Some topological invariants of isolated hypersurface singularities

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§0. Introduction.

This manuscript was prepared for the students of the EMS Summer School, Eger (Hungary, 1996). It contains the five lectures of the author, but also a lot of additional material. Each section ends with a list of open problems/conjectures and with a large number of exercises (some of them with hint or with solution). The exercises constitute an important part of the note, many key results of the hypersurface singularities are presented here. The reader can find here some important properties of Brieskorn singularities, quasi–homogeneous singularities and some quotient singularities; some historically crucial examples (like A'Campo's example for a singularity with non–finite monodromy); the description of the invariants in terms of the resolution graphs; connections with classical topology, realization of the Poincaré sphere, lens spaces, and some exotic spheres; discussions about Casson invariant and some applications of the Dedekind sums; and many others. Many exercises, and especially the last sections, deal with plane curve singularities.

The theorems/remarks/exercises with * are considered difficult, or are presented without any proof. We included them in order to provide a more correct global picture of the theory.

On the other hand, this note is very far to cover all important aspects of the theory of hypersurface singularities. Nevertheless, we hope that it will become a useful guide for beginners.

Contents:

§1. The analytic and topological type. The link.

1.1. The basic object of our series of lectures is a germ of an analytic function $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. The germ f defines its zero set $(V_f, 0) = (\{f = 0\}, 0) \subset (\mathbf{C}^{n+1}, 0)$, which is a germ of an analytic set.

In the theory of hypersurface singularities, the most common equivalence relation is the $\mathcal{R}$ (=right)-equivalence.

1.2. Definition. Two germs $g,\ f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ are called $\mathcal{R}$-equivalent, if there exists a biholomorphic germ $\varphi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ such that $f = g \circ \varphi$. Notation: $f \overset{\mathcal{R}}{\sim} g$.

1.3. Example. Assume that $\partial f(0) := (\partial_1 f(0), \dots, \partial_{n+1} f(0)) \neq 0$. (Here $\partial_i f = \partial f / \partial z_i$.) Then, by the submersion theorem, $f \overset{\mathcal{R}}{\sim} (z \to z_1)$. These germs are exactly the "smooth germs", i.e. if $\partial f(0) \neq 0$, then 0 is a smooth point of V_f.

A germ f with $\partial f(0) = 0$ is called "singular germ".

The singular locus of the space–germ $(V_f, 0)$ is given by $Sing(V_f) := \{z : \partial f(z) = 0\}$. The germ f is called **isolated singularity** if $Sing(V_f) = \{0\}$. This is the case if and only if $\dim \mathcal{O}/(\partial f) < \infty$, where $\mathcal{O}$ is the local $\mathbf{C}$–algebra of analytic germs $(\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$, and (∂f) is the ideal generated by the partial derivatives $\partial_i f$, $i = 1, \dots, n+1$. (cf. Exercise 1.2)

1.4. Definition. The algebra $M(f) := \mathcal{O}/(\partial f)$ is called the Milnor algebra of the isolated singularity f, and its dimension $\mu(f)$ is the Milnor number of f.

1.5. Some examples of isolated hypersurface singularities. $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is called:

a) Brieskorn singularity if $f = \sum_{i=1}^{n+1} z_i^{a_i}$ $(a_i \geq 2)$;

b) homogeneous singularity of degree d if $f = \sum_\alpha a_\alpha z^\alpha$, where $z^\alpha = z_1^{\alpha_1} \cdots z_{n+1}^{\alpha_{n+1}}$, and $\sum \alpha_i = d$ if $a_\alpha \neq 0$.

c) quasi–homogeneous singularity of weights $w_1, \dots, w_{n+1}$ $(w_i \in \mathbf{Q},\ w_i \geq 2)$ if $f = \sum_\alpha a_\alpha z^\alpha$, such that if $a_\alpha \neq 0$ then $\sum_{i=1}^{n+1} \alpha_i / w_i = 1$.

Now, we will introduce another equivalence relation.

1.6. Definition. We say that two isolated singularities f and g have the same analytic type if there is a biholomorphic map $\varphi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ such that $\varphi(V_f) = V_g$.

Its topological analogue is the following:

1.7. Definition. We say that two isolated hypersurface germs f and g have the same topological type if there is a homeomorphism $\varphi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ such that $\varphi(V_f) = V_g$; i.e. if $(\mathbf{C}^{n+1}, V_f, 0)$ is homeomorphic to $(\mathbf{C}^{n+1}, V_g, 0)$.

Some major questions in the theory of hypersurface singularities are:

1.8. Problems.

a) Classify the germs modulo $\mathcal{R}$–equivalence;

b) Give simple algebraic criterion for $(\mathbf{C}^{n+1}, V_f, 0)$ to be biholomorphic (respectively, homeomorphic) to $(\mathbf{C}^{n+1}, V_g, 0)$.

c) (Or, at least) find invariants of the equivalence classes.

1.9. Examples.*

Set $A_1(f) = M(f)$, $A_2(f) = \mathcal{O}/(\partial f) + (f)$, and $A_3(f) = \mathcal{O}/$integral closure of (∂f). By a theorem of Briançon and Skoda [8], $f^{n+1} \in \overline{(\partial f)}$, in particular, $A_1(f) = M(f)$ is a $\mathbf{C}\{t\}/(t^{n+1})$–algebra via the multiplication $t \cdot \bar{u} = \overline{fu}$.

a) [Yau] [56], (see also [Scherk] [51]) If $A_1(f)$ is isomorphic to $A_1(g)$ as $\mathbf{C}\{t\}/(t^{n+1})$-algebra, then $f \overset{\mathcal{R}}{\sim} g$.

b) [Mather and Yau] [25] If $A_2(f)$ is isomorphic to $A_2(g)$ as $\mathbf{C}$–algebra, then f and g have the same analytic type.

c) [Lê–Ramanujan] [21] If for some i $(1 \leq i \leq 3)$ $A_i(f)$ is isomorphic to $A_i(g)$ as a $\mathbf{C}$–algebra, then f and g have the same topological type.

d) [Lê–Ramanujan] [21] Let f_t $(t \in [0,1])$ be a C^∞–family of isolated singularities, such that $\mu(f_t)$ is constant. Then the topological type of f_t is constant, provided that $n \neq 2$.

The first geometrical invariant (in any classification) is the **link of** f. We will use the following notations: $B_\epsilon = \{z \in \mathbf{C}^{n+1} : ||z|| \leq \epsilon\}$, and $S_\epsilon = \partial B_\epsilon$ (where $\epsilon > 0$).

1.10. Theorem. [Milnor] [26]

Fix a germ f as above which defines an isolated singularity at the origin. Then there exists an $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ one has:

a) The sphere S_ϵ meets V_f transversely, in particular, $K_f = S_\epsilon \cap V_f$ does not depend on the choice of ϵ, it is a $(2n-1)$–dimensional C^∞–manifold. By definition, K_f is the "link of f".

b) $(B_\epsilon, B_\epsilon \cap V_f)$ is homeomorphic to $(cone(S_\epsilon), cone(K_f))$; in particular, the embedding $K_f \subset S_\epsilon$ determines the topological type of f completely.

(Above $cone(Z) := [0,1] \times Z/(0,z) \sim (0,z')$.)

Proof. For simplicity, we will assume that f is a polynomial function (for the analytic case see Exercise 1.10).

Consider the function $r : V_f \to [0,\infty)$ given by $r(z) = ||z||^2$. Then the set of critical values of r is a real algebraic (constructible) subset of $\mathbf{R}$ of dimension zero, hence it is finite. Set ϵ_0 such that $0 < \epsilon_0^2 <$the smallest critical value of r. Hence a) follows. For b), we can construct a vector field $v(z)$ on $B_{\epsilon_0} \setminus \{0\}$ (first locally, then gluing the local vector fields by a partition of unity), such that $\langle v(z), z\rangle > 0$, and for $z \in V_f \setminus \{0\}$ the vector field $v(z)$ is tangent to V_f. Then, we normalize it by $w(z) = v(z)/\langle 2z, v(z)\rangle$. Consider the differential equation $dz/dt = w(z)$. Given a solution $p(t)$, one has $dp(t)/dt = w(p(t))$, hence $dr(p(t))/dt = 1$, so we can choose the parameter t such that $r(p(t)) = t$. For $a \in S_{\epsilon_0}$, let $p(a,t)$ be the solution which satisfies the initial condition $p(a, \epsilon_0^2) = a$. Then $p : S_{\epsilon_0} \times (0, \epsilon_0^2] \to B_{\epsilon_0} \setminus \{0\}$ is a diffeomorphism such that $p(K_f \times (0, \epsilon_0^2]) = V_f \cap B_{\epsilon_0} \setminus \{0\}$ $(K_f = S_{\epsilon_0} \cap V_f)$. Since $p(a,t)$ tends uniformly to zero as $t \to 0$, p extends to a homeomorphism $cone(S_{\epsilon_0}) \to B_{\epsilon_0}$ via $[s,a] \mapsto p(a, s\epsilon_0^2)$ which identifies $cone(K_f)$ with $V_f \cap B_{\epsilon_0}$. □

1.11. Definition. Two isolated singularities f and g are called link–equivalent if (S_ϵ, K_f)

is homeomorphic to $(S_{\epsilon'}, K_g)$ for all ϵ and ϵ' sufficiently small.

1.12. Remark.*

a) By Exercise 1.10.c, if $f \overset{\mathcal{R}}{\sim} g$, or if f and g have the same analytic type, then f and g are link–equivalent.

b) Obviously, by theorem 1.10, link–equivalence implies the topological equivalence. Saeki in [49] proved that also the converse is true; in particular, the topological type of f is equivalent to the embedding $K_f = V_f \cap S_\epsilon \hookrightarrow S_\epsilon$.

1.13. Example.* The case $n = 1$.

a) Assume that $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is an irreducible germ. Let $n : (D, 0) \to (V_f, 0)$ be the normalization of $(V_f, 0)$. With a good choice of the local coordinates t of $(D, 0)$ and (x, y) of $(\mathbf{C}^2, 0)$ one has: $n(t) = (x(t), y(t)) = (t^n, \sum_{i>n} a_i t^i)$. Therefore, we can write:

$$y = \sum_{i>n} a_i x^{i/n} \quad \text{(the Puiseux expansion of } f\text{)}.$$

Consider the fractions $\{i/n\}_{i>n, a_i \neq 0}$, and rewrite them in an increasing order, in the form:

$$\underset{\text{integers}}{\cdots} < \frac{m_1}{n_1} < \underset{\text{type } q/n_1}{\cdots} < \frac{m_2}{n_1 n_2} < \underset{\text{type } q/n_1 n_2}{\cdots\cdots} < \cdots < \frac{m_g}{n_1 \cdots n_g} < \underset{\text{type } q/n_1 \cdots n_g}{\cdots}$$

where $g.c.d.(m_i, n_i) = 1$ and $n_i > 1$ for all $i = 1, \ldots, g$.

The pairs of integers $(m_i, n_i)_{i=1}^g$ are called the Puiseux pairs of f. Let f_P be the germ which has the "standard expansion" $y(x) = \sum_{i=1}^g x^{m_i/n_1 \cdots n_i}$.

Then, by a result of K. Brauner [7], W. Burau [11] and O. Zariski [57]: f has the same topological type as f_P, and f_P (i.e. the set of Puiseux pairs) depends only on the topological type of f.

The link K_f is diffeomorphic to S^1, in particular the abstract manifold K_f contains very little information about the topological type. On the other hand, the knot $K_f \subset S_\epsilon$ contains all the information about the topological type: $K_f \subset S_\epsilon$ is an iterated torus knot (associated with the Puiseux pairs of f).

b) If f has r irreducible components, then K_f =disjoint union of r copies of S^1's. By a result of M. Lejeune [22] and O. Zariski: the topological type of f is determined by the topological type of each irreducible component of f, and by all the intersection multiplicities of the pairs of these components.

To complete the discussion, we mention a theorem of J. Reeve [48], which asserts that the intersection multiplicities at the origin of two germs is exactly the linking number (in S_ϵ) of their links (which is obviously a topological invariant).

1.14. Resolution graphs.

We generalize the notion of isolated singularity of plane curves $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ to the following situation. Let (X, x) be a normal surface singularity, and let $f : (X, x) \to (\mathbf{C}, 0)$ be the germ of an analytic function which defines a one–dimensional isolated singularity. We consider an embedded resolution $\phi : (\mathcal{Y}, D) \to (X, f^{-1}(0))$ of $(f^{-1}(0), x) \subset (X, x)$. This means that the restriction $\phi : \mathcal{Y} \setminus \phi^{-1}(x) \to X \setminus \{x\}$ is biholomorphic and the divisor $\phi^{-1}(f^{-1}(0))$ in $\mathcal{Y}$ has only normal crossing singularities. Let $E=\phi^{-1}(x)$ be the exceptional divisor and $E=\cup_{w\in\mathcal{W}}E_w$ be its decomposition in irreducible divisors. Similarly, let $\cup_{a\in\mathcal{A}}S_a$ be the irreducible decomposition of the strict transform S of $f^{-1}(0)$. Then $D=E\cup S$. Let G_f be the resolution graph of f (associated with ϕ), i.e. its vertices $\mathcal{V}=\mathcal{W}\coprod\mathcal{A}$ consist of the "nonarrowhead" vertices $\mathcal{W}$ (corresponding to E_w's), and "arrowhead" vertices $\mathcal{A}$ (corresponding to S_a's). We will assume that no E_w has self intersection, $\mathcal{W}\neq\emptyset$, and $\#E_\alpha\cap E_\beta\leq 1$ for any $\alpha,\beta\in\mathcal{W}$, $\alpha\neq\beta$. If two irreducible divisors, corresponding to $v_1,v_2\in\mathcal{V}$ have an intersection point, then $(v_1,v_2)=(v_2,v_1)$ is an edge of G_f connecting v_1 and v_2. The set of edges connecting two nonarrowheads is denoted by $\mathcal{E}$. For any $w\in\mathcal{W}$, we denote by $\mathcal{V}_w$ the set of vertices $v\in\mathcal{V}$ adjacent to w.

The graph G_f is decorated by the self intersection (or Euler–) numbers $e_w:=E_w\cdot E_w$ for any $w\in\mathcal{W}$. The genus of E_w is denoted by g_w ($w\in\mathcal{W}$).

The **intersection matrix** $(A_{\alpha\beta})$ $(\alpha,\beta\in\mathcal{W})$ defined by $A_{\alpha\beta}=e_\alpha$ if $\alpha=\beta$, and $=1$ if $\alpha\neq\beta$, $(\alpha,\beta)\in\mathcal{E}$; and $=0$ otherwise. By a result of Grauert [17]:

(**1.15**) $(A_{\alpha\beta})$ is negative definite (in particular, non-degenerate).

For any $v\in\mathcal{V}$, let m_v be the order of vanishing of $f\circ\phi$ along the irreducible divisor corresponding to v. In particular, $m_a=1$ for any $a\in\mathcal{A}$. These multiplicities, for any $w\in\mathcal{W}$, satisfy:

$$(\mathbf{1.16})\qquad e_w\cdot m_w+\sum_{v\in\mathcal{V}_w}m_v=0.$$

By (1.15) and (1.16), the Euler numbers $\{e_w\}_w$ determine the multiplicities completely.

The resolution graph $G(X)$ of (X,x) (without a map–germ f) is obtained from (any) G_f by deleting its arrows and the multiplicity structure $\{m_w\}_w$.

1.13'. Example.* **The case** $n=1$ (continued).

The following pieces of data, concerning of a plane curve singularity, are equivalent:

(a) the topological type – characterized by the iterated torus knots corresponding to the irreducible components of f, and their linking numbers;

(b) the Puiseux pairs of the irreducible components and the intersection multiplicities of the irreducible components with each other;

(c) any embedded resolution graph of f.

For plane curve singularities the resolution graph is special, e.g.: each E_w is rational, G_f is a tree, and $(A_{\alpha\beta})$ is unimodular (cf. Exercise 1.16).

1.17. Example.* **The case** $n=2$.

The germ $(V_f,0)\subset(\mathbf{C}^3,0)$ is a normal surface singularity (cf. E.1.14), in particular has a resolution graph $G(V_f)$. By a topological construction, the link K_f can be recovered from the graph $G(V_f)$: K_f can be constructed as a (connected) plumbing manifold where $G(V_f)$ is exactly its plumbing graph. Conversely, by a result of W. Neumann [42], the oriented homeomorphism type of K_f determines $G(V_f)$ (up to blowing ups). (Actually, by another result of W. Neumann [loc.cit.], already $\pi_1(K_f)$ determines $G(V_f)$, excepting some cases which are classified. This generalizes a result of D. Mumford [30] which says that $\partial f(0) = 0$ implies $\pi_1(K_f) \neq 1$.)

On the other hand, the graph $G(V_f)$ says very little about the embedding $K_f \subset S_\epsilon$ (cf. exercises E.1.15 and E.3.3.c).

For an algorithm of construction of the graph $G(V_f)$ for germs of type $f(x, y, z) = g(x, y) + z^n$, see Appendix.

1.18.* K_f **in the general case.**

By a result of J. Milnor [26], the $(2n-1)$–manifold K_f is $(n-2)$–connected. In particular, K_f is a (rational) homology sphere if and only if $H_{n-1}(K_f, \mathbf{Z}) = 0$ (resp. $H_{n-1}(K_f, \mathbf{Q}) = 0$). If $n \geq 3$, then $\pi_1(K_f) = 1$, hence $H_{n-1}(K_f, \mathbf{Z}) = 0$ implies that K_f is a homotopy sphere, and by a result of S. Smale follows that K_f is *homeomorphic* to S^{2n-1} (but maybe not diffeomorphic) (cf. E.3.11).

