ELSEVIER

Topology and its Applications 67 (1995) 95–111

TOPOLOGY AND ITS APPLICATIONS

The eta-invariant of variation structures I

András Némethi [a,b,1]

[a] IMAR, Bucharest, Romania

[b] The Ohio State University, Columbus, OH 43210-1174, USA

Received 22 April 1994; revised 1 February 1995

Abstract

The main motivation of the paper is the signature of the Milnor fiber of the germ $f_1 + f_2^q$, where (f_1, f_2) is an isolated complete intersection singularity. This is computed in terms of a new invariant of the pair (f_1, f_2), which corresponds to the algebraic version of the eta-invariant, introduced by Atiyah, Patodi and Singer, for a hermitian flat bundle over the circle.

Keywords: Singularities; Milnor fiber; Signature; Eta-invariant

AMS classification: 32S05; 32S25; 32S55; 11E39

1. Introduction

Let $f_1 : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with one-dimensional singular locus Σ. Consider a germ f_2 so that the pair $\phi = (f_1, f_2)$ defines an isolated complete intersection singularity (ICIS). Our goal is to compute the correction term $\sigma(f_1 + f_2^q) - \sigma(f_1)$, where $\sigma(f)$ denotes the signature of the Milnor fiber of the germ f.

Let G be the local fundamental group of the complement of the discriminant locus of the ICIS. Then ϕ defines a representation $\rho : G \to \mathrm{Aut}(U; b)$, where U is the middle homology group of the fiber with intersection form b. Moreover, since the geometric monodromy of ϕ at the boundary of the fiber is trivial, for each $g \in G$, we have a variation map $V(g) : U^* \to U$.

Any $g \in G$ defines a spectral decomposition of the system

$$(U; b, \rho(g), V(g)) = \bigoplus_{\chi} (U_\chi; b_\chi, \rho_\chi(g), V_\chi(g));$$

[1] Partially supported by NSF grant No. DMS-9203482. E-mail: nemethi@math.ohio-state.edu.

SSDI 0166-8641(95)00021-6

(here U_χ are the generalized eigenspaces associated with $\rho(g)$). Define $\eta(g) = \sum_\chi \eta(g)_\chi$, where:

$$\eta(g)_\chi = \begin{cases} (1-2a)\,\mathrm{sign}\, b_\chi & \text{if } \chi = \mathrm{e}^{-2\pi i a};\ 0 < a < 1, \\ \mathrm{sign}(I + \rho_1(g^{-1}))V_1(g) & \text{if } \chi = 1. \end{cases}$$

In fact, η is well defined on the conjugacy classes of G. Let M, respectively L, be the topological standard meridian, respectively the longitude of the discriminant component $\phi(\Sigma)$.

The main result of this paper is the following (Theorem 5.2):

Theorem. *For $q \gg 0$, the following relation holds:*

$$\sigma(f_1 + f_2^q) - \sigma(f_1) = q \cdot \eta(M) + \eta(L) - \eta(L + qM).$$

In general, the question is the following. Let inv be any invariant of the analytic germs. For example, $\chi(f)$ = the Euler characteristic of the Milnor fiber, $\zeta(f)$ = the zeta function of f, $\mathrm{Sp}(f)$ = the spectrum of f, $\sigma(f)$ = the signature, etc.

Question: Can the correction term $\mathrm{inv}(f_1 + f_2^q) - \mathrm{inv}(f_1)$ be computed in terms of some information provided by the restriction of ϕ above a small neighbourhood of the link of $\phi(\Sigma)$? If the answer is yes, then what is this correction term?

For $\chi(f)$, $\zeta(f)$, $\mathrm{Sp}(f)$ the answer is yes. For $\chi(f)$ the correction term depends on the transversal Milnor numbers of Σ [18], for $\zeta(f)$ (respectively, $\mathrm{Sp}(f)$) it depends on the monodromy representations (respectively, the mixed Hodge module $\phi_*\mathbb{Q}^H_{(\mathbb{C},0)}$) in the neighbourhood of $\phi(\Sigma)$ [16,8] (respectively, [14]). (Note, that these invariants are "additive" invariants.)

Our result shows that for $\sigma(f)$ the answer is again yes. The correction term depends on the monodromy representation and the variation map (we prefer to call this pair variation structure), and it is expressed in terms of some "eta-invariants". (For another presentation of these eta-invariants and their relations with the spectral pairs, see [9].)

The attentive reader can realize that our eta-invariant is, in some sense, a generalized version of the eta-invariant (associated with the circle and with a hermitian representation) defined in [1]. The clarification of this relation is the subject of another paper.

Our approach is algebraic, in Sections 2–4 we extend Meyer's result [5] to the case of variation structures. In Section 2, we study abstract variation structures $(U; b, \rho, V)$ and in Section 3 we determine their Witt group when $G = \mathbb{Z}^k$. The object of Section 4 is a special 2-cocycle of G which measures the nonadditivity of the signature of two-dimensional manifolds (with boundary) with coefficients in a variation structure. This generalizes the Meyer's cocycles [5], which were defined for nondegenerate representations. (See 2.6 for a more detailed explication.) In the classical case, for abelian representations, the Meyer's cocycle is the coboundary of the classical eta-invariant cocycle. This fact is generalized in Section 4 for the degenerate case. Section 5 contains the main application: the computation of the correction term of the signature in the case of the Yomdin series.

(In this paper, the signature of a (-1)-hermitian form b is $\mathrm{sign}\, b = \mathrm{sign}(i \cdot b)$.)

2. Hermitian variation structures

2.1. If U is a finite dimensional $\mathbb{C}$-vector space, we denote its dual $\text{Hom}(U,\mathbb{C})$ by U^*. There is a natural identification $\theta: U \to U^{**}$ given by $\theta(u)(\varphi) = \varphi(u)$.

Any $\mathbb{C}$-linear endomorphism $b: U \to U^*$ with $\overline{b^* \circ \theta} = \varepsilon b$ defines an ε-hermitian form on U. (Here $\varepsilon = \pm 1$ and $\bar{\ }$ is the complex conjugation.) We will use the notation

$$B: U \otimes U \to \mathbb{C}, \quad B(u,v) = b(u)(\overline{v}),$$

too. The automorphisms $h: U \to U$ with $\overline{h}^* \circ b \circ h = b$ form the orthogonal group $\text{Aut}(U;b)$. Any group endomorphism $\rho: G \to \text{Aut}(U;b)$ is called a b-representation of the group G.

Any representation $\rho: G \to \text{Aut}(U)$ defines a left action of G on $\text{Hom}(U^*,U)$ by $g * \varphi = \rho(g) \circ \varphi$. Then, by definition, a twisted homomorphism is a map

$$V: G \to \text{Hom}(U^*,U)$$

with $V(gh) = \rho(g) \circ V(h) + V(g)$.

Definition 2.2. An ε-hermitian variation structure of the group G is a system $\mathcal{V} = (U; b, \rho, V)$ with the following properties:

(a) b is an ε-hermitian form on (the finite dimensional space) U,

(b) ρ is a b-representation of G,

(c) V is a twisted homomorphism (with respect to the left action of G), with the following properties:

(i) $\overline{\theta^{-1} \circ V(g)}^* = -\varepsilon V(g)\overline{\rho(g)}^*$, and

(ii) $V(g) \circ b = \rho(g) - I$.

If b is nondegenerate then $\mathcal{V}$ is called nondegenerate too.

It is not difficult to verify that an ε-hermitian variation structure satisfies also the following supplementary

Properties 2.3. (a) For any $g \in G$, one has $b \circ V(g) = \overline{\rho(g)}^{*,-1} - I$.

(b) Consider the right action of G on $\text{Hom}(U^*,U)$ defined by $\varphi * g = \varphi \circ \overline{\rho(g)}^{*,-1}$. Then V is a twisted homomorpism with respect to the right action, too, i.e., for any g and $h \in G$, one has $V(gh) = V(g) \circ \overline{\rho(h)}^{*,-1} + V(h)$.

