Sel. math., New ser. 5 (1999) 161 – 179
1022–1824/99/010161–19$1.50 + 0.20/0

Selecta Mathematica, New Series

Dedekind sums and the signature of $f(x,y)+z^N$, II

András Némethi

Mathematics Subject Classification (1991). 32S.

Key words. Milnor fiber, signature, eta invariant, Dedekind sums, singularities.

1. Introduction

The present paper has two goals.

First, we compute the signature of the Milnor fiber associated with a singularity of type $f(x,y)+z^N$, and we prove some conjectures of A. Durfee.

A. Durfee in [4] listed some conjectures about the signature σ, the Milnor number μ, and the geometric genus p_g of an isolated complete intersection singularity $g:(\mathbf{C}^{k+2},0)\to(\mathbf{C}^k,0)$. Namely, he conjectured that $p_g\leq\mu/6$ (Conjecture 5.3 in [4], in the sequel $(*)$), which is equivalent to $-3\sigma\geq\mu+3\mu_0$ (where μ_0 is the rank of the kernel of the Milnor lattice). In particular $(*)$ implies a weaker conjecture (5.2 in [4]): the negativity of the signature. On the other hand, J. Wahl in [17] constructed a smoothing of a non-complete intersection with positive signature, showing the subtility of the problem. But, even in the hypersurface case, the conjectures still resist persistent attempts at proof (see [22]).

In a series of papers, the author studied suspensions of hypersurface singularities (i.e., germs of type $f(x_1,\dots,x_n)+z_{n+1}^N$). In particular, in [7], the signature of $f+z^N$ is computed in terms of eta-invariants $\eta(f;N)$ associated with (the variation structure of) f and the number $N:\sigma(f+z^N)=\eta(f;N)-N\cdot\eta(f;1)$. In [10], the eta-invariant $\eta(f;N)$ of a plane curve singularity f is expressed in terms of generalized Dedekind sums associated with the embedded resolution graph G_f of f (we recall this here in 2.3). This is a powerful relation; for example, in the particular case of Brieskorn polynomials, it is equivalent to the computation of the lattice points in a tetrahedron in terms of Dedekind sums, solved in particular cases by Mordell [6], and in general by Pommersheim [13]. This formula generalizes a result (Proposition (2.5) [12]) of W. Neumann and J. Wahl, where the signature is computed when the link of $g=f+z^N$ is an integer homology sphere. Using this relation (2.3) in [10] we prove that for an irreducible germ f, its eta-invariant and the signature of $f+z^N$ are additive with respect to the splice decomposition of the graph G_f. In particular, we prove that the inequality $(*)$ is valid for these type of germs. Unfortunately, additivity results as in [10] (or as in [12]) are not true if f is not irreducible.

In this paper, by a different method, we attack the same problem for arbitrary f. We prove $(*)$ for $f+z^N$ with reducible f and with the following additional restriction: $gcd(m_w, N)=1$ for all multiplicities m_w of the irreducible exceptional divisors in the minimal embedded resolution of f. Actually, we provide a large list of inequalities: we compare the signature with the Milnor number μ_f of f (topological inequality), with the multiplicity ν_f of f (algebraic inequality) and the size of G_f (combinatorial inequality) (cf. section 5). (Using any of these inequalities one can prove the negativity of σ.)

The method is the following. We start with the relation which describes the signature in terms of Dedekind sums (cf. 2.3). Using some properties of these Dedekind sums, in section 3 we transform this relation in some inequalities expressed in terms of the combinatorics of the embedded resolution graph G_f of f. These inequalities are verified in section 4.

Now we come to our second point. This is less precise but more important (for the author). We are searching for an answer to the following question: what is special in the hypersurface case? Why does the Milnor fiber of a hypersurface have negative signature and a non-hypersurface maybe not? From the present paper we learn that the answer is in the particular form of the embedded resolutions of hypersurface singularities. In the two dimensional case (i.e., when $f:(X,x)\to(\mathbf{C},0)$, with (X,x) normal surface singularity) we can be more precise: all the combinatorial relations (which imply our inequalities) proved in section 4 for plane curve singularities (i.e., with X smooth) distinguish these hypersurface germs from the general class of germs of functions defined on a normal surface singularity (cf. 4.25). The author expects that similar relations are valid in higher dimension, and they are responsible for the particular behaviour of hypersurface singularities.

Something more about the relation $\sigma(f+z^N)=\eta(f;N)-N\eta(f;1)$. Since $\eta(f;N)$ is periodic in N, $\lim_{n\to\infty}(\sigma(f+z^N)/N)=-\eta(f;1)$, and any estimate for $\eta(f;1)$ will provide inequalities for σ when N is large. In particular, any inequality of type $\sigma(f+z^N)\leq c\cdot N$ (c constant) with $c>-\eta(f;1)$ is "weak" for large N, but can be very difficult for small N, because of the very irregular behaviour of $\eta(f;N)$ (i.e., of the Dedekind sums). Moreover, any inequality of type $\sigma(f+z^N)\leq c\cdot I_f$, (where $I_f=\mu_f$, or ν_f or any invariant of f), can be very weak for f with I_f large, but very sharp for germs at the beginning of the classification list. For germs with I_f small, in order to verify an inequality as above, the periodic term $\eta(f;N)$ must be really computed. This explains the form of our inequalities in section 4 (and the difficulties in their proofs).

Finally, we mention the papers of Y. Xu and S. S.-T. Yau [19], [20], where they verified the inequality $(*)$ (actually, even a stronger version) in the case of quasi-homogeneous germs $g : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$.

2. σ_N in terms of the embedded resolution graph of f

In this section we introduce some notations and we recall one of the main results of [10].

Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be a germ of an analytic function which defines an isolated singularity at the origin. We consider an embedded resolution $\phi : (\mathcal{Y}, D) \to (\mathbf{C}^2, f^{-1}(0))$ of $(f^{-1}(0), 0) \subset (\mathbf{C}^2, 0)$ (here $D = \phi^{-1}(0))$). Let $E = \phi^{-1}(0)$ be the exceptional divisor and let $E = \cup_{w\in\mathcal{W}} E_w$ be its decomposition in irreducible divisors. If $f = \prod_{a\in\mathcal{A}} f_a$ is the irreducible decomposition of f, then $D = E \cup \cup_{a\in\mathcal{A}} S_a$, where S_a is the strict transform of $f_a^{-1}(0)$. Let G_f be the resolution graph of f, i.e., its vertices $\mathcal{V} = \mathcal{W} \coprod \mathcal{A}$ consist of the non-arrowhead vertices $\mathcal{W}$ (corresponding to the irreducible exceptional divisors), and arrowhead vertices $\mathcal{A}$ (corresponding to the strict transform divisors of D). We will assume that no irreducible exceptional divisor has an autointersection and $\mathcal{W} \neq 0$. If two irreducible divisors corresponding to $v_1, v_2 \in \mathcal{V}$ have an intersection point, then (v_1, v_2) $(= (v_2, v_1))$ is an edge of G_f. The set of edges is denoted by $\mathcal{E}$. Since G_f is a tree, one has

$$\#\mathcal{W} + \#\mathcal{A} = \#\mathcal{E} + 1. \tag{2.1}$$

For any $w \in \mathcal{W}$, we denote by $\mathcal{V}_w$ the set of vertices $v \in \mathcal{V}$ adjacent to w. The graph G_f is decorated by the self-intersection (or Euler) numbers $e_w := E_w \cdot E_w$ for any $w \in \mathcal{W}$.