Open problems and conjectures. (see also **(1.8)**)

1.19. [58] **The Zariski's multiplicity problem:** The multiplicity $m(f)$ of f is determined by the topological type of f.

Here, if $f = f_d + f_{d+1} + \cdots$ is the decomposition of f in homogeneous terms f_i of degree i, (with $f_d \neq 0$), then $m(f) = d$. In the case $n = 1$, the conjecture has a positive answer. Indeed, for irreducible f with Puiseux pairs $(m_i, n_i)_{i=1}^g$, the multiplicity is $m(f) = n_1 \cdots n_g$. But already the case $n = 2$ appears to be a very difficult problem.

1.20. Two–dimensional singularities. [H. Laufer]

Let $G(X)$ be the resolution graph of a normal surface singularity (X, x).

(a) Give necessary conditions on $G(X)$ for (X, x) to be a hypersurface singularity (i.e. $(X, x) = (V_f, 0)$ for some f).

(b) Give sufficient conditions on a graph G for a hypersurface singularity f to exist with $G = G(V_f)$.

Exercises for the first section.

E.1.1. (Morse Lemma) A singular germ (i.e. $\partial f(0) = 0$) $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is called non–degenerate if its Hessian $\det(\partial^2_{ij}(f)(0)) \neq 0$. For such a germ f prove that $f \overset{\mathcal{R}}{\sim} (z \mapsto \sum_{i=1}^{n+1} z_i^2)$.

Hint: First write $f = \sum_{i,j} a_{ij}(z) z_i z_j$, where $a_{ij}(z) = a_{ji}(z)$, e.g. $a_{ij}(z) = -\int_0^1 \partial_j \partial_i f(tz)(t-1)dt$. Then use the diagonalization method similarly as for quadratic forms.

E.1.2. Let $f \in \mathcal{O}$. Prove that the following facts are equivalent:

(a) there is a neighborhood $0 \in U \subset \mathbf{C}^{n+1}$ such that $V_f \cap U \setminus \{0\}$ is smooth;

(b) $(\partial f) \supset m^k$ for some k;

(c) $(\partial f) + (f) \supset m^k$ for some k;

(d) $\dim \mathcal{O}/(\partial f) < \infty$;

(e) $\dim \mathcal{O}/(\partial f) + (f) < \infty$.

Hint (cf. e.g. [24], page 2): Use the local analytic Nullstellensatz.

E.1.3. Let f and g be isolated singularities. Then the following statements are equivalent:

(a) f and g are analytically equivalent, i.e. $(\mathbf{C}^{n+1}, V_f, 0)$ and $(\mathbf{C}^{n+1}, V_g, 0)$ are biholomorphic.

(b) The germs $(V_f, 0)$ and $(V_g, 0)$ are isomorphic as analytic sets, i.e. there is a $\mathbf{C}$–algebra isomorphism $\mathcal{O}/(f) \to \mathcal{O}/(g)$.

(c) There is a biholomorphic isomorphism $\varphi : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ such that $\varphi^*((g)) = (f)$.

E.1.4. Let f and g be isolated singularities. Then the following statements are **not** equivalent (give examples!):

(a) f and g are topological equivalent,

(b) $(V_f, 0)$ and $(V_g, 0)$ are homeomorphic.

E.1.5. (a) The local degree of a germ $G : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ (with $G^{-1}(0) = 0$), by definition, is the number of solutions of the equation $G(z) = a$ for a generic. Prove that the local degree of G is exactly the dimension of $\mathcal{O}/(G_1, \ldots, G_{n+1})$ where G_i is the i^{th}–component of G.

(Notice that this fact in the case of real analytic functions is not true; take e.g. $G : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ given by $(x, y) \mapsto (x^3 + xy^2, y)$.)

(b) If G is as above, and G_i is a homogeneous polynomial of degree d_i, then the local degree of G is $\prod_i d_i$.

E.1.6. The topological degree of $\partial f/||\partial f||$.

Fix an isolated singularity f. Choose $\epsilon > 0$ with $\{z \in B_\epsilon : \partial f(z) = 0\} = \{0\}$. Define $G = \partial f/||\partial f|| : S^{2n+1}_\epsilon \to S^{2n+1}_1$. Prove that $\mu(f) = \deg G$ (where $\deg G$ is the topological degree of G), in the following steps:

(a) First prove the assertion for non–degenerate germs (cf. E.1.1).

(b) Let f_a be a deformation of f (morsification) such that $f_a(z) = f(z) + \sum_i a_i z_i$, where $\{a_i\}_{i=1}^{n+1}$ are generic, small coefficients such that all the critical points of f_a (in B_ϵ) are non-degenerate. Prove that $\deg G = \deg G_a$, where $G_a = \partial f_a/||\partial f_a||$.

(c) The critical points $\{z_i\}_i$ of f_a satisfy the equation $\partial f(z) + a = 0$, hence their number is exactly $\mu(f)$ (cf. E.1.5).

(d) Fix small balls B_i around z_i, and notice that G_a is well–defined in $B_\epsilon \setminus \cup_i B_i$, hence by a property of the degree, one has: $\deg G_a|B_\epsilon = \sum_i \deg G_a|B_i$. Now, apply (a) for the points z_i.

E.1.7. Assume that $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defines an isolated singularity.

(a) If $f = \sum z_i^{a_i}$, then $\mu(f) = \prod_i (a_i - 1)$.

(b) If f is homogeneous of degree d, then $\mu(f) = (d - 1)^{n+1}$.

(c)* [Milnor–Orlik] If f is quasi–homogeneous with weights $\{w_i\}_i$, then $\mu(f) = \prod_i (w_i - 1)$.

(d) Construct rational numbers $w_1, \ldots, w_{n+1}$ ($w_i \geq 2$) such that there is no quasi–homogeneous isolated singularity of weights $\{w_i\}_i$.

Hint (cf. [28]): Use E.1.5.b. In the case (c), construct a covering $G : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ such that $f \circ G$ has homogeneous components, and use the multiplicativity of the degree with respect to the composed maps.

E.1.8. Assume that f **is a quasi–homogeneous isolated singularity** of weights

$w_1, \dots, w_{n+1}$. Then:

(a) $M(f) = \mathcal{O}/(\partial f)$ has a base $\{\hat{m}_i\}_{i=1}^{\mu(f)}$, where m_i are monomials.

(b) Let $l(\alpha) = \sum_i (\alpha_i + 1)/w_i$. Then $\mathcal{O}$ has a grading $\mathcal{O} = \oplus_l \mathcal{O}_l$ such that $\mathcal{O}_l = \mathbf{C}\langle z^\alpha : l(\alpha) = l\rangle$. This induces a grading $M(f) = \oplus_l M(f)_l$.

(This grading is the shifted version of the "canonical grading" provided by $l'(\alpha) = \sum_i \alpha_i/w_i$. The disadvantage of our grading provided by l is that $M_l \cdot M_{l'} \not\subset M_{l+l'}$. Nevertheless, we prefer this one bacause of the special form of $P_{M(f)}(t)$, cf. E.2.14*.)

(c) The Poincaré series $P_A(t)$ of a graded $\mathbf{C}$–algebra $A = \oplus_l A_l$ is $P_A(t) := \sum_l \dim A_l \cdot t^l$. Prove that the Poincaré series $P^{\mathbf{w}}_{\mathcal{O}}(t)$ of $\mathcal{O}$ (with the above grading) is:

$$P^{\mathbf{w}}_{\mathcal{O}}(t) = \prod_i P^{w_i}_{\mathbf{C}\{z\}}(t) = \prod_i \frac{t^{1/w_i}}{1 - t^{1/w_i}}.$$

(d) Let $l_0 = \sum_i 1/w_i$, and assume that $f_i \in \mathcal{O}_{l_i + l_0}$, $i = 1, \dots, s$. Then, using the exact sequence:

$$0 \to \mathcal{O}/(f_1, \dots, f_{i-1})(-l_i) \xrightarrow{\cdot f_i} \mathcal{O}/(f_1, \dots, f_{i-1}) \to \mathcal{O}/(f_1, \dots, f_i) \to 0,$$

show that the Poincaré series of $A = \mathcal{O}/(f_1, \dots, f_s)$ is

$$P_A(t) = P^{\mathbf{w}}_{\mathcal{O}}(t) \cdot \prod_{i=1}^{s} (1 - t^{l_i}).$$

(e) Using $\partial_i f \in \mathcal{O}_{1 + l_0 - 1/w_i}$, show that

$$P_{M(f)}(t) = \prod_{i=1}^{n+1} \frac{t^{1/w_i} - t}{1 - t^{1/w_i}}.$$

E.1.9. Using the result (1.9.d) of Lê–Ramanujan, show that:

(a) the topological type of a quasi–homogeneous isolated singularity depends only on its weights;

(b) if f is homogeneous of degree d, then it has the same topological type as $\sum_{i=1}^{n+1} z_i^d$.

In the following exercise the following lemma is useful:

Curve Selection Lemma:* [Milnor] [26] *Let V be an open neighbourhood of $p \in \mathbf{R}^m$ and let $f_1, \dots, f_k, g_1, \dots, g_l$ be real analytic functions on V such that p is in the closure of $Z := \{x \in V : f_i(x) = 0, i = 1, \dots, k; g_j > 0, j = 1, \dots, l\}$. Then there exists a real analytic curve $\gamma : [0, \delta) \to V$ with $\gamma(0) = p$ and $\gamma(t) \in Z$ for $t > 0$.*

E.1.10.* (see, e.g. [24].) Let $(X, x) \subset (\mathbf{C}^N, 0)$ be an analytic space–germ such that $X \setminus \{x\}$ is smooth. (E.g. $(V_f, 0) \subset (\mathbf{C}^{n+1}, 0)$ for some isolated singularity f.)

(a) Let $r : X \to [0, \infty)$ be the restriction of a real analytic function $\tilde{r}$ defined on a neighbourhood of $x \in X$ such that $r^{-1}(0) = \{x\}$. Then 0 is not an accumulation point of the critical values of $r|X \setminus \{x\}$.

(b) Define $X_{r \le \epsilon} = \{y \in X : r(y) \le \epsilon\}$ and similarly $X_{r=\epsilon}$. Let $r : X \to [0, \infty)$ be as above and ϵ be such that $X_{r \le \epsilon}$ is compact and $r|X \setminus \{x\}$ has no critical values in $(0, \epsilon]$.

Then $X_{r=\epsilon}$ is a compact real analytic submanifold of X and there exists a homeomorphism H of the cone of $X_{r=\epsilon}$ onto $X_{r\leq\epsilon}$ such that $r\circ H/\epsilon$ is the projection onto $[0,1]$.

(c) The diffeomorphism type of $X_{r=\epsilon}$ (ϵ small) does not depend on the choice of the function r. In particular, the diffeomorphic type of the link $X_{r=\epsilon}$ of X at x depends only on the abstract analytic space–germ (X,x).

(d) Extend the above results for the pair $(\mathbf{C}^{n+1},V_f,0)$ instead of $(X,x)=(V_f,0)$.

E.1.11. If the isolated singularity $f:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ has r irreducible components then its link K_f is the disjoint union of r copies of S^1's.

E.1.12. Find the link $K_f\subset S^3_\epsilon$ of

(a) $f=x^2+y^2$; $f=x^2+y^3$; $f=x^a+y^b$;

(b) $y=x^{3/2}+x^{7/4}$.

E.1.13. Find a link $K\subset S^3$ which is not algebraic (i.e. does not exist f such that (S^3,K) is homeomorphic to (S^3_ϵ,K_f)).

E.1.14. If $f:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ is an isolated singular germ, then $(V_f,0)$ is a normal surface space–germ.
Hint: Use Serre's criterion.

E.1.15. Find the resolution graph $G(V_f)$ of V_f, where:

(a) f is a homogeneous isolated singularity of degree d (cf. E.5.6.c);

(b) Using the Appendix, prove that K_f is diffeomorphic to K_h, where $f=x^2+y^7+z^{14}$ and $h=x^3+y^4+z^{12}$. Notice that $\mu(f)\neq\mu(h)$.
Hint: $G(V_f)=G(V_h)$ contains only one vertex with $e=-1$ and $g=3$.

E.1.16. Let $G(V_f)$ be the resolution graph of $(V_f,0)$ $(n=2)$. Then:

(a) $H_1(K_f,\mathbf{Q})=H_1(E,\mathbf{Q})$. In particular, K_f is a rational homology sphere if and only if $G(V_f)$ is a tree and E_w is rational for all $w\in\mathcal{W}$.

(b) The order of the torsion part of $H_1(K_f,\mathbf{Z})$ is $|\det(A_{\alpha\beta})|$. (Obviously, $H_2(K_f,\mathbf{Z})$ is free.)

Hint: Consider a resolution $(U,E)\to(V_f,0)$. Then $\partial U=K_f$, and U and the exceptional curve E have the same homotopy type. In particular, $H_3(U)=0$. Now consider the exact sequence of the pair (U,K_f):

$$H_2(U)\to H_2(U,K_f)\to H_1(K_f)\to H_1(U)\to H_1(U,K_f).$$

By duality: $H_q(U,K_f)=H_{4-q}(U)^*$, and by this identification, the first arrow of the sequence is exactly $A:H_2(U)\to H_2(U)^*$ (which has trivial kernel because A is nondegenerate, cf. 1.15). Therefore, one has the exact sequence:

$$0\to\mathrm{coker}A\to H_1(K_f,\mathbf{Z})\to H_1(E,\mathbf{Z})\to 0.$$

E.1.17. Some quotient singularities.

(a) Consider the subgroup $\mathbf{Z}_a\subset SU(2)$, given by the diagonal matrices $diag(\zeta,\zeta^{-1})$, where ζ runs through all a^{th}–roots of unity. If u and v are the coordinates on $\mathbf{C}^2$, then $x=uv$, $y=u^a$ and $z=-v^a$ are generators of the invariant subalgebra $\mathbf{C}[u,v]^{\mathbf{Z}_a}$, and satisfy the relation $x^a+yz=0$.

The map $q:\mathbf{C}^2\to\mathbf{C}^3$ defined by $q(u,v)=(x,y,z)$ induces a topological covering $S^3\to K_f$, where $f=x^a+yz$ with Galois group $\mathbf{Z}_a$. Deduce that $\pi_1(K_f)=\mathbf{Z}_a$.

(b)* Extend the above computation for all finite subgroups of the special unitary group $SU(2)$ computing $\pi_1(K_f)$ for all quotient hypersurface singularities $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ (which are exactly the "simple singularities" in the sense of Arnold). E.g. the binary icosahedral group (the inverse image of the icosahedral group under the surjection $SU(2) \to SO(3)$) is the fundamental group of V_f, where $f = x^2 + y^3 + z^5$. This group has 120 elements. (cf. Exercise 3.10).

E.1.18. Consider a homogeneous polynomial $f : \mathbf{C}^{n+1} \to \mathbf{C}$ of degree d, which defines an isolated singularity at the origin. We can construct two spaces from f. First, the zero set $Z(f)$ of f in the projective space $\mathbf{CP}^n$, and also the link K_f. What is the relation between these two spaces?

Hint: There is a natural projection $K_f \to Z(f)$ which is an S^1-fiber bundle. Actually, this is the S_1-bundle associated with $\mathcal{O}(-d)$.

E.1.19. Prove that $f \stackrel{\mathcal{R}}{\sim} g \;\Rightarrow\; f \stackrel{an}{\sim} g \;\Rightarrow\; f \stackrel{top}{\sim} g$. Constructing examples, show that these equivalence relations are not identical.

Our last exercise has an informative role, it illustrates that any analytic germ is $\mathcal{R}$–equivalent to an algebraic germ.

E.1.20.* Finite determinacy property.

Let $f \in \mathcal{O}$ such that $m^k \subset m \cdot (\partial f)$, then $f \stackrel{\mathcal{R}}{\sim} j_k f$, where $j_k f$ is the k^{th}–jet of f (namely if $f = \sum f_i$, where f_i is homogeneous of degree i, then $j_k f = \sum_{i \leq k} f_i$).

Hint (for details, see [Gibson] [16], page 116):

We have to show that $j_k f = j_k g$ implies $f \sim g$. Take $F(z,t) = f_t(z) = (1-t)f(z) + tg(z)$ ($t \in \mathbf{R}$). It is enough to prove that $f_t \sim f_s$ for s fixed and t closed to s. Hence, it is enough to construct a germ $H : (\mathbf{C}^{n+1} \times \mathbf{R}, (0,s)) \to (\mathbf{C}^{n+1}, 0)$ such that (a) $H(z,s) = z$, (b) $H(0,t) = 0$, and (c) $F(H(z,t),t) = F(z,s)$. Notice that (c) is automatically satisfied for $t = s$, so we can replace (c) by the relation that its left hand side does not depend on t: i.e. by (c'): $\sum_i \partial_t H_i(z,t) \cdot \partial_i F(H(z,t),t) + \partial_t F(H(z,t),t) = 0$. If we succeed to construct $\xi : (\mathbf{C}^{n+1} \times \mathbf{R}, (0,s)) \to (\mathbf{C}^{n+1}, 0)$ such that (d) $\sum_i \xi_i \partial_i F = -\partial_t F$ and (e) $\xi_i(0,t) = 0$, then the flow associated with ξ satisfies (a-b-c'). Since $\partial_t F \in m^{k+1}$, for (d-e) we need to show $m^{k+1} \subset m \cdot (\partial F)$. Since $\partial_i F - \partial_i f \in t \cdot m^k$, one has $m^{k+1} \subset m \cdot (\partial F) + t \cdot m^{k+1}$, hence Nakayama's Lemma (in the variables (z,t)) finishes the proof.

