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Casson invariant of cyclic coverings via eta-invariant and Dedekind sums

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Abstract

Let Σ be a 3-dimensional oriented manifold and let $K \subset \Sigma$ be a knot. We assume that Σ is an integer homology sphere and (Σ, K) has a plumbing representation. We denote the cyclic n-fold covering of Σ branched along K by $\Sigma(K, n)$, and we assume that this manifold is integer homology sphere as well. If λ denotes the Casson invariant, then we show that $\lambda(\Sigma(K,n)) - n \cdot \lambda(\Sigma)$ can be computed from homological information only. More precisely, we compute in terms of an eta-type-invariant associated with the isometric structure of the knot. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and the main result

If Σ is a 3-dimensional oriented manifold such that $H_*(\Sigma, \mathbb{Z}) = H_*(S^3, \mathbb{Z})$, we denote its Casson invariant (see [1]) by $\lambda(\Sigma)$. If $K \subset \Sigma$ is a knot in Σ, then the n-fold cyclic covering of Σ branched along K is denoted by $\Sigma(K, n)$. We would like to compute the expression $\lambda(\Sigma(K,n)) - n \cdot \lambda(\Sigma)$ in terms of homological invariants of the covering, provided that $\Sigma(K, n)$ is an integer homology sphere as well.

In this note, if the group of coefficients of a homology group is not specified, then it is $\mathbb{C}$. Set $X = \Sigma \setminus T$, where T is an open tubular neighborhood of K, and let $\widetilde{X}$ be the infinite cyclic covering of X determined by the natural homomorphism $\pi_1(X) \to H_1(X, \mathbb{Z}) = \mathbb{Z}$. In particular, $\mathbb{Z}$ acts freely as the group of covering transformations of $\widetilde{X}$. Its generator $1_{\mathbb{Z}}$ induces the monodromy transformation $t_*: H_1(\widetilde{X}) \to H_1(\widetilde{X})$. On

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the other hand, there exists a natural skew-symmetric non-singular cup product pairing $H^1(\widetilde{X}, \partial\widetilde{X})^{\otimes 2} \to H^2(\widetilde{X}, \partial\widetilde{X}) = \mathbb{C}$ [9]. By the duality $H^1(\widetilde{X}, \partial\widetilde{X}) = H_1(\widetilde{X})$, we can consider its dual $b: H_1(\widetilde{X}) \otimes H_1(\widetilde{X}) \to \mathbb{C}$. Moreover $b(t_* x, t_* y) = b(x, y)$ for any x and y, in particular, the system $\mathcal{I} = (H_1(\widetilde{X}), b, t_*)$ constitutes an isometric structure. Notice also that t_* has no eigenvalue equal to one. The isometric structure $\mathcal{I}$ defines a flat hermitian bundle over the circle S^1 with hermitian form $\sqrt{-1} \cdot b$ and monodromy t_*. In particular, following Atiyah, Patodi and Singer [2], we can define its eta-invariant. Actually, for any integer n with the property $\det(t_*^n - 1) \neq 0$ (or equivalently $H_1(\Sigma(K, n)) = 0$), we define $\eta(\mathcal{I}; n)$ in the following way. Consider the spectral decomposition

$$\bigl(H_1(\widetilde{X}), b\bigr) = \bigoplus_{\chi} \bigl(H_1(\widetilde{X})_\chi, b_\chi\bigr)$$

provided by the operator t_*^n (i.e., $(H_1)_\chi$ is the generalized χ-eigenspace of t_*^n). Then the eta-invariant $\eta(\mathcal{I}; n)$ is defined by the sum $\sum_\chi \eta(\mathcal{I}; n)_\chi$, where

$$\eta(\mathcal{I}; n)_\chi = \begin{cases} (1 - 2c) \cdot \operatorname{signature}(\mathrm{i} b_\chi) & \text{if } \chi = \mathrm{e}^{2\pi \mathrm{i} c};\ 0 < c < 1, \\ 0 & \text{if } |\chi| \neq 1. \end{cases}$$

(In the above definition we used the fact that $\chi = 1$ is not an eigenvalue of t_*^n. For the definition of η in the general case, see [14]. Actually, for $\chi \neq 1$, $\eta(\mathcal{I}; n)_\chi$ is the eta-invariant of Atiyah, Patodi and Singer associated with the circle S^1 and the flat bundle over S^1 with monodromy χ and hermitian form $\mathrm{i} b_\chi$.)

The main result of this note is the following:

Theorem 1. *Assume that the pair (Σ, K) has a plumbing representation* (*or equivalently, if (Σ, K) is a graph knot, in the sense of Eisenbud and Neumann's book* [5]). *If the n-fold cyclic covering of Σ, branched along K, is integer homology sphere as well, then*

$$\lambda\bigl(\Sigma(K, n)\bigr) - n \cdot \lambda(\Sigma) = \tfrac{1}{8}\bigl(\eta(\mathcal{I}; n) - n \cdot \eta(\mathcal{I}; 1)\bigr). \tag{$*$}$$

In the plumbing representation we allow non-connected plumbing graphs as well, or equivalently, we allow disjoint union of splice diagrams.