For any $v \in \mathcal{V}$, let m_v be the multiplicity of $f \circ \phi$ along the irreducible divisor corresponding to v. In particular, for any $a \in \mathcal{A}$, one has $m_a = 1$. The multiplicities satisfy the following relations. For any $w \in \mathcal{W}$ one has

$$e_w m_w + \sum_{v\in\mathcal{V}_w} m_v = 0. \tag{2.2}$$

These relations determine the multiplicities $\{m_w\}_{w\in\mathcal{W}}$ in terms of the self-intersection numbers $\{e_w\}_w$.

It is convenient to use the following notations:

(a) for any $w \in \mathcal{W}$, we define $M_w := gcd\,(m_w, m_{v_1}, \ldots, m_{v_t})$, where $\mathcal{V}_w = \{v_1, \ldots, v_t\}$; and
(b) for any $e = (v_1, v_2) \in \mathcal{E}$, we define $m_e := gcd\,(m_{v_1}, m_{v_2})$.

For any $a \in \mathcal{A}$, there exists exactly one $w_a \in \mathcal{W}$ such that $(a, w_a) \in \mathcal{E}$.

With these notations one has (cf. also [12], Proposition 2.5).

2.3. Theorem. [10] *Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be an isolated plane curve singularity as above. Then the signature σ_N of the Milnor fiber of the suspension $f(x, y) + z^N$ is*

$$\sigma_N = \eta(f; N) - N \cdot \eta(f; 1),$$

where

$$\eta(f;K) = \#\mathcal{A} - 1 + \sum_{e\in\mathcal{E}} \big((K, m_e) - 1\big) - \sum_{w\in\mathcal{W}} \big((K, M_w) - 1\big) +$$
$$+4 \cdot \sum_{w\in\mathcal{W}} \sum_{v\in\mathcal{V}_w} \sum_{k=1}^{m_w} \left(\left(\frac{km_v}{m_w}\right)\right) \cdot \left(\left(\frac{kK}{m_w}\right)\right).$$

Notice also that $K \mapsto \eta(f; K)$ is a periodic function.

3. σ_N via the reciprocity law of Dedekind sums

In this section we will use the generalization of the reciprocity law of Dedekind given by Rademacher [14], [21] in order to rewrite theorem (2.3). In the new expression all the Dedekind sums will have the integer N in the denominator. We separate the needed notations and facts about the Dedekind sums in the appendix.

We start with the following easy remark. If $g : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$ is an arbitrary function, then

$$\sum_{w\in\mathcal{W}} \sum_{v\in\mathcal{V}_w} g(w, v) + \sum_{a\in\mathcal{A}} g(a, w_a) = \sum_{(u,v)\in\mathcal{E}} \big(g(u, v) + g(v, u)\big). \tag{3.1}$$

Since $s(m_{w_a}, N; m_a) = 0$ for any $a \in \mathcal{A}$ (because $m_a = 1$), (3.1) gives

$$4 \sum_{w\in\mathcal{W}} \sum_{v\in\mathcal{V}_w} s(m_v, N; m_w) = 4 \sum_{(u,v)\in\mathcal{E}} \big[s(m_v, N; m_u) + s(m_u, N; m_v)\big]. \tag{3.2}$$

Using the reciprocity law (A.2), this expression is equal to

$$\sum_{e=(u,v)\in\mathcal{E}} \left[-4s(m_u, m_v; N) - (N, m_e) + \frac{N^2m_e^2 + m_u^2(N, m_v)^2 + m_v^2(N, m_u)^2}{3Nm_um_v}\right].$$

Using this equality, the formula of the eta-invariant $\eta(f; N)$ given in (2.3) can be transformed in

$$\eta(f; N) = \#\mathcal{A} - 1 - \#\mathcal{E} - \sum_{w\in\mathcal{W}} \big((N, M_w) - 1\big) - 4 \sum_{(u,v)\in\mathcal{E}} s(m_u, m_v; N)+$$
$$\sum_{(u,v)\in\mathcal{E}} \frac{N}{3} \frac{m_e^2}{m_um_v} + \frac{1}{3N} \cdot \left[\sum_{(u,v)\in\mathcal{E}} \frac{m_u(N, m_v)^2}{m_v} + \frac{m_v(N, m_u)^2}{m_u}\right]. \tag{3.3}$$

For the last sum we apply (3.1), (2.2) (and $m_a = 1$), hence it can be replaced by

$$\sum_{w\in\mathcal{W}} \sum_{v\in\mathcal{V}_w} \frac{m_v(N,m_w)^2}{m_w} + \sum_{a\in\mathcal{A}} \frac{m_{w_a}(N,m_a)^2}{m_a} = \sum_{w\in\mathcal{W}} (-e_w)(N,m_w)^2 + \sum_{a\in\mathcal{A}} m_{w_a}.$$

Using (2.1) and the above identity, (3.3) reads

$$\begin{aligned} \eta(f;N) = -\#\mathcal{W} - \sum_{w\in\mathcal{W}} \big((N,M_w)-1\big) - 4\sum_{(u,v)\in\mathcal{E}} s(m_u,m_v;N) + \\ \sum_{(u,v)\in\mathcal{E}} \frac{N}{3}\,\frac{m_e^2}{m_u m_v} + \frac{1}{3N}\left[\sum_{w\in\mathcal{W}}(-e_w)(N,m_w)^2 + \sum_{a\in\mathcal{A}} m_{w_a}\right]. \end{aligned} \tag{3.4}$$

The periodicity of $\eta(f;N)$ and (3.4) applied for $N=1$ provide:

3.5. Theorem.

$$\lim_{N\to\infty} \frac{-\sigma_N}{N} = \eta(f;1) = -\#\mathcal{W} + \frac{1}{3}\sum_{(u,v)\in\mathcal{E}} \frac{m_e^2}{m_u m_v} + \frac{1}{3}\left[\sum_{w\in\mathcal{W}}(-e_w) + \sum_{a\in\mathcal{A}} m_{w_a}\right].$$

The expression $\sigma_N = \eta(f;N) - N\cdot\eta(f;1)$ of Theorem (2.3) transforms into the following formula:

3.6. Theorem.

$$\begin{aligned} \sigma_N = (N-1)\cdot\#\mathcal{W} - \sum_{w\in\mathcal{W}} \big((N,M_w)-1\big) - 4\sum_{(u,v)\in\mathcal{E}} s(m_u,m_w;N) + \\ \frac{1}{3N}\cdot\sum_{w\in\mathcal{W}}(-e_w)\left[(N,m_w)^2 - N^2\right] + \frac{1-N^2}{3N}\sum_{a\in\mathcal{A}} m_{w_a}. \end{aligned}$$

It is interesting to consider some particular cases:

3.7. Corollary.

a) *Assume that* $(N,m_w)=1$ *for any* $w\in\mathcal{W}$*. Then*

$$\begin{aligned} \sigma_N = (N-1)\cdot\#\mathcal{W} - 4\sum_{(u,v)\in\mathcal{E}} s(m_u,m_v;N) + \\ \frac{1-N^2}{3N}\left[\sum_{w\in\mathcal{W}}(-e_w) + \sum_{a\in\mathcal{A}} m_{w_a}\right]. \end{aligned}$$

b) *Assume that* $N=2$*. Then*

$$\sigma_2 = \#\{w\in\mathcal{W} : 2\nmid M_w\} - \frac{1}{2}\left[\sum_{\substack{w\in\mathcal{W}\\ 2\nmid m_w}}(-e_w) + \sum_{a\in\mathcal{A}} m_{w_a}\right].$$

