_1$, then $\tilde{\mathcal{V}}_1^{k+1}(x)$ is generated by $e_k\otimes f_2$ over $\mathbf{C}[h^{\otimes}]$, where $h^{\otimes}=J_k\otimes J_2$. Therefore $xi^{-(n+m+1)^2-(k+1)+2} = sign(V^{\otimes})^{-1}(e_k\otimes f_2)((\log J_k\otimes J_2)^k(e_k\otimes f_2)) = sign(V^{\otimes})^{-1}(e_k\otimes f_2)(e_1\otimes f_1) = (-1)^{(n+1)(m+1)}(\tilde{V}_u^k)^{-1}_{k,1}\cdot(\tilde{V}_v^2)^{-1}_{2,1}$. Therefore $x=uv$.

Now, by $b=V^{-1}(h-I)$, we deduce that $(\tilde{V}_x^{k+1})^{-1}_{i,j}=0$ if $i+j\leq k+1$. Therefore, for any $z\in\tilde{\mathcal{V}}_1^{k+1}(x)$ with the property $(\log J_{k+1})^{k-1}z=0$, we have $(\tilde{V}_x^{k+1})^{-1}(z,(\log J_{k+1})^{k-2}z)=0$. On the other hand, if $z=\sum_{i=1}^{k-1}a_ie_i\in\tilde{\mathcal{V}}_1^{k-1}(y)$, then $(\tilde{V}_y^{k-1})^{-1}(z,(\log J_{k-1})^{k-2}z)=a_{k-1}^2(\tilde{V}_y^{k-1})^{-1}_{k,1}=a_{k-1}^2yi^{-(n+m+1)^2-(k-1)+2}$. Therefore, we can recover y, if we find a $z\in\mathbf{ker}(\log h^{\otimes})^{k-1}$ so that $z^{\otimes}=(V^{\otimes})^{-1}(z,(\log J_k\otimes J_2)^{k-2}z)\neq 0$. Take $z=e_{k-1}\otimes f_2-(k-1)e_k\otimes f_1$. Then $(\log J_k\otimes J_2)^{k-2}z=e_1\otimes f_2-e_2\otimes f_1$; thus $(-1)^{(n+1)(m+1)}z^{\otimes}=(\tilde{V}_u^k)^{-1}_{k-1,1}\cdot(\tilde{V}_v^2)^{-1}_{2,2}-(\tilde{V}_u^k)^{-1}_{k-1,2}\cdot(\tilde{V}_v^2)^{-1}_{2,1}-(k-1)(\tilde{V}_u^k)^{-1}_{k,1}\cdot(\tilde{V}_v^2)^{-1}_{1,2}+(k-1)(\tilde{V}_u^k)^{-1}_{k,2}\cdot(\tilde{V}_v^2)^{-1}_{1,1}$. Since $(\tilde{V}_{\pm}^k)^{-1}_{i,j}=0$ if $i+j\leq k$, the first and the last term is zero. Moreover, $(\tilde{V}_{\pm}^k)^{-1}_{k-1,2}=-(\tilde{V}_{\pm}^k)^{-1}_{k,1}$, therefore $z^{\otimes}/k=yi^{-(n+m+1)^2-k+3}=(-1)^{(n+1)(m+1)}uvi^{-n^2-k+2-m^2}$, i.e. $y=-uv$.
c) For the general case we can assume that $3\leq l\leq k$. By the case $l=2$ we have: $(\tilde{\mathcal{V}}_1^{l-1}(v))_m\otimes_s(\tilde{\mathcal{V}}_1^2(+1))_1=(\tilde{\mathcal{V}}_1^l(v))_{m+1}\oplus(\tilde{\mathcal{V}}_1^{l-2}(-v))_{m+1}$. Now, multiply this by $(\tilde{V}_1^k(u))_{n+1}$ and use the obtained identity as an inductive step.
Step 3: The case $\lambda\neq 1,\ \eta=1$.
By Step 1: $(\tilde{\mathcal{V}}_\lambda^k(u))_{n+1}=(\mathcal{V}_\lambda^1(+1))_1\otimes_s(\tilde{\mathcal{V}}_1^k(u))_n$. Now use Step 2 and Step 1 again.
Step 4: $(\mathcal{V}_\lambda^1(+1))_1\otimes_s(\mathcal{V}_\eta^1(+1))_1=(\mathcal{W}_{\lambda\eta}^1(s(\lambda,\eta))_2$; where $\lambda\neq 1$ and $\eta\neq 1$.
a) The case $\lambda\eta=1$. Recall : $\mathcal{V}_\lambda^1(+1)=(\mathbf{C};1,\lambda,\lambda-1)$. Therefore $V^{\otimes}=-(\lambda-1)(\eta-1)=2(\mathbf{Re}(\lambda)-1)<0$. But $s(\lambda,\eta)=-1$, so we are finished.
b) If $\lambda\eta\neq 1$, then the left hand side is $\mathcal{V}_{\lambda\eta}^1(x)$ for a suitable x. Now, $b^{\otimes}=-\frac{\lambda\eta-1}{(\lambda-1)(\eta-1)}$ therefore $x=\mathbf{sign}(i\,b^{\otimes})=s(\lambda,\eta)$.
Step 5: For the last case $\lambda\neq 1$ and $\eta\neq 1$, use the "decomposition" of $\mathcal{V}_\lambda^k$ resp. of $\mathcal{V}_\eta^l$ as in the proof of Step 3, and use Step 4.
3.6. The formula (3.5.c) suggests that the four types $\mathcal{W}_\xi^k(\pm 1)$ $(\xi\in S^1)$ fit in a single parameter space. In order to see this, fix $\xi=e^{-2\pi i\beta}$ $(0\leq\beta<1)$ on S^1, and consider the result of the multiplication of $\mathcal{V}_\xi^k(+1)$ with $\mathcal{V}_{e^{-2\pi i\alpha}}(+1)$