This shows that in the above cases the Casson invariant behaves as a secondary invariant with respect to the n-fold cyclic (branched) coverings.

Remarks.

(1) The above theorem gives in homological terms the Casson invariant of any cyclic covering of S^3 branched along a knot $K \subset S^3$, which can be represented by plumbing (or splice) diagram (since $\lambda(S^3) = 0$).

(2) The formula from Theorem 2 shows that the function $n \mapsto \eta(\mathcal{I}; n)$ (n with $\det(t_*^n - 1) \neq 0$) is periodic (cf. [14, Section 5]). Therefore, if (Σ, K) is a graph knot, then $n \mapsto \lambda(\Sigma(K, n))$ is quasi-periodic (i.e., is a sum of a linear function and a periodic function).

(3) The relation $(*)$ makes sense even in the case of rational homology spheres (for the left hand side, see generalization by Walker [26]). Hence, one can expect that it is true even in that case.

The first part of the proof of the theorem contains the reduction to the Seifert graph case. The key steps of the proof are the relations (1) and (3) (proved in [6] and [14], respectively). The first one relates the Casson invariant to the signature of the Milnor fiber of a hypersurface singularity. The second one expresses this signature in terms of eta-invariant, this is an index-theoretical result about the signature defect.

In Sections 3 and 4, we emphasize the arithmetical aspects. If a 3-manifold Σ can be represented by a splice diagram, then there is an arithmetical formula of $\lambda(\Sigma)$ in terms of the weights (see, for example, [13]). In this paper we give a different set of relations.

In Section 3, we express the eta-invariant $\eta(\mathcal{I};n)$ in terms of generalized Dedekind sums [22,27] associated with the multiplicities of the plumbing graph of the pair (Σ,K) (Theorem 2). This relation makes the connection between the Casson invariant, eta-invariant and the Dedekind sums. In Section 4 we apply the Rademacher–Dedekind reciprocity law in order to transform our relation into a deeper formula which describes $\lambda(\Sigma(K,n))$ in terms of the combinatorics of the graph and Dedekind sums (Theorem 3). Here we discuss also the particular case of algebraic knots (Σ,K). This makes the connection with the signature of hypersurface singularities and the result of Mordell [10] about the number of lattice points in a (three-dimensional) tetrahedron.

2. Proof of Theorem 1

In the case of integer homology spheres, the plumbing representation is equivalent to the representation of (Σ,K) in terms of splice diagrams [5]. Since the latter one is more concise, in this proof we prefer this representation.

Reduction to the irreducible case. Assume that the splice diagram Γ of (Σ,K) is not connected: Γ is the disjoint union $\Gamma_1\cup\Gamma_2$, where Γ_1 is connected and contains the arrow corresponding to the knot K. This means that Σ is the disjoint sum $\Sigma_1\#\Sigma_2$, where $K\subset\Sigma_1$. Moreover, $\Sigma(K,n)=\Sigma_1(K,n)\#\Sigma_2\#\cdots\#\Sigma_2$ (n copies of Σ_2), hence by the additivity theorem of Casson (see [1]):

$$\lambda\bigl(\Sigma(K,n)\bigr)-n\lambda(\Sigma)=\lambda\bigl(\Sigma_1(K,n)\bigr)-n\lambda(\Sigma_1).$$

On the other hand, the isometric structure $\mathcal{I}(\Sigma,K)$ associated with the pair (Σ,K) is isomorphic to the isometric structure $\mathcal{I}(\Sigma_1,K)$, hence the relation $(*)$ is true if and only if it is true for connected diagrams.

In the sequel we assume that the splice diagram Γ of (Σ,K) is connected and $n>1$.

The irreducible case. Call a weight on an edge of Γ "near" or "far" according to whether it is on the end of the edge nearest to or further from the (unique) arrowhead of Γ (representing K). We recall the following results. (In the following two facts we assume that Γ is minimal.)

Fact 1 [13]. *The n-fold cyclic cover of Σ branched along K is an integer homology sphere $\Sigma(K,n)$ if and only if n is prime to all near weights in Γ, and $\Sigma(K,n)$ is then represented by the splice diagram obtained from Γ by multiplying each far weight by n.*

Fact 2 [5, (11.2)]. *(Σ, K) is a fibered knot if and only if the multiplicities of all nodes are non-zero, or equivalently, all the near weights in Γ are non-zero.*

These give the following:

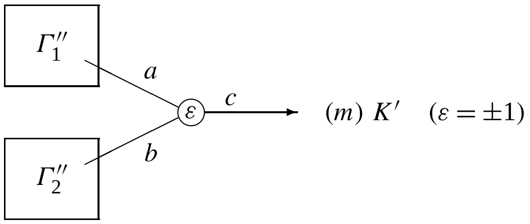
Corollary 1. *The irreducible graph knot (Σ, K) is fiberable if and only if there exists $n > 1$ such that the n-fold cyclic cover $\Sigma(K, n)$ is an integer homology sphere. In particular, we obtain the fiberability of (Σ, K) in the main theorem.*

If F denotes the fiber of $\Sigma \setminus T \to S^1$ (which is isotopic to the minimal Seifert surface of K), then $\widetilde{X} \sim F \times \mathbb{R}$, hence: $\mathcal{I} = (H_1(F); \langle,\rangle; h)$, where $\langle,\rangle$ is the skew-symmetric intersection form on the 2-dimensional manifold F, and h is the monodromy transformation of the fibration.