§2. The Milnor fiber, the monodromy and the variation map.

2.1. Fix an isolated singularity $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$, and fix $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$, the sphere S_ϵ meets V_f transversely (cf. 1.10). (This means that at any point $z \in S_\epsilon \cap V_f$ the tangent spaces satisfy: $T_z S_\epsilon + T_z V_f = T_z \mathbf{C}^{n+1}$. Notation: $S_\epsilon \pitchfork V_f$.)

Consider the restriction of f on B_{ϵ_0}. Since $\{c \in \mathbf{C} : f^{-1}(c) \text{ has singular points}\} \subset \mathbf{C}$ is a finite (analytic) set, there is $\delta_1 > 0$ such that for any $c \in D_{\delta_1} \setminus \{0\}$ (where D_δ is the disc $\{c : |c| \leq \delta\}$):

$$f^{-1}(c) \cap B_{\epsilon_0} \text{ is smooth.} \tag{2.2}$$

Moreover, since the transversality is an open property, there is a $\delta_2 > 0$ such that for any $c \in D_{\delta_2}$:

$$f^{-1}(c) \pitchfork S_{\epsilon_0}, \tag{2.3}$$

(or equivalently, $f|f^{-1}(D_{\delta_2}) \cap S_{\epsilon_0}$ has no critical points). Set $\delta = \min(\delta_1, \delta_2)$.
Now, we recall:

2.4. The relative Ehresmann's Theorem. *Let $(E, \partial E)$ be a C^∞–manifold with boundary and $f : (E, \partial E) \to B$ a proper C^∞–map. Assume that f (on E) and $f|\partial E : \partial E \to B$ have no critical points. Then f is a C^∞ locally trivial fibration of pair of spaces.*

(2.2), (2.3) and Ehresmann's theorem give:

2.5. Theorem.

(a) For $0 < \delta << \epsilon_0 << 1$:

$$(f^{-1}(D_\delta \setminus \{0\}) \cap B_{\epsilon_0}, f^{-1}(D_\delta \setminus \{0\}) \cap S_{\epsilon_0}) \xrightarrow{f} D_\delta \setminus \{0\}$$

is a C^∞ locally trivial fibration of pair of spaces (called the "local fibration of f"). Its fiber $(F, \partial F)$ is called the Milnor fiber of f (actually, it is the local "nearby" fiber of the local "central fiber" $f^{-1}(0)$).

(b) The fibration $f : f^{-1}(D_\delta \setminus \{0\}) \cap S_{\epsilon_0} \to D_\delta \setminus \{0\}$ extends to a C^∞ fibration $f : f^{-1}(D_\delta) \cap S_{\epsilon_0} \to D_\delta$ (with fiber ∂F). Since D_δ is contractible, this latter fibration is a C^∞ trivial fibration.

An immediate consequence of (b) is that for any $c \in D_\delta$, $\partial(f^{-1}(c) \cap B_{\epsilon_0})$ is diffeomorphic to $\partial(f^{-1}(0) \cap B_{\epsilon_0}) = K_f$; i.e.:

$$\partial F = K_f. \tag{2.6}$$

A crucial result of Milnor asserts:

2.7. Theorem. [Milnor] [26] *F has the homotopy type of a bouquet of (μ copies of) n–spheres $\bigvee_\mu S^n$, where μ is the Milnor number (cf. 1.4).*

2.8. Remark. By elementary algebraic topology, it is not difficult to prove that $H_*(F,\mathbf{Z}) = H_*(\bigvee_\mu S^n,\mathbf{Z})$ (cf. E.2.4). But for $n\geq 2$, the vanishing of $\pi_1(F)$ needs some Morse theory (the core of Milnor's proof). These two facts imply Milnor's theorem (2.7) via Whitehead theorem (for Whitehead theorem, see, e.g. [54], page 399).

By (2.5), F is a $(2n)$–dimensional real manifold with boundary, with a natural orientation. Therefore, by Lefschetz duality $H_q(F,\partial F,\mathbf{Z}) = H^{2n-q}(F,\mathbf{Z})$ (see, e.g. [54], page 298). Hence, via (2.7), one has:

$$(\mathbf{2.9})\qquad \tilde{H}_q(F,\mathbf{Z}) = \tilde{H}_q(F,\partial F,\mathbf{Z}) = \begin{cases} \mathbf{Z}^\mu & \text{if } q=n \\ 0 & \text{otherwise.}\end{cases}$$

Notational Convention: In order to simplify the notations, we write $H_n(F,\mathbf{Z})$ instead of $\tilde{H}_n(F,\mathbf{Z})$. This is important only if $n=0$ (e.g. in E.2.11 or E.3.6). For singularities $f:(\mathbf{C},0)\to(\mathbf{C},0)$, the reader must replace $H_n(F,\mathbf{Z})$ by $\tilde{H}_n(F,\mathbf{Z})$.

2.10. Pairings. The perfect pairing $<,>$: $H_n(F,\partial F,\mathbf{Z})\otimes H_n(F,\mathbf{Z})\to\mathbf{Z}$ is induced by the algebraic intersections of type $<\alpha,\beta>$, where α (resp. β) is a relative (resp. absolute) oriented n–cycle. It is convenient to introduce its "opposite" intersection pairing as well: $<,>'$: $H_n(F,\mathbf{Z})\otimes H_n(F,\partial F,\mathbf{Z})\to\mathbf{Z}$ induced by $<\beta,\alpha>$. Obviously, $<\alpha,\beta> = (-1)^n<\beta,\alpha>'$. The non–degenerate pairing $<,>$ identifies $H_n(F,\partial F,\mathbf{Z})$ with $H_n(F,\mathbf{Z})^* = Hom_{\mathbf{Z}}(H_n(F,\mathbf{Z}),\mathbf{Z})$ via $[\alpha]\mapsto<\alpha,\cdot>$. Then the natural inclusion $j:H_n(F,\mathbf{Z})\to H_n(F,\partial F,\mathbf{Z})$ is identified with $b:H_n(F,\mathbf{Z})\to H_n(F,\mathbf{Z})^*$, $b(\beta)=(\beta,\cdot)$, where $(,):H_n(F,\mathbf{Z})\otimes H_n(F,\mathbf{Z})\to\mathbf{Z}$ is the intersection form on $H_n(F,\mathbf{Z})$ (induced by the algebraic intersection of absolute n–cycles). Notice that $(,)$ is $(-1)^n$–symmetric, i.e. $(a,b)=(-1)^n(b,a)$.

2.11. ker(,). In general $(,)$ is degenerate. In fact, by the homology long exact sequence of the pair $(F,\partial F)$, by (2.9), and by the above identification (2.10), one has (cf. also (1.18)):

$$(\mathbf{2.12})\qquad 0\to H_n(K_f,\mathbf{Z})\to H_n(F,\mathbf{Z})\xrightarrow{b} H_n(F,\mathbf{Z})^*\to H_{n-1}(K_f,\mathbf{Z})\to 0.$$

2.13. Corollary. [Milnor] [26]

(a) $H_n(K_f,\mathbf{Z}) = ker(,)\subset H_n(F,\mathbf{Z})$*, in particular,* K_f *is a rational homology sphere if and only if* $(,)$ *is non–degenerate;*

(b) K_f *is an integer homology sphere if and only if* $(,)$ *is unimodular.*

2.14. Monodromy. Now, we return to the theorem (2.5), and we consider the locally trivial fibration $(f^{-1}(\partial D_\delta)\cap B_{\epsilon_0}, f^{-1}(\partial D_\delta)\cap S_{\epsilon_0})\to\partial D_\delta = S^1_\delta$. Any C^∞ fibration with fiber $(F,\partial F)$ over S^1 is given (modulo an isomorphism) by a "characteristic map" (= geometric monodromy) $m:(F,\partial F)\to(F,\partial F)$ (well–defined modulo an isotopy), such that the fibration identifies with

$$(F,\partial F)\times[0,1]/_{(x,1)\sim(m(x),0)}\to[0,1]/_{0\sim 1}=S^1.$$

(2.5.b) assures that in our case, the geometric monodromy m can be chosen with:

$$\textbf{(2.15)} \qquad m|\partial F = id_{\partial F}.$$

The algebraic monodromies $h : H_n(F, \mathbf{Z}) \to H_n(F, \mathbf{Z})$ and $h^r : H_n(F, \partial F, \mathbf{Z}) \to H_n(F, \partial F, \mathbf{Z})$ are induced by m.

Since m preserves the intersection of the cycles, one has:

$$\textbf{(2.16)} \qquad < h^r\alpha, h\beta > = < \alpha, \beta >, \quad (h\beta, h\beta') = (\beta, \beta').$$

The identification $H_n(F, \partial F, \mathbf{Z}) = H_n(F, \mathbf{Z})^*$ (cf. 2.10), and (2.16) give:

$$\textbf{(2.17)} \qquad h^r = (h^*)^{-1}.$$

(In matrix notation, h^* is the transpose of h.)
The **characteristic polynomial** $\det(t - h)$ of h is denoted by $\Delta(t)$.

Since our local fibration is provided by algebraic (or analytic) maps, it satisfies the following:
2.18. Monodromy Theorem* [Grothendieck, Brieskorn] [10]

(a) The eigenvalues of the monodromy operator h are roots of unity.

(b) The nilpotent part h_{nil} of h satisfies $(h_{nil})^{n+1} = 0$, i.e. the size of any Jordan block is $\leq n + 1$.

(c) the size of any Jordan block of $h_1 := h|H_n(F, \mathbf{Z})_1$ is $\leq n$.

(Here $H_n(F, \mathbf{Z})_1$ is the generalized 1–eigenspace of $H_n(F, \mathbf{Z})$, see below).

The above bounds for the Jordan blocks are optimal. On the other hand, in the case of some special singularities, h can have only small Jordan blocks (cf. E.2.2). For example:
2.19. Theorem. [Lê] [20] (For a generalization, see [Némethi–Steenbrink] [41].)

If $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ is irreducible, then $h_{nil} = 0$.

2.20. The equivariant signature.

If $A : \mathbf{R}^\mu \otimes \mathbf{R}^\mu \to \mathbf{R}$ is a symmetric bilinear form, then let μ_+ (resp. μ_-) be the dimension of a maximal subspace of $\mathbf{R}^\mu$, where A is positive(resp. negative) definite. The signature of A is defined by $\sigma(A) := \mu_+ - \mu_-$. There is a similar definition for (complex) hermitian forms (i.e. if $\bar{A}^t = A$).

If A is a complex form with $\bar{A}^t = -A$, then we define $\sigma(A)$ via $\sigma(iA)$ (because: $\overline{iA}^t = iA$).

In our situation, consider the generalized eigenspace decomposition $\oplus_\lambda H_n(F, \mathbf{C})_\lambda$ of $H_n(F, \mathbf{C})$ (i.e. $H_n(F)_\lambda = \{v \in H_n(F, \mathbf{C}) : (h \otimes 1_\mathbf{C} - \lambda)^k v = 0 \text{ for some } k\}$). By (2.16), the decomposition is compatible with $(\,,)$. Extend $(\,,)$ to an $(-1)^n$–hermitian form on $H_n(F, \mathbf{C})$. Then:

$$(H_n(F, \mathbf{C}), (\,,)) = \oplus_\lambda (H_n(F, \mathbf{C})_\lambda, (\,,)_\lambda),$$

and $(a, b)_\lambda = (-1)^n \overline{(b, a)_\lambda}$. We define the equivariant signature $\sigma_\lambda(f)$ (respectively the signature $\sigma(f)$) by $\sigma((\,,)_\lambda)$ (resp. $\sigma(\,,)$). Obviously $\sigma(f) = \sum_\lambda \sigma_\lambda(f)$.

If n is odd then $\sigma(f)=0$, but even in this case the equivariant signatures are non-zero in general. They satisfy:

$$\sigma_{\bar{\lambda}}(f)=(-1)^n\sigma_\lambda(f).$$

2.21. The variation map.

(2.15) guarantees the existence of another operator $V : H_n(F,\partial F,\mathbf{Z}) \to H_n(F,\mathbf{Z})$, called the variation map. Its construction follows.

If α is a relative n–cycle (with $\partial\alpha \in \partial F = K_f$), then $m(\alpha)$ satisfies $\partial(m(\alpha)-\alpha) = m(\partial\alpha)-\partial\alpha = 0$, hence $m(\alpha)-\alpha$ is an absolute n–cycle. The correspondence $[\alpha] \in H_n(F,\partial F,\mathbf{Z}) \xrightarrow{V} [m(\alpha)-\alpha] \in H_n(F,\mathbf{Z})$ is a well–defined map, and by its very construction, it makes the following diagram commutative:

$$\textbf{(2.22)}\qquad \begin{array}{ccc} H_n(F,\mathbf{Z}) & \xrightarrow{\ j\ } & H_n(F,\partial F,\mathbf{Z}) \\ {\scriptstyle h-1}\downarrow & \swarrow{\scriptstyle V} & \downarrow{\scriptstyle h^r-1} \\ H_n(F,\mathbf{Z}) & \xrightarrow{\ j\ } & H_n(F,\partial F,\mathbf{Z}) \end{array}$$

Moreover, V satisfies also the following relations.

2.23. Lemma. *For any $x_1,x_2 \in H_n(F,\partial F,\mathbf{Z})$ one has:*

(a) $(Vx_1,Vx_2)+<x_1,Vx_2>+<Vx_1,x_2>'=0$,

(b) $<h^rx_1,Vx_2>+<Vx_1,x_2>'=0$.

Proof. (a) Let α_i be relative n–cycles representing x_i $(i=1,2)$, and with $\partial\alpha_1\cap\partial\alpha_2=\emptyset$. Recall that $Vx_i=[m\alpha_i-\alpha_i]$. Therefore, the left hand side of (a) is (where now the bracket denotes the intersection of the cycles): $<m(\alpha_1)-\alpha_1,m(\alpha_2)-\alpha_2>+<\alpha_1,m(\alpha_2)-\alpha_2>$ $+<m(\alpha_1)-\alpha_1,\alpha_2>=<m(\alpha_1),m(\alpha_2)>-<\alpha_1,\alpha_2>=0$. For (b) we use $h^r=id+j\circ V$ (cf. 2.22), therefore its left hand side is: $<\alpha_1+j(m(\alpha_1)-\alpha_1),m(\alpha_2)-\alpha_2>+<m(\alpha_1)-\alpha_1,\alpha_2>=0$. $\square$

2.24. Remark. In the next section we will prove that V is an isomorphism. Then (a) (resp. (b)) will provide $(,)$ (resp. h and h^r) in terms of V. It turns out that the (integer) variation map is an extremly strong invariant of f.

Open problems and conjectures.

2.25. Given an isolated singularity f, find an algorithm for the computation of the (integer) operators h and/or V.

2.26. [Durfee] [14] Assume that $n=2k$. Prove that $(-1)^k\sigma(f)>0$.

2.27. [Durfee] [14] Assume that $n=2$. Prove that $\sigma(f)\leq-\mu(f)/3$.

2.28. [Durfee] [14] If the germ f degenerates to the germ g, then $|\sigma(f)| \leq |\sigma(g)|$. (Remark: the Milnor numbers satisfy $\mu(f) \leq \mu(g)$.)

Exercises for the second section.

E.2.1. Let $f_i : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ $(i = 1, 2)$ be two isolated singularities such that $V_{f_1} \cap V_{f_2} = \{0\}$. Set $f = f_1 \cdot f_2$. Prove that:

$$\mu(f) = \mu(f_1) + \mu(f_2) + 2 \cdot i_0(f_1, f_2) - 1.$$

(Here i_0 denotes the intersection multiplicity at the origin.)

E.2.2. Let f be a **quasi–homogeneous isolated singularity** of weights $w_1, \ldots, w_{n+1}$. Set $E = f^{-1}(\partial D_\delta) \cap (B_\epsilon \setminus S_\epsilon)$ and $F = f^{-1}(\delta) \cap (B_\epsilon \setminus S_\epsilon)$.

(a) Using the natural $\mathbf{C}^*$–action, find the geometric monodromy $m : F \to F$ (without the property (2.15) !) as follows. Consider the map $F \times [0,1] \xrightarrow{T} E$ defined by $T(z,t) = (z_1 e^{2\pi i t/w_1}, \ldots, z_{n+1} e^{2\pi i t/w_{n+1}})$. Then $f(T(z,t)) = \delta e^{2\pi i t}$, hence $m = T(\cdot, 1)$.

(b) Conclude that h has a finite order.

(c) The (open) Milnor fiber F of f is diffeomorphic to $F_1 := \{z \in \mathbf{C}^{n+1} : f(z) = 1\}$. Moreover, the local fibration $f^{-1}(\partial D_\delta) \cap (B_\epsilon \setminus S_\epsilon) \to \partial D_\delta$ is isomorphic (as a fiber bundle) to $E := \{z \in \mathbf{C}^{n+1} : |f(z)| = 1\} \xrightarrow{f} \partial D_1$.

E.2.3. The non-degenerate singularity (see [24], pages 36-41).