Now, for (3.7.a) we can apply the inequality (A.3) and (2.1), and we obtain:

3.8. Corollary. *Assume that* $(N, m_w) = 1$ *for any* $w \in \mathcal{W}$. *Then*

$$\sigma_N \le \frac{1-N}{N}\big(\#\mathcal{W} + \#\mathcal{E}\big) + \frac{1-N^2}{3N}\left[-4\#\mathcal{W} - \#\mathcal{A} + 1 + \sum_{w\in\mathcal{W}}(-e_w) + \sum_{a\in\mathcal{A}} m_{w_a}\right].$$

We would like to have a similar inequality for arbitrary N. We start with the following lemma:

3.9. Lemma. *There exists a one-to-one function* $l : \mathcal{W} \to \mathcal{E}$ *such that* $l(w) = (w, v)$ *for some* $v \in \mathcal{V}$.

The proof is easy and is left to the reader.

Notice that (A.4) provides for $l(w) = (w, v)$

$$-4 \cdot s(m_w, m_v; N) + \frac{1}{3N}\big((N, m_w)^2 - N^2\big) \le 0. \tag{3.10}$$

Therefore, by (3.10) and (A.3) one has

$$-4 \sum_{(u,v)\in\mathcal{E}} s(m_u, m_v; N) + \frac{1}{3N}\big((N, m_v)^2 - N^2\big) \le$$
$$-4 \cdot \sum_{\substack{(u,v)\in\mathcal{E}\\ (u,v)\notin l(\mathcal{W})}} s(m_u, m_v; N) \le \frac{(N-1)(N-2)}{3N}\,(\#\mathcal{A} - 1).$$

This inequality and (3.6) give:

3.11. Corollary. *For any* N *the following inequality holds:*

$$\sigma_N \le -\sum_{w\in\mathcal{W}}\big((N, M_w) - 1\big) + \frac{1}{3N} \cdot \sum_{w\in\mathcal{W}}(-e_w - 1)\big[(N, m_w)^2 - N^2\big]$$
$$+\frac{1-N}{N}\#\mathcal{E} + \frac{1-N^2}{3N}\Big[-3\#\mathcal{W} - \#\mathcal{A} + 1 + \sum_{a\in\mathcal{A}} m_{w_a}\Big].$$

This inequality simplified reads as:

3.12. Corollary.

a) *For any* $N \ge 1$

$$\sigma_N \le \frac{1-N}{N}\#\mathcal{E} + \frac{1-N^2}{3N}\Big[-3\#\mathcal{W} - \#\mathcal{A} + 1 + \sum_{a\in\mathcal{A}} m_{w_a}\Big].$$

b) *If* $N-1 \geq B(f) := \Big[\sum_{w\in\mathcal{W}}(-e_w-1)(m_w^2-1)\Big]/(3\#\mathcal{E})$ *then:*

$$\sigma_N \leq \frac{1-N^2}{3N}\Big[-4\#\mathcal{W} - \#\mathcal{A} + 1 + \sum_{w\in\mathcal{W}}(-e_w) + \sum_{a\in\mathcal{A}} m_{w_a}\Big].$$

Proof. Use $-e_w - 1 \geq 0$, and $(N, m_w)^2 - N^2 \leq (m_w^2 - 1 + (1-N^2)$. □

All our inequalities ((3.8), (3.12), cf. also (3.5) and (3.7.b)) give estimates of type $(-\sigma_N)/(N-1) \geq \{$combinatorial expression in terms of $G_f\}$. In the next section we will study these expressions.

4. Inequalities satisfied by the minimal resolution graph of f

I. Preliminaries

Let $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ be an isolated plane curve singularity. Let G_f be its *minimal* embedded resolution, with the convention that $\#\mathcal{W} \neq \emptyset$, (which in the sequel will be shortly called resolution). We will keep the notations of section 2 for the numerical invariants of G_f. Recall that, for any arrow $a \in \mathcal{A}$, $w_a \in \mathcal{W}$ is the unique vertex adjacent with a, and m_{w_a} is its multiplicity.

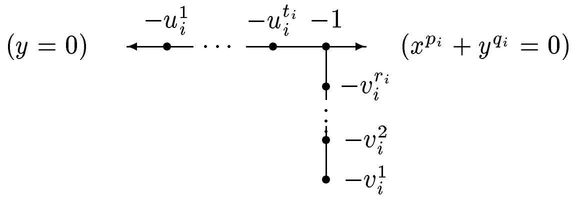
In this section we will present some properties of

$$M_f^r := \sum_{a\in\mathcal{A}} m_{w_a} - r \cdot \#\mathcal{W}(G_f) \quad (r = 2, 3, 4),$$

which distinguish the plane curve singularities among the singular germs $f : (X, x) \to (\mathbf{C}, 0)$ defined on a normal surface singularity (X, x) (cf. 4.32). We will compare M_f^r with the Milnor number μ_f of f (topological inequality), with the multiplicity ν_f of f (algebraic inequality), and with the size of G_f (combinatorial inequality).

It is convenient to use the following notation: if two germs f and g have the same topological type, in particular, the same minimal embedded resolution graph $G_f = G_g$, then we write $f \sim g$. (Notice that f can have large modularity, but G_f depends only on the topological embedding $(f^{-1}(0), 0) \subset (\mathbf{C}^2, 0)$.)

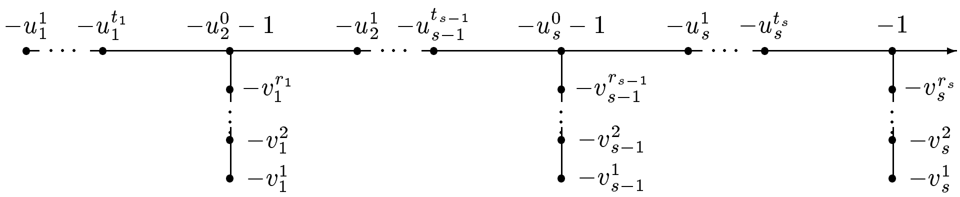
We recall the structure of G_f when f **is irreducible**. Assume that f has Newton pairs $(p_i, q_i)_{i=1}^s$ ($p_i \geq 2$, $q_i \geq 1$, $q_1 > p_1$). The minimal embedded resolution graph of $y(x^{p_i} + y^{q_i})$ ($q_i \geq 1$, $p_i \geq 2$) has the following form:



where u_i^l and v_i^l ($u_i^0, v_i^0 \geq 1$, and $u_i^l, v_i^l \geq 2$ for $l > 0$) are given by the continuous fractions:

$$\frac{p_i}{q_i} = u_i^0 - \cfrac{1}{u_i^1 - \cfrac{1}{\ddots - \cfrac{1}{u_i^{t_i}}}}\ ; \qquad \frac{q_i}{p_i} = v_i^0 - \cfrac{1}{v_i^1 - \cfrac{1}{\ddots - \cfrac{1}{v_i^{r_i}}}}\ .$$

The graph G_f can be reconstructed (by splicing) from the graphs of $y(x^{p_i} + y^{q_i})$ and the numbers u_i^0 as follows ([5] Appendix of chap. 1, and section 22):