(c) If $gh = hg$, then $\rho(h) \circ V(g) \circ \overline{\rho(h)}^* = V(g)$. (Write $V(gh) = V(hg)$ and use Definition 2.2(c) and (b).)

Definition 2.4. Two ε-hermitian structures $(U; b, \rho, V)$ and $(U'; b', \rho', V')$ of the group G are isomorphic if there exists an isomorphism $\varphi: U \to U'$ such that $b = \bar{\varphi}^* b' \varphi$, $\rho(g) = \varphi^{-1}\rho'(g)\varphi$, and $V(g) = \varphi^{-1}V'(g)(\bar{\varphi}^*)^{-1}$ for any $g \in G$.

The set of isomorphism classes of ε-hermitian variation structures is denoted by $HV_\varepsilon(G)$. The natural direct sum determines a semigroup structure on it.

Examples 2.5. (1) If b is nondegenerate then $V(g) = (\rho(g) - I)b^{-1}$, i.e., the semigroup of the nondegenerate variation structures is equivalent to the semigroup $HI_\varepsilon(G)$ of ε-hermitian isometric structures (systems $(U; b, \rho)$ with nondegenerate form b and with axioms (a) and (b)).

(2) If $V(g)$ is an isomorphism, then $\rho(g) = -\varepsilon V(g)(\overline{\theta^{-1} \circ V(g)^*})^{-1}$, and $b = -V(g)^{-1} - \varepsilon(\overline{\theta^{-1} \circ V(g)^*})^{-1}$. In particular, if $G = \mathbb{Z}$, then the sub-semigroup $HV^s_\varepsilon(\mathbb{Z}) = \{\mathcal{V} \subset HV_\varepsilon(\mathbb{Z});\ V(1) \text{ isomorphism}\}$ is equivalent to the semigroup of the sesqui-linear forms over $\mathbb{C}$.

(3) An important element in $HV^s_\varepsilon(\mathbb{Z})$, provided by an isolated hypersurface singularity f, is $\mathcal{V}(f)$ generated by $(U; b, h(1), V(1))$ = (middle homology of the Milnor fiber of f; intersection form, monodromy, variation map). Notice that the variation map $V(1)$ of f can be identified with the inverse of the Seifert form (up to a sign) [3], in particular, $V(1)$ is an isomorphism.

(4) Consider an isolated complete intersection singularity $f:(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$ $(n > 1)$. Let $\phi: (\mathcal{X}, 0) \to (S, 0)$ be a "good representative" of f with discriminant locus $\Delta \subset S$. Consider a base-point $* \in S - \Delta$. The homology of the fiber $F = \phi^{-1}(*)$ is concentrated in $U = H_{n-1}(F, \mathbb{C})$. Identify its dual U^* with $H_{n-1}(F, \partial F; \mathbb{C})$, and extend the real intersection form to a *hermitian* form $b: U \to U^*$. The monodromy representation $\rho: G = \pi_1(S - \Delta, *) \to \mathrm{Aut}(U; b)$, and the variation map $V: G \to \mathrm{Hom}(U^*, U)$ constitute a system $\mathcal{V}(\phi) = (U; b, \rho, V)$ which is our basic example of $(-1)^{n-1}$-hermitian variation structure.

Remarks 2.6. Even for very simple groups, the classification of the variation structures is nontrivial. In the case $G = \mathbb{Z}$, the elements of two important sub-semigroups of $HV_\varepsilon(\mathbb{Z})$ are classified. Namely, the nondegenerate structures (i.e., the isometric ones) are classified by Milnor [6], the semigroup $HV^s_\varepsilon(\mathbb{Z})$ is determined in [10]. (For the semi-ring structure of $HV^s_\varepsilon(\mathbb{Z})$, see [11].)

But in the general case, without any supplementary assumption about the variation structures, even in the case $G = \mathbb{Z}$, the classification is not known (for the author). (For the type of anomalies, which may occur already in the case $G = \mathbb{Z}$, see Example 2.7.9 in [10].)

The classification of the elements of $HV^s_\varepsilon(\mathbb{Z})$ gives a nice description and organization of the topological invariants of an isolated singularity f. For this, and for the relation of $\mathcal{V}(f)$ to the spectral pairs of f, see [9,10].

In the case of $G = \mathbb{Z}^k$ $(k \geqslant 2)$, even the underlying problem of classification of endomorphisms $\rho: \mathbb{Z}^k \to GL_n(\mathbb{C})$ is open.

Geometric examples 2.7. Relation to the signature and motivations for further definitions.

(a) Let $(B, \partial B)$ be a two-dimensional, connected manifold with boundary. Assume that $p: \mathcal{X} \to B$ is a locally trivial fibration with fiber F, a $(4k-2)$-dimensional manifold without boundary.

By a result of Meyer [5], the signature $\sigma(\mathcal{X})$ of $\mathcal{X}$ can be computed by the isometric structure $(U; b, \rho)$ associated with p. More precisely: let b be the (-1)-symmetric hermitian (intersection) form on $U = H_{2k-1}(F, \mathbb{C})$, and let $\rho : \pi_1(B) \to \mathrm{Aut}(U; b)$ be the monodromy representation of p. Then the twisted cohomology group $H^1(B, \partial B; \rho)$ carries a symmetric hermitian form with signature $\sigma(\rho)$. Meyer proved that $\sigma(\mathcal{X}) = \sigma(\rho)$.

Moreover, even if $\partial F \neq \emptyset$ (in this case $\mathcal{X}$ is a manifold with corners), but b is nondegenerate, the above identity still holds.

(b) If B and F both have boundaries, and if the intersection forms on $H_1(B)$ respectively, on $H_{2k-1}(F)$ are both degenerate, then the signature $\sigma(\mathcal{X})$ can not be reduced to $\sigma(\rho)$. The form b and the representation ρ have no control over the geometric behaviour of the fibration in the neighbourhood of the boundary of the fibers (∂F, in this algebraic sense, corresponds to $\ker b$).

Assume that the restriction of $p : \partial\mathcal{X} \cap p^{-1}(B - \partial B) \to B - \partial B$ is a trivial fibration (cf. 2.5.4). Then, we can associate with p a variation structure $\mathcal{V}(p) = (U; b, \rho, V)$ (where V is the variation map of p). Now, V preserve a lot of information from the geometry of the fibration in the neighbourhood of the boundary. It turns out that the variation structure $\mathcal{V}(p)$ takes the role of the (nondegenerate) isometric structures in the case when b is degenerate.

(c) Assume again that b is nondegenerate. Then there exist some methods for the computation of $\sigma(\rho)$. The question is: can these methods be generalized to the case of variation structures (i.e., to the nondegenerate case)?

The first remark is that $\sigma(\rho)$ vanishes on the hyperbolic structures, therefore it depends only on the class of $(U; b, \rho)$ in the corresponding Witt group.

Here arises the first question: when is a variation structure "hyperbolic", and how can the definition of Witt groups be extended to the case of variation structures. This problem is solved in Section 3, and the Witt group of $G = \mathbb{Z}^k$ is computed. (In our case, the signature will not vanish on the hyperbolic structures, but some other invariants will vanish, see Section 4.)

(d) Meyer computed $\sigma(\rho)$, when b is nondegenerate, using his "Meyer cocycles" [5]. This gives a motivation to find the corresponding "Meyer cocycles", when b is degenerate, in terms of the variation structure. This construction is given in Section 4.