Assume that $f = \sum z_i^2$. Write $z = x + iy$. Then the Milnor fiber $f^{-1}(\delta) \cap B_\epsilon$ is $F = \{x + iy \in \mathbf{C}^{n+1} : |x|^2 + |y|^2 \leq \epsilon^2,\ |x|^2 - |y|^2 = \delta,\ (x,y) = 0\}$. With the parametrization $u = x/|x|,\ v = c \cdot y$ $(c = 2/(\epsilon^2 - \delta), c > 0)$, F can be identified with the unit disk bundle E of the tangent bundle of S^n:

$$E = \{u + iv \in \mathbf{C}^{n+1} : |u| = 1,\ |v| \leq 1,\ (u,v) = 0\}.$$

This diffeomorphism $h_0 : E \to F$ is given by: $h_0(u + iv) = (|v|/c + \delta)u + iv/c$. Now, consider the family of homeomorphisms $g_\theta : E \to E$ $(\theta \in \mathbf{R})$:

$$g_\theta(u + iv) = [u \cos(\pi|v|\theta) + \frac{v}{|v|} \sin(\pi|v|\theta)] + i[-u|v| \sin(\pi|v|\theta) + v \cos(\pi|v|\theta)].$$

Set $h_\theta := e^{\pi i \theta} h_0 \circ g_\theta : E \to f^{-1}(D_\delta \cap B_\epsilon$. Then h_θ is a homeomorphism onto $f^{-1}(\delta e^{2\pi i \theta}) \cap B_\epsilon$. Notice that h_0 and h_1 coincide on ∂E, thus defining a trivialization on $f^{-1}(D_\delta) \cap S_\epsilon$ (note that this trivialization extends over D_δ). So, the automorphism $h_1 \circ h_0^{-1}$ of F represents the geometric monodromy m (with $m|\partial F = id_{\partial F}$). Under the identification of F with E via h_0, this corresponds to $h_0^{-1} \circ h_1 = -g_1$ on E.

(b) $H_n(E) = \mathbf{Z}$ is generated by $[S^n]$, where $S^n = \{v = 0\}$. We orient S^n by letting $(e_1, \ldots, e_n)$ be an orientation basis of $T_{e_0} S^n$, where $(e_0, \ldots, e_n)$ is the standard basis in $\mathbf{R}^{n+1}$. Notice that $-g_1|S^n : S^n \to S^n$ is $-id_{S^n}$. Therefore, $h = (-1)^{n+1}$. The intersection $([S^n],[S^n])_E$, with respect to the tangent bundle orientation, is the Euler–number $e(S^n)=1+(-1)^n$. Since the complex orientation differs from the tangent bundle orientation by the sign of the permutation $(e_1,\ldots,e_n,ie_1,\ldots,ie_n)\mapsto(e_1,ie_1,\ldots,e_n,ie_n)$, one has: $([S^n],[S^n])_F=(-1)^{n(n-1)/2}(1+(-1)^n)$.

(c) Fix $u^0\in S^n=\{v=0\}\subset E$, and let D^n be the fiber in E over u^0, i.e. $D^n=\{u^0+iv:(u^0,v)=0,\ |v|\leq 1\}$. It is oriented by $(ie_1,\ldots,ie_n)$. Then $[D^n,\partial D^n]$ is a base for $H_n(E,\partial E)$. Moreover $<[D^n,\partial D^n],[S^n]>$ is the sign of the permutation $(ie_1,\ldots,ie_n,e_1,\ldots,e_n)\to(e_1,ie_1,\ldots)$, which is $(-1)^{n(n+1)/2}$. Take $\delta^*=(-1)^{n(n+1)/2}[D^n,\partial D^n]$, and $\delta=[S^n]$, then $<\delta^*,\delta>=1$.

(d) Prove that $V(\delta^*)=(-1)^{(n+1)(n+2)/2}\delta$.

(e) Consider the case $n=1$. Then $(,)=0$, $h=id_{\mathbf{Z}}$ (i.e. both invariants are trivial). But V is a non-trivial endomorphism, in fact it is an isomorphism.

E.2.4. With the notations of (2.5), prove that $F=f^{-1}(\delta)\cap B_\epsilon$ satisfies: $H_*(F,\mathbf{Z})=H_*(\bigvee_\mu S^n,\mathbf{Z})$ (cf. 2.8) in the following steps:

(a) For $0<\delta<<\epsilon<<1$ prove that $B_{\epsilon,\delta}:=f^{-1}(D_\delta)\cap B_\epsilon$ is contractible. Hint: Using the Curve Selection Lemma, prove that the differentials of $r(z)=||z||^2$ and of $z\mapsto|f(z)|$ do not point in opposite direction. Hence, there is a vector field whose integral curves contract $B_{\epsilon,\delta}$ to the origin.

(b) From the long exact sequence of the pair $(B_{\epsilon,\delta},F)$, prove that it is enough to verify that $(*)$ $H_q(B_{\epsilon,\delta},F,\mathbf{Z})=\mathbf{Z}^\mu$ if $q=n$, and $=0$ otherwise.

(c) Verify $(*)$ for a non–degenerate singularity.

(d) Consider a morsification f_a of f as in (E.1.6). Let $\{z_i\}_{i=1}^{\mu(f)}$ be the set of critical points and $c_i=f(z_i)\in int(D_\delta)$ the set of critical values. Let $B_i(\epsilon')$ be a Milnor ball of f with center z_i, $D_i=D_i(\delta')$ a small disc with center c_i $(0<\delta'<<\epsilon')$. Take $c_i'\in\partial D_i$ and $\gamma_i:[0,1]\to D_\delta$ $(i=1,\ldots,\mu)$ such that $\gamma_i(0)=\delta$, $\gamma_i(1)=c_i'$, and $\Gamma:=\cup_i(im\gamma_i)\cup(\cup_i D_i)\hookrightarrow D_\delta$ admits a strong deformation retract r. Use Ehresmann's Theorem to prove that $f_a:f_a^{-1}(D_\delta\setminus\cup_i\{c_i'\})\cap B_\epsilon\to D_\delta\setminus\cup_i\{c_i'\}$ is a fiber bundle, hence the deformation retract r can be lifted. Notice also that $(f_a^{-1}(D_i)\cap B_\epsilon)\setminus B_i\to D_i$ is a trivial fiber bundle. Therefore: $H_*(B_{\epsilon,\delta},F)=$

$$\begin{aligned}
&= H_*(f_a^{-1}(D_\delta)\cap B_\epsilon, f_a^{-1}(\delta)\cap B_\epsilon) && \text{(transversality)}\\
&= H_*(f_a^{-1}(\Gamma)\cap B_\epsilon, f_a^{-1}(\delta)\cap B_\epsilon) && \text{(deformation retract)}\\
&= H_*(f_a^{-1}(\Gamma)\cap B_\epsilon, f_a^{-1}(\cup_i im\gamma_i)\cap B_\epsilon) && \text{(deformation retract)}\\
&= \oplus_i\ H_*(f_a^{-1}(D_i)\cap B_\epsilon, f_a^{-1}(c_i')\cap B_\epsilon) && \text{(excision)}\\
&= \oplus_i\ H_*(f_a^{-1}(D_i)\cap B_\epsilon, f_a^{-1}(c_i')\cup(f_a^{-1}(D_i)\setminus B_i)) && \text{(deformation retract)}\\
&= \oplus_i\ H_*(f_a^{-1}(D_i)\cap B_i, f_a^{-1}(c_i')\cap B_i)) && \text{(excision).}
\end{aligned}$$

Now, apply (c).

E.2.5. Assume that $n=0$ and $f:(\mathbf{C},0)\to(\mathbf{C},0)$ is given by $f(z)=z^a$. Find F, m, h and prove that $\Delta(t)=(t^a-1)/(t-1)$.

E.2.6. A'Campo's formula. [2] Assume that $n=1$ (A'Campo's result is for arbitrary n). As above, let G_f be the embedded resolution graph of f. For each $w\in\mathcal{W}$ let

$\delta_w = \#\mathcal{V}_w$. Prove that

$$\frac{t-1}{\Delta(t)} = \prod_{w\in\mathcal{W}} (t^{m_w}-1)^{2-\delta_w}.$$

(This proves the first part (2.18.a) of the Monodromy theorem.) In particular, $\chi(F) = 1-\mu(f) = \sum_w m_w(2-\delta_w)$.
Hint: If ϕ is as in (1.14), consider $F' = \phi^{-1}(F) \subset \mathcal{Y}$, which is diffeomorphic to F. Then there exists a map $\pi : F' \to D$ such that for any singular point P of D: $\pi^{-1}(P)$ is a disjoint union of S^1's (which do not have any contribution in the Euler–characteristic or in the characteristic polynomial); and $\pi^{-1}(E_w^{reg}) \to E_w^{reg}$ (E_w^{reg}=regular part of E_w) is an m_w–covering with a Galois action corresponding to the monodromy ($w \in \mathcal{W}$). Notice also that $\chi(E_w^{reg}) = 2-\delta_w$. (cf. 4.3 and 4.11)
E.2.7. [A'Campo][3] Assume $n = 1$. Using (E.2.6), prove that if f is singular (i.e. $\partial f(0) = 0$) then trace$(h) = \pm 1$.
E.2.8. Recall that $cone(Z) = [0,1]\times Z/(0,z) \sim (0,z')$, and there is a natural inclusion $Z \hookrightarrow cone(Z),\ z \mapsto [1,z]$.
(a) Prove that $f^{-1}([0,\delta]) \cap B_\epsilon$ ($0<\delta<<\epsilon<<1$) is contractible.
(b) Prove that there exists a homotopy equivalence:

$$\varphi : (cone(F), F) \to (f^{-1}([0,\delta])\cap B_\epsilon, f^{-1}(\delta)\cap B_\epsilon)$$

(where $F = f^{-1}(\delta)\cap B_\epsilon$), such that $f(\varphi([t,z])) = t\delta$.
Hint: Use the Curve Selection Lemma (cf. the hint of E.2.4.a).
E.2.9. The join space.
Let X and Y be two spaces. The join space $X*Y$ is defined by $X\times[0,1]\times Y/\sim$, where $(x,0,y)\sim(x,0,y')$ and $(x,1,y)\sim(x',1,y)$. There is a natural projection $p : X*Y \to [0,1]$ $[x,t,y]\mapsto t$.
(a) By the identifications $p^{-1}(1/2) = X\times Y$, $p^{-1}([0,1/2]) = X\times cone(Y)$, ($[x,t,y]\mapsto (x,[2t,y])$), and $p^{-1}([1/2,1]) = cone(X)\times Y$, prove that

$$X*Y = X\times cone(Y) \bigcup_{X\times Y} cone(X)\times Y.$$

(b) [Milnor] By the Künneth formula and the Mayer–Vietoris exact sequence of the above decomposition, prove that:

$$(*) \qquad \tilde H_r(X*Y,\mathbf{R}) = \bigoplus_{p+q=r-1} \tilde H_p(X,\mathbf{R})\otimes \tilde H_q(Y,\mathbf{R}).$$

Moreover, if either $H_*(X,\mathbf{Z})$ or $H_*(Y,\mathbf{Z})$ is free, then $(*)$ is valid even with $\mathbf{Z}$–coefficients.
(c) If $m_X : X\to X$ and $m_Y : Y\to Y$ are homeomorphism, then $m_X * m_Y$ is defined via $[x,t,y]\mapsto[m_X(x),t,m_Y(y)]$. Show that $(*)$ is compatible with the morphisms, i.e. $(m_X*m_Y)_{*,r} = \oplus_{p+q=r-1}(m_X)_{*,p}\otimes(m_Y)_{*,q}$.
E.2.10. Prove the following homeomorphisms:
(a) $S^n * S^m = S^{n+m+1}$;
(b) $S^0 * X =$ suspension of X;

(c) Prove that $(\bigvee_\alpha S^n) * (\bigvee_\beta S^m)$ and $\bigvee_{\alpha\beta} S^{n+m+1}$ have the same homotopy types.

E.2.11. Sebastiani–Thom theorem.

Let $g : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ and $h : (\mathbf{C}^{m+1}, 0) \to (\mathbf{C}, 0)$ be isolated singularities. Define $f : (\mathbf{C}^{n+1} \times \mathbf{C}^{m+1}, 0) \to (\mathbf{C}, 0)$ by $f(x, y) = g(x) + h(y)$.

(a) Prove that f is an isolated singularity, and $M(f) = M(g) \otimes M(h)$. In particular, $\mu(f) = \mu(g) \cdot \mu(h)$.

(b) Consider the map $u : (\mathbf{C}^{n+1} \times \mathbf{C}^{m+1}, 0) \to (\mathbf{C} \times \mathbf{C}, 0)$, $u(x, y) = (g(x), h(y))$. Then (in a good neighborhood system, whose description is left to the reader) u is a C^∞ locally trivial fibration over $\mathbf{C} \times \mathbf{C} \setminus \{cd = 0\}$, with fiber $F_g \times F_h$.

(c) For any $\theta \in [0, 2\pi]$, consider $L_\theta = \{(c, d) \in \mathbf{C}^2; c + d = \delta e^{2\pi i\theta}\} \cap B_r$ ($\delta << r << 1$). Then $u^{-1}(L_\theta)$ can be identified with $f^{-1}(\delta e^{2\pi i\theta})$.

(d) Set $P_\theta = L_\theta \cap \{c = 0\}$ and $Q_\theta = L_\theta \cap \{d = 0\}$, and let S_θ be the segment $[P_\theta, Q_\theta] \subset L_\theta$. Then the inclusion $S_\theta \subset L_\theta$ admits a strong deformation retract, which (by (b)) can be lifted to u. Therefore, F_f has the same homotopy type as $u^{-1}(S_\theta)$.

(e) Let M_θ be the midpoint of S_θ. Verify that $u^{-1}(M_\theta) = F_g \times F_h$, $u^{-1}([P_\theta, M_\theta]) = (cone(F_g)) \times F_h$, $u^{-1}([M_\theta, Q_\theta]) = F_g \times cone(F_h)$ (cf E.2.8). Therefore, $u^{-1}(S_\theta) = F_g * F_h$.

(f) $u^{-1}(M_\theta) = g^{-1}(\delta e^{2\pi i\theta}/2) \times h^{-1}(\delta e^{2\pi i\theta}/2)$, θ moving in $[0, 1]$ induces on the right hand side of this identity the monodromy $m_g \times m_h$, hence (cf. E.2.9.c) m_f can be identified with $m_g * m_h$.

(g) Using (E.2.9), prove that one has the following identifications (this is called Sebastiani-Thom formula [53]): $H_{n+m+1}(F_f, \mathbf{Z}) = \tilde{H}_n(F_g, \mathbf{Z}) \otimes \tilde{H}_m(F_h, \mathbf{Z})$ and $h_f = h_g \otimes h_h$.

(h) If $\Delta_f(t) = \prod_{i=1}^{\mu(f)} (t - \zeta_i)$ ($\zeta_i \in S^1$), is the characteristic polynomial of the germ f, then we denote $Div(f) := \sum_i (\zeta_i) \in \mathbf{Z}[S^1]$. Note that the free abelian group $\mathbf{Z}[S^1]$ generated by the elements of S^1 is a ring by $(\zeta)(\zeta') = (\zeta\zeta')$.

With the notation of this exercise, prove that: $Div(f) = Div(g) \cdot Div(h)$.

E.2.12. Brieskorn polynomials. Let $\Lambda_a = \sum_{j=1}^{a} (e^{2\pi ij/a}) \in \mathbf{Z}[S^1]$. (For the definition of $Div(f)$, see 2.11.h.)

(a) If $f : (\mathbf{C}, 0) \to (\mathbf{C}, 0)$, $f(z) = z^a$, then $Div(f) = \Lambda_a - (1)$.

(b) If $f = \sum_i z_i^{a_i}$, then $Div(f) = \prod_i (\Lambda_{a_i} - (1))$.

(c) $\Lambda_a \Lambda_b = (a, b)\Lambda_{[a,b]}$.

Remark.* [Milnor–Orlik] [28] If f is **quasi–homogeneous isolated singularity** of weights $\{w_i\}_i$, then the following formula of Milnor–Orlik gives $Div(f)$ (hence $\Delta(t)$ too) in terms of weights.:

$$Div(f) = \prod_i (v_i^{-1} \Lambda_{u_i} - (1)),$$

where $w_i = u_i/v_i$ and u_i, v_i are positive relatively prime integers.

E.2.13. Consider the following two germs in two variables: $f = xy^3 + x^8 y$ and $g = xy^4 + x^6 y$. Prove that their Div is the same $(\Lambda_{23} + (1))$, their Milnor number is the same (=24), the number of irreducible components is the same (=3), but their topological type is different.

E.2.14.* Let $P_{M(f)}(t)$ be the Poincaré polynomial of a quasi–homogeneous singularity (cf. E.1.8). $P_{M(f)}(t) = \sum_{i=1}^{\mu(f)} t^{r_i}$ can be codified (because $r_i \in \mathbf{Q}$, cf. 2.18) in $\mathbf{Z}[\mathbf{Q}]$ in the form $\sum_{i=1}^{\mu(f)}(r_i)$. On the other hand, there is a natural map $q : \mathbf{Z}[\mathbf{Q}] \to \mathbf{Z}[\mathbf{Q}/\mathbf{Z}] \to \mathbf{Z}[S^1]$ induced by $(r) \mapsto (e^{2\pi i r})$. Prove that $q(\sum_i(r_i)) = Div(f)$, i.e. by (E.1.8) and by the remark at the end of (E.2.12), one has:

$$q\Big(\prod_i \frac{t^{1/w_i} - t}{1 - t^{1/w_i}}\Big) = \prod_i (v_i^{-1}\Lambda_{u_i} - (1)).$$

§3. The Milnor fibration and the Seifert form.