In particular

$$\#\mathcal{W}(G_f) = \sum_{i=1}^{s} \#\mathcal{W}(G_{y(x^{p_i}+y^{q_i})}). \tag{4.1}$$

If $q_i = 1$, then

$$\#\mathcal{W}(G_{y(x^{p_i}+y^1)}) = p_i. \tag{4.2}$$

For $q_i \geq 2$ we have the following estimate:

4.3. Lemma. *Assume that $f \sim x^p + y^q$ or $f \sim y(x^p + y^q)$, with $(p, q) = 1$, $p \geq 2$, $q \geq 2$. Then*

$$\#\mathcal{W}(G_f) \leq (p + q + 1)/2.$$

Proof. First notice that $G_{x^p+y^q}$ and $G_{y(x^p+y^q)}$ differ only by an arrow, so the number of their vertices agrees. We will use the induction over the pair (p, q). Assume that $q > p$. If $q = p + 1$, then $2\#\mathcal{W}(G_f) = p + q + 1$. Assume $q \geq p + 2$. Consider the blowing up $(u, v) \xrightarrow{\phi} (x, y) = (uv, u)$. Then $\tilde{f} = f \circ \phi = u^p(v^p + u^{q-p})$.

Now, by the inductive step and the above remark: $\#\mathcal{W}(G_f) \leq 1+(p+q-p+1)/2 \leq (p+q+1)/2$. $\square$

The unique arrow of G_f corresponds to $\mathcal{A} = \{a\}$, and

$$m_{w_a} = a_s p_s, \tag{4.4}$$

where a_s can be computed inductively as follows:

$$a_1 = q_1, \quad \text{and for} \quad i \geq 1 : a_{i+1} = q_{i+1} + p_i p_{i+1} a_i. \tag{4.5}$$

Now, assume that f **is not irreducible**: $f = \prod_{a\in\mathcal{A}} f_a$, where f_a's are irreducible and $\#\mathcal{A}_f > 1$ ($\mathcal{A}_f := \mathcal{A}(G_f)$). It is convenient to fix an ordering of the set $\mathcal{A}$.

In the next lemmas, we would like to compare the graphs G_f and G_{f_a}'s. Recall that for any germ g, G_f denotes its *minimal* embedded resolution graph. If we want to emphasize that a certain invariant is considered in a graph G, then we put G in a parenthesis near the corresponding invariant.

If E is one of the irreducible exceptional divisors of the embedded resolution $\phi : (\mathcal{Y}, D) \to (\mathbf{C}^2, f^{-1}(0))$ of $f^{-1}(0)$, and $g : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is any germ, then $m_E(g, G_f)$ denotes the multiplicity of $g\circ\phi$ along E. In particular, if E_a corresponds to $w_a \in \mathcal{W}$ in the graph G_f, then $m_{E_a}(f, G_f) = m_{w_a}(G_f) = m_{w_a}$. If $a \neq a'$ ($a, a' \in \mathcal{A}_f$), then (cf. e.g. [3]): $m_{E_a}(f_{a'}) = \nu(f_a, f_{a'})$, where $\nu(\, , \,)$ denotes the intersection multiplicity at the origin. With these notations one has

$$\sum_{a\in\mathcal{A}} m_{w_a}(G_f) = \sum_{a\in\mathcal{A}} m_{E_a}(f_a, G_f) + 2\sum_{a<a'} \nu(f_a, f_{a'}). \tag{4.6}$$

Let G_{f_a} denote the minimal resolution graph of f_a. By our convention $m_{w_a}(G_{f_a})$ is the multiplicity of f_a along the irreducible divisor (in G_{f_a}) which intersects the strict transform of $f_a = 0$.

We say that two germs are "tangent" if their tangent cones at the origin have a common line.

4.7. Lemma. *Assume that $f = g \cdot f_0$ and $f' = g \cdot f_0'$ such that $f_0 \sim f_0'$ are topologically equivalent irreducible germs, and f_0' and g are not "tangent". Then*

(a)
$$\begin{aligned}&\sum_{a\in\mathcal{A}_f} m_{E_a}(f_a, G_f) - 2\#\mathcal{W}(G_f) \geq \\ &\sum_{a\in\mathcal{A}_{f'}} m_{E_a}(f_a', G_{f'}) - 2\#\mathcal{W}(G_g) - \\ &\quad 2\#\mathcal{W}(G_{f_0}) + 2 + 2T + T',\end{aligned}$$

(b)
$$\sum_{a\in\mathcal{A}_f} m_{E_a}(f_a, G_f) + 2\nu(g, f_0) - r\#\mathcal{W}(G_f) \geq$$
$$\sum_{a\in\mathcal{A}_{f'}} m_{E_a}(f'_a, G_{f'}) + 2\nu(g)\nu(f_0) -$$
$$r\#\mathcal{W}(G_g) - r\#\mathcal{W}(G_{f_0}) + r + (r+2)T + 2T',$$

where $r = 3$ or 4, and $T = T' = 0$, unless f_0 is a non-smooth (resp. smooth) component tangent with a non-smooth component of g, in which case $T = 1$ (resp. $T' = 1$).

Proof. Let $\phi_1 : (\mathcal{Y}, E) \to (\mathbf{C}^2, g^{-1}(0))$ be the minimal resolution of $g^{-1}(0)$ with exceptional divisor E (and $\mathcal{W} \neq \emptyset$). Let X_0 be the strict transform of $f_0^{-1}(0)$ via ϕ_1. Let ϕ_2 be the minimal resolution of $(\mathcal{Y}, E \cup X_0)$ at the point $E \cap X_0$. Notice that ϕ_2 can be considered as a "part" of the minimal resolution tower of $(\mathbf{C}^2, f_0^{-1}(0))$. Set $\phi = \phi_1 \circ \phi_2$, and let Z_0 (resp. Z_a) be the strict transform of $\{f_0 = 0\}$ (resp. of the irreducible component $\{g_a = 0\}$ of g) via ϕ.

Assume that $Z_0 \cap Z_a = \emptyset$ for any $a \in \mathcal{A}_g$. First notice that ϕ provides G_f, ϕ_1 provides G_g, and the number of irreducible exceptional divisors of ϕ_2 is $\leq \#\mathcal{W}(G_{f_0}) - 1$.

Now we claim that the following inequalities hold (with $k = 0$):

(i) $$\sum_{a\in\mathcal{A}_f} m_{E_a}(f_a, G_f) \geq \sum_{a\in\mathcal{A}_{f'}} m_{E_a}(f'_a, G_{f'}) + T' + 2k$$

(ii) $$\#\mathcal{W}(G_f) \leq \#\mathcal{W}(G_g) + \#\mathcal{W}(G_{f_0}) - 1 - T + k$$

(iii) $$\nu(g, f_0) \geq \nu(g)\nu(f_0) + T + T' + k.$$

Indeed, in general $\sum_{a\in\mathcal{A}_f} m_{E_a}(f_a, G_f) \geq \sum_{a\in\mathcal{A}_{f'}} m_{E_a}(f'_a, G_{f'})$, but if f_0 is smooth tangent with g, then $m_{E_{f'_0}}(f'_0, G_{f'}) = 1$. But $m_{E_{f_0}}(f_0, G_f) \geq 2$, which gives (i). For (ii) notice that $\#\mathcal{W}(G_f) \leq \#\mathcal{W}(G_g) + \#\mathcal{W}(G_{f_0}) - 1$ (see above), but if two non-smooth components are tangent, then they have at least two common infinitely near points, so the minimal resolution graph satisfies (ii); (iii) is obvious.