(e) By the index theory, $\sigma(\rho)$ can be computed as follows (b is nondegenerate again). The representation ρ is equivalent to a flat bundle Γ. Since b is (-1)-symmetric, there is a natural complex structure (Γ, J) on Γ. The signature bundle $\mathrm{sign}(\Gamma)$ is defined by $(\Gamma, J)^* - (\Gamma, J)$. Let c_1 be its first Chern class (this is, in fact, a pull-back of the cohomology class given by the Meyer cocycles). Then $\sigma(\rho) = \sigma(B, \mathrm{sign}(\Gamma))$, and by [1]

$$\sigma\big(B, \mathrm{sign}(\Gamma)\big) = \int_B c_1 - \frac{1}{2}\eta\big(\partial B, \mathrm{sign}(\Gamma)\big). \qquad (*)$$

Here the last term is the eta-invariant associated with the signature operator of ∂B and the signature bundle $\mathrm{sign}(\Gamma)$.

Assume that the representation ρ is abelian. Then $c_1 = 0$, i.e., the primary invariant $\int c_1$ vanishes, and $\sigma(\rho)$ can be computed by the eta-invariant associated with $\rho|\partial B$.

This motivates one more question: assume that b is degenerate and the variation structure is abelian. Can the signature $\sigma(\mathcal{X})$ be computed only by the restriction of the variation structure on ∂B? What is the corresponding expression of this new "eta-invariant" in terms of this restriction?

Our basic, new definition is exactly this new "eta-invariant". Our main application (Theorem 5.2) is, in fact, a result of type (∗).

3. The Witt group of variation structures

Definition 3.1. A hermitian variation structure is *hyperbolic* if there exists a kernel $K \subset U$, i.e., a subset K such that

(a) $\dim K = \frac{1}{2}\dim U$,

(b) $K \subset K^{\perp} = \{x \mid B(x,y) = 0 \text{ for any } y \in K\}$,

(c) for any $g \in G$ one has $\rho(g)(K) \subset K$ and $V(g)(K^*) \subset K$ where $K^* = \{\varphi \in U^* \mid \overline{\varphi}(K) = 0\}$.

Examples 3.2. (1) If b is nondegenerate then $\mathcal{V}$ is hyperbolic if and only if the isometric structure $(U; b, \rho)$ is hyperbolic (i.e., there exists a ρ-invariant K with $K = K^{\perp}$).

(2) Consider $-\mathcal{V} = (U; -b, \rho, -V)$. Then $\mathcal{V} \oplus (-\mathcal{V})$ is hyperbolic with kernel $K = \Delta U = \{(x,x) \mid x \in U\}$.

3.3. By 3.2(2) the semigroup $WV_\varepsilon(G) = (HV_\varepsilon(G)/\{\text{hyperbolic structures}\}, \oplus)$ is a group. It is called the Witt group of the variation structures of G.

Our first goal is the computation of $WV_\varepsilon(\mathbb{Z}^k)$. For any $\chi \in \operatorname{Hom}(\mathbb{Z}^k, \mathbb{C}^*)$, we define the generalized eigenspace

$$U_\chi = \left\{x \in U \mid \left(\rho(g) - \chi(g)\right)^N x = 0 \text{ for some } N \text{ and any } g \in G\right\}.$$

$\hat{G} = \operatorname{Hom}(\mathbb{Z}^k, S^1)$ denotes the group of characters.

The verification of the following lemma is left to the reader.

Lemma 3.4. *Let $G = \mathbb{Z}^k$. Then:*

(a) *There is a direct sum decomposition:*

$$(U; b, \rho, V) = (U'; b', \rho', V') \oplus \bigoplus_{\chi \in \hat{G}} (U_\chi; b_\chi, \rho_\chi, V_\chi),$$

where $U' = \bigoplus_{\chi \notin \hat{G}} U_\chi$. Moreover, $(U'; b', \rho', V')$ is hyperbolic.

(b) *$(U; b, \rho, V)$ is hyperbolic if and only if $(U_\chi; b_\chi, \rho_\chi, V_\chi)$ is hyperbolic for any $\chi \in \hat{G}$, in particular*

$$WV_\varepsilon(G) = \bigoplus_{\chi \in \hat{G}} WV_\varepsilon(G)_\chi.$$

Here $WV_\varepsilon(G)_\chi$ is the Witt group of variation structures $\mathcal{V}$ with $\rho(g) - \chi(g)I$ nilpotent for any $g \in G$.

These Witt groups are given by the following proposition:

Proposition 3.5.

$$WV_\varepsilon(\mathbb{Z}^k)_\chi = \begin{cases} \mathbb{Z} & \textit{if } \chi \in \hat{G} - \{1\}, \\ \mathbb{Z}_2 & \textit{if } \chi = 1. \end{cases}$$

The generators are $(\mathbb{C}; \pm \mathrm{i}^{(1-\varepsilon)/2}, \chi, \pm(\chi - 1)\mathrm{i}^{(\varepsilon-1)/2})$ *if* $\chi \neq 1$, *and* $(\mathbb{C}; 0, 1, 0)$ *if* $\chi = 1$.

Proof. If $\chi \neq 1$, then $WV_\varepsilon(\mathbb{Z}^k)_\chi$ is the Witt group of the corresponding isometric structures. Thus the isomorphism is given by $\mathcal{V} \mapsto \text{signature}(\mathrm{i}^{(1-\varepsilon)/2} b_\chi)$.

Assume that $\chi = 1$. Fix a nonzero element $g_0 \in \mathbb{Z}^k$. Consider the weight filtration $W^\bullet$ on U_1, centered at zero, induced by g_0; i.e., the unique filtration $W^\bullet$ with the following properties:

(a) $W^k \subset W^{k+1}$; $N(W^k) \subset W^{k-2}$,

(b) $N^k : Gr^k_W U_1 \to Gr^{-k}_W U_1$ is an isomorphism, where $Gr^i_W U_1 = W^i / W^{i-1}$ and $N = \log \rho(g_0)$.

Set $m \in \mathbb{N}$ such that $W^m = U_1$. The dual filtration $W^*_\bullet$ is given by

$$W^*_k = \{\varphi \in U^*_1 \mid \overline{\varphi}(W^{k-1}) = 0\}.$$

Then $W^*_{-m} = U^*_1$; $W^*_k \subset W^*_{k-1}$. Moreover, $\overline{N}^* W^*_k \subset W^*_{k+2}$ and

$$(\overline{N}^*)^k : Gr^{W^*}_{-k} U^*_1 \to Gr^{W^*}_k U^*_1$$

is an isomorphism (here $\overline{N}^* = \log \overline{\rho(g_0)}^*$).

Now, since $\rho(g_0) \in \mathrm{Aut}(U_1; b_1)$ one has $B_1(W^k, W^l) = 0$ for $k + l < 0$. Indeed, take $x \in W_k$ and $y \in W_l$. Consider a maximal $s \geqslant 0$ (respectively, $t \geqslant 0$) such that $x = N^s x'$ (respectively, $y = N^t y'$). Then $N^{2s+k+1} x' = 0$ respectively, $N^{2t+l+1} y' = 0$. Therefore $B(x, y) = B(N^s x', N^t y') = 0$ if either $s + t \geqslant 2s + k + 1$ or $s + t \geqslant 2t + l + 1$. If both inequalities fail, then $s + t \leqslant 2s + k$ and $s + t \leqslant 2t + l$. By taking their sum: $0 \leqslant k + l$ which is a contradiction. The vanishing of $B(W^k, W^l)$ for $k + l < 0$ is equivalent to $b_1(W^k) \subset W^*_{-k}$ for any $-m \leqslant k \leqslant m$.

We want similar restrictions for $V(g)$ where $g \in G$ is an arbitrary element of G. Since G is abelian,

$$\rho(g)V(h) + V(g) = V(gh) = V(hg) = V(h)\overline{\rho(g)}^{*,-1} + V(g).$$

In particular, for any $g \in G$ one has $N \circ V(g) = -V(g)\overline{N}^*$. By a similar argument as above: $V(g)(W^*_k) \subset V_{-k}$.