3.1. Consider the local fibration (2.5):

$$(E, \partial E) := (f^{-1}(\partial D_\delta) \cap B_\epsilon, f^{-1}(\partial D_\delta) \cap S_\epsilon) \xrightarrow{f} \partial D_\delta = S^1$$

with fiber $(F, \partial F)$. The Wang exact sequence provides:

$$(\mathbf{3.2}) \qquad 0 \to H_{n+1}(E) \to H_n(F) \xrightarrow{h-id} H_n(F) \to H_n(E) \to 0.$$

But this exact sequence gives very little supplementary information about the local structure. More precisely: (3.2) gives the homology groups of E, but provides no new relation, restriction for $(F, \partial F)$ or h. The next theorem identifies E with $S^{2n+1}_\epsilon \setminus \partial F$, then via this identification (3.2) becomes a very strong exact sequence.

We start with a definition.

3.3. Definition. A "fibered knot $K \subset S^{2n+1}$" (or open book decomposition of S^{2n+1}) is an embedding of an $(n-2)$–connected $(2n-1)$–manifold K in S^{2n+1} (where K is empty if $n = 0$), together with a smooth fiber bundle $\phi : S^{2n+1} \setminus K \to S^1$ that has the following properties:

(i) There is a tubular (disk) neighborhood T of K and a bundle equivalence α of T to the trivial bundle $K \times D$ with

$$\begin{array}{ccc} T \setminus K & \xrightarrow{\alpha|T \setminus K} & K \times (D \setminus \{0\}) \\ & {\scriptstyle \phi|T\setminus K} \searrow \quad \swarrow {\scriptstyle (x,c) \mapsto c/|c|} & \\ & S^1 & \end{array}$$

(ii) for all θ, the fiber $F^o_\theta = \phi^{-1}(e^{2\pi i\theta})$ is $(n-1)$–connected. (Its closure F_θ is a $(2n)$–manifold with boundary K by (i).)

(Above S^1 and S^{2n+1} are considered with their natural orientations.)

3.4. Example.* In the classical knot–theory, by a result of Stallings and Neuwirth, the following facts are equivalent:

(a) $K \subset S^3$ is a fibered knot;

(b) the commutator subgroup $[G, G]$ of $G := \pi_1(S^3 \setminus K)$ is a free group;

(c) the commutator group $[G, G]$ is a finitely generated group.

(cf. Exercise 3.2)

3.5. Theorem. [Milnor] [26] *Fix an isolated singularity f. (Recall $K_f = f^{-1}(0) \cap S_\epsilon$.) Then:*

(a) For ϵ sufficiently small $\phi = f/|f| : S_\epsilon \setminus K_f \to S^1$ is a C^∞ locally trivial fibration (called the Milnor fibration of f), which defines an open book decomposition of S_ϵ.

(b) The local fibration $f : f^{-1}(\partial D_\delta) \cap B_\epsilon \to \partial D_\delta$ and the Milnor fibration are bundle isomorphic. Actually, if we consider $\mathcal{F}_\theta := f^{-1}(e^{2\pi i\theta}[\delta, \infty)) \cap B_\epsilon$, then $f^{-1}(\delta e^{2\pi i\theta}) \cap B_\epsilon \subset \mathcal{F}_\theta$ can be "pushed out" by a flow (along $\mathcal{F}_\theta$), isotopically to $\mathcal{F}_\theta \cap S_\epsilon = \phi^{-1}(e^{2\pi i\theta})$.
(In the sequel, we will denote both fibers $f^{-1}(\delta e^{2\pi i\theta})$ and $\phi^{-1}(e^{2\pi i\theta})$ simply by F_θ.)

3.6. The Wang exact sequence revisited. By (3.5), the space E in (3.2) is $S_\epsilon \setminus K_f$. Notice that:

$$\begin{aligned} H_q(S_\epsilon \setminus K_f, \mathbf{Z}) &= H^{2n-q}(K_f, \mathbf{Z}) \text{ (by Alexander duality)} \\ &= H_{q-1}(K_f, \mathbf{Z}) \text{ (by Poincaré duality).} \end{aligned}$$

Therefore, (3.2) reads as:

$$0 \to H_n(K_f, \mathbf{Z}) \to H_n(F, \mathbf{Z}) \xrightarrow{h-id} H_n(F, \mathbf{Z}) \to H_{n-1}(K_f, \mathbf{Z}) \to 0.$$

3.7. Corollary.

(a) rank $H_n(K_f) =$ rank $H_{n-1}(K_f) = \dim\ker(h - id)$, in particular, K_f is a rational homology sphere if and only if $\Delta(1) = \det(id - h) \neq 0$.

(b) K_f is an integer homology sphere if and only if $\Delta(1) = \pm 1$.

3.8. Corollary (3.7) is very useful for the study of the nilpotent part of h as well. Notice that the dimension of the generalized 1–eigenspace $\dim H_n(F)_1$ can be determined from $\Delta(t)$ (i.e. it is the exponent of $(t-1)$ in $\Delta(t)$). On the other hand, $\operatorname{rank} H_n(K_f) = \dim\ker(h-1) \leq \dim H_n(F)_1$, and the inequality is strict if and only if there exists at least one Jordan block of h with eigenvalue 1 and size ≥ 2. (If we want to test the Jordan blocks corresponding to $\lambda = e^{2\pi ik/l}$, with $k/l \notin \mathbf{Z}$, then we replace f by $f + z^l$, where z is a new variable. Then by Sebastiani–Thom theorem, it is enough to test the $\lambda = 1$ eigenvalue for this new germ. Cf. E.3.12.)

3.9. The exact sequences (2.12) and (3.2) (or 3.6) have big similarities. We will show that they can be identified via the variation map (and some duality isomorphisms).

The (Gysin) isomorphism $\alpha : H_q(S_\epsilon \setminus K_f, \mathbf{Z}) \to H_{q-1}(K_f, \mathbf{Z})$ and its inverse $\beta : H_{q-1}(K_f, \mathbf{Z}) \to H_q(S_\epsilon \setminus K_f, \mathbf{Z})$ are defined as follows. If c is a closed (oriented) q–cycle in $S_\epsilon \setminus K_f$, consider a $(q+1)$–cycle d in S_ϵ with $\partial d = c$ and $d \pitchfork K_f$. Then $\alpha([c]) = [d \cap K_f]$. Conversely, if e is a closed $(q-1)$–cycle in K_f, let $\pi : T \to K_f$ be the projection of a tubular neighbourhood of K_f onto K_f, then $\beta([e]) = [\partial\pi^{-1}e] = [\pi^{-1}(e) \cap \partial T]$. Now, $\alpha = \beta^{-1}$, and the following diagram is commutative (please, verify!):

$$\begin{array}{ccccccc}
0 \to H_{n+1}(S_\epsilon \setminus K_f) & \xrightarrow{\partial} & H_n(F) & \xrightarrow{h-1} & H_n(F) & \longrightarrow & H_n(S_\epsilon \setminus K_f) \to 0 \\
\simeq \uparrow \beta & & \| & & \uparrow V & & \simeq \uparrow \beta \\
0 \to H_n(K_f) & \longrightarrow & H_n(F) & \xrightarrow{j} & H_n(F, K_f) & \longrightarrow & H_{n-1}(K_f) \to 0 \\
 & & \| & & \| & & \\
 & & H_n(F) & \xrightarrow{b} & H_n(F)^* & &
\end{array}$$

Now, the Five lemma implies:

3.10. Theorem. $V : H_n(F, K_f, \mathbf{Z}) \to H_n(F, \mathbf{Z})$ *is an isomorphism.*

3.11. Corollary of (2.23) and (3.10). *Let* $\theta : H_n(F, \mathbf{Z}) \to H_n(F, \mathbf{Z})^{**}$, $\theta(u)(\varphi) = \varphi(u)$ *be the natural isomorphism. Then* h *and* $(,)$ *can be computed from* V *as follows:*

$$h = -(-1)^n V(\theta^{-1} \circ V^*)^{-1},$$

$$b = -V^{-1} - (-1)^n (\theta^{-1} \circ V^*)^{-1}.$$

3.12. Corollary.

(a) $\ker b = \ker(h-1) = H_n(K, \mathbf{Z})$, *(recall 2.22:* $V \circ b = h - 1$*),*

(b) if b *is non-degenerate (equivalently, if* $\Delta(1) \neq 0$*), then* $|\det(,)| = |\det(h-1)| = |\Delta(1)|$.

3.13. Example. Let $f = z_1^2 + z_2^3 + z_3^5$. Then (cf. E.2.12) $Div(f) = (\Lambda_2 - 1)(\Lambda_3 - 1)(\Lambda_5 - 1) = \Lambda_{30} - \Lambda_6 - \Lambda_{10} - \Lambda_{15} + \Lambda_2 + \Lambda_3 + \Lambda_5 - 1$, hence

$$\Delta(t) = \frac{(t^{30}-1)(t^2-1)(t^3-1)(t^5-1)}{(t^6-1)(t^{10}-1)(t^{15}-1)(t-1)} = \frac{t^{10} - t^5 + 1}{t^2 - t + 1},$$

and $\Delta(1) = 1$, hence $(,)$ is unimodular (see also E.3.10).

3.14. The Seifert form.

Recall the notations: $F_\theta = f^{-1}(e^{2\pi i\theta}[0,\infty)) \cap S_\epsilon$, $\partial F_\theta = K_f = f^{-1}(0) \cap S_\epsilon$, $F_\theta^o = F_\theta \setminus K_f = \phi^{-1}(e^{2\pi i\theta})$. The inclusion $F_\theta^o \hookrightarrow F_\theta$ identifies their homology. Let g be a continuous map $[0,1] \times F_\theta^o \to S_\epsilon \setminus K_f$ such that $g_\theta = g(\theta, \cdot)$ maps F_0^o homeomorphically onto F_θ^o and $g_0 = id_{F_0^o}$. Via g_θ we can identify F_θ^o and F_0^o provided that $0 \leq \theta < 1$, and g_1 is a candidate for the geometric monodromy m.

The Seifert form $S : H_n(F_0, \mathbf{Z}) \otimes H_n(F_0, \mathbf{Z}) \to \mathbf{Z}$ is defined by

$$S(\beta, \beta') = \text{linking number}(\beta, (g_{1/2})_* \beta'),$$

where the linking number is considered in S_ϵ: linking number(β, β')=algebraic intersection number (α, β'), where α is an $(n+1)$–cycle in S_ϵ with $\partial\alpha = \beta$.

3.15.* S can be identified with the following sequence of isomorphisms (all the groups are considered with $\mathbf{Z}$–coefficients):

$$H_n(F_0) \overset{\partial^{-1}}{\to} H_{n+1}(S_\epsilon, F_0) \overset{(1)}{=} H_{n+1}(S_\epsilon, \cup_{0 \leq \theta \leq 1/2} F_\theta)$$

$$\overset{(2)}{=} H_{n+1}(\cup_{1/2 \leq \theta \leq 1} F_\theta, F_{1/2} \cup F_1) \overset{(3)}{=} H_n(\cup_{1/2 \leq \theta \leq 1} F_\theta)^* \overset{(1)}{=} H_n(F_0)^*,$$

where (1) is induced by the inclusion $F_0 \hookrightarrow \cup_{0 \leq \theta \leq 1/2} F_\theta$ (which admits a strong deformation retract), (2) is given by excision, and (3) by Poincaré duality. In particular, S is unimodular. This fact will follow also from (3.16).

In fact, the Seifert form and the variation map satisfy the following relation:

3.16. Theorem. *If* $\alpha \in H_n(F, \partial F, \mathbf{Z})$, $\beta \in H_n(F, \mathbf{Z})$, *(with* $F = F_0$*) then* $S(V\alpha, \beta) =$

$< \alpha, \beta >$.

Proof.

$$\begin{aligned} S(V\alpha,\beta) &= \text{linking number}(m\alpha-\alpha,(g_{1/2})_*\beta)\\ &= \text{linking number}(\partial \cup_{0\leq\theta\leq 1}(g_\theta)_*\alpha,(g_{1/2})_*\beta)\\ &= \text{intersection number}(\cup_{0\leq\theta\leq 1}(g_\theta)_*\alpha,(g_{1/2})_*\beta)\\ &=< (g_{1/2})_*\alpha,(g_{1/2})_*\beta >_{F_{1/2}}=< \alpha,\beta >_{F_0}=< \alpha,\beta > . \end{aligned}$$

3.17. Corollary. *$S(a,b) =< V^{-1}a, b >$. In particular, via the identification in (2.10), S can be identified with $(V^{-1})^t$. In matrix notations, the objects S, $(,)$, h and h^r satisfy:*

$$(,) = -S + (-1)^{n+1}S^t$$

$$h = (-1)^{n+1}(S^t)^{-1}S$$

$$h^r = (-1)^{n+1}S(S^t)^{-1}.$$

(Notice that in matrix notations $(,) = b^t$.)

3.18. Some (difficult) results about the Seifert form.*

(a) By a result of J. Levine [23], the Seifert form of a fibered knot $K \subset S^{2n+1}$ is an invariant of the embedding $K_f \hookrightarrow S^{2n+1}$.

(b) Moreover, for $n \geq 3$, A. Durfee [12] and M. Kato [18] independently proved (based on some results of Levine) that the set of isomorphism classes of fibered knots in S^{2n+1} is equivalent (via their Seifert form) to the set of conjugacy classes of unimodular bilinear forms over $\mathbf{Z}$.

In particular, the topological type of f is completely determined by the Seifert form S of f, (and it determines S modulo a conjugacy), provided that $n \geq 3$.

(c) It was conjectured that in the case $n = 1$, the real Seifert form S of f determines the topological type of f. (This is true for irreducible germs, since by a result of Lê, already the characteristic polynomial $\Delta(t)$ determines the topological type of irreducible plane curve singularities. cf. E.3.3.b). The conjecture was disproved in the Schrauwen–Steenbrink–Stevens paper [52]. Actually, P. Du Bois and F. Michel [13] constructed two germs with different topological type, but with the same *integer* Seifert form. Artal–Bartolo consider the duble suspension of these curve singularities in order to construct two germs in the case $n = 2$ with the same integer Seifert forms but different links (in particular, with different topological types) [5].

(d) S. S.-T. Yau conjectured that the topological type of an isolated singularity $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ is determined by the characteristic polynomial $\Delta(t)$ of its monodromy operator, and by the fundamental group $\pi_1(K_f)$ of its link. (Notice that both $\Delta(t)$ and $\pi(K_f)$ depend only on the topological type of f, cf. E.3.3). The conjecture was verified in the case of quasi–homogeneous singularities by Xu and S. S.-T. Yau. This conjecture was also disproved by a counterexample constructed by Artal–Bartolo in his thesis [6].

Open problems and conjectures.

3.19. Given an isolated singularity f, find a method, or algorithm for the computation of its Seifert form S.
3.20. In the correspondence Durfee–Kato (3.18.b), characterize those Seifert forms (i.e. unimodular matrices) which correspond to the algebraic knots (i.e. $(S^{2n+1},K)=(S_\epsilon,K_f)$ for some germ f).
3.21. Given a quasi–homogeneous isolated singularity, find its integer Seifert form (in terms of its weights, cf. E.1.9).

Exercises for the third section.

E.3.1. Prove J. Reeve's theorem (cf. 1.13.b), namely that the intersection multiplicity of two plane curve singularities (without common component) is the linking number of their links in S_ϵ.
E.3.2. Assume that f is an irreducible plane curve singularity. If $G:=\pi_1(S_\epsilon\setminus K_f)$, prove that the commutator subgroup $[G,G]$ is free. (cf. 3.4)
Hint: use the homotopy long exact sequence of the Milnor fibration.
E.3.3. (a) [Teissier] $\mu(f)$ depends only on the topological type of f.
(b) [Lê] $\Delta(t)$ depends only on the topological type of f.
(c) Let $f=x^2+y^7+z^{14}$ and $g=x^3+y^4+z^{12}$. Show that their links are diffeomorphic, but their topological type is different (cf. E.1.15).
Hint for (a-b): we assume that $n\geq 2$ (the case $n=1$ can be proved either by similar argument, or by the classification result 1.13). Then by the homotopy exact sequence of the Milnor fibration one has: $\phi_*:\pi_1(S_\epsilon\setminus K_f)\xrightarrow{\sim}\pi_1(S^1)=\mathbf{Z}$. The universal covering $(S_\epsilon\setminus K_f)\tilde{}$ of $S_\epsilon\setminus K_f$ can be (homotopicaly) identified with F, and $\mu(f)$ is the rank of its n^{th}–homology. Moreover, $\Delta(t)$ can be described using the Galois action of $\pi_1(S_\epsilon\setminus K_f)$ on $H_n(S_\epsilon\setminus K_f)\tilde{}$. Actually, it is its Reidemeister torsion associated with the representation $\pi_1(S_\epsilon\setminus K_f)=\mathbf{Z}\to\mathbf{C}((t)),\ 1\mapsto t$.
We mention here that in the case $n=1$, $\Delta(t)$ can be identified with the Alexander polynomial of the link $K_f\subset S_\epsilon$ (modulo $\pm t^{\pm i}$).
E.3.4. If $f=\sum z_i^{a_i}$, with $(a_i,a_j)=1$ for $i\neq j$, then $\Delta(1)=1$. In particular, K_f is an integer homology sphere (cf. 3.7.b).
E.3.5. Deligne–Sakamoto theorem* (independently). [50]
(Continuation of Sebastiani–Thom theorem (E.2.11).)
With the notations of (E.2.11), prove that the Seifert forms satisfy:

$$S_f=(-1)^{(n+1)(m+1)}S_g\otimes S_h.$$

(For a proof in the spirit of (2.11), rewrite [40] for the local case.)
E.3.6. Consider $f:(\mathbf{C},0)\to(\mathbf{C},0)$ given by $f(z)=z^a$ (cf. E.2.5 and E.2.12.a). Show

that its Seifert form (in a special base) is:

$$S(z \mapsto z^a) = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}$$

Hint: The goemetric Milnor fiber can be identified with a points: the a^{th}–roots of unity $\{P_0, \dots, P_{a-1}\}$ in the complex plane. The geometric monodromy is $P_0 \mapsto P_1 \mapsto \dots \mapsto P_{a-1} \mapsto P_0$. Moreover, $\tilde{H}_0(F, \mathbf{Z}) = \{\sum_i n_i P_i : \sum_i n_i = 0\}$. A base is given by $e_i := P_i - P_{i-1}$, $i = 1, 2, \dots, a-1$. In this base S has the wanted form.