for $\alpha \in (0,1)$:

$$\mathcal{V}^1_{e^{-2\pi i\alpha}}(+1) \otimes_s \mathcal{V}^k_{\xi}(+1) = \begin{cases} \mathcal{V}^k_{\xi e^{-2\pi i\alpha}}(+1) & 0 < \alpha < 1-\beta \\ \tilde{\mathcal{V}}^k_1(-1) & \alpha+\beta = 1 \\ \mathcal{V}^k_{\xi e^{-2\pi i\alpha}}(-1) & 1-\beta < \alpha < 1. \end{cases}$$

So $\{\mathcal{W}^k_{e^{-2\pi i\alpha}}(\pm 1)\}_{\alpha,\pm 1}$ fit in the following parameter space:

$$\begin{array}{cccccc} \alpha: & 0 & & 1 & & 2 \\ \hline & \tilde{\mathcal{V}}^k_1(\pm 1) & \mathcal{V}^k_{e^{-2\pi i\alpha}}(\pm 1) & \tilde{\mathcal{V}}^k_1(\mp 1) & \mathcal{V}^k_{e^{-2\pi i\alpha}}(\mp 1) & \tilde{\mathcal{V}}^k_1(\pm 1) \end{array}$$

Define $\mathcal{V}^k(\alpha) = \mathcal{W}^k_{e^{-2\pi i\alpha}}((-1)^{[\alpha]})$. Then (3.5.c) has the following form:

3.7. **Proposition.**

$$(\mathcal{V}^k(\alpha))_{n+1} \otimes_s (\mathcal{V}^l(\beta))_{m+1} = \bigoplus_{t=1}^{\min(k,l)} \mathcal{V}^{k+l+1-2t}(\alpha+\beta+t+1).$$

3.8. Fix $k \geq 1$. Let $D^* = \{z \in \mathbf{C}^*,\ |z| < 1\}$, the open punctured disc, and $\overline{D}^* = \{z \in \mathbf{C}^*,\ |z| \leq 1\}$ the closed one. By the above discussion, the parameter space of the indecomposable nondegenerated sesquilinear forms of type $\mathcal{V}^{2k}_\lambda$, $(\lambda \notin S^1)$, and $\mathcal{W}^k_\lambda(\pm 1)$, $(\lambda \in S^1)$, is $D^* \cup \mathbf{R}/2\mathbf{Z}$. In order to see the right topology on this (i.e."$lim_{\lambda \to e^{-2\pi i\alpha}, |\lambda|<1} = \{\alpha, \alpha+1\}$"), we make the following construction. We define an equivalence relation on $\overline{D}^*$, generated by: $z_1 \sim z_2$ if $z_1, z_2 \in D^*$ and $z_1^2 = z_2^2$. The factor set $\mathcal{M} = \overline{D}^*/_\sim$ (with the factor topology) is the moduli space of our forms.