Consider a base in U_1 (and a dual base in U^*_1) compatible with the filtration $W^\bullet$. This gives the splittings $W^k = W^{k-1} \oplus W^k_c$. Our maps have the following block-decomposition:

$$b = \begin{pmatrix} 0 & & b^m \\ & \cdot & \\ b^{-m} & & * \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho^m & & * \\ & \ddots & \\ 0 & & \rho^{-m} \end{pmatrix}, \quad V = \begin{pmatrix} * & & V^m \\ & \cdot & \\ V^{-m} & & 0 \end{pmatrix}.$$

(The dimension of the i-block is $n_i = \dim Gr^i_W U_1$, where $-m \leqslant i \leqslant m$.)

Since the weight filtration can be characterized completely by kernels and images of powers of N, the commutativity of G implies that the weight filtration $W^\bullet$ is ρ-invariant. Consider $-Gr^0_W\mathcal{V}_1 = (Gr^0_W U_1; [-b^0], [\rho^0], [-V^0])$ and the direct sum $\mathcal{H} = \mathcal{V}_1 \oplus (-Gr^0_W \mathcal{V}_1)$. Define $K \subset U_1 \oplus Gr^0_W U_1$ by $K = (W_{-1} \oplus \{0\}) \oplus \{(x, [x]) \mid x \in W^k_c\}$. Then K is a kernel (use the above matrix forms) and defines a hyperbolic structure on $\mathcal{H}$. Consequently, in the Witt group, $\mathcal{V}_1$ can be replaced with $Gr^0_W \mathcal{V}_1$. The representation ρ^0 of this latter structure has the property $\rho^0(g_0) = $ identity. Using induction over the generators of G, we obtain that the Witt group is generated by structures with $\rho(g) = 1$ for any $g \in G$. Now, since $\overline{b}^* = \varepsilon b$, $\overline{V}^* = -\varepsilon V$ and $V \circ b = 0$, we deduce that, in fact, it is generated by the structures $(\mathbb{C}; \pm \mathrm{i}^{(1-\varepsilon)/2}, 1, 0)$, $(\mathbb{C}; 0, 1, \pm \mathrm{i}^{(1+\varepsilon)/2})$ and $\mathcal{T} = (\mathbb{C}; 0, 1, 0)$. But the direct sum of any of these with $\mathcal{T}$, is hyperbolic, in particular $WV(G)_1$ is generated by $\mathcal{T}$ and $2\mathcal{T} = 0$. On the other hand $\mathcal{T}$ is not zero in the Witt group because the dimension of a hyperbolic space is even. □

Remark 3.6. In the classical theory the following statement is true: If an isometric structure $\mathcal{I} = (U; b, \rho)$ (b nondegenerate) is zero in the Witt group $WU(G)$, then $\mathcal{I}$ is hyperbolic. In our case this is *not* true. Take for example $\mathcal{V} = 2(\mathbb{C}; 1, 1, 0)$. Moreover, this example shows that the signature does not vanish on the set of hyperbolic structures, that is, the set of hyperbolic structures is too large apparently. This is true, but exactly for this reason it will be helpful in our discussion. In fact, we want to define some cocycles on $WV_\varepsilon(G)$ (as secondary invariants) rather than elements in the cohomology of G (i.e., primary characteristic classes).

4. The cocycle associated to the nonadditivity of the signature

4.1. Let $\mathcal{V} = (U; b, \rho, V)$ be an ε-hermitian variation structure of G. Then b defines an ε-hermitian nondegenerate form Φ on $U^* \oplus U$ by

$$\Phi\big((\varphi, u), (\psi, v)\big) = \varepsilon\overline{\psi}(u) + \varphi(\overline{v}) + b(u)(\overline{v}).$$

Any $g \in G$ defines two maps $s_r(g), s_l(g) : U^* \oplus U \to U^* \oplus U$ defined by

$$s_l(g)(\varphi, u) = \big(\varphi, \rho(g)u - \rho(g)V(g^{-1})\varphi\big),$$

$$s_r(g)(\varphi, u) = \Big(\overline{\rho(g)}^{*,-1}\varphi, -V(g)\varphi + u\Big).$$

By a straightforward verification (use 2×2 blocks computation), we obtain that s_l respectively, s_r are representations of the group G in the orthogonal group $O(\Phi)$ of Φ.

4.2. For any $g \in G$, define $K_g = \{(\varphi, u) \in U^* \oplus U \mid V(g)\varphi = u\}$. It is not hard to verify that K_g is a Φ-kernel, i.e., $K_g = K_g^\perp$ (the latter one is the Φ-orthogonal). Moreover, for any g and h:

4.3. $s_l(h)K_g = K_{hg}$ and $s_r(h)K_g = K_{gh^{-1}}$.

4.4. Any three kernels K_i $(i = 1,2,3)$ in $U^* \oplus U$ define an $(-\varepsilon)$-hermitian form [17]. We recall this construction. On the space

$$K_1 \cap (K_2 + K_3) = \{x_1 \in K_1 \mid \text{there exist } x_2 \in K_2,\ x_3 \in K_3 \text{ with } x_1 + x_2 + x_3 = 0\}$$

define the sesqui-linear form $\Psi(x_1, x_1') = \Phi(x_1, x_2')$ (where $x_1' + x_2' + x_3' = 0$ and $x_2' \in K_2$, $x_3' \in K_3$). Then Ψ is $(-\varepsilon)$-hermitian with kernel $\mathrm{Ker} = K_1 \cap K_2 + K_1 \cap K_3$. We define $\sigma(\mathcal{V}; K_1, K_2, K_3)$ as the signature of the induced nondegenerate form on $K_{1,2,3} = K_1 \cap (K_2 + K_3)/\mathrm{Ker}$ multiplied by ε (if there is no danger of confusion then it is denoted by $\sigma(K_1, K_2, K_3)$). If $K_i = K_j$ for some pair (i, j), then $K_{1,2,3} = 0$, hence $\sigma(K_1, K_2, K_3) = 0$.

Lemma 4.5. (a) $\sigma(K_{\tau(1)}, K_{\tau(2)}, K_{\tau(3)}) = \mathrm{sign}(\tau)\sigma(K_1, K_2, K_3)$ *for any permutation* $\tau \in \mathcal{S}_3$; *(here* $\mathrm{sign}(\tau) \in \{\pm 1\}$ *is the sign of* τ*).*

(b) $\sigma(K_1, K_2, K_3) - \sigma(K_0, K_2, K_3) + \sigma(K_0, K_1, K_3) - \sigma(K_0, K_1, K_2) = 0$ *for any kernels* K_i $(i = 1,2,3,4)$.

(c) *If* $o \in O(\Phi)$ *is an orthogonal automorphism, then* $\sigma(o(K_1), o(K_2), o(K_3)) = \sigma(K_1, K_2, K_3)$. *In particular,* $\sigma(K_{hg_1}, K_{hg_2}, K_{hg_3}) = \sigma(K_{g_1h}, K_{g_2h}, K_{g_3h}) = \sigma(K_{g_1}, K_{g_2}, K_{g_3})$.

Proof. (a) follows from [17], (b) from [13]. If $o \in O(\Phi)$ then o induces an isometry of the corresponding Ψ forms. This with 4.2–4.3 proves (c). $\square$

4.6. By 4.5, $\sigma(\mathcal{V}; K_1, K_2, K_3)$ defines a homogeneous cocycle of the group G in $\mathbb{Z}$ ($\mathbb{Z}$ is considered with the trivial G-action). The corresponding nonhomogeneous cocycle is $\sigma(g, h) = \sigma(K_e, K_g, K_{gh})$ (e is the neutral element of G). This cocycle is a coboundary if there exists a function $f : G \to \mathbb{Z}$ such that $\sigma(g, h) = f(g) + f(h) - f(gh)$; i.e., if $\sigma(K_e, K_g, K_h) = f(g) + f(g^{-1}h) - f(h)$. The semigroup morphism

$$\Theta_I : HV_\varepsilon(G) \to H^2(G, \mathbb{Z}), \quad \Theta_I(\mathcal{V}) = \sigma(\mathcal{V}; \cdot, \cdot, \cdot)$$

is not trivial in general. For example, if G is the mapping class group Γ_g and

$$\mathcal{V} = \left(\mathbb{C}^{2g}; \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \ \rho, \ (\rho - I)b^{-1}\right)$$

where ρ is given by $\Gamma_g \to \mathrm{Sp}(2g, \mathbb{C})$ ("passing to the homology"), then $\Theta_I(\mathcal{V})$ is the generator of $H^2(G, \mathbb{Z}) = \mathbb{Z}$ [4].