Now assume that $Z_0 \cap Z_a = P$ for some $a \in \mathcal{A}_g$. Let D be the irreducible exceptional divisor of ϕ with $P \in D$. If the smooth germs Z_0 and Z_a have contact k ($k \geq 1$), then we need k more blow ups. The last newly created exceptional divisor is denoted by D'. Then we claim that the inequalities (i)–(iii) are valid. Indeed: $m_{D'}(g_a) = m_D(g_a) + k$ and $m_{D'}(f_0) = m_D(f_0) + k$, which proves (i), (ii) follows by the same argument as above, and finally, (iii) follows, for example, from Max Noether's theorem (cf. e.g., [3]) which describes the intersection multiplicity in terms of the multiplicity sequence.

Finally, notice that (i)–(iii) implies the lemma. $\square$

4.8. Lemma. *Let $f=\prod_{a\in\mathcal{A}_f} f_a$ be the irreducible decomposition of f. Then*

(a)
$$\sum_a m_{E_a}(f_a,G_f)-2\#W(G_f)\geq$$
$$\sum_a \left[m_{w_a}(G_{f_a})-2\#\mathcal{W}(G_{f_a})\right]+2(\#\mathcal{A}_f-1)+2T+T',$$

(b)
$$\sum_a m_{w_a}(G_f)-r\#\mathcal{W}(G_f)\geq$$
$$\sum_a \left[m_{w_a}(G_{f_a})-r\#\mathcal{W}(G_{f_a})\right]+r(\#\mathcal{A}_f-1)+$$
$$2\sum_{a<a'}\nu(f_a)\cdot\nu(f_{a'})+(r+2)T+2T',$$

where $r=3$ or 4, and $T=T'=0$ unless f has a non-smooth irreducible component tangent to some other non-smooth (resp. smooth) component, in which case $T=1$ (resp. $T'=1$.)

Proof. The proof is over induction. We replace step by step all the irreducible components f_a of f (starting with the smooth ones) by f'_a such that $f_a\sim f'_a$ and the tangent cone of f'_a is in a generic position with respect to the tangent cone of the other components. The inductive step is provided by (4.7). With the notation of (4.7), notice that

$$\#\mathcal{W}(G_{f'})=\#\mathcal{W}(G_g)+\#\mathcal{W}(G_{f'_0})-1. \tag{4.9}$$

Therefore, (4.7.a) reads

$$\begin{gathered}\sum_a m_{E_a}(f_a,G_f)-2\#\mathcal{W}(G_f)\geq\\ \sum_a m_{E_a}(f'_a,G_{f'})-2\#\mathcal{W}(G_{f'})+2T+T'.\end{gathered} \tag{4.10}$$

Moreover, using (4.6) and (4.9), (4.7.b) reads

$$\begin{gathered}\sum_a m_{w_a}(G_f)-r\#\mathcal{W}(G_f)\geq\\ \sum_a m_{w_a}(G_{f'})-r\#\mathcal{W}(G_{f'})+(r+2)T+2T'.\end{gathered} \tag{4.11}$$

So induction can be applied. Finally, notice that if the components of f have all distinct tangent cones, then $\nu(f_a,f_{a'})=\nu(f_a)\cdot\nu(f_{a'})$, $m_{E_a}(f_a,G_f)=m_{w_a}(G_{f_a})$, and $\#\mathcal{W}(G_f)=\sum_a\#\mathcal{W}(G_{f_a})-(\#\mathcal{A}_f-1)$. Hence the lemma follows (again by 4.6). $\square$

II. Examples

We are interested in the following expressions:

$$E_f^{\rm top} := \sum_a m_{w_a} - 2\#\mathcal{W}(G_f) - \#\mathcal{A} + 1 - (\mu_f - 2),$$
$$E_f^{\rm alg} := \sum_a m_{w_a} - 3\#\mathcal{W}(G_f) - (\nu_f^2 - 2\nu_f - 3),$$
$$E_f^{\rm com} := \sum_a m_{w_a} - 4\#\mathcal{W}(G_f) - \#\mathcal{A} + 1.$$

The following table gives these invariants of the following plane curve singularities: $f \sim x$ (i.e. f smooth), $f \sim x^2 + y^{2k+1}$, (A_{2k}, $k \geq 1$), $f \sim x^3 + y^{3k+1}$, (E_{6k}, $k \geq 1$), $f \sim x^3 + y^{3k+2}$, (E_{6k+2}, $k \geq 1$), $f \sim x^2 + y^{2k}$, (A_{2k-1}, $k \geq 1$), $f \sim (x^2 + y^{2k+1})(y^2 + x^{2l+1})$, ($A_{2k,2l}$, $k \geq 1$, $l \geq 1$), $f \sim y(x^2 + y^{2k+1})$, (D_{2k+3}, $k \geq 1$).

$f \sim$	$\#\mathcal{A}$	$\sum_a m_{w_a}$	$\#\mathcal{W}$	$E_f^{\rm top}$	$E_f^{\rm alg}$	$E_f^{\rm com}$	$\sum(-e_w)$
smooth	1	1	1	1	2	-3	1
A_{2k}	1	$2(2k+1)$	$k+2$	0	$k-1$	-6	$2k+4$
E_{6k}	1	$3(3k+1)$	$k+3$	$k-1$	$6k-6$	$5k-9$	$2k+7$
E_{6k+2}	1	$3(3k+2)$	$k+3$	k	$6k-3$	$5k-6$	$2k+7$
A_{2k-1}	2	$4k$	k	2	$k+3$	-1	$2k-1$
$A_{2k,2l}$	2	$4l+4k+12$	$k+l+3$	0	$k+l+2$	-1	$2k+2l+7$
D_{2k+3}	2	$4k+7$	$k+2$	1	$k+1$	-2	$2k+4$

The Milnor number in the case of $A_{2k,2l}$ is $2k+2l+7$, in other cases is exactly the corresponding index.

III. The topological inequality

4.12. Theorem. *If G_f is the minimal embedded resolution graph of f, and μ is the Milnor number of f, then*

(a)
$$E_f^{\rm top} := \sum_{a\in\mathcal{A}} m_{w_a} - 2\#\mathcal{W}(G_f) - \#\mathcal{A} + 1 - (\mu - 2) \geq 0.$$

If $f \not\sim A_n$ $(n > 1)$, $A_{2k,2l}$ $(k \geq 1,\ l \geq 1)$, D_{2k+3} $(k \geq 1)$, E_6, then

(b)
$$\sum_{a\in\mathcal{A}} m_{w_a} - 4\#\mathcal{W}(G_f) - \#\mathcal{A} + 1 + \sum_{w\in\mathcal{W}} (-e_w) \geq \mu.$$

Proof. Assume that f is irreducible with $s = 1$ and Newton pair (p,q). By (4.3) and (4.4), $m_{w_a} - 2\#\mathcal{W} \geq pq - (p+q+1) = \mu - 2$. Assume that $s \geq 2$ (cf. I). Let $f_{(l)}$ be a germ with Newton pairs $(p_i,q_i)_{1=1}^l$, $(1 \leq 1 \leq s)$. Notice that

$$\mu(f_{(l)}) = (a_l - 1)(p_l - 1) + p_l \cdot \mu(f_{(l-1)}). \tag{4.13}$$

First consider the expression $P_{(l)} := p_l a_l - \mu(f_{(l)})$. Then $P_{(1)} \geq 4$ and for $l \geq 2$: $P_{(l)} - p_l P_{(l-1)} = p_l + q_l - 1 \geq 2$. Therefore $P_{(l)} \geq P_{(l-1)} + 2$, hence by induction $P_{(l)} \geq 2 + 2l$ for any $1 \leq l < s$.