3.9. If X is a set, let $\mathbf{N}[X]$ be the free abelian semi-group generated by (the base) X: $\mathbf{N}[X] = \{\text{finite sums } \sum n_i x_i;\ n_i \in \mathbf{N}, x_i \in X\}$. On $\mathbf{N}[\mathcal{M}]$ we define the following multiplication, given on generators:

$$[z_1] * [z_2] = \begin{cases} [z_1 z_2] & \text{if } |z_1| = 1 \text{ or } |z_2| = 1 \\ [z_1 z_2] + [z_1/\overline{z}_2] & \text{if } |z_1| < |z_2| < 1 \\ [z_1 z_2] + [z_1/\overline{z}_2] + [-z_1/\overline{z}_2] & \text{if } |z_1| = |z_2| < 1 \end{cases}$$

The resulted semi–ring $(\mathbf{N}[\mathcal{M}], +, *)$ is denoted by $\mathcal{H}$.

If $\mathcal{R}$ is a semi–ring then the additive structure of $\mathcal{R}[\mathbf{N}^*] = \{\sum_{i \geq 1} r_i X^i : r_i \in \mathcal{R}\}$ can be completed in several ways to a semi–ring structure. One of them is the polynomial structure, denoted by $\mathcal{R}[X]$, given by $r_i X^i \cdot r_j X^j =$

$r_i r_j X^{i+j}$. We define another one as follows. Let $A : \mathcal{R} \to \mathcal{R}$ be an automorphism of $\mathcal{R}$. We define the "*twisted Clebsch–Gordan* "multiplication on $\mathcal{R}$ by:

$$r_k Y^k \bullet r_l Y^l = \sum_{t=1}^{\min(k,l)} A^{t+1}(r_k r_l) Y^{k+l+1-2t}.$$

In the case of $\mathcal{R} = \mathcal{H}$ we have a natural automorphism induced by $z \mapsto -z$ ($z \in \overline{D}^*$). The *twisted Clebsch–Gordan semi-ring structure* $\mathcal{H}[tcg]$, by definition, is $(\mathcal{H}[\mathbf{N}^*], +, \bullet)$. Its multiplication is :

$$[z]Y^k \bullet [w]Y^l = \sum_{t=1}^{\min(k,l)} [(-1)^{t+1} zw] Y^{k+l+1-2t}.$$

The inclusion $\mathbf{R}/2\mathbf{Z} \subset \overline{D}^*$, given by $\alpha \mapsto e^{-\pi i \alpha}$, induces the corresponding sub-semi-ring structures on $\mathbf{N}[\mathbf{R}/2\mathbf{Z}]$ (with multiplication: $[\alpha] * [\beta] = [\alpha + \beta]$), and on $\mathbf{N}[\mathbf{R}/2\mathbf{Z}][tcg][X]$ (with automorphism $A[\alpha] = [\alpha + 1]$).

Denote $GHVS(S^1) = \{\mathcal{V} \in GHVS : \text{ the eigenvalues of } h \text{ lie on } S^1\}$.

3.10. **Theorem.** *We have the following semi–ring isomorphisms:*

a)

$$(\mathcal{H}[tcg][X], +, \bullet) \approx (GHVS, \oplus, \otimes_s);$$

b)

$$(\mathbf{N}[\mathbf{R}/2\mathbf{Z}][tcg][X], +, \bullet) \approx (GHVS(S^1), \oplus, \otimes_s).$$

c)

$$\begin{aligned}(\mathcal{H}[tcg], +, \bullet) &\approx \{\text{simple } \varepsilon\text{–HVS}, \oplus, \otimes\} \text{ (with fixed } \varepsilon)\\ &\approx \{\text{nondegenerated sesqui-linear forms}, \oplus, \otimes\};\end{aligned}$$

Proof. Use the correspondence $(\mathcal{V}_\lambda^{2k})_{n+1} \longleftrightarrow [\sqrt{\lambda}] Y^k X^{n+1}$, ($\lambda \in D^*$), and $(\mathcal{V}^k(\alpha))_{n+1} \longleftrightarrow [e^{-\pi i \alpha}] Y^k X^{n+1}$.