4.7. A kernel $\tilde{K} \subset U^* \oplus U$ is called $\mathcal{V}$-invariant if for any $g \in G$ $s_l(g)\tilde{K} = \tilde{K}$. For example, if $K \subset U$ is a kernel of the variation structure $\mathcal{V}$ (see 3.1), then $\tilde{K} = K^* \oplus K$ is an invariant kernel in $U^* \oplus U$.

If $\tilde{K}$ is an invariant kernel, then 4.5(b)–(c) gives for $\tilde{K}$, K_e, K_g and K_h:

4.8. $\sigma(K_e, K_g, K_h) = \sigma(\tilde{K}, K_e, K_g) + \sigma(\tilde{K}, K_e, K_{g^{-1}h}) - \sigma(\tilde{K}, K_e, K_h)$. In particular, $\sigma(\mathcal{V})$ is a coboundary. (Moreover, $f(\mathcal{V}; g) = \sigma(\mathcal{V}; \tilde{K}, K_e, K_g)$ is independent of the choice of the invariant kernel $\tilde{K}$; to see this, consider 4.5(b) for $\tilde{K}_1$, $\tilde{K}_2$, K_e and K_g.)

Therefore Θ_I induces a map (denoted in the same way):

$$\Theta_I : WV_\varepsilon(G) \to H^2(G, \mathbb{Z}).$$

4.9. Set $\mathcal{V} \in HV_\varepsilon(G)$. Any $g \in G$ defines a morphism $\mathbb{Z} \to G$ by $n \mapsto g^n$. This induces a structure $g^*\mathcal{V} = (U; b, g^*\rho, g^*V) \in HV_\varepsilon(\mathbb{Z})$ where $g^*\rho(n) = \rho(g^n)$ and $g^*V(n) = V(g^n)$. This is an abelian structure, hence

$$g^*\mathcal{V} = \bigoplus_\chi (g^*\mathcal{V})_\chi = (U'; b', \rho', V') \oplus \bigoplus_{\chi \in \hat{\mathbb{Z}}} (U_\chi; b_\chi, \rho_\chi, V_\chi),$$

where $U' = \bigoplus_{\chi \notin \hat{\mathbb{Z}}} U_\chi$ (cf. 3.4).

We define $\eta : G \to \mathbb{Z}$ by

$$\eta(\mathcal{V}; g) = \eta(g^*\mathcal{V}) = \sum_\chi \eta(g^*\mathcal{V})_\chi, \quad \text{where}$$

$$\eta(g^*\mathcal{V})_\chi = \begin{cases} 0 & \text{if } \chi \notin \hat{\mathbb{Z}}, \\ (1-2a)\,\mathrm{sign}(b_\chi) & \text{if } \chi(g) = \mathrm{e}^{-2\pi i a};\ 0 < a < 1, \\ \mathrm{sign}[(I + \rho_1(g^{-1}))V_1(g)] & \text{if } \chi = 1. \end{cases}$$

(Notice that b_χ is ε-hermitian, and $(I + \rho(g^{-1}))V(g)$ is $(-\varepsilon)$-hermitian form.)

By definition $\eta(g)$ depends only on the variation $g^*\mathcal{V} \in HV_\varepsilon(\mathbb{Z})$.

The first result of this section is:

Proposition 4.10. *Suppose that $\mathcal{V}$ is hyperbolic, i.e., it has a kernel $K \subset U$. If $\tilde{K} = K^* \oplus K$ denotes the corresponding invariant kernel in $U^* \oplus U$, then*

$$\sigma(\tilde{K}, K_e, K_g) = \eta(g).$$

In particular, $\sigma(\tilde{K}, K_e, K_g)$ depends only on $g^*\mathcal{V}$.

Proof. In order to simplify the notations, we will assume that the eigenvalues of g lie on the unit circle; the proof in the general case is the same.

The element g defines a decomposition $\bigoplus_\chi (U_\chi; b_\chi, \rho_\chi, V_\chi)$ of $g^*\mathcal{V}$. Notice, that

$$U_\chi^* = \{\varphi \in U^* \mid \overline{\varphi}(U_{\chi'}) = 0 \text{ if } \chi' \neq \chi\}.$$

In particular, the kernel K_g has a decomposition $\bigoplus_\chi (K_g)_\chi$, where

$$(K_g)_\chi = \{(\varphi, u) \in U_\chi^* \oplus U_\chi \mid V_\chi \varphi = u\}.$$

On the other hand, $K_e = U^*$ obviously has a decomposition $K_e = \bigoplus_\chi U_\chi^* = \bigoplus_\chi (K_e)_\chi$, too.

Since the kernel $K \subset U$ is $\rho(g)$-invariant, one has $K = \bigoplus_\chi K_\chi$, where $K_\chi = K \cap U_\chi$. Obviously $K_\chi \subset K_\chi^\perp$. If $\chi \neq 1$ then b_χ is nondegenerate and V_χ is an isomorphism.

Therefore K_χ is a kernel for $\chi \neq 1$. Now, from a dimension computation we get this for $\chi = 1$ too. In particular, $\tilde{K}_\chi = K_\chi^* \oplus K_\chi$ is a kernel in $U_\chi^* \oplus U_\chi$ and

$$\sigma(\tilde{K}, K_e, K_g) = \sum_\chi \sigma(\tilde{K}_\chi, (K_e)_\chi, (K_g)_\chi).$$

Therefore, we can assume that $U = U_\chi$ for some χ. Consider $\chi \neq 1$. Then $\operatorname{sign} b_\chi = 0$ because b_χ is nondegenerate and $(U_\chi; b_\chi)$ is hyperbolic. On the other hand, Ψ_χ is the zero form (because $V(g)^{-1}(K) = K^*$).

Assume that $\chi = 1$. We want to compute $\sigma(\tilde{K}, K_e, K_g) = \sigma(K_g, \tilde{K}, K_e)$. Consider the map $\Lambda : K_g \cap (\tilde{K} + K_e) \to V(g)^{-1}(K)$ given by

$$\{(\alpha, v) \mid V\alpha = v,\ \alpha = -\varphi - \psi,\ \varphi \in K^*,\ v = -k, \text{ where } k \in K\} \mapsto \alpha.$$

Λ defines an isomorphism, so we can transfer the form Ψ on $V(g)^{-1}(K)$. Notice that we can choose $\varphi = 0$. Therefore

$$\begin{aligned}\Psi((\alpha, v), (\alpha', v')) &= \Phi((\alpha, v), (\varphi', k')) = \Phi((\alpha, v), (0, -V(g)\alpha')) \\ &= -\alpha(\overline{V(g)\alpha'}) - b(V(g)\alpha)(\overline{V(g)\alpha'}).\end{aligned}$$

Since $V(g)\alpha$ and $V(g)\alpha'$ are in K, and $K \subset K^\perp$, the latter term vanishes. In conclusion, Ψ on $V(g)^{-1}(K)$ has the form $\Psi(\alpha, \alpha') = -\alpha(\overline{V(g)\alpha'})$.

Consider the sesqui-linear form $\Psi_e = -\frac{1}{2}(\Psi - \varepsilon\overline{\Psi}^*) = -\frac{1}{2}(I + \overline{\rho(g)}^{-1})\overline{V(g)}$ on U^*. Obviously, Ψ_e is $(-\varepsilon)$-hermitian, and it extends Ψ.