Now, by (4.1): $E^{\mathrm{top}}_{f(l)} - E^{\mathrm{top}}_{f(l-1)} = p_l a_l - p_{l-1}a_{l-1} - 2\#\mathcal{W}(G_{y(x^{p_l}+y^{q_l})}) - \mu(f_{(l)}) + \mu(f_{(l-1)})$. If $q_l \geq 2$ then by (4.3), (4.5) and (4.13): $E^{\mathrm{top}}_{f(l)} - E^{\mathrm{top}}_{f(l-1)} \geq (p_l - 1)P_{(l-1)} - 2 \geq P_{(l-1)} - 2$. The same inequality is valid in the case $q_l = 1$ (use (4.2) instead of (4.3)). Therefore $E^{\mathrm{top}}_{f(l)} - E^{\mathrm{top}}_{f(l-1)} \geq 2l - 2$ for any $2 \leq l \leq s$, which gives:

$$\text{for irreducible } f \text{ one has: } E^{\mathrm{top}}_f \geq s(s-1). \tag{4.14}$$

Now assume that $f = \prod_a f_a$. First recall that

$$\mu(f) = \sum_a \mu(f_a) + 2\sum_{a<a'} \nu(f_a, f_{a'}) - \#\mathcal{A} + 1. \tag{4.15}$$

Now, (4.6), (4.8.a) and (4.15) gives:

$$E^{\mathrm{top}}_f \geq \sum_a E^{\mathrm{top}}_{f_a} + 2T + T' \quad \text{(cf. (4.8) for notations).} \tag{4.16}$$

Therefore (a) follows from (4.14) and (4.15). In order to prove (b) we need to verify that

$$E^{\mathrm{top}}_f + \sum_{w\in\mathcal{W}} (-e_w) - 2\#\mathcal{W} \geq 2, \tag{4.17}$$

excepting the four cases given in the hypothesis. Consider the invariant $I(G) = \sum_w(-e_w) - 2\#\mathcal{W}$ associated with any graph G. After a blow up, $\#\mathcal{W}$ increases 1, and $\sum(-e_w)$ with 2 or 3. Therefore, for any G, $I(G) \geq -1$, but if we blow up at least one node of the exceptional divisor, then $I(G) \geq 0$. Therefore, if f has at least one factor with $\nu(f_a) \geq 2$, then

$$I(G_f) \geq 0; \quad \text{and} \quad I(G_f) = 0 \Longleftrightarrow f \sim A_{2k}. \tag{4.18}$$

Assume that (4.17) is not true for f. If f is irreducible, then $\nu(f) \geq 2$, hence by (4.14) and (4.18) $s = 1$. For $\nu(f) \geq 4$, one has $I(G_f) \geq 2$, hence by (II) $f \sim A_{2k}$ or E_6. (In fact, if $s = 1$, then $I(G_f) = \#\mathcal{W} - [q/p] - 2$.)

Assume that $\#\mathcal{A} > 1$, and let S be the number of smooth components. Since $E^{\mathrm{top}}_{\mathrm{smooth}} = 1$, one has $S \leq 2$. If $S = 2$, then by (4.18) we can have no other component, hence $f \sim A_{2k-1}$. If $S \leq 1$, then by (4.18) $I(G_f) \geq 0$, hence all the tangent lines are distinct (by 4.16). In this case

$$I(G_f) = \sum_a I(G_{f_a}) + \#\mathcal{A} - 1. \tag{4.19}$$

So, if $S = 1$, then (4.18), (4.19) and $E^{\mathrm{top}}_{\mathrm{smooth}} = 1$ implies $f \sim D_{2k+3}$. If $S = 0$, then similarly: $f \sim A_{2k,2l}$. $\square$

IV. The algebraic inequality

4.20. Theorem. *If G_f is the minimal embedded resolution graph of f and ν is the multiplicity of f, then*

(a) $$E_f^{\mathrm{alg}} := \sum_{a\in\mathcal{A}} m_{w_a} - 3\#\mathcal{W}(G_f) - (\nu^2 - 2\nu - 3) \geq 0,$$

(b) $$\sum_{a\in\mathcal{A}} m_{w_a} - 3\#\mathcal{W}(G_f) - \#\mathcal{A} + 1 \geq 0.$$

Proof. The proof is similar as the proof of (4.12). First recall that if f is irreducible, then $\nu = p_1 \dots p_s$ (recall $p_1 < q_1$). If $s+1$, then $E^{\mathrm{alg}} \geq 0$ by (II) if $p_1 = 2$ or 3, and by (4.3) if $p_1 \geq 4$. If $s \geq 2$, then $E^{\mathrm{alg}}_{f_{(l)}} \geq E^{\mathrm{alg}}_{f(l-1)}$ by a similar argument as in (4.12). If $f = \prod f_a$, then by (4.8.b) $E^{\mathrm{alg}}_f \geq \sum_a E^{\mathrm{alg}}_{f_a}$. This last inequality, together with table (II), shows (b) in the case $\nu \leq 3$. If $\nu > 3$, then $\nu^2 - 2\nu - 3 \geq \#\mathcal{A} - 1$. (The details are left to the reader.) $\square$

V. The combinatorial inequality

4.21. Theorem. *Let G_f be the minimal embedded resolution graph of f.*

(a) *Assume that $f \not\sim A_n$ $(n \geq 1)$, E_6, E_8, $A_{2k,kl}$ $(k \geq 1,\ l \geq 1)$, D_{2k+3} $(k\geq 1)$. Then*
$$E_f^{\mathrm{com}} := \sum_{a\in\mathcal{A}} m_{w_a} - 4\#\mathcal{W}(G_f) - \#\mathcal{A} + 1 \geq 0.$$

(b) *$E^{\mathrm{com}}_f \geq -6$ for any f, and $E^{\mathrm{com}}_f \geq \#\mathcal{A}(\#\mathcal{A}-1) - 4$ if $\#\mathcal{A} \geq 2$.*

Proof. For irreducible f with $s = 1$: $E^{\mathrm{com}}_f \geq -6$, and for $p_1 \geq 4$: $E^{\mathrm{com}}_f \geq 0$ (use the table (II) and (4.3)). By a similar inductive step as in (4.12) $E^{\mathrm{com}}_{f_{(l)}} \geq E^{\mathrm{com}}_{f_{(l-1)}} + 6$. Therefore
$$E^{\mathrm{com}}_f \geq 6(s-2) \quad \text{for irreducible} \quad f. \tag{4.22}$$
If $\#\mathcal{A} \geq 2$, write $f = g \cdot f_0$ with f_0 irreducible. With the notation of (4.7.b), if $f'_a \neq f'_0$, then $m_{E_a}(f'_a, G_{f'}) = m_{E_a}(f'_a, G_g)$, hence by (4.6) and (4.7.b):
$$E^{\mathrm{com}}_f \geq E^{\mathrm{com}}_g + E^{\mathrm{com}}_{f_0} + 2\nu(g)\cdot\nu(f_0) + 3 + 6T + 2T'. \tag{4.23}$$
Notice that $E^{\mathrm{com}}_{\mathrm{smooth}} = -3$. Let S be the number of smooth components of f.