3.11. **Remark.** The complex conjugation (cf. 2.5.6) has the following effect:

$$([z]Y^k X^{n+1})^- = \begin{cases} [\overline{z}]\, Y^k X^{n+1} & \text{if } |z| < 1 \\ \left[e^{-\pi i \beta}\right] Y^k X^{n+1} & \text{if } z = e^{-\pi i \alpha} \text{ and } \beta = 2[\alpha] - \alpha + n + k \end{cases}$$

4 Hodge numbers and spectral pairs associated with variation structures

Let $\mathcal{V} = (U; b, h, V)$ be a *simple* variation structure.

In the sequel, we assume that the eigenvalues of our structures are on the unit circle. Recall that $s = s(\lambda) = 0$ if $\lambda \neq 1$ and $= 1$ otherwise.

We want to construct a weight filtration W on U^*_λ and a mod–2 decomposition on $Gr^W_* U^*_\lambda$. By the decomposition theorem, it is enough to define them on the indecomposable homogeneous elements. The weight filtration is given by $\bar{h}^{*,-1}$, the center of the filtration associated to the homogeneous element $(\mathcal{W}_\lambda)_{n+1}$, is, by definition, $n+s$. In fact, it is the unique filtration with center $n+s$ and with the properties: $\mathbf{dim} Gr^W_l(\mathcal{W}^{r+1}_\lambda(u)) = 1$ if $l = n+s-r, \ldots, n+s+r$, and $\log \overline{J}^{*,-1}_{r+1}(W_l) \subset W_{l-2}$.

The $\mathbf{Z}_2$–decomposition

$$Gr^W_l \mathcal{W}^{r+1}_\lambda(u) = F_+ Gr^W_l \mathcal{W}^{r+1}_\lambda(u) \oplus F_- Gr^W_l \mathcal{W}^{r+1}_\lambda(u)$$

is given by:

$$\mathbf{dim} F_v Gr^W_{n+s+r-t} \mathcal{W}^{r+1}_\lambda(u) = \begin{cases} 1 & \text{if } uv(-1)^t = 1 \\ 0 & \text{otherwise} \end{cases}$$

In other words: $Gr^W_{n+s+r-t}\mathcal{W}^{r+1}_\lambda(u) = F_{(-1)^t u} Gr^W_{n+s+r-t}\mathcal{W}^{r+1}_\lambda(u)$.

If $\mathcal{V}_\lambda = \sum_{u=\pm 1, r\geq 0} p^{r+1}_\lambda(u)\mathcal{W}^{r+1}_\lambda(u)$ then we redefine $p^{n+s+r,u}_\lambda = p^{r+1}_\lambda(u)$, $u = \pm 1, r \geq 0$; ($n+s+r$ is the weight of the "primitive element" of $\mathcal{W}^{r+1}_\lambda(u)$); and define $h^{w,u}_\lambda = \mathbf{dim} F_u Gr^W_w U^*_\lambda$. By these notations, we have the following relations:

$$p^{w,u}_\lambda = h^{w,-u}_\lambda - h^{w+2,u}_\lambda \quad w \geq n+s;$$

$$h^{w,u}_\lambda = \sum_{l\geq 0} p^{w+2l,(-1)^l u}_\lambda \quad w \geq n+s;$$

$$h^{n+s-k,u}_\lambda = h^{n+s+k,(-1)^k u}_\lambda.$$

In particular, $\mathcal{V} = \sum p^{r+1}_\lambda(u)\mathcal{W}^{r+1}_\lambda(u)$ is completely determined by the numbers $\{p^{w,u}_\lambda;\ w \geq n+s\}$ or by $\{h^{w,u}_\lambda\}$.