Fact. *The Ψ_e-orthogonal of $V(g)^{-1}(K)$ in U^* is in $V(g)^{-1}(K)$.*

Proof of Fact. Set $\alpha \in V(g)^{-1}(K)^\perp$ such that $\Psi_e(\alpha, \alpha') = 0$ for any $\alpha' \in V(g)^{-1}(K)$. Then $\alpha(I + \overline{\rho(g)}^{-1})(\bar{k}) = 0$ for any $k \in K$. But, $I + \rho(g)^{-1}$ is an isomorphism and preserves K, thus $\alpha \in K^*$. The inclusion $K^* \subset V(g)^{-1}(K)$ ends the proof of Fact.

Now, by a well-known result, $\operatorname{sign} \Psi_e = \operatorname{sign} \Psi_e | V(g)^{-1}(K) = \operatorname{sign} \Psi$.

The verification $\varepsilon \operatorname{sign} \Psi_e = \eta(g)$ is easy, considering the two cases $\varepsilon = \pm 1$. (Recall that for $\varepsilon = 1$, $\operatorname{sign} b = \operatorname{sign} \bar{b}$, and for $\varepsilon = -1$, $\operatorname{sign} b = \operatorname{sign}(ib)$.) □

Example 4.11. Assume that b_1 is nondegenerate. Then

$$\eta_1(g) = \operatorname{sign}\{[\rho_1(g) - \rho_1(g)^{-1}]b_1^{-1}\} = \varepsilon \cdot \operatorname{sign}\{b_1[\rho_1(g) - \rho_1(g)^{-1}]\}.$$

The latter equality follows, for example, by a verification on the generators of $HI_\varepsilon(\mathbb{Z})$ (for their description, see for example [9, (3.18)]). Notice, that even in this nondegenerate case, $\eta_1(g)$ does not vanish, in general, on hyperbolic spaces. In fact, it vanishes on odd-dimensional indecomposable structures, and it is nonzero on even-dimensional ones.

4.12. Denote the 2-dimensional G-cocycles (respectively, coboundaries) by $Z^2(G, \mathbb{Z})$ (respectively, $B^2(G, \mathbb{Z})$). For arbitrary $\mathcal{V} \in HV_\varepsilon(G)$, define the following 2-cocycle:

$$\theta(\mathcal{V}; g, h) = \sigma(\mathcal{V}; K_e, K_g, K_h) - \eta(g) - \eta(g^{-1}h) + \eta(h).$$

This defines a map $\Theta_{II}: HV_\varepsilon(G) \to Z^2(G,\mathbb{Z})$. Notice that Θ_{II} factorized in $H^2(G,\mathbb{Z})$ is exactly Θ_I. Now, Θ_{II} vanishes on hyperbolic structures (by 4.8 and 4.10). The induced map, still denoted by Θ_{II}, is: $WV_\varepsilon(G) \to Z^2(G,\mathbb{Z})$.

Notice that there are examples with $\Theta_{II} \neq 0$ (even with $\Theta_I = 0$, for example when $G = \{\text{free group}\}$). For the interested reader, we sketch out one example. Consider a flat bundle Γ_ρ with a nondegenerate flat hermitian form b over an orientable surface S with genus $g \geqslant 2$. Assume that $\sigma(S;\Gamma_\rho) \neq 0$. Consider an embedded (open) disk $D^2 \subset S$. Set $G = \pi_1(S - D^2)$ and $\mathcal{V} = (U; b, \rho)$ the induced structure associated with G. Then, by the general index theory, $\Theta_{II}(\mathcal{V}) \neq 0$ (even if we replace η with any $f(\mathcal{V})$ which depends only on $g^*\mathcal{V}$).

The vanishing of Θ_I is equivalent to $\operatorname{im}\Theta_{II} \subset B^2(G,\mathbb{Z})$; i.e., $\sigma(\mathcal{V}; K_e, K_g, K_h) = f(\mathcal{V}; g) + f(\mathcal{V}; g^{-1}h) - f(\mathcal{V}; h)$ for some $f(\mathcal{V})$. In general, $f(\mathcal{V}; g)$ depends on $\mathcal{V}$ and g, and not only on $g^*\mathcal{V}$. The main question is: can $f(\mathcal{V})$ be constructed "independently of $\mathcal{V}$", i.e., so that $f(\mathcal{V}; g)$ depends only on $g^*\mathcal{V}$? A candidate for such an f is η. The vanishing of Θ_{II} gives a positive answer to this question with $f(\mathcal{V}) = \eta$.

Theorem 4.13. *If G is a (finitely generated) abelian group then Θ_{II} vanishes. In particular,*

$$\sigma(\mathcal{V}; K_e, K_g, K_h) = \eta(g) + \eta(g^{-1}h) - \eta(h).$$

Proof. Since we can construct an epimorphism $\mathbb{Z}^k \to G$, it is enough to verify the theorem for $G = \mathbb{Z}^k$. If $\chi = 1$, then $WV_1(\mathbb{Z}^k) = \mathbb{Z}_2$ is generated by $\mathcal{T}$ (3.5), and $\Theta_{II}\mathcal{T} = 0$.

Assume that $\chi \in \hat{G} - \{1\}$. Then, by 3.5, we have to verify the identity only for $\mathcal{V} = (\mathbb{C}; b, \chi, V)$, where $b \neq 0$, $\bar{b} = \varepsilon b$ and $V(g') = (\chi(g') - 1)b^{-1}$ for any $g' \in G$.

If $\chi(g) = 1$, then $V(g) = 0$ and $K_g = K_e$, hence $\sigma(K_e, K_g, K_h) = 0$. On the other hand, $\eta(g) = \eta_1(g) = 0$ and $\eta(g^{-1}h) = \eta(h)$. Similarly, if $\chi(h) = 1$ or $\chi(g^{-1}h) = 1$, both sides of the identity vanish. So, we can assume that $1 \neq \chi(g) = \mathrm{e}^{-2\pi \mathrm{i} a} \neq \chi(h) = \mathrm{e}^{-2\pi \mathrm{i} c} \neq 1$, where $0 < a < 1$, $0 < c < 1$. Then the form Ψ is

$$\frac{[1 - \chi(g)][1 - \chi(h)]}{[\chi(h) - \chi(g)]b\varepsilon}.$$

Therefore,

$$\varepsilon \cdot \operatorname{sign}\Psi = \operatorname{sign}(\sin(c-a)\pi) \cdot \operatorname{sign}[-\mathrm{i}/b] = \operatorname{sign} b \cdot \operatorname{sign}(\sin(c-a)\pi).$$

Now, the verification is elementary. $\square$

5. The signature of the Yomdin series

5.1. Let $f_1: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ $(n > 1)$ be a nonisolated singularity with one-dimensional singular locus Σ. Consider a germ $f_2: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ so that the pair (f_1, f_2) forms an isolated complete intersection singularity. Let $\phi: (\mathcal{X}, 0) \to (D^2, 0)$ be a "good representative" of (f_1, f_2). The irreducible components of the discriminant locus Δ are $\Delta_1, \ldots, \Delta_s$. Here $\Delta_1 = \phi(\Sigma)$.

Now, ϕ defines two sets of invariants. The first one is the associated $(-1)^{n-1}$-hermitian variation structure $\mathcal{V}(\phi) = (U; b, \rho, V)$ as in 2.5(4). The second one is a set of η-invariants, as it will be defined in the sequel.

Any $g \in G$ defines a structure $g^*\mathcal{V}(\phi) \in HV_{(-1)^{n-1}}(\mathbb{Z})$, in particular a number $\eta(g) = \eta(g^*\mathcal{V}(\phi))$ as in 4.9. It is not hard to verify that $\eta(g^{-1}) = -\eta(g)$ and $\eta(h^{-1}gh) = \eta(g)$. This shows that η is well defined on the conjugacy classes of G. In particular, $\eta(S^1)$ is well defined for any (oriented) embedding $S^1 \hookrightarrow D^2 - \Delta$.