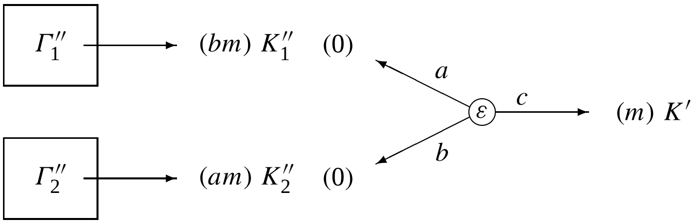
Notice that the graph Γ is not unique, but equivalent graphs [5, 8.1] give the same invariants λ and η, so we can choose (by [5, 8.1, Property 6]) a diagram Γ such that each vertex has a degree less than or equal to three (see also [13, p. 60]).

Now, if we have a multilink $(\Sigma', K'(m))$, represented by the graph Γ':

Γ':

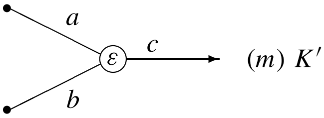


then we can consider its splice decomposition:



and the splice diagram:

$\Gamma_{a,b,c}(\varepsilon, m)$:



(Sometimes it is convenient to write $a = a_1$, $b = a_2$. Above, if Γ_i'' ($i = 1, 2$) is only a vertex, then the diagram $\Gamma_i'' \to (a_i m)$ is empty.)

Using the additivity of the Casson invariant under splicing, proved independently by Akbulut and McCarthy [1], Boyer and Nicas [4] and Fukuhara and Maruyama (according to [4]) (see also [13, p. 60]) one has:

$$\begin{aligned}\lambda\big(\Sigma'(K', n)\big) - n\lambda(\Sigma') &= \lambda\big(\Sigma(a, b, c)(K', n)\big) - n\lambda\big(\Sigma(a, b, c)\big)\\ &\quad + \sum_{i=1}^{2} \lambda\big(\Sigma''(K_i'', n)\big) - n\lambda(\Sigma_i'').\end{aligned}$$

On the other hand, by a simple Mayer–Vietoris argument, the isometric structure $\mathcal{I}(\Gamma', K'(m))$ splits in the direct (orthogonal) sum

$$\bigoplus_{i=1}^{2} \mathcal{I}\big(\Gamma_i'', K_i''(a_i m)\big) \oplus \mathcal{I}\big(\Gamma_{a,b,c}(\varepsilon, m), K'(m)\big)$$

(see, for example, [5, pp. 114–116]). Therefore, in order to prove ($*$), it is enough to prove it for diagrams $\Gamma_{a,b,c}(\varepsilon, m)$, where $(m, n) = 1$ (cf. Fact 1), and the isometric structure is provided by the fibration of the multilink structure $K'(m)$.

Using [5, 8.1, Property 2], we can assume that $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$. Fact 2 gives that $a > 0$ and $b > 0$. If $c = 0$, then $(a, b, c) = (1, 1, 0)$ and (by the classification theorem of [5]) Σ is S^3 and the link is trivially embedded. Hence both sides of ($*$) are zero.

In the sequel assume that $a > 0$, $b > 0$, $c > 0$. By [5, 8.1, Property 1], we can assume that $m > 0$ as well. In the diagram $\varepsilon = \pm 1$ denotes the orientation class of Σ, if we change ε into $-\varepsilon$ then in ($*$) both sides will change their sign, so we can assume that $\varepsilon = +1$ (cf. [5, p. 119]).

Notice that the left hand side of ($*$), applied for the diagram $\Gamma_{a,b,c}(1, m)$ is independent of (the multilink structure) m. In the following lemma we prove the similar fact for the right hand side.

Let F be the fiber of $(\Gamma_{a,b,c}(1, 1), K(1))$, b the intersection form, h the monodromy operator, and set $\mathcal{I} = (H_1(F, k), b, h)$. Then the fiber of the multilink $(\Gamma_{a,b,c}(1, m), K(m))$ is the disjoint union $F \cup \cdots \cup F$ (m copies) and the corresponding isometric structure $\mathcal{I}_m$ is [5, p. 115]: $\mathcal{I}_m = (H_1(F, k)^{\oplus m}, b^{\oplus m}, h_m)$, where $h_m(x_1, \ldots, x_m) = (x_2, x_3, \ldots, h(x_1))$.