The $\mathbf{Z}_2$–spectral pairs lies in the semiring

$$\mathbf{N}[(\mathbf{R}/2\mathbf{Z})\times\mathbf{N}] = \{\sum_{(\alpha,w)}(\alpha,w),\ \alpha\in\mathbf{R}/2\mathbf{Z}, w\in\mathbf{N}\},$$

where the additive structure is the natural one, but the multiplicative one is given by:

$$(\alpha,w)*(\beta,\omega)=(\alpha+\beta+1,w+\omega+1).$$

The system of equations:

$$\begin{cases} e^{-2\pi i\alpha}=\lambda \\ (-1)^{w-n-[-\alpha]}=u \end{cases}$$

has exactly one solution $\alpha=\alpha_{\lambda,w,u}\in\mathbf{R}/2\mathbf{Z}$. We associate with the space $F_uGr_w^WU_\lambda^*$ the spectral pairs $(\alpha,w-s(\lambda))$ with multiplicity $h_\lambda^{w,u}=\dim F_uGr_w^WU_\lambda^*$. The collection of the $\mathbf{Z}_2$–spectral pairs of $\mathcal{V}$ is:

$$Spp_{mod-2}(\mathcal{V})=\sum_{\lambda,w,u}h_\lambda^{w,u}(\alpha_{\lambda,w,u},w-s(\lambda)).$$

It is clear that passing to the spectral pairs we do not lose any information: we can recuperate $\mathcal{V}$ from its spectral invariants. Moreover, $Spp_{mod-2}(\mathcal{V}_1\oplus\mathcal{V}_2)=Spp_{mod-2}(\mathcal{V}_1)+Spp_{mod-2}(\mathcal{V}_2)$.

The symmetry of the weight filtration gives the invariance of $Spp_{mod-2}(f)$ with respect to the transformation $(\alpha,n+k)\longleftrightarrow(\alpha-k,n-k)$. If the structure comes from a real one, the stability with respect to the complex conjugation gives an addition invariance with respect $(\alpha,n+k)\longleftrightarrow(n-1-\alpha,n-k)$.

4.1. **Theorem.**

$$h_\lambda^{w,u}(\mathcal{V}\otimes_s\mathcal{W})=\sum h_\xi^{p,v}(\mathcal{V})h_\eta^{q,z}(\mathcal{W})$$

where the sum is over the following combinations:

$\lambda=\xi\eta,\ w=p+q+1-s(\xi)-s(\eta)+s(\xi\eta),\ u=vz\cdot s(\xi,\eta).$

Proof. It is enough to verify the theorem only on the indecomposable homogeneous elements. Instead of a direct verification, we give the following argument. The relation $\lambda=\xi\eta$ is clear. The weights (eigenvalues of the diagonal element) of the tensor product of two irreducible $sl_2(\mathbf{C})$–representations are the corresponding sums of the factors' weights. Since our centers are shifted with $n+s(\xi),\ m+s(\eta)$ resp. $n+m+1+s(\xi\eta)$ the second relation

follows. For the third one, it is enough to verify the sign behaviour only for the top weights (thanks to the compatibility of the twisted Clebsch–Gordan multiplication).
4.2. This theorem can be considered as the topologic version of (8.11–8.12) in [7]. That result works in analytic context and its proof is based on the theory of D–modules. In fact, our basic motivation to prove theorem (4.1) was the decision to give a topological support to that result.
4.3. Apparently, introducing different centers and separating the cases $\lambda \neq 1$ and $\lambda = 1$, we mixed up the multiplication structure comletely. A not very careful analysis of theorem (4.1) suggests the same fact. But, hopefully, an appropiate organisation of the informations will give a very nice multiplication law.
4.4. **Theorem.**

$$Spp_{mod-2}(\mathcal{V}_1 \otimes_s \mathcal{V}_2) = Spp_{mod-2}(\mathcal{V}_1) * Spp_{mod-2}(\mathcal{V}_2).$$

Proof. Again, it is enough to verify the relation only on the indecomposable homogeneous elements. For these, $Spp((\mathcal{W}_\lambda^{r+1}(u))_{n+1}) = \sum_{k=0}^r (\alpha - k, n + r - 2k)$ where $\lambda = e^{-2\pi i\alpha}$ and $(-1)^{s(\lambda)+r-[-\alpha]} = u$. Now, use the identity $s(\lambda) - [-\alpha] = [\alpha] + 1$ and $\cup_{k=0}^r \cup_{j=0}^l \{k + j\} = \cup_{t=0}^{\min(r,l)} \cup_{i=0}^{l+r-2t} \{t + i\}$.
4.5. **Corollary.**

$$Spp_{mod-2} : (GHVS(S^1), \oplus, \otimes_s) \to (\mathbf{N}[(\mathbf{R}/2\mathbf{Z}) \times \mathbf{N}], +, *)$$

is a monomorphism of semi-rings.

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