Let T be a closed tubular neighbourhood of $\Delta_1 \cap \partial D^2$ in $S^3 = \partial D^2$. Let L and M be the standard topological longitude and meridian in ∂T. They can be represented by embeddings $S^1 \to \partial T$. Moreover, for any $q \in \mathbb{Z}$, the homology class $[L + qM] \in H_1(\partial T, \mathbb{Z})$ can also be represented (essentially, in a unique way, up to isotopy) by an $S^1 \hookrightarrow \partial T$. In particular, the numbers $\eta(L)$, $\eta(M)$, and $\eta(L + qM)$ are well defined.

Theorem 5.2. *For $q \in \mathbb{N}$ sufficiently large:*

$$\sigma(f_1 + f_2^q) - \sigma(f_1) = q \cdot \eta(M) + \eta(L) - \eta(L + qM).$$

Proof.

Step 1. Let $(F_i, \partial F_i)$ $(i = 1, 2)$ be two copies of (the fiber of ϕ) $(F, \partial F)$. Assume that the orientation of $(F_1, \partial F_1)$ (respectively, of $(F_2, \partial F_2)$) is the orientation (respectively, inverse orientation) of $(F, \partial F)$. Glue F_1 and F_2 along $\partial F_1 = -\partial F_2$. The result is Z, an oriented $2(n-1)$-dimensional manifold. Consider the maps $\beta_1 : H_{n-1}(Z) \to H_{n-1}(Z, F_2)$ induced by the injection $(Z, \emptyset) \to (Z, F_2)$, and $\beta_2 : H_{n-1}(Z) \to H_{n-1}(F_1)$ induced by the projection $Z \to F_1$ (given by the identifications $F_i \to F_1$). Then $(\beta_1, \beta_2) : H_n(Z) \to U^* \oplus U$ is an isomorphism. This identifies a cycle $\varphi \in U^* = H_{n-1}(F, \partial F)$ with the cycle $\varphi \cup_{\partial\varphi} (-\varphi)$ in Z, where φ is considered in F_1 and $-\varphi$ in F_2, and the gluing is along $\partial\varphi$. A cycle $u \in U = H_{n-1}(F)$ is identified with the corresponding cycle imbedded in F_1. Now, it is easy to verify that the $(-1)^{n-1}$-hermitian intersection form on Z, identified on $U^* \oplus U$, is exactly Φ.

Step 2. Consider the closed unit disc D_1 in $\mathbb{C}$ with its natural orientation and boundary S^1. Let $S^1_+ = \{z \in S^1 \mid \operatorname{Re} z > 0\}$ be a segment in S^1 with its natural orientation.

Set $g \in G$ and consider a representative of its geometric monodromy $m_g : (F, \partial F) \to (F, \partial F)$ so that $m_g|\partial F = 1_{\partial F}$. Construct $E_g = (F, \partial F) \times [0, 1]/_{(x,0)\sim(m_g(x),1)}$, and identify $[0, 1]/_\sim$ with S^1. Then E_g is a locally trivial fibration of pairs of spaces over S^1 with fiber $(F, \partial F)$ and characteristic map m_g. Since S^1_+ is contractible, we can assume that this fibration has a product structure $F \times S^1_+$ over S^1_+. Since $\partial E_g = \partial F \times S^1$, we can construct $X_g = E_g \cup (\partial F \times D_1)$ by gluing along $K_g \times S^1$. Now we cut a window in X_g: consider $Z_g = X_g - (F - \partial F) \times S^1_+$. This Z_g is a $(2n-1)$-dimensional manifold with boundary Z (its corners can be smoothed). Moreover, the kernel $\ker\{H_{n-1}(Z) \to H_{n-1}(Z_g)\}$ is exactly $K_g \in U^* \oplus U$.

Step 3. We are in the following geometric situation. (See, for example [16,7,10].) There is an embedding j of the closed disc $D_2 = \{|z| \leqslant 2\}$ in D^2 (in fact $j(D_2) = \{(c,d) \in D_2 \mid c + d^q = \text{small constant}\}$), so that:

(a) $\partial(j(D_2)) \cap \Delta = \emptyset$, $\partial(j(D_1)) \cap \Delta = \emptyset$, where $D_1 \subset D_2$ is the closed unit disc,

(b) $\phi^{-1}(j(D_2))$ (respectively, $\phi^{-1}(j(D_1))$) can be identified with the Milnor fiber of $f + f_2^q$ (respectively, of f),

(c) $j(D_2) \cap \Delta_1$ is a subset of $j(D_2 - D_1)$, and it contains exactly q points; the small, positively oriented circles in $j(D_2)$ around these intersection points correspond to the meridian M of Δ_1. Moreover, the oriented boundary $\partial j(D_1)$ is the longitude L of Δ_1.

In the sequel, we identify D_2 with $j(D_2)$. Consider an embedded path $\gamma : [0,1] \to D_2$ with endpoints $\gamma(0) \neq \gamma(1)$, $\gamma(0), \gamma(1) \in \partial D_2$, such that it separates the points $D_2 \cap \Delta_1$ from the point of D_1. $D_2 - \operatorname{im}\gamma$ has two connected components, D' and D''_q with $D_1 \subset D'$ and $D''_q \cap \Delta = D_2 \cap \Delta_1$. In particular $\phi^{-1}(D')$ is diffeomorphic to the Milnor fiber of f. Applying Wall's theorem [17] (for the skew-symmetric case, see [5]), we get

$$\sigma(f_1 + f_2^q) = \sigma(f_1) + \operatorname{sign}\phi^{-1}(D''_q) + \sigma(K_e, K_{qM}, K_{-L}).$$

Now, if $D(j)$ is a small disc which contains an intersection point $z_j \in D_2 \cap \Delta_1$, then $\operatorname{sign}\phi^{-1}(D(j)) = 0$, because for an arbitrary point $z \in D(j) - \{z_j\}$, the natural map $H_{n-1}(\phi^{-1}(z)) \to H_{n-1}(\phi^{-1}(D(j)))$ is onto. Cutting $\phi^{-1}(D''_q)$ $q-1$ times by paths as above, we obtain:

$$\operatorname{sign}\phi^{-1}(D''_q) = \sum_{k=1}^{q-1} \sigma(K_e, K_M, K_{-(q-k)M}).$$

Since $ML = LM$, by 3.13, and by $\eta(-g) = -\eta(g)$, we obtain the desired relation. □

5.3. In [9] the equivariant version of 5.2 is proved by a different method. Moreover, also the quasiperiodicity of the expression $\{q\eta(M) - \eta(L + qM)\}_q$ is proved.

Examples 5.4 (The suspension case). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. Consider $f_1 : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ defined by $f_1(z, z_{n+1}) = f(z)$. Set $f_2 = z_{n+1}$ and $\phi = (f_1, f_2)$ as above. The singular locus of the ICIS ϕ is $\Sigma = \{z_1 = \cdots = z_n = 0\}$, and the discriminant locus Δ contains only one irreducible component $\Delta = \Delta_1$, which is smooth. In particular, $G = \mathbb{Z}$ and $L = 0$, and the (generally very rich) structure provided by the variation map and the monodromy representation of ϕ collapses in the variation structure $\mathcal{V}(f)$ of f. Therefore, there is a huge qualitative and technical difference between the suspension case and our case of the Yomdin series.

Since $\sigma(f_1) = 0$, Theorem 5.2, for suspensions, gives:

$$\sigma(f + z_{n+1}^q) = q \cdot \eta(\mathcal{V}(\phi); M) - \eta(\mathcal{V}(\phi); qM) = q \cdot \eta(\mathcal{V}(f); 1) - \eta(\mathcal{V}(f); q).$$

5.5. In the following, we compute the expression $q \cdot \eta(\mathcal{V}; 1) - \eta(\mathcal{V}; q)$ for some (-1)-hermitian variation structure of the group $\mathbb{Z}$. In particular, we will reprove Brieskorn's signature formula [2] for his singularities $z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}}$ (n even).