Lemma 1. *Assume that* $m > 0$, $(m, n) = 1$, *and* $\det(h^n - 1) \neq 0$. *Then* $\eta(\mathcal{I}_m; n) = \eta(\mathcal{I}; n)$, *in particular*:

$$\eta(\mathcal{I}_m; n) - n \cdot \eta(\mathcal{I}_m; 1) = \eta(\mathcal{I}; n) - n \cdot \eta(\mathcal{I}; 1).$$

Proof. Notice that the monodromy h has distinct eigenvalues hence, over $\mathbb{C}$, the isometric structure $\mathcal{I}$ decomposes in a sum of one-dimensional hermitian isometric structures [8]. Therefore, it is enough to verify the above equality for a hermitian isometric structure $\mathcal{I} = (\mathbb{C}, \mathrm{i}, \mathrm{e}^{2\pi \mathrm{i} c})$, where $0 < c < 1$, and $nc \notin \mathbb{Z}$. In this case:

$$\mathcal{I}_m = \bigoplus_{j=0}^{m-1} \big(\mathbb{C}, \mathrm{i}, \mathrm{e}^{2\pi \mathrm{i}(c+j)/m}\big),$$

hence:

$$\eta(\mathcal{I}_m; n) = \sum_{j=0}^{m-1}\left(1 - 2\left\{n \cdot \frac{c+j}{m}\right\}\right) = 1 - 2\{nc\} = \eta(\mathcal{I}; n).$$

(Cf. (A.1) or Lemma 1 [23]. Above $\{x\}$ denotes the fractional part of x.) □

Remark 1. It is not difficult to verify that for any Γ and $m > 0$ one has:

$$\eta\big(\mathcal{I}\big(\Gamma, K(-m)\big); n\big) = \eta\big(\mathcal{I}\big(\Gamma, K(m)\big); n\big).$$

Indeed, the modification $m \mapsto -m$ in the multilink structure provides the modification $(b, t_*) \mapsto (-b, t_*^{-1})$ in the isometric structure. In particular, $\eta(\mathcal{I}(\Gamma, K(m)); n)$ is independent of the choice of $m \in \mathbb{Z} \setminus \{0\}$.

By Lemma 1, we can assume that $m = 1$, i.e., $(\Sigma = \Sigma(a, b, c), K)$ is the Seifert knot given by the diagram $\Gamma_{a,b,c}(1, 1)$, where a, b, c are relative prime integers. The n-fold cyclic covering $\Sigma(K, n)$ is exactly $\Sigma(a, b, cn)$ (where $(a, n) = (b, n) = 1$). Now, by [6, (2.10)]:

$$8 \cdot \lambda\big(\Sigma(a, b, N)\big) = \text{signature } \sigma \text{ of the Milnor fiber of the hypersurface singularity } x^a + y^b + z^N. \tag{1}$$

Let $\mathcal{I}_{a,b} = (H, b_{a,b}, h_{a,b})$ be the isometric structure associated with the knot $(S^3 = \Sigma(a, b, 1), K_{a,b})$ (given by the diagram $\Gamma_{a,b,1}(1, 1)$). Then (see, for example, [7]) $(\Sigma(a, b, N), K)$ is the N-fold cyclic covering of $(S^3, K_{a,b})$, in particular:

$$\mathcal{I}\big(\Sigma(a, b, N), K\big) = \big(H, b_{a,b}, h_{a,b}^N\big). \tag{2}$$

Now, by [14, (5.22)]:

$$\sigma(x^a + y^b + z^N) = \eta(\mathcal{I}_{a,b}; N) - N \cdot \eta(\mathcal{I}_{a,b}; 1). \tag{3}$$

Therefore:

$$\begin{aligned}
&8\big[\lambda(\Sigma(K, n)) - n\lambda(\Sigma)\big]\\
&\quad\overset{(1)}{=} \sigma(x^a + y^b + z^{nc}) - n \cdot \sigma(x^a + y^b + z^c)\\
&\quad\overset{(3)}{=} \eta(\mathcal{I}_{a,b}; nc) - nc\eta(\mathcal{I}_{a,b}; 1) - n\big[\eta(\mathcal{I}_{a,b}; c) - c\eta(\mathcal{I}_{a,b}; 1)\big]\\
&\quad= \eta(\mathcal{I}_{a,b}; nc) - n\eta(\mathcal{I}_{a,b}; c)\\
&\quad\overset{(2)}{=} \eta(\mathcal{I}; n) - n\eta(\mathcal{I}; 1).
\end{aligned}$$

This ends the proof of Theorem 1. □

3. The eta-invariant via the plumbing graph

In this section we compute the eta-invariant $\eta(\mathcal{I}; n)$ of the isometric structure $\mathcal{I} = \mathcal{I}(\Sigma, K)$ in terms of the combinatorial data of a plumbing graph G of (Σ, K).

First, we introduce some notations. Let $\mathcal{W}$ be the set of non-arrowhead vertices of G. The knot K is represented in G by an arrow attached to the vertex $w_0 \in \mathcal{W}$. Let $\mathcal{V}$ be the set of vertices: $\mathcal{V} = \mathcal{W} \sqcup \{v_0\}$, where v_0 is the arrowhead corresponding to K. For any $w \in \mathcal{W}$, we denote by $\mathcal{V}_w$ the set of vertices $v \in \mathcal{V}$ adjacent to w. Set $\delta_w = \#\mathcal{V}_w$ for any $w \in \mathcal{W}$. If $\delta_w > 2$, then $w \in \mathcal{W}$ is called "rupture point". The set of rupture points is denoted by $\mathcal{R}$. Let $\mathcal{E}$ be the set of edges (i.e., the set of non-ordered pairs (u, v), $u, v \in \mathcal{V}$, such that u adjacent to v).