E.3.7. The non–degenerate case. Using (E.3.5-E.3.6), prove that the Seifert form of non–degenerate germ $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is $S = (-1)^{(n+1)(n+2)/2}$ (compare with the formula of the variation map (E.2.3.d)).

E.3.8. Prove that if f is an isolated singularity, z_1 and z_2 are new varibles, and $f' = f + z_1^2 + z_2^2$, then $S_{f'} = -S_f$; $(,)_{f'} = -(,)_f$ and $h_{f'} = h_f$.

E.3.9.

(a) Using (E.3.5-E.3.6) and (3.17), compute the intersection form $(,)$ for the following germs: $f = x^2 + y^2 + z^a$ and $g = x^2 + y^3 + z^5$.

Prove that $(,)$ is negative definite (for f one can use also (3.8)). Notice that $(,)_f$ is the negative of the A_{a-1}–Cartan matrix. Compute $\det(,)_f$ and deduce that $|TorsH_1(K_f, \mathbf{Z})| = a$ (cf. E.1.16.b). Verify this by computing $\Delta(1)$ too. (In fact $H_1(K_f, \mathbf{Z}) = \pi_1(K_f) = \mathbf{Z}_a$, cf. E.1.17.a).

(b) Prove that for any f the intersection form $(,)$ is *even*, i.e. $(v, v) \in 2\mathbf{Z}$ for any v.

(c)* Identify $(,)_g$ with the negative of the E_8–Cartan matrix (the unique unimodular even form over $\mathbf{Z}$ with rank 8 and signature -8).

E.3.10. The Poincaré sphere. Consider the link K_f of $f = x^2 + y^3 + z^5$. Show that K_f is an integer homology sphere (cf. 3.13 and 3.7), but it is not a homotopy sphere (cf. E.1.17.b*, or Mumford's theorem in 1.17).

E.3.11. Exotic spheres on Brieskorn manifolds. [Brieskorn, Hirzebruch, Mayer] (The abstract link K_f of a Brieskorn singularity f is called Brieskorn manifold.)

Consider the Brieskorn singularity $f = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^3 + z_7^5$.

(a) Using (E.3.8-E.3.9), show that $(,)$ is negative definite, in particular $\sigma(f) = -\mu(f) = -8$.

(b) Using (3.13) and (E.3.8) deduce that $\Delta(1) = 1$, hence K_f is an integer homology sphere. Therefore, by (1.18), $\partial F = K_f$ is homeomorphic to S^{11}.

(c) By (b), (and using that F is 5–connected), deduce that the quotient space $F/\partial F$ is a 12–dimensional topological manifold, without boundary, it is 5-connected and its signature is -8.

(d) If X is a differentiable 12–dimensional manifold, then by the Hirzebruch signature theorem its signature can be computed from its L–class, namely

$$\text{signature}(X) = \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3)[X],$$

where $p_i \in H^{4i}(X, \mathbf{Z})$ are the Pontrjagin classes. Deduce that if X is 5–connected (hence $p_1 = 0$), then the signature of X is divizible by 62.

(e) Deduce that the topological manifold $F/\partial F$ does not carry any differentiable structure.

(f) Deduce that ∂F is an exotic sphere (i.e. it is a differentiable manifold, homeomorphic to the standard sphere but not diffeomorphic to it.) (If ∂F were the standard sphere S^{11}, which is the boundary of the ball B^{12}, then $F \cup_{S^{11}} B^{12} = F/\partial F$ would have a differentiable structure.)

(g)* We can find exotic spheres among Brieskorn manifolds even in smaller dimension. Consider $f_k = z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1}$. Then, for $k = 1, 2, \ldots, 28$, one obtains all 28 possible differentiable structures on the 7–sphere (Milnor's 7–spheres). (But the proof of this fact is more complicated.)

E.3.12. A'Campo's example of non–finite monodromy. [1] (cf. 3.8.)

Consider the plane curve singularity $g = (x^2 + y^3)(x^3 + y^2)$.

(a) Computing the resolution graph G_g of g (cf. 4.5), and using A'Campo's theorem (E.2.6), deduce that $\Delta_g(t) = (t-1)(t^5+1)^2$.

(b) Consider $f = g + z^2$, where z is a new variable. Using Sebastiani–Thom property, prove that $\Delta_f(t) = (t+1)(t^5-1)^2$. In particular, the generalized 1–eigenspace $H_2(F_f)_1$ has dimension 2.

(c) Using the Appendix, find the resolution graph of $(V_f, 0)$. (The minimal resolution looks as follows: there are only two irreducible exceptional divisors, both rational with self intersection $e = -3$, without auto intersection points, and they intersect each other in two points. Cf. Appendix.)

(d) Using (E.1.16.a), show that $H_1(K_f)$ has rank one, therefore (by 3.7) $\dim \ker(h_f - 1) = 1$. Deduce that h_f has a (2×2)–Jordan block with eigenvalue 1 (cf. 3.8).

(e) Deduce that the monodromy h_g of g has a (2×2)–Jordan block with eigenvalue -1.

§4. The equivariant signature of plane curve singularities. ϵ–hermitian variation structures.

4.1. In this section we discuss the topological invariants of isolated plane curve singularities $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$. Recall (cf. 1.13') that the topological type of f is completely determined by the embedded resolution graph G_f of f. For G_f, we will use the notations of (1.14). Actually, in this section we will learn how one can read (some of) the topological invariants from G_f.

4.2. The number r of irreducible components is the same as the number of link components of $K_f \subset S^3_\epsilon$ (cf. E.1.11). On the graph, it is the number of arrowhead vertices $\#\mathcal{A}$.

4.3. The characteristic polynomial $\Delta(t)$ is given by A'Campo's formula (cf. E.2.6):

$$\Delta(t) = (t-1) \prod_{w \in \mathcal{W}} (t^{m_w} - 1)^{\delta_w - 2},$$

where $\delta_w = \#\mathcal{V}_w$.

This gives the semisimple part of the complexified monodromy h. The above formula shows also that if N is multiple of all the multiplicities m_w ($w \in \mathcal{W}$), then $\lambda^N = 1$ for all the eigenvalues λ of h (cf. 2.18.a).

This, together with the Monodromy Theorem (2.18) gives that $(h^N - 1)^2 = 0$. (Actually, the monodromy restricted on the generalized 1–eigenspace is the identity.)

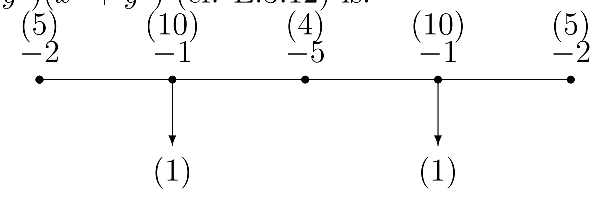
For the completeness of the discussion, we present here a result of W. Neumann (without proof). Neumann used the notation $\Delta^1(t)$ for the characteristic polynomial of the restriction $h|\ker(h^N - 1)$. Notice that $\Delta(t)$ and $\Delta^1(t)$ determine the Jordan normal form of h.

4.4. Theorem.* [Neumann] [43]

$$\Delta^1(t) = (t-1) \prod_{e \in \mathcal{E}} (t^{d_e} - 1) / \prod_{w \in \mathcal{W}} (t^{d_w} - 1),$$

where for any $e = (w_1, w_2) \in \mathcal{E}$ we let $d_e := g.c.d.(m_{w_1}, m_{w_2})$; and for any $w \in \mathcal{W}$ we let $d_w := g.c.d.(m_v|\ v \in \mathcal{V}_w \cup \{w\})$.

4.5. Example. The minimal resolution graph of A'Campo's plane curve singularity $f = (x^2 + y^3)(x^3 + y^2)$ (cf. E.3.12) is:



Then, by (4.4), $\Delta^1(t) = t + 1$, hence there is exactly one Jordan block of size 2, and its eigenvalue is -1 (cf. E.3.12).

4.6. The Milnor fiber, together with its intersection form, can be easily determined. Indeed, as an abstract 2–dimensional real, oriented surface with r boundary components, it is completely determined by r and and the rank $\mu(f)$ of $H_1(F)$. $\mu(f)$ reads from the graph as follows (cf. E.2.6 or 4.3):

$$1-\mu(f)=\sum_{w\in\mathcal{W}} m_w(2-\delta_w).$$

$(,)$ is skew–symmetric, $\dim\ker(,)=r-1$, and $H_1(F,\mathbf{Z})=\ker(,)\oplus H$, where $(,)$ restricted on H is unimodular, i.e. represented in a "good base" has the form $\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$.

The abstract Milnor fiber says very little about the topological type of f. For example, $f=x^3+y^5$ and $g=x^2+h^9$ have diffeomorphic Milnor fibers, but obviously their open book decomposition (or link) is different (compute e.g. their characteristic polynomial.).

4.7. On the other hand, the **isometric structure** $\mathcal{I}=(H_1(F,\mathbf{Z});h,(,))$ (i.e. h considered as an element of the orthogonal group of $(,)$) is a rather strong invariant of f. In E.4.3, we will give an example of two germs for which the invariants $\Delta(t)$ and $\Delta^1(t)$ (hence the Jordan normal forms) are the same, r and μ (hence the Milnor fibers and the intersection forms) are the same, but their isometric structure is not the same, even over $\mathbf{C}$, (i.e. the monodromy operators are not conjugate in the orthogonal group of $(,)$). The complexified isometric structure $\mathcal{I}_{\mathbf{C}}$ (where $(,)_{\mathbf{C}}$ is a (-1)–hermitian with $(a,b)=-\overline{(b,a)}$) has a direct sum decomposition $\oplus(H_n(F,\mathbf{C})_\lambda,(,)_\lambda)$ corresponding to the generalized eigenspaces of the monodromy operator. As we will see later, if h is diagonalizable, then $(H_n(F,\mathbf{C})_\lambda,(,)_\lambda)$ is completely determined by $\mu_\lambda:=\dim H_n(F,\mathbf{C})_\lambda$ (i.e. by the exponent of $t-\lambda$ in $\Delta(t)$) and σ_λ. In general, $\mathcal{I}_{\mathbf{C}}$ is characterized by $\Delta(t)$, $\Delta^1(t)$ and the collection of equivariant signatures $\{\sigma_\lambda\}_\lambda$. Actually, even the real Seifert form can be deduced from these index type invariants. If $(,)$ is nondegenerate (i.e. $r=1$) then $V=(h-1)\circ b^{-1}$ (cf. 2.22), but even in general, by a result of Neumann, $\Delta(t)$, $\Delta^1(t)$ and $\{\sigma_\lambda\}_\lambda$ determine the real Seifert form. This shows the importance of these invariants.

In the following theorem, we compute the equivariant signature in terms of the resolution graph. A vertex $w\in\mathcal{W}$ is called **rupture point** if $\delta_w\geq 3$. The set of rupture points is denoted by $\mathcal{R}$. We introduce also the following number theoretical function: $((x))=1/2-\{x\}$ in $x\notin\mathbf{Z}$, and $=0$ if $x\in\mathbf{Z}$, where $\{x\}$ denotes the fractional part of x. (In some papers, $((\cdot))$ is introduced with the opposite sign convention.)

4.8. Theorem. [W. Neumann] [43] *Fix* $\lambda=e^{2\pi ip/q}$, *where* $g.c.d(p,q)=1$. *Then* $\sigma_\lambda=\sum_{w\in\mathcal{R}}\sigma^w_\lambda$, *where for* $w\in\mathcal{R}$ *:*

$$\sigma^w_\lambda=\begin{cases} 0 & \text{if } q\not| m_w \\ 2\sum_{v\in\mathcal{V}_w}((\frac{m_v\cdot p}{q})) & \text{if } q|m_w. \end{cases}$$

4.9. Remark. The original proof of W. Neumann is based on a difficult index theoretical result of Atiyah–Patodi–Singer, namely on the theory of eta–invariants. The theorem can be reproved using Hodge theory as follows: σ_λ can be described from the equivariant

Hodge numbers associated with the mixed Hodge structure on $H^1(F,\mathbf{C})$ by a result of J. Steenbrink, and these numbers are computed from G_f by Schrauwen–Steenbrink–Stevens. We present here a new, completely elementary proof.

Before we start the proof of (4.8), we recall the following additivity property of the signature with respect to pasting along boundary components.

4.10. Novikov's additivity property:
Notice that the Euler–characteristic, or characteristic polynomial is additive in exact sequences (e.g. with respect to the decomposition of spaces, by a Mayer–Vietoris argument). The behavior of the signature is more complicated. Nevertheless, we have the following additivity theorem for manifolds. We recall that if X is a $(2k)$–manifold (maybe with boundary), then $H_k(X,\mathbf{R})$ carries a $(-1)^k$–symmetric intersection form $(\,,\,)$, and the signature of X is defined as the signature of this form. If $m: X\to X$ is a diffeomorphism then it induces at homology level an automorphism h of $H_k(X)$, which is in the orthogonal group of $(\,,\,)$, and the equivariant signatures are the signatures of the hermitian form obtained by restricting the intersection form to the generalized eigenspaces of h.

Let X_i be two oriented manifolds with boundary ∂X_i $(i=1,2)$. Assume that the disjoint union B_1 of some of the connected components of ∂X_1 is diffeomorphic to the disjoint union B_2 of some of the connected components of ∂X_2 with opposite orientation. We construct the manifold X by gluing X_1 and X_2 along boundary components B_1 and $-B_2$ (via that diffeomorphism). Novikov's theorem asserts that: $\sigma(X)=\sigma(X_1)+\sigma(X_2)$. If $m: X\to X$ is a diffeomorphism such that $m(X_i)=X_i$ $(i=1,2)$, then:

$$\sigma_\lambda(X)=\sigma_\lambda(X_1)+\sigma_\lambda(X_2).$$

4.11. Proof of 4.8. Consider an embedded resolution $\phi:(\mathcal{Y},D)\to(\mathbf{C}^2,f^{-1}(0))$ as in (1.14). Replace it by one of its representatives $\phi:\mathcal{Y}\to B_\epsilon$, where B_ϵ is also a Milnor ball of f. Let $F=f^{-1}(\delta)\cap B_\epsilon$ and take $F':=\phi^{-1}(F)$. Obviously, the restriction $\phi:F'\to F$ is an isomorphism. F' is the nearby fiber in $\mathcal{Y}$ of the central fiber $D=(f\circ\phi)^{-1}(0)\cap B_\epsilon$. In particular, there exists a continuous surjection $\pi:F'\to D$, such that, for any singular point P of D, $\pi^{-1}(P)$ is a union of S^1's, and if E_w^{reg} denotes the regular part of E_w, then $F'_w:=\pi^{-1}(E_w^{reg})\to E_w^{reg}$ is an m_w–fold regular covering whose Galois action corresponds to the restriction of the geometric monodromy m to F'_w. Similarly, $\pi:F'_a:=\pi^{-1}(S_a^{reg})\to S_a^{reg}$ (for any $a\in\mathcal{A}$) is an m_a–covering. (Since $m_a=1$, actually this is an isomorphism.) (Fore some of the arguments, see below.)

Now, consider the decomposition $F'=(\cup_w F'_w)\cup(\cup_a F'_a)$, which is compatible with the geometric monodromy action m. Therefore, by Novikov's theorem, $\sigma_\lambda(f)=\sum_w\sigma_\lambda(F'_w)+\sum_a\sigma_\lambda(F'_a)$. Since F'_a is a punctured disc with a trivial action, its equivariant signature is trivial. Moreover, if $\delta_w\leq 2$, then F'_w is either a disk (if $\delta_w=1$) or a punctured disk (if $\delta_w=2$), in both cases they carry no equivariant signature (because $(\,,\,)$ is trivial). Therefore $\sigma_\lambda(f)=\sum_{w\in\mathcal{R}}\sigma_\lambda(F'_w)$.

Now, we will study more carefully the covering $F'_w\to E_w^{reg}$ $(w\in\mathcal{R})$. If P is a regular point of D on E_w then F'_w in a neighborhood U_P of P has local equation $x^{m_w}=\delta$, hence π is an m_w–fold covering. The monodromy action is the cyclic permutation of the connected components of $F'_w \cap U_P$, which is exactly the Galois action of the covering group $\mathbf{Z}_{m_w}$. We denote the corresponding representation by $\rho : \pi_1(E_w^{reg}) \to \mathbf{Z}_{m_w}$, where $\hat{1}$ corresponds to the Galois action. If P is a singular point of D on E_w, then F'_w locally is given by $x^{m_w}y^{m_v} = \delta$, where $\{x = 0\}$ is the local equation of E_w, and the projection π is $(x, y) \mapsto y$. If $(x, y) = (0, \eta e^{2\pi it})$, $0 \leq t \leq 1$, is a small oriented loop γ around P in E_w, then $\pi^{-1}(\gamma)$ is given by $x^{m_w} = \delta\eta^{-m_w}e^{-2\pi itm_w}$. Therefore, $\rho([\gamma]) = \widehat{-m_w}$.