We start with a one-dimensional structure $\mathcal{V}(\alpha) \in HV^s_{-1}(\mathbb{Z})$, generated by $V(1) = e^{\pi i\alpha}$, where $\alpha \in \mathbb{R}$. Then by 2.5(2), $b = 2i\sin\pi\alpha$, and for any $n \in \mathbb{Z}$ the representation is $\rho(n) = e^{2\pi i n\alpha}$, and

$$V(n) = \begin{cases} ne^{\pi i\alpha} & \text{if } \alpha \in \mathbb{Z}, \\ b^{-1}[\rho(n) - 1] = \frac{\sin\pi n\alpha}{\sin\pi\alpha} e^{\pi i n\alpha} & \text{if } \alpha \notin \mathbb{Z}. \end{cases}$$

The signature $\sigma(\mathcal{V}(\alpha); K_0, K_1, K_{n+1})$ $(n \in \mathbb{Z})$, can be computed directly from the definition (see 4.4). If $K_{n+1} = K_0$ then this signature is zero, otherwise it is equal to $\operatorname{sign}\frac{V(1)\cdot V(n+1)}{V(n+1)-V(1)}$. This gives for $n \geqslant 1$:

$$\sigma\big(\mathcal{V}(\alpha); K_0, K_1, K_{n+1}\big) = \begin{cases} (-1)^\alpha & \text{if } \alpha \in \mathbb{Z}, \\ \operatorname{sign}(\sin\pi n\alpha \cdot \sin\pi(n+1)\alpha) & \text{if } \alpha \notin \mathbb{Z}. \end{cases}$$

The right hand side of the above identity has a completely different presentation, too. Define the function $h : \mathbb{R} \to \{-1, 0, 1\}$ by $h(\alpha) = \operatorname{sign}\sin\pi\alpha$ and

$$F_n(\alpha) = \sum_{k=1}^{n-1} h\left(\alpha + \frac{k}{n}\right)$$

for $n \geqslant 2$ and $F_1(\alpha) = 0$.

Lemma 5.6. *Let $n \geqslant 1$. Then:*

$$F_{n+1}(\alpha) - F_n(\alpha) = \begin{cases} (-1)^\alpha & \text{if } \alpha \in \mathbb{Z}, \\ \operatorname{sign}(\sin\pi n\alpha \cdot \sin\pi(n+1)\alpha) & \text{if } \alpha \notin \mathbb{Z}. \end{cases}$$

Proof. The verification is elementary, and it is left to the reader. □

Corollary 5.7. *Fix $q \geqslant 1$. Then:*

$$q \cdot \eta\big(\mathcal{V}(\alpha); 1\big) - \eta\big(\mathcal{V}(\alpha); q\big) = F_q(\alpha).$$

Proof.

$$\begin{aligned} q \cdot \eta\big(\mathcal{V}(\alpha); 1\big) - \eta\big(\mathcal{V}(\alpha); q\big) &= \sum_{n=1}^{q-1} \sigma\big(\mathcal{V}(\alpha); K_0, K_1, K_{n+1}\big) \\ &= \sum_{n=1}^{q-1} \big(F_{n+1}(\alpha) - F_n(\alpha)\big) = F_q(\alpha). \end{aligned}$$ □

Example 5.8 (The signature of the Brieskorn's singularity). Set $f = z_1^{a_1} + \cdots + z_n^{a_n}$ where n is even. The variation map $V(f)$ is the inverse of the Seifert form of f (see [3]). This latter one, by a result of Sakamoto [15], is $(-1)^{(n-1)n/2}\Gamma_{a_1} \otimes \cdots \otimes \Gamma_{a_n}$, where Γ_a is an $(a-1)$-dimensional form with $(\Gamma_a)_{ij} = 1$ if $i = j$ or $i + 1 = j$, and $= 0$

otherwise. Now, if we write Γ_a in the base given by the eigenvectors of the monodromy $-\Gamma_a^{-1}\Gamma_a^*$, then Γ_a is diagonal. By a computation we obtain that

$$\Gamma_a^{-1} = \bigoplus_{k=1}^{a-1}(1 - e^{2\pi i k/a}).$$

Therefore:

$$V(f) = V(1) = (-1)^{n(n-1)/2}(-2i)^n \bigoplus_{k_1=1}^{a_1-1} \cdots \bigoplus_{k_n=1}^{a_n-1} e^{\pi i \sum_{j=1}^{n} k_j/a_j} \cdot \prod_{j=1}^{n} \sin\frac{\pi k_j}{a_j}$$

which is equivalent (as a sesqui-linear form) to

$$\bigoplus_{k_1=1}^{a_1-1} \cdots \bigoplus_{k_n=1}^{a_n-1} \mathcal{V}\Big(\sum_{j=1}^{n} k_j/a_j\Big).$$

By 5.7,

$$a_{n+1}\cdot\eta\big(\mathcal{V}(f);1\big) - \eta\big(\mathcal{V}(f);a_{n+1}\big) = \sum_{k_1=1}^{a_1-1}\cdots\sum_{k_n=1}^{a_n-1} F_{a_{n+1}}\left(\sum_{j=1}^{n} k_j/a_j\right).$$

Therefore, we obtain Brieskorn's formula:

$$\sigma\big(z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}}\big) = \sum_{k_1=1}^{a_1-1}\cdots\sum_{k_{n+1}=1}^{a_{n+1}-1} \operatorname{sign}\sin\left(\pi\sum_{j=1}^{n+1} k_j/a_j\right).$$

References

[1] M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry I; II; III, Math. Proc. Cambridge Philos. Soc. 77 (1975) 53–69; 78 (1975) 405–432; 79 (1976) 71–99.

[2] E. Brieskorn, Beispiele zur Differentialtopologie von Singularitäten, Invent. Math. 2 (1966) 1–14.

[3] A. Durfee and L. Kaufman, Periodicity of branched cyclic covers, Math. Ann. 218 (1975) 157–174.

[4] J. Harer, The second cohomology group of the mapping class group of an orientable surface, Invent. Math. 72 (1983) 221–239.

[5] W. Meyer, Die Signatur von Localen Koeffizientensystemen und Faserbündeln, Bonner Mathematische Schriften 53 (University of Bonn, Bonn, 1972).

[6] J. Milnor, On isometries of inner product spaces, Invent. Math. 8 (1969) 83–97.

[7] A. Némethi, The Milnor fiber and the zeta function of the singularities of type $f = P(h, g)$, Compositio Math. 79 (1991) 63–97.

[8] A. Némethi, The zeta function of singularities, J. Algebraic Geom. 2 (1993) 1–23.

[9] A. Némethi, The equivariant signature of hypersurface singularities and eta-invariant, Topology 34 (1995) 243–259.

[10] A. Némethi, The real Seifert form and the spectral pairs of isolated hypersurface singularities, Compositio Math. 98 (1995) 23–41.

[11] A. Némethi, The semi-ring structure and the spectral pairs of sesqui-linear forms, Algebra Colloq. 1 (1994) 85–95.

[12] W. Neumann, Cyclic suspension of knots and periodicity of signature for singularities, Bull. Amer. Math. Soc. 80 (1974) 977–981.
[13] W. Neumann, Signature related invariants of manifolds I, Monodromy and γ-invariants, Topology 18 (1979) 147–172.
[14] M. Saito, The Steenbrink's conjecture, Math. Ann. 289 (1991) 543–572.
[15] K. Sakamoto, Milnor fiberings and their characteristic maps, in: Proceedings International Conference on Manifolds and Related Topics in Topology, Tokyo (1973).
[16] D. Siersma, The monodromy of a series of hypersurface singularities, Comment. Math. Helv. 65 (1990) 181–197.
[17] C.T.C. Wall, Non-additivity of the signature, Invent. Math. 7 (1969) 269–274.
[18] I.N. Yomdin, Complex surfaces with a one dimensional set of singularities, Sibirsk. Mat. Z. 15 (1974) 1061–1081.