Since Σ is a homology sphere, G is a tree. It is decorated by the Euler-numbers e_w $(w \in \mathcal{W})$. We recall that the plumbing construction provides a canonical orientation of Σ, and an orientation of K: K is a fiber of the oriented circle bundle corresponding to w_0.

Recall that any $\boldsymbol{m} \in H_1(\Sigma \setminus K, \mathbb{Z})^*$ defines a multilink structure of the pair (Σ, K) (cf. [5, pp. 136–137]). If $M = M_{v_0}$ is the standard topological meridian of K (i.e., $l(M, K) = 1$, where l denotes the linking number), then $\boldsymbol{m}(M) = m_{v_0}$ (which sometimes is denoted only by m) is the multiplicity of the knot K; and the map $[S] \mapsto \boldsymbol{m}([S])$ is defined by $\boldsymbol{m}([S]) = m \cdot l(S, K)$. For any $w \in \mathcal{W}$ let M_w be an oriented fiber of the oriented circle bundle (used in the plumbing construction) corresponding to w. Then the "multiplicity of w" is:

$$m_w = \boldsymbol{m}\big([M_w]\big) = m \cdot l(M_w, K). \tag{3.1}$$

They satisfy the following relations: for any $w \in \mathcal{W}$ one has

$$e_w \cdot m_w + \sum_{v \in \mathcal{V}_w} m_v = 0. \tag{3.2}$$

Lemma 2. *Let (Σ, K) and n be as in the introduction, G a plumbing graph of (Σ, K), and $m = m_{v_0} = \pm 1$. Then one has*:

(a) $(n, m_w) = 1$ *for any* $w \in \mathcal{R}$*;*
(b) $(n, m_u, m_v) = 1$ *for any* $(u, v) \in \mathcal{E}$*; and*
(c) *if G is minimal* (*see, for example,* [5]), *then $m_v \neq 0$ for any $v \in \mathcal{V}$.*

Proof. (a) The rupture points correspond to the Seifert components in a splice decomposition. By the algorithm given in [5, p. 84] m_w is a product of near weights. Now apply Fact 1.

(b) Consider a chain $\{u_1, u_2, \ldots, u_s\}$ of G (i.e., $u_{i+1} \in \mathcal{V}_{u_i}$ for $i = 1, \ldots, s-1$; $\delta_{u_i} = 2$ for $i = 2, \ldots, s-1$) such that $\{u_1, u_2\} = \{u, v\}$ and $u_s \in \mathcal{R}$. Let $d = (n, m_u, m_v)$. Then (3.2) applied for the nodes $u_2, \ldots, u_{s-1}$ gives that $d | m_{u_s}$. Therefore $d = 1$ by (a).

(c) This follows from the fiberability condition; it is a reformulation of Fact 2 for plumbing graphs. $\square$

In the sequel it is convenient to use the following classical notation:

$$\big((x)\big) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Theorem 2. *Let (Σ, K) be as above, and $\mathcal{I}$ its isometric structure. Let G be a plumbing graph of (Σ, K) such that $m_v \neq 0$ for any $v \in \mathcal{V}$. Fix a multilink structure $\boldsymbol{m}$ with $m \neq 0$. Let n be an integer such that $\Sigma(K, n)$ is a rational homology sphere. Then*:

$$\eta(\mathcal{I}, n) = 4 \cdot \sum_{w \in \mathcal{W}} S_w \quad \textit{where } S_w = \sum_{v \in \mathcal{V}_w} \sum_{k=1}^{|m_w|-1} \left(\left(\frac{k m_v}{m_w}\right)\right) \cdot \left(\left(\frac{kn|m|}{|m_w|}\right)\right). \qquad (**)$$

Remark 2.

(a) Notice that by (3.1) S_w is independent of the choice of $\boldsymbol{m}$. (This fact is consistent with Lemma 1 and Remark 1.)

(b) Equivalent graphs (with the property $m_w \neq 0$ for any $w \in \mathcal{W}$) provide the same expression $\sum_w S_w$, i.e., the operations described on page 140 in [5] do not alter the right hand side of $(**)$.

(c) By $(**)$, the function $n \mapsto \eta(\mathcal{I}; n)$ is periodic.

Proof of Theorem 2. The relations (3.2) imply that for w with $\delta_w \leqslant 2$ one has $S_w = 0$. Therefore, the right hand side $R(G, n) = 4 \sum_{w \in \mathcal{W}} S_w$ of $(**)$ is $4 \cdot \sum_{w \in \mathcal{R}} S_w$. This shows that $R(G, n)$ is additive with respect to splicing. On the other hand, we can repeat the additivity argument of the eta-invariant used in the proof of Theorem 1 (namely [5, pp. 114–116]). These, and Remark 2(a), imply that it is enough the verify the identity $\eta(\mathcal{I}, n) = R(G, n)$ only for plumbing graphs corresponding to Seifert knots $(\varepsilon\Sigma(a_1, \ldots, a_\delta), K(1))$, where $a_2, \ldots, a_\delta > 0$, $a_1 \geqslant 0$, and a_1 is the far-weight of the edge which has the arrow corresponding to K.