Assume $\#\mathcal{A} = 2$. If $S = 2$, then $f \sim A_{2k-1}$. (4.22) and (4.23) implies that if $S = 1$, then either $E^{\mathrm{com}}_f \geq 0$ or $f \sim D_{2k+3}$, and if $S = 0$, then either $E^{\mathrm{com}}_f \geq 0$ or $f \sim A_{2k,2l}$. In all cases $E^{\mathrm{com}}_f \geq -2$.

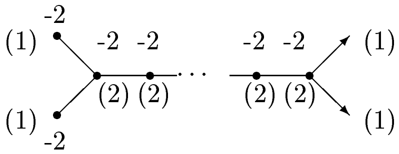
Assume that $\#\mathcal{A} > 2$. Notice that $E^{\mathrm{com}}_{f_0} + 2\nu(g)\nu(f_0) \geq 2\#\mathcal{A}_f - 5$, hence (4.23) gives $E^{\mathrm{com}}_f \geq E^{\mathrm{com}}_g + 2(\#\mathcal{A}_f - 1)$, which provides (b). $\square$

VI. Remarks

4.24. In the previous inequalities the minimality of the resolution graph G_f is necessary. With some additional blow ups, we may keep the invariants μ, ν, m_{w_a} constant, but increase $\#\mathcal{W}$ arbitrarily high.

4.25. If we want to formulate in a very simple form the above results proved for plane curve singularities, then we can say that $\sum_a m_{w_a}$ is "large" with respect to the other invariants. On the other hand, for general germs $f : (X,x) \to (\mathbf{C},0)$ (where (X,x) is a normal surface singularity) it is no longer true that $\sum_a m_{w_a}$ is large. In particular, all the inequalities presented in this section are specific for plane curve singularities, and the negativity of $\sigma(f+z^N)$ is provided exactly by the existence of this type of relations in G_f. We expect that similar relations exist for the embedded resolution of higher dimensional hypersurface singularities.

To see that in general $\sum m_{w_a}$ can be small, consider the following diagram with $\sum m_{w_a} = 4$, but with $\#\mathcal{W}$ arbitrary high.



5. Inequalities satisfied by σ_N

For the convenience of the reader we recall the notations: $\#\mathcal{W}$ (resp. $\#\mathcal{E}$) denotes the number of vertices (resp. edges) of the minimal embedded resolution graph G_f of f, $\mathcal{A}_f$ is the number of irreducible components of f, ν_f is its multiplicity and μ_f is its Milnor number.

I. Topological inequalities

Let ϵ_f be 1 if $f \sim A_n$ $(n \geq 1)$, $A_{2k,2l}$ $(k \geq 1, l \geq 1)$, D_{2k+3} $(k \geq 1)$, E_6, otherwise $\epsilon_f = 0$.

5.1 Theorem.

(a)
$$\sigma(f+z^2) \leq -\frac{1}{2}\big(\mu_f + \#\mathcal{A} - 1\big).$$

(b) *For any f and N:*
$$\sigma\big(f+z^N\big) \leq \frac{1-N}{N}\#\mathcal{E} + \frac{1-N^2}{3N}\big(\mu_f - \#\mathcal{W} - 2\big).$$

(c) *If* $N \geq B(f)+1$ *(cf.* 3.12.b*), then*

$$\sigma\big(f+z^N\big) \leq \frac{1-N^2}{3N}\big(\mu_f - 3\epsilon_f\big).$$

(d) *If* $(N, m_w)=1$ *for any* $w \in \mathcal{W}(G_f)$*, then*

$$\sigma\big(f+z^N\big) \leq \frac{1-N}{N}\big(\#\mathcal{W}+\#\mathcal{E}\big)+\frac{1-N^2}{3N}\big(\mu_f - 3\epsilon_f\big)$$

and with the same assumption (below p_g *is the geometric genus):*

$$\begin{aligned} \sigma(f+z^N) &\leq -\mu(f+z^N)/3 \quad \textit{or, equivalently:}\\ p_g(f+z^N) &\leq \mu(f+z^N)/6. \end{aligned}$$

Notice that in the inequality (5.1.a) the coefficient $1/2$ of μ is the same as in the inequality $\sigma \leq -\mu/2-\mu_0$ of Tomari [16].

Proof. By (2.2) there are at least two vertices w such that m_w is odd, hence (3.7.b) and (4.12.a) give (a). (b–c) follows from (4.12) and (3.12); and the first part of (d) from (4.12) and (3.8).

Recall the standard relations (for a germ $g:(\mathbf{C}^3,0)\to(\mathbf{C},0)$: $\mu=\mu_0+\mu_++\mu_-$, $\sigma=\mu_+-\mu_-$ and $2p_g=\mu_0+\mu_+$. The hypothesis $(N,m_w)=1$ for any w implies that $\mu_0(f+z^N)=0$ (use, e.g. A'Campo's theorem about the characteristic polynomial of f in terms of m_w's). Therefore $\sigma\leq-\mu/3$ is equivalent to $p_g\leq\mu/6$. We already know from the first part of (d) that these inequalities are valid for $f\not\sim A_n$, $A_{2k,2l}$, D_{2k+3}, E_6. In the sequel we verify these cases.

Let $g:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ be an isolated singularity with Milnor lattice L_g. Let g_t be a deformation of g such that $g_0=g$ and g_t, for $t\neq0$ small, has k singular points with Milnor lattices $L_1,\dots,L_k$. Then there is a natural embedding $\oplus_i L_i\hookrightarrow L_g$. If c is the codimension of this embedding, then $\sigma(g)\leq c+\sum_i\sigma(L_i)$. In the suspension case, any deformation f_t of f provides an embedding in $L(f+z^N)$.

Now we verify the exceptional cases. If $f\sim A_n$, then $\sigma\leq-\mu/3$ follows from (5.1.a). Actually, (5.1.a) implies that $(*)$ $\sigma(x^2+y^{2k+1}+z^N)\leq-k(N-1)$. If $f\sim D_{2k+3}$, then if we put the components in generic position, we obtain an embedding $A_{2k}\oplus A_1\oplus A_1\hookrightarrow D_{2k+3}$. Therefore $\sigma(D_{2k+3}+z^N)\leq\sigma(A_{2k}+z^N)+\sigma(A_1+z^N)+(N-1)\leq-k(N-1)-2(N-1)+(N-1)<\mu/3$ (here we used $(*)$). If $f\sim A_{2k,2l}$, one has the embedding $A_{2k}\oplus A_{2l}\oplus 4A_1\hookrightarrow A_{2k,2l}$. Recall that $\mu(A_{2k,2l})=2k+2l+7$. Now $(*)$ gives easily the result if $k+l\geq4$, otherwise we use $\sigma(x^2+y^3+z^N)\leq-4(N-1)/3$ (provided that $(N,2)=1$) which can be easily verified by Brieskorn's formula [2]. The case $f\sim E_6$ is again Brieskorn type and its verification is left to the reader. $\square$

II. Algebraic and combinatorial inequalities, and the limit $\lim_{n\to\infty} -\sigma_N/N$

Similarly, as in the "topological case", we can give a list of inequalities corresponding to the different cases. In the next theorem we present only a few; the interested reader can formulate all the others.