For simplicity, we will assume that $\delta_w = 3$ for any $w \in \mathcal{R}$. (Actually, this is the case, for example, for the minimal resolution graph of any irreducible plane curve singularity.) The reader can try to extend the proof to the general case.

In the sequel, we will fix a vertex $w \in \mathcal{R}$ and we will use the notations $m := m_w$, and $\{m_1, m_2, m_3\} = \{m_v\}_{v \in \mathcal{V}_w}$. E_w can be diffeomorphically identified with $\mathbf{C} \setminus \{\pm 2\}$. Then π is an m–fold covering $\pi : X \to B$, given by the representation $\rho : L(\partial^-, \partial^+) \to \mathbf{Z}_m$, $\rho(\partial^-) = \widehat{-m_1}$, $\rho(\partial^+) = \widehat{-m_2}$ and Galois action corresponding to $\hat{1}$. Here $L(\partial^-, \partial^+) = \pi_1(B)$ is the free group generated by two generators $\partial^{\mp} = [t \mapsto \mp 2 + e^{2\pi it}]$.

We will assume also that $g.c.d.(m, m_1, m_2, m_3) = 1$, the general case follows easily from this one.

Set $B' = \{z \in B : Re(z) \leq 0\}$, $B'' = \{z \in B : Re(z) \geq 0\}$, $C = B' \cap B''$, $X' = \pi^{-1}(B')$, $X'' = \pi^{-1}(B'')$, $Y = \pi^{-1}(C)$, $P = (0, i)$, $Q = (0, -i)$. Since $Y \to C$ is a trivial fibration, we can fix a trivialization $T : \{1, 2, \ldots, m\} \times C \to Y$, and let $P_i = T(\{i\} \times P)$, $Q_i = T(\{i\} \times Q)$.

Let γ'_P (resp. γ'_Q) be an oriented closed path in B' with endpoints at P (resp. Q) homotop to ∂^-. Similarly, let γ''_P (resp. γ''_Q) be a closed path in B'' with endpoints at P (resp. Q) homotop to ∂^+. Let e_i^P (resp. e_i^Q) an oriented path in X', lifting of γ'_P (resp γ'_Q) with starting point P_i (resp Q_i). Similarly, let f_i^P (resp. f_i^Q) be the oriented path in X'' with starting point P_i (resp Q_i) which is the lifting of γ''_P (resp. of γ''_Q).

Consider the homology exact sequence (over $\mathbf{Z}$) of the pair (X, Y):

$$(*) \qquad 0 \to H_1(X) \to H(X, Y) \xrightarrow{\partial} H_0(Y) \to H_0(X) \to 0.$$

Now, $H_0(X) = \mathbf{Z}$, because $g.c.d.(m_1, m_2, m) = 1$. $H_0(Y) = \mathbf{Z}^m$, where a base is given by the points $\{P_i\}_{i=1}^m$. The monodromy action is $P_i \mapsto P_{i+1}$. (In our notation an index i is always identified with $i + km$, $k \in \mathbf{Z}$, i.e. the index, in fact, lives in $\mathbf{Z}_m$.) Therefore, $h(\sum_i a_i P_i) = \sum_i a_i P_{i+1} = \sum_i a_{i-1} P_i$, hence the monodromy action is:

$$h : \mathbf{Z}^m \to \mathbf{Z}^m, \quad h(a_1, \ldots, a_m) = (a_m, a_1, \ldots, a_{m-1}).$$

Notice that $H_1(X, Y) = H_1(X', Y) \oplus H_1(X'', Y)$. $H_1(X', Y) = \mathbf{Z}^m$, where a base is given by the class of paths $\{e_i^P\}_i$ (notice the identity of the relative homology classes $[e_i^P] = [e_i^Q]$). Since $\partial[e_i^P] = P_{i-m_1} - P_i$, one has: $\partial(\sum a_i e_i^P) = \sum a_i(P_{i-m_1} - P_i) = \sum(a_{i+m_1} - a_i)P_i$, hence $\partial(a) = (h^{-m_1} - 1)a$.

Similarly, $H_1(X'', Y) = \mathbf{Z}^m$ is generated by $[f_i^P]_i$, and $\partial[f_i^P] = P_{i-m_2} - P_i$, hence $\partial(a) = (h^{-m_2} - 1)a$. Therefore $(*)$ reads as:

$$(**) \qquad 0 \to H_1(X) \to \mathbf{Z}^m \oplus \mathbf{Z}^m \xrightarrow{\partial} \mathbf{Z}^m \to \mathbf{Z} \to 0,$$

$$\partial(a,b)=(h^{-m_1}-1)a+(h^{-m_2}-1)b,$$

where h is the permutation $h(a_1,\dots,a_m)=(a_m,a_1,\dots,a_{m-1})$.

If $(a,b)\in\ker\partial$, then $\sum a_ie_i^P+\sum b_if_i^P$ is a closed 1–cycle in X, representing an element in $H_1(X)$. We want to describe the intersection form $(,)$ on $H_1(X)$ (i.e. on $\ker\partial$) in terms of (a,b). Let (a,b) and (a',b') be elements of $\ker\partial$. It is convenient to represent (a,b) as $\sum a_ie_i^P+\sum b_if_i^P$ and (a',b') as $\sum a'_ie_i^Q+\sum b'_if_i^Q$. Now, notice that one has the following intersection of cycles:

$$(e_i^P,e_j^Q)=\begin{cases}-1 & \text{if } \hat{j}=\hat{i}\\ +1 & \text{if } \hat{j}=\widehat{i-m_1}\\ 0 & \text{otherwise.}\end{cases}$$

$$(f_i^P,f_j^Q)=\begin{cases}+1 & \text{if } \hat{j}=\hat{i}\\ -1 & \text{if } \hat{j}=\widehat{i+m_2}\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, $(\sum a_ie_i^P,\sum a'_je_j^Q)=\sum_i a_i(-a'_i+a'_{i-m_1})=<a,(h^{m_1}-1)a'>_s$, where $<x,y>_s$ is the standard inner product $\sum x_iy_i$ (or the standard hermitian product $\sum x_i\overline{y_i}$). Similarly, $(b_if_i^P,\sum b'_jf_j^Q)=\sum b_i(b'_i-b'_{i+m_2})=-<b,(h^{-m_2}-1)b'>_s$. Therefore, the pairing:

$$(***)\qquad ((a,b),(a',b'))=<a,(h^{m_1}-1)a'>_s-<b,(h^{-m_2}-1)b'>_s$$

restricted on $\ker\partial$ provides the intersection form $(,)$ of $H_1(X)$.

Now, we can consider the eigenspace decomposition of the monodromy action h (∂_λ can be obtained from ∂ by replacing h by λ).

If $\lambda=1$, then $(**)$ becomes:

$$0\to H_1(X,\mathbf{C})_1\to\mathbf{C}\oplus\mathbf{C}\xrightarrow{\partial_1}\mathbf{C}\to\mathbf{C}\to 0,$$

where $\partial_1=0$ and $(,)_1=0$. In particular, $\sigma_1(F'_w)=0$ (cf. E.4.2).

If $\lambda\neq 1$, $\lambda=e^{2\pi il/m}$, then $(**)$ gives:

$$0\to H_1(X,\mathbf{C})_\lambda\to\mathbf{C}\oplus\mathbf{C}\xrightarrow{\partial_\lambda}\mathbf{C}\to 0,$$

where $\partial_\lambda(a,b)=(\lambda^{-m_1}-1)a+(\lambda^{-m_2}-1)b$. Then $\ker\partial_\lambda$ is generated by $(u,v)=(-(\lambda^{-m_2}-1),\lambda^{-m_1}-1)$. By a computation: $((u,v),(u,v))_\lambda=<u,(\lambda^{m_1}-1)u>_s-<v,(\lambda^{-m_2}-1)v>_s=(\lambda^{-m_1}-1)(\lambda^{-m_2}-1)(\lambda^{m_1+m_2}-1)=-8i\sin(\pi m_1l/m)\sin(\pi m_2l/m)\sin(\pi(m_1+m_2)l/m)$. Therefore:

$$\sigma_\lambda(F'_w)=sign\Big(\sin(\pi m_1l/m)\sin(\pi m_2l/m)\sin(\pi(m_1+m_2)l/m)\Big).$$

Now use $((-x))=-((x))$, the fact that $m_1+m_2+m_3\equiv 0(\text{mod } m)$, and (cf. E.4.8):

$$2((s))+2((t))-2((s+t))=sign\Big(\sin(\pi s)\sin(\pi t)\sin(\pi(s+t))\Big).$$

This ends the proof. $\square$

4.12. ϵ**-hermitian variation structures** ($\epsilon = \pm 1$)**.** [34, 31]

In this subsection U is a complex vector space with a complex conjugation, denoted by $\bar{\cdot}$. U^* denotes its dual $Hom_{\mathbf{C}}(U,\mathbf{C})$. There is a natural isomorphism $\theta : U \to U^{**}$, given by $\overline{\theta(u)(\varphi) = \varphi(u)}$. If $\varphi \in Hom_{\mathbf{C}}(U,U')$, then $\bar{\varphi} \in Hom_{\mathbf{C}}(U,U')$ is defined by $\bar{\varphi}(x) := \overline{\varphi(\bar{x})}$. The dual $\varphi^* : U'^* \to U^*$ of φ is defined by $\varphi^*(\psi) = \psi \circ \varphi$.

4.13. Definition. An ϵ–hermitian variation structure is a system $(U; b, h, V)$, where U is as above, and:

(a) $b : U \to U^*$ is a $\mathbf{C}$–linear endomorphism with $\overline{b^* \circ \theta} = \epsilon b$;

(b) h is a b–orthogonal automorphism of U, i.e. $\overline{h^*} \circ b \circ h = b$;

(c) $V : U^* \to U$ is a $\mathbf{C}$–linear endomorphism, with $\overline{\theta^{-1} \circ V^*} = -\epsilon V \circ \bar{h}^*$, and $V \circ b = h - id$.

If b is an isomorphism, then $V = (h - id)b^{-1}$, in particular, the structure $(U; b, h, V)$ is completely determined by the isometric structure $(U; b, h)$.

If V is an isomorphism, then $h = -\epsilon V \overline{(\theta^{-1} \circ V^*)}^{-1}$, and $b = -V^{-1} - \epsilon \overline{(\theta^{-1} \circ V^*)}^{-1}$. The structures with this property are called "simple variation structures". For the classification of (non-degenerate) isometric structures, see [Milnor] [27], and for the classification of simple variation structures, see [Némethi] [34].

Any base $\{e_i\}_i$ of U defines a dual base $\{e_i^*\}_i$ of U^* by $e_j^*(\bar{e}_i) = 1$ if $j = i$ and $= 0$ otherwise. It is convenient in the description of the variation structures to use the matrix representation in a convenient base of U and its dual base. Then θ corresponds to the identity matrix. If an endomorphism $\varphi : U \to U'$, in a given base, has matrix representation A, then $\bar{\varphi}^*$ in the dual base is represented by the transposed matrix $\bar{A}^t$.

4.14. Example/Definition of $\mathcal{V}(f)$. If $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ is an isolated singularity, then by the results of section 2, ($U = H_n(F, \mathbf{C})$, b =the complex $(-1)^n$-symmetric hermitian intersection form, h=the complexified monodromy operator, and V=the complexified variation map) form a $(-1)^n$-hermitian variation structure, which is simple, by (3.10), (cf. (3.11) as well). This variation structure is denoted by $\mathcal{V}(f)$.

For simplicity of our discussion, in all our examples and result in the sequel, we will assume that $h_{nil} = 0$, i.e. the complex monodromy operator is diagonalizable.

4.15. Examples of simple ϵ–hermitian variation structures: [34] (see also [44, 43].) We will use the notation $\epsilon = (-1)^n$ (cf. 4.14). For $|\lambda| = 1$, we define:

$$\mathcal{W}_\lambda(\pm 1) = (\mathbf{C}; \pm i^{-n^2}, \lambda, \pm(\lambda - 1)i^{n^2}), \quad \text{if } \lambda \neq 1, \text{ and}$$

$$\mathcal{W}_1(\pm 1) = (\mathbf{C}; 0, 1_{\mathbf{C}}, \pm i^{n^2+1}) \quad \text{if } \lambda = 1.$$

It is easy to verify that they are $(-1)^n$–hermitian variation structures.

4.16. Theorem. [34] *Any simple ϵ–hermitian variation structure, with $h_{nil} = 0$ and eigenvalues of h on the unit circle S^1, is uniquely expressible as a direct sum of indecomposable ones up to order of summands and isomorphism. The indecomposable structures are: $\mathcal{W}_\lambda(\pm 1)$ $(\lambda \in S^1)$.*

Proof. First put h in the Jordan normal form. Then, via the relations (4.13), the classification is just the classification of hermitian forms. $\square$

Now, we will consider the ($\epsilon = -1$)–variation structure $\mathcal{V}(f)$ associated with an isolated plane curve singularity with $h_{nil} = 0$ (cf. 4.14). By (4.16) one has:

$$(\mathbf{4.17}) \qquad \mathcal{V}(f) = \bigoplus_{\lambda} p_{\lambda}^{+} \cdot \mathcal{W}_{\lambda}(+1) \oplus \bigoplus_{\lambda} p_{\lambda}^{-} \cdot \mathcal{W}_{\lambda}(-1).$$

The question is: can any combination, as represented in the right hand side of (4.17), be realized as the variation structure $\mathcal{V}(f)$ of some f? The answer is no. The first restriction appears immediately. Since $\mathcal{V}(f)$ is the complexification of a real structure, it is stable with respect to the complex conjugation. Therefore (cf. E.4.2):

$$(\mathbf{4.18}) \qquad p_{\lambda}^{+} = p_{\bar{\lambda}}^{-} \text{ provided that } \lambda \neq 1.$$

The next restriction is not so evident. By (4.13.c), the restriction of V on $\ker(h^{*} - 1)$ satisfies $\overline{V}^{*} = V$, hence it is a symmetric form. It can be diagonalized with p_1^{+} entries of -1 and p_1^{-} entries of $+1$ on the diagonal.

4.19. Proposition. [Neumann] [43] $p_1^{-} = 0$, *or equivalently, the Seifert form* S *restricted on* $\ker(h-1) = \ker(,)$ *is a symmetric negative definite form. In particular,* $p_1^{+} = \dim\ker(h-1) = r-1$.
Proof. The Milnor fiber has r oriented boundary components corresponding to the irreducible components $\{f_i\}_{i=1}^{r}$ of f. Their homology classes in $H_1(F, \mathbf{Z})$ are denoted by $\{d_i\}_i$. They generate $\ker(h-1) = \ker(,)$ with the relation $\sum_i d_i = 0$. Therefore, $V(\sum_i a_i d_i, \sum_j a_j d_j) = \sum_i a_i V(d_i, \sum_{j\neq i}(a_j - a_i)d_j) = \sum_{i\neq j} a_i(a_j - a_i)V(d_i, d_j)$. Now, $V(d_i, d_j) = V(d_j, d_i) = i_0(f_i, f_j) > 0$ (cf. Reeve's theorem 1.13 or E.3.1). Hence: $V(\sum a_i d_i, \sum a_i d_i) = -\sum_{i<j}(a_i - a_j)^2 \cdot i_0(f_i, f_j)$. $\square$
4.20. Let $\mu_\lambda = \dim H_1(F, \mathbf{Z})_\lambda$ and σ_λ the corresponding equivariant signature. Then, for $\lambda \neq 1$ and $\epsilon = (-1)^n = -1$, one has: $\sigma_\lambda(\mathcal{W}_\lambda(\pm 1)) = \sigma_\lambda(\pm i^{-1}) = \pm 1$. Therefore:

$$(\mathbf{4.21}) \qquad \mu_\lambda = p_\lambda^{+} + p_\lambda^{-}, \quad \sigma_\lambda = p_\lambda^{+} - p_\lambda^{-}.$$

4.22. Corollary. [Neumann] *Assume that* $f : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ *satisfies* $h_{nil} = 0$. *Then:*

$$\mathcal{V}(f) = (r-1) \cdot \mathcal{W}_1(+1) \oplus \bigoplus_{\lambda \neq 1} \frac{\mu_\lambda + \sigma_\lambda}{2} \cdot \mathcal{W}_\lambda(+1) \oplus \bigoplus_{\lambda \neq 1} \frac{\mu_\lambda - \sigma_\lambda}{2} \cdot \mathcal{W}_\lambda(-1).$$

(Notice that for $\lambda \neq 1$, the relation $p_\lambda^{+} = p_{\bar{\lambda}}^{-}$ corresponds to $\mu_\lambda = \mu_{\bar{\lambda}}$ and $\sigma_\lambda = -\sigma_{\bar{\lambda}}$.)

Open problems and conjectures.

4.23. Classify the ϵ–hermitian variation structures.
(The problem is important for the following reason. Let $f : (X, x) \to (\mathbf{C}, 0)$ be an analytic germ such that both (X, x) and $(f^{-1}(0), 0)$ have only isolated singularities (and the

dimension is arbitrary). Then, similarly as in the hypersurface case, f defines a variation structure, which is simple if and only if the link of (X,x) is a rational homology sphere, which is not the case in general.) In order to understand the difficulties of the classification, see E.4.9.