If $a_1 = 0$, then $a_i = 1$ for $i > 1$ and $\eta(\mathcal{I}; n) = 0$. But the multiplicity of the unique rupture point is ± 1, so $R(G, n) = 0$ too.

In the sequel we assume that $a_1, \ldots, a_\delta > 0$. In the next paragraph we prove that the case $\varepsilon = -1$ follows from the case $\varepsilon = +1$.

Indeed, if $G(\{e_w\}_{w \in \mathcal{W}}, K(1))$ is the decorated plumbing graph associated with $(+\Sigma(a_1, \ldots, a_\delta), K(1))$, then $G(\{-e_w\}_{w \in \mathcal{W}}, K((-1)^r))$ is a possible decorated graph of $(-\Sigma(a_1, \ldots, a_\delta), K(1))$, where r is the number of vertices in the chain (strict) between the rupture point and the arrow-head [11, (3.3)]. If $m_v^{\pm}$ denotes the corresponding multiplicities for $\varepsilon = \pm 1$, then by (3.2), $m_v^- \in \{m_v^+, -m_v^+\}$ for any $v \in \mathcal{V}$, and

$$\frac{m_u^-}{m_v^-} = -\frac{m_u^+}{m_v^+} \quad \text{for any } (u, v) \in \mathcal{E}.$$

Therefore $R(G(-\varepsilon), n) = -R(G(+\varepsilon), n)$. But $\eta(\mathcal{I}(-\Sigma, K); n) = -\eta(\mathcal{I}(\Sigma, K); n)$ (because the change of the orientation provides the modification $(b, t_*) \mapsto (b, t_*^{-1})$). Hence the reduction follows.

Now consider the knot $(\Sigma(a_1, \ldots, a_\delta), K(1))$. This is an algebraic knot (see, for example, [5, p. 62]): there exist an analytic normal surface singularity $(\mathcal{X}, x)$ and an analytic germ $f : (\mathcal{X}, x) \to (\mathbb{C}, 0)$ such that $(\mathcal{X} \cap S_r, f^{-1}(0) \cap S_r)$ is diffeomorphic to (Σ, K), where $S_r = \rho^{-1}(r)$ for a sufficiently small $r > 0$ and real analytic map $\rho : \mathcal{X} \to [0, \infty)$ with $\rho^{-1}(0) = \{x\}$.

In the sequel we would like to apply the results of [14, Sections 5.16–5.22] and [15]. These are formulated for hypersurface singularities. In their proofs we used two ingredients: the variation map associated with the germ f is an isomorphism (over real numbers), and the polarization properties of the limit mixed Hodge structure of the vanishing cohomology of f. In our new situation here, $\mathcal{X} \cap S_r = \Sigma$ is an integer homology sphere, therefore the variation map of f (which is equivalent to the Seifert form of $K \subset \Sigma$ by Alexander duality) is unimodular. On the other hand, all the polarization properties of the mixed Hodge structure are valid in this case as well, see, for example, [25,17, 15]. (Actually, in our case the monodromy operator has no eigenvalues $= 1$, so the limit mixed Hodge structure of the vanishing cohomology has the same nice polarization properties as the limit mixed Hodge structure associated with degeneration of projective fibers.) In particular, the results described in [15] and [14, (5.16–5.22)] are true for germs $f : (\mathcal{X}, x) \to (\mathbb{C}, 0)$ as well, provided that $\mathcal{X} \cap S_r$ is a homology sphere.

Notice that $\det(t_* - 1) \neq 0$ and $\det(t_*^n - 1) \neq 0$, hence (5.20–5.21) in [14] reads as:

$$\eta(\mathcal{I}; n) = 2 \cdot \sum \big((nc)\big)\big(\Sigma p_{\lambda,-}(f) - \Sigma p_{\lambda,+}(f)\big),$$

where the sum is over all eigenvalues $\lambda = \mathrm{e}^{-2\pi \mathrm{i} c}$ $(0 < c < 1)$ of the monodromy operator; and

$$\Sigma p_{\lambda,\pm}(f) = \#\big\{c\colon c \text{ is a spectral number of } f \text{ with } \lambda = \mathrm{e}^{-2\pi \mathrm{i} c},\ (-1)^{[c]} = \pm 1\big\}.$$