5.2. Theorem. *For any f and N:*

$$\sigma(f+z^N) \le \frac{1-N}{N}\#\mathcal{E} + \frac{1-N^2}{3N}(\nu_f^2 - 2\nu_f - 3 - \#\mathcal{A} + 1).$$
$$\sigma(f+z^N) \le \frac{1-N}{N}\#\mathcal{E} + \frac{1-N^2}{3N}(\#\mathcal{W} + \#\mathcal{A}(\#\mathcal{A}-1) - 6).$$
$$\sigma(f+z^N) \le \frac{1-N}{N}\#\mathcal{E} \le 0.$$

Proof. Use (3.12), (4.20) and (4.21).

Our first inequality in (5.2) is similar to the main result of [1].

5.3. Theorem. *The limit* $\lim_{N\to\infty}(-\sigma_N)/N = \eta(f;1)$ *satisfies*

$$\eta(f;1) \ge (\mu_f + \#\mathcal{W} + \#\mathcal{A} - 3 + B)/3,$$

where $B := \sum_{w\in\mathcal{W}}(-e_w) - 2\#\mathcal{W} \ge -1$*; and also:*

$$\eta(f;1) \ge (3\#\mathcal{W} + \#\mathcal{A} - 8)/3.$$

Proof. Use (3.5) and (4.12).

III. Conjectures

For any $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ and N conjecturally, the following inequalities hold:

1. $\sigma(f+z^N) \le -\dfrac{N+1}{3N}\mu(f+z^N).$

2. $\eta(f;1) + \eta(f;N) - \eta(f;N+1) \ge 0,$

(which would imply the monotonity: $\sigma_{N+1} \le \sigma_N$).

3. $\eta(f;1) + \eta(f;N) - \eta(f;N+1) \ge \mu_f/3,$

(which would imply, for example, the inequality $\sigma \le -\mu/3$ for $f + z^N$).

Appendix

In this appendix we separate some properties of the generalized Dedekind sums [14], [21]. They are defined for arbitrary non-zero integers a, b, c by

$$s(b,c;a)=\sum_{k=1}^{a}\left(\left(\frac{kb}{a}\right)\right)\left(\left(\frac{kc}{a}\right)\right),$$

where $((x))$ is the usual function defined via the fractional part {x} as

$$((x))=\begin{cases}\{x\}-1/2 & \text{if } x\notin \mathbf{Z}\\ 0 & \text{otherwise.}\end{cases}$$

Below $(a_1,\dots,a_n)$ denotes the greatest common divisor of these numbers. By a similar argument as in [10] (A.1), one can prove that

$$s(b,c;a)=(a,b,c)\cdot s\left(\frac{b(a,b,c)}{(a,b)(b,c)},\frac{c(a,b,c)}{(a,c)(b,c)};\frac{a(a,b,c)}{(a,b)(a,c)}\right). \tag{A.1}$$

The famous generalization of the reciprocity law of Dedekind, given by Rademacher [14] (see also [21]), asserts that if a, b, c are strict positive, mutually coprime integers, then $s(b,c;a)+s(c,a;b)+s(b,a;c)=-1/4+(a^2+b^2+c^2)/12abc$. Using (A.1), this relation for arbitrary, strict positive integers reads

$$\begin{aligned}&s(b,c;a)+s(c,a;b)+s(b,a;c)=\\&\qquad -\frac{(a,b,c)}{4}+\frac{a^2(b,c)^2+b^2(a,c)^2+c^2(b,a)^2}{12abc}.\end{aligned} \tag{A.2}$$

The inequality (A.3) of [10] together with (A.1) of this paper provide:

Corollary. *For any integers a, b, c ($a>0$) one has:*

$$\mid s(b,c;a)\mid \le \frac{(a-1)(a-2)}{12a}. \tag{A.3}$$

$$\mid s(b,c;a)\mid \le \frac{a^2-(a,b)^2}{12a}. \tag{A.4}$$

References

[1] T. Ashikaga. The Signature of the Milnor fiber of Complex Surface Singularities on Cyclic Coverings. Preprint (1995).

[2] E. Brieskorn. Beispiele zur Differentialtopologie von Singularitäten. *Invent. Math.* **2** (1966), 1–14.

[3] E. Brieskorn and H. Knörrer. *Plane algebraic curves.* Birkhäuser, 1986.
[4] A. Durfee. The Signature of Smoothings of Complex Surface Singularities. *Math. Ann.* **232** (1978), 85–98.
[5] D. Eisenbud and W. Neumann. *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities.* Ann. of Math. Studies **110**. Princeton University Press, 1985.
[6] L.J. Mordell. Lattice points in a tetrahedron and generalized Dedekind sums. *J. Indian Math.* **15** (1951), 41–46.
[7] A. Némethi. The equivariant signature of hypersurface singularities and etainvariant. *Topology* **34** (1995), 243–259.
[8] A. Némethi. The eta-invariant of variation structures I. *Topology and its Applications* **67** (1995), 95–111.
[9] A. Némethi. The real Seifert form and the spectral pairs of isolated hypersurface singularities. *Compositio Math.* **98** (1995), 23–41.
[10] A. Némethi. Dedekind sums and the signature of $f(x,y)+z^N$. *Selecta Mathematica, New ser.* **4** (1998), 361–376.
[11] W. Neumann. Splicing Algebraic Links. *Advanced Studies in Pure Math.* **8** (1986), 349–361, (Proc U.S.-Japan Seminar on Singularities 1984).
[12] W. Neumann and J. Wahl. Casson invariant of links of singularities. *Comment. Math. Helv.* **65** (1991), 58–78.
[13] J.E. Pommersheim. Toric varieties, lattice points and Dedekind sums. *Math. Ann.* **295** (1993), 1–24.
[14] H. Rademacher. Generalization of the Reciprocity formula for Dedekind sums. *Duke Math. Journal* **21** (1954), 391–397.
[15] H. Rademacher and E. Grosswald. Dedekind sums. *The Carus Math. Monographs* **16** (1972).
[16] M. Tomari. The inequality $8p_g \leq \mu$ for hypersurface two-dimensional isolated double points. *Math. Nachr.* **164** (1993), 37–48.
[17] J. Wahl. Smoothings of normal surface singularities. *Topology* **20** (1981), 219–246.
[18] Y. Xu and S.S.-T. Yau. The inequality $\mu \geq 12p_g - 4$ for hypersurface weakly elliptic singularities. *Contemporary Math.* **90** (1989), 375–344.
[19] Y. Xu and S.S.-T. Yau. Durfee's conjecture and coordinate free characterization of homogeneous singularities. *J. Diff. Geometry* **37** (1993), 375–396.
[20] Y. Xu and S.S.-T. Yau. A sharp estimate of the number of integral points in a tetrahedron. *J. reine angew. Math.* **423** (1992), 199–219.
[21] D. Zagier. Higher dimensional Dedekind sums. *Math. Ann.* **202** (1973), 149–172.
[22] C. Weber. (Ed.) *Noeuds, tresses et singularités.* Monographie No. 31 de L'Enseignement Mathématique, 1983, (Problem C. of W. Neumann).

A. Némethi
The Ohio State University
Dept. of Mathematics
231 West 18th Avenue
Columbus, OH 43210
USA
e-mail: nemethi@math.ohio-state.edu