4.24. Generalize (4.8) for arbitrary dimensions.

4.25. Find algebraicity restrictions, as in (4.19), for higher dimensional variation structures (cf. [34]).

Exercises for the fourth section.

E.4.1. (a) Compute $\{\sigma_\lambda(f)\}_\lambda$ for $f=x^2+y^3$. Prove that $\mathcal{V}(x^2+y^3)=\mathcal{W}_{\exp(\pi i/3)}(+1)\oplus\mathcal{W}_{\exp(-\pi i/3)}(-1)$.

(b) Prove that $\mathcal{V}(x^2+y^2)=\mathcal{W}_1(+1)$.

E.4.2. (a) Prove that $(\,,\,)|\ker(h-1)$ is trivial.

(b) Prove that $\sigma_\lambda=(-1)^n\sigma_{\bar\lambda}$ (for any λ), and $p^-_\lambda=p^+_{\bar\lambda}$ (for $\lambda\neq 1$, cf. 4.18).

(c) Using either (a) or (b), prove that $\sigma_1=0$ provided that $n=1$.

(d) If $n=2$, then in general $\sigma_1\neq 0$. Isn't this in contradiction with (a)?

(e) Prove that $\sigma_1\neq 0$ if $f=(x^2+y^3)(x^3+y^2)+z^2$ (cf. E.3.12 and 4.5).

Hint for (e): Notice that the different generalized eigenspaces are orthogonal. In our case $H_1(F)_1$ is two-dimensional, and $\ker(\,,\,)$ is its one–dimensional subspace. Therefore $\sigma_1=0$ would imply that $(\,,\,)_1$ is trivial, hence $H_1(F)_1=\ker(\,,\,)$, which is false.

Remark.* Actually, from Hodge theory, (see, e.g. [34]), for any isolated hypersurface singularity with $n=2$, the equivariant signature $\sigma_1(f)$ is non-negative. Therefore, in our case (E.4.2.e): $\sigma_1(f)=1$.

E.4.3.* Marie-Claire Grima's family. (Cf. [44].)

Assume that $p+r=q+s$ and $ps<qr$, and $g.c.d.(pe,qf)=g.c.d.(re,sf)=g.c.d.(pf,qe)=g.c.d.(rf,se)=1$. Set $f=(x^{pe}+y^{qf})(x^{re}+y^{sf})$ and $g=(x^{pf}+y^{qe})(x^{rf}+y^{se})$. Then f and g have the same $\Delta(t)$, $\Delta^1(t)$, μ, r, $h\otimes\mathbf{C}$, F, $(\,,\,)$. But, their equivariant signatures are different.

Compute the case $(p,q,r,s,e,f)=(1,3,5,3,2,1)$.

E.4.4.* The quasi–homogeneous case. The complex (real) Seifert form. [Némethi] [34]

Assume that $f:(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$ is quasi–homogeneous of weights $\{w_i\}_{i=1}^{n+1}$. Fix a set of monomials $\{z^\alpha:\alpha\in\Lambda\subset\mathbf{N}^{n+1}\}$, such that $\{[z^\alpha]\}_\alpha$ is a base of $M(f)$ (cf. E.1.8.a). We define (cf. E.1.8.b) $l(\alpha)=\sum_{i=1}^{n+1}(\alpha_i+1)/w_i$. Then

$$\mathcal{V}(f)=\bigoplus_{\alpha\in\Lambda}\mathcal{W}_{\exp(2\pi i l(\alpha))}((-1)^{[l(\alpha)]+n}),$$

where $[\cdot]$ is the integer part function.

Remark. The codification of $\{z^\alpha\}_{\alpha\in\Lambda}$ in $\mathbf{Z}[\mathbf{Q}]$ via $z^\alpha\mapsto(l(\alpha))$ gives $P_{M(f)}(t)$; its codification in $\mathbf{Z}[\mathbf{Q}/2\mathbf{Z}]$ via $z^\alpha\mapsto\mathcal{W}_{\exp(2\pi i l(\alpha))}((-1)^{[l(\alpha)]+n})$ gives $\mathcal{V}(f)$, and finally the codification in $\mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ via $z^\alpha\mapsto(\exp(2\pi i l(\alpha)))$ provides $\Delta(t)$. The natural projections

$\mathbf{Z}[\mathbf{Q}] \to \mathbf{Z}[\mathbf{Q}/2\mathbf{Z}] \to \mathbf{Z}[\mathbf{Q}/\mathbf{Z}]$ give the corresponding relations between the Poincaré polynomial, (real) variation map (= real Seifert form) and characteristic polynomial.

E.4.5. (a) Prove that $\mathcal{V}(x^2+y^2+z^a) = \oplus_{k=1}^{a-1}\mathcal{W}_{\exp(2\pi ik/a)}(-1)$.

(b) Using (E.4.4), compute $\mathcal{V}(x^2+y^3+z^5)$.

(c) Notice that in both cases $l(\alpha)\in(1,2)$ for any $\alpha\in\Lambda$, hence $(,)$ is negative definite.

Remark. If $n=2$, f quasi–homogeneous, then $l(\alpha)\in(1,2)$ for any $\alpha\in\Lambda$ if and only if $(f^{-1}(0),0)$ is a rational normal surface singularity. Using Hodge theory (namely the spectrum of f), this can be generalized for any hypersurface singularity.

E.4.6. Non–additivity of the signature.

Consider the space X from (4.11) together with the monodromy action $X\to X$ and its decomposition $X = X'\cup X''$, which is compatible with the monodromy action. Then $\sigma_\lambda(X') = \sigma_\lambda(X'') = 0$ for any λ, but $\sigma_\lambda(X)\neq 0$ in general. Actually, the important invariant $\sigma_\lambda(f)$ appears exactly as the measure of this non–additivity.

Is this in contradiction with Novikov's additivity theorem?

(In general, the non–additivity defect was computed by C.T.C. Wall.)

E.4.7. (a) The function $((x))$ is odd, and its Fourier sine expansion is:

$$((x)) = \frac{1}{\pi}\sum_{k\geq 1}\frac{1}{k}\sin(2\pi kx).$$

(b) For any $x\in\mathbf{R}$, $l,k\in\mathbf{N}$ (with $g.c.d.(k,l)=d$) one has:

$$\sum_{j=0}^{l-1}((k\cdot\frac{x+j}{l})) = d\cdot((kx/d)).$$

E.4.8. Prove that

$$2((s))+2((t))-2((s+t)) = sign\Big(\sin(\pi s)\sin(\pi t)\sin(\pi(s+t))\Big).$$

E.4.9. The following matrices define an indecomposable (+1)-variation structure, but h has two Jordan blocks:

$$b=\begin{pmatrix}0&0&1\\0&0&0\\1&0&1\end{pmatrix},\quad h=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\quad V=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

§5. The signature and Dedekind sums.

5.1. In this section we will present Brieskorn's formula for the signature of the Milnor fiber of Brieskorn singularities (in arbitrary dimension), and we will relate it with a famous, classical lattice point problem. In the second part, we will compute the signature of singularities of type $f(x,y,z) = g(x,y) + z^a$ in terms of generalized Dedekind sums, showing the deep arithmetical nature of the signature.

We hope also, that this section will convince the reader about the elegance of the variation structures.

Since the definition of a variation structure $\mathcal{W}_\lambda(\pm 1)$ depends on $\epsilon = (-1)^n$, sometimes we write $\mathcal{W}_\lambda(\pm 1)_n$.

5.2. Consider a Brieskorn singularity $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ given by $f = \sum_{i=1}^{n+1} z_i^{a_i}$, and assume that n is even. Then the intersection form $(\,,)$ of the Milnor fiber is symmetric. We will compute its signature $\sigma(f)$ in terms of $\{a_i\}_i$.

Start with the singularity $z \mapsto z^a$ (i.e. $n = 0$). Then:

$$(\mathbf{5.3}) \qquad \mathcal{V}(z^a) = \bigoplus_{k=1}^{a-1} \mathcal{W}_{\exp(2\pi i k/a)}(+1)_{n=0}.$$

Since $\mathcal{W}_\xi(+1)_{n=0} = (\mathbf{C}; 1, \xi, \xi - 1)$, this is equivalent to:

$$(\mathbf{5.4}) \qquad \mathcal{V}(z^a) = \bigoplus_{k=1}^{a-1} (\mathbf{C}; 1, e^{2\pi i k/a}, e^{2\pi i k/a} - 1).$$

By Sebastiani–Thom and Deligne–Sakamoto theorem (cf. E.2.11 and E.3.5), the variation map $V(f)$ and the monodromy $h(f)$ of $f = \sum_i z_i^{a_i}$ satisfy:

$$(V(f), h(f)) = \oplus'(V_{\mathbf{k}}, h_{\mathbf{k}}),$$

where $\oplus' = \oplus_{k_1=1}^{a_1-1} \cdots \oplus_{k_{n+1}=1}^{a_{n+1}-1}$, $\mathbf{k} = (k_1, \ldots, k_{n+1})$, and:

$$V_{\mathbf{k}} = (-1)^{n(n+1)/2} (e^{2\pi i k_1/a_1} - 1) \cdots (e^{2\pi i k_{n+1}/a_{n+1}} - 1), \quad \text{and } h_{\mathbf{k}} = e^{2\pi i \sum_{j=1}^{n+1} k_j/a_j}.$$

Now, since $b = (h-1)V^{-1}$, one has $b(f) = \oplus' b_{\mathbf{k}}$, where $b_{\mathbf{k}} = (h_{\mathbf{k}} - 1)/V_{\mathbf{k}}$. By a computation:

$$b_{\mathbf{k}} = \frac{\sin(\pi \sum_j k_j/a_j)}{2^n \prod_j \sin(\pi k_j/a_j)},$$

which has signature *sign* $\sin(\pi \sum_j k_j/a_j)$. Therefore:

5.5. Theorem. [Brieskorn] [9] *If n is even, then:*

$$\sigma\Big(\sum_{i=1}^{n+1} z_i^{a_i}\Big) = \sum_{k_1=1}^{a_1-1} \cdots \sum_{k_{n+1}=1}^{a_{n+1}-1} \mathit{sign}\ \sin\Big(\pi \sum_{j=1}^{n+1} k_j/a_j\Big).$$

5.6. Corollary. *Consider the following lattice point counting:*

$$S_t := \#\{\mathbf{k} \in \mathbf{Z}^{n+1} : 1 \le k_j \le a_j - 1, (1 \le j \le n+1), t < \sum_{j=1}^{n+1} k_j/a_j < t+1\}.$$

Then: $\sigma(\sum_j z_j^{a_j}) = \sum_t (-1)^t S_t$.

5.7. If $n = 2$ then this formula is much simpler. First notice that, in general:

$$\#\{(k_1, \dots, k_{n+1}) \in \mathbf{Z}^{n+1};\ \sum_j k_j/a_j \in \mathbf{Z}\} = \mu_1,$$

where μ_1 is the dimension of the 1–eigenspace $H_n(F)_1$. This follows from (E.2.12). Moreover, that exercise also shows that in our case ($n = 2$) one has:

$$\mu_1 = \frac{(a_1, a_2)(a_2, a_3)(a_3, a_1)}{(a_1, a_2, a_3)} - (a_1, a_2) - (a_2, a_3) - (a_3, a_1) + 2.$$

Now, if $n = 2$ and $t \notin \{0, 1, 2\}$ then $S_t = 0$. Moreover:

$$S_0 = S_2 \text{ by the symmetry of the lattice points,}$$

$$S_0 + S_1 + S_2 = \mu_{\neq 1}(f) = (a_1 - 1)(a_2 - 1)(a_3 - 1) - \mu_1,$$
$$S_0 - S_1 + S_2 = \sigma(f).$$

Hence, (if $n = 2$):

$$(\mathbf{5.8}) \qquad 4 \cdot S_0 = \sigma(f) + (a_1 - 1)(a_2 - 1)(a_3 - 1) - \mu_1.$$

Therefore, the computation of the signature is equivalent to the counting of the lattice points in the open tetrahedron $T(a_1, a_2, a_3)$ with vertices $(0,0,0)$, $(a_1, 0, 0)$, $(0, a_2, 0)$, and $(0, 0, a_3)$. In particular cases (e.g. when a_1, a_2, a_3 are small numbers) it is easy to count these lattice points, but in general it is not. There was a famous classical problem to give a precise formula of the number of these lattice points (i.e. of S_0) in terms of the numbers a_1, a_2 and a_3. The problem was solved by L. J. Mordell [29] when the numbers $\{a_i\}$ are pairwise relative prime; the general case was solved recently by Pommersheim. The formula provides S_0 in terms of some Dedekind sums involving the a_j's. In (5.18), we will see that our theorem (5.15) applied in the particular case of Brieskorn singularities will provide Mordell's theorem.

5.9. Now, we will start to compute the signature of $f(x, y, z) = g(x, y) + z^a$, where g defines an isolated plane curve singularity (with $h_{nil}(g) = 0$) and a is an arbitrary integer.

Again, we will use Deligne–Sakamoto result: $V(f) = V(g) \otimes V(z^a)$. By theorem (4.22), we have to compute only tensor products of type: $\mathcal{W}_\lambda(\pm 1)_{n=1} \otimes \mathcal{V}(z^a)_{n=0}$ (and $\mathcal{V}(z^a)$ has the decomposition (5.4), hence we need to compute only one–dimensional tensor products). We proceed as follows: we make the product of the variation maps and eigenvalues, and compute b from the relation $(\lambda-1)/V$.

5.10. Lemma. (a)

$$\sigma(\mathcal{W}_1(+1)_{n=1}\otimes\mathcal{V}(z^a))=-a.$$

(b) *Let* $\lambda=e^{2\pi i\alpha}\neq 1$, *then:*

$$b\Big(\mathcal{W}_\lambda(\pm1)_{n=1}\otimes\mathcal{W}_{\exp(2\pi ik/a)}(+1)_{n=0}\Big)_{n=2}=\mp\frac{1}{2}\cdot\frac{\sin\pi(\alpha+k/a)}{\sin(\pi k/a)\cdot\sin(\pi\alpha)}.$$

Proof. (a) Notice that $V(\mathcal{W}_1(+1))=-1$, hence tensorized with $(e^{2\pi ik/a}-1)$ gives $V=-(e^{2\pi ik/a}-1)$ and $h=e^{2\pi ik/a}$, hence $b=(h-1)/V=-1$ for any $k=1,\dots,a-1$. In the case of (b), the variation map of the tensor product is $\pm i(\lambda-1)(e^{2\pi ik/a}-1)$, and its monodromy is $\lambda e^{2\pi ik/a}$. □

The expression of the right hand side of (b) can be transformed as follows:

5.11. Lemma.

$$\sum_{k=1}^{a-1} sign\,\frac{\sin\pi(\alpha+\frac{k}{a})}{\sin\pi\alpha}=2a((\alpha))-2((a\alpha)).$$

Proof. Consider the two cases $l/a<\alpha<(l+1)/a$ and $\alpha=l/a$. □

Now, (4.22), (5.10) and (5.11) give:

5.12. Theorem. [Némethi] [35, 36, 37] *Assume that* $h_{nil}(g)=0$. *Then, with the notation* $\lambda=e^{2\pi i\alpha}$, *one has:*

$$\sigma(g(x,y)+z^a)=-(r_g-1)(a-1)+2\sum_{\lambda\neq1}\sigma_\lambda(g)\cdot\Big(((a\alpha))-a((\alpha))\Big).$$

5.13. Example.

(a) Let $g=x^2+y^2$. Then $\mathcal{V}(g)=\mathcal{W}_1(+1)$, hence $\sigma(g+z^a)=-(a-1)$ (cf. E.3.9 and E.4.5).

(b) Let $g=x^2+y^3$. Then $r_g=1$, and $\sigma_\lambda(g)=\pm1$ if $\lambda=\exp(\pm2\pi i/6)$ (cf. E.4.1). Therefore:

$$\sigma(x^2+y^3+z^a)=4\cdot\Big(((a/6))-a((1/6))\Big).$$

Therefore, if $1\leq a<6$ then $\sigma(f)=-\mu(f)=2(1-a)$, hence $(,)_f$ is negative definite. If $a=6$, then $\sigma(f)=-8$, $\mu(f)=10$ and $\mu_1(f)=2$, hence $(,)_f$ is negative semi–definite. If $a=7$, then $(,)_f$ is non–degenerate with $\mu=12$ and $\sigma=-8$. Actually, if $a=6t+1$, then $\sigma(f)=-8t$.

Now, using theorem (4.8), the signature $\sigma(f)$ of $g+z^a$ can be computed from the embedded resolution graph of g as follows:

$$\sigma(g+z^a)=-(r_g-1)(a-1)+4\sum_{w\in\mathcal{R}}\sum_{v\in\mathcal{V}_w}\sum_{l=1}^{m_w-1}((\frac{lm_v}{m_w}))\cdot\Big(((\frac{la}{m_w}))-a((\frac{l}{m_w}))\Big).$$

5.14. Definition. [Dedekind, Rademacher] [47, 46] For integers a, b, c $(a \neq 0)$ the generalized Dedekind sum is defined as

$$s(b, c; a) = \sum_{k=1}^{a-1} ((\frac{kb}{a})) \cdot ((\frac{kc}{a})).$$

The classical Dedekind sum of the pair (a, b) is $s(b, 1; a)$.

5.15. Theorem. [Némethi] [36, 37] *Assume that $h_{nil}(g) = 0$. From the embedded resolution graph G_g of g, the signature of $g + z^a$ reads as follows:*

$$\sigma(g + z^a) = -(r_g - 1)(a - 1) + 4 