On the other hand, the set of spectral numbers (or the characteristic numbers) $Sp(f) \in \mathbb{Z}[\mathbb{Q}]$ associated with f is computed from the plumbing graph (or equivalently from the embedded resolution graph $(\mathcal{X}, f^{-1}(0))$) in [24,21]. Proposition 6.5 and Remark 2.11 of [21] give:

$$Sp(f) = \sum_{e \in \mathcal{E}} \sum_{k=1}^{m_e - 1} \left(\left(-\frac{k}{m_e}\right) + \left(\frac{k}{m_e}\right)\right) + \sum_{w \in \mathcal{W}} \sum_{k=1}^{m_w - 1} \left(\sum_{v \in \mathcal{V}_w} \left\{\frac{k m_v}{m_w}\right\} - 1\right)\left(\left(1 - \frac{k}{m_w}\right) + \left(-1 + \frac{k}{m_w}\right)\right),$$

where $m_e = (m_u, m_v)$ for any $e = (u, v) \in \mathcal{E}$. Since

$$\sum_{k=1}^{m-1} \left(\left(\frac{kn}{m}\right)\right) = 0, \tag{3.3}$$

the contribution from $\mathcal{E}$ is zero in $\eta(\mathcal{I}; n)$, and by a computation:

$$\eta(\mathcal{I}; n) = 4 \cdot \sum_{w \in \mathcal{W}} \sum_{v \in \mathcal{V}_w} \sum_{k=1}^{m_w - 1} \left\{\frac{k m_v}{m_w}\right\}\left(\left(\frac{kn}{m_w}\right)\right).$$

This is equivalent to $(**)$ because of (3.3). □

Remark 3. Assume that (Σ, K) is represented by a splice diagram Γ. Then the right hand side of $(**)$ can be computed (without the construction of the whole plumbing graph) as follows [12]. We need for any $w \in \mathcal{R}$ the multiplicities $\{m_v\}_{v\in\mathcal{V}_w}$ (modulo m_w). The rupture points of the plumbing diagram, which can be constructed by the algorithm described in [5], correspond to the Seifert components of Γ. If $w \in \mathcal{R}$ corresponds to the Seifert component $(\Sigma(a_1,\dots,a_\delta);\ m_1K_1 \cup \dots \cup m_\delta K_\delta)$, then $m_w = \sum_{j=1}^{\delta} a_1 \cdots \hat{a}_j \cdots a_\delta m_j$, and the neighbor vertex corresponding to the edge marked with a_j $(j = 1,\dots,\delta)$ has a multiplicity, which is modulo m_w equal to $(m_j - \beta_j m_w)/a_j$, where the numbers $\{\beta_j\}_j$ satisfy $\beta_j a_1 \cdots \hat{a}_j \cdots a_\delta \equiv 1 \pmod{a_j}$.

4. The Casson and eta-invariant via generalized Dedekind sums

For arbitrary non-zero integers $a,\ b,\ c$, we consider the generalized Dedekind sum (cf. [22,27]):

$$s(b,c;a) = \sum_{k=1}^{|a|} \left(\left(\frac{kb}{a}\right)\right)\left(\left(\frac{kc}{a}\right)\right).$$

Using this notation, Theorem 2 reads as:

Theorem 2′. *With the choice $m = 1$ one has*:

$$\eta(\mathcal{I};n) = 4 \sum_{w\in\mathcal{R}} \sum_{v\in\mathcal{V}_w} \operatorname{sign}(m_w)\cdot s(m_v,n;m_w).$$

Now, we recall the famous generalization of the reciprocity law of Dedekind given by Rademacher [22] (see also [27]). If $a,\ b,\ c$ are strict positive, mutually coprime integers, then:

$$s(b,c;a) + s(c,a;b) + s(b,a;c) = -\frac{1}{4} + \frac{a^2+b^2+c^2}{12abc}.$$

Now, for any a, b and c with $(a,b,c) = 1$:

$$s(b,c;a) = s\left(\frac{b}{(a,b)(b,c)}, \frac{c}{(a,c)(b,c)}; \frac{a}{(a,b)(a,c)}\right).$$

Therefore, Rademacher's result reads as:

Reciprocity Law. *Let $a,\ b,\ c$ be strict positive integers such that $(a,b,c) = 1$. Then the following relation holds*:

$$s(b,c;a) + s(c,a;b) + s(a,b;c) = -\frac{1}{4} + \frac{a^2(b,c)^2 + b^2(a,c)^2 + c^2(b,a)^2}{12abc}. \quad \text{(RL)}$$

Using this, in the next theorem we express the eta-invariant in terms of Dedekind sums with denominator n.

Theorem 3. *Let (Σ, K), $\mathcal{I}$ and n be as in the introduction. Fix a plumbing graph G with a multilink structure given by $m_{v_0} = 1$, such that $m_w \neq 0$ for any $m_w \in \mathcal{W}$ (e.g., take G minimal). Then, with the notations of Section* 3, *one has*:

$$\eta(\mathcal{I};n)=-4\sum_{(u,v)\in\mathcal{E}}s(m_u,m_v;n)-\sum_{(u,v)\in\mathcal{E}}\operatorname{sign}(m_um_v)+n\cdot\sum_{(u,v)\in\mathcal{E}}\frac{(m_u,m_v)^2}{3m_um_v}$$
$$+\frac{1}{3n}\biggl(m_{w_0}+\sum_{w\in\mathcal{W}}(n,m_w)^2(-e_w)\biggr).$$

Proof. First notice that $(m_u,m_v,n)=1$ for any $(u,v)\in\mathcal{E}$ (cf. Lemma 2b). Then:

$$\eta(\mathcal{I};n)=4\sum_{w\in\mathcal{W}}\sum_{v\in\mathcal{V}_w}\operatorname{sign}(m_wm_v)\cdot s\bigl(|m_v|,n;|m_w|\bigr)$$
$$=4\sum_{(u,v)\in\mathcal{E}}\operatorname{sign}(m_um_v)\bigl(s\bigl(|m_v|,n;|m_u|\bigr)+s\bigl(|m_u|,n;|m_v|\bigr)\bigr).$$

Now apply (RL), and notice the following:

$$\sum_{(u,v)\in\mathcal{E}}\frac{m_u^2(n,m_v)^2+m_v^2(n,m_u)^2}{m_um_v}$$
$$=\frac{(n,1)^2m_{w_0}^2}{1\cdot m_{w_0}}+\sum_{w\in\mathcal{W}}\sum_{v\in\mathcal{V}_w}\frac{(n,m_w)^2m_v^2}{m_wm_v}$$
$$=m_{w_0}+\sum_{w\in\mathcal{W}}\frac{(n,m_w)^2}{m_w}\sum_{v\in\mathcal{V}_w}m_v\overset{(3.2)}{=}m_{w_0}+\sum_{w\in\mathcal{W}}(n,m_w)^2(-e_w).\qquad\square$$

Corollary 2. *With the above notations, one has*:

$$8\cdot\bigl[\lambda(\Sigma(K,n))-n\cdot\lambda(\Sigma)\bigr]$$
$$=-4\cdot\sum_{(u,v)\in\mathcal{E}}s(m_u,m_v;n)-(1-n)\cdot\sum_{(u,v)\in\mathcal{E}}\operatorname{sign}(m_um_v)$$
$$+\frac{1-n^2}{3n}m_{w_0}+\sum_{w\in\mathcal{W}}\frac{(n,m_w)^2-n^2}{3n}(-e_w).$$

Example. If $\Sigma(K,2)$ is an integer homology sphere, then:

$$8\cdot\bigl[\lambda\bigl(\Sigma(K,2)\bigr)-2\cdot\lambda(\Sigma)\bigr]=\sum_{(u,v)\in\mathcal{E}}\operatorname{sign}(m_um_v)+\frac{1}{2}\biggl(-m_{w_0}+\sum_{\substack{w\in\mathcal{W}\\ m_w\text{ odd}}}e_w\biggr).$$

Remark 4. $\eta(\mathcal{I};n)$ even for $n=1$ is important, because:

$$(-8)\cdot\lim_{n\to\infty}\frac{\lambda(\Sigma(K,n))-n\cdot\lambda(\Sigma)}{n}=\eta(\mathcal{I};1).$$

(Notice that in the expression of $\eta(\mathcal{I};1)$ (in Theorem 3) $s(m_u,m_v;1)=0$.)

The algebraic case. If (Σ,K) is algebraic (i.e., it is the link of the pair $(\mathcal{X},f^{-1}(0))$, where $(\mathcal{X},x)$ is a normal surface singularity, and $f:(\mathcal{X},0)\to(\mathbb{C},0)$ is an analytic germ), then the embedded resolution graphs of $(\mathcal{X}, f^{-1}(0))$ provide nice plumbing diagrams. For example, they satisfy: $e_w < 0$ and $m_w > 0$ for any $w \in \mathcal{W}$ (here $m = m_{v_0} = 1$).

If $\mathcal{X}$ is smooth, then $\Sigma = S^3$ and $K \subset S^3$ is the link of an irreducible plane curve singularity f. Let $\mathcal{I}(f)$ be its isometric structure (i.e., $\mathcal{I}(f) = \mathcal{I}(S^3, K)$). By [7], $\Sigma(K,n)$ is exactly the link of the singularity $\{f(x,y) + z^n\}$. By Theorem 1:

$$8 \cdot \lambda\big(\Sigma(K,n)\big) = \eta\big(\mathcal{I}(f); n\big) - n\eta\big(\mathcal{I}(f); 1\big).$$

But, by [14], the right hand side of the last equality is exactly the signature $\sigma(f + z^n)$ of the Milnor fiber of $f(x,y) + z^n$. In particular, we obtain:

$$8 \cdot \lambda(\text{link of } f + z^n) = \text{signature of } f + z^n. \tag{$***$}$$

This equality was proved by Neumann and Wahl in [13] and it was one of the leading relations what the author wanted to understand.

Now, using $(***)$, all the results of [18] about the signature of $f + z^N$ can be transformed for the Casson invariant of the link of $\{f + z^N = 0\}$.

Above, if $f(x,y) = x^a + y^b$, then the signature of $x^a + y^b + z^n$ can be computed by Brieskorn formula [3] from the number of lattice points in the tetrahedron $(0,0,0), (0,0,a), (0,b,0), (n,0,0)$. Now, if we apply for the graph of $x^a + y^b$ (which has only one rupture point) the result of Theorem $2'$, we obtain the number of these lattice points in terms of Dedekind sums. This is exactly the famous Mordell's formula [10] (for details, see [19,20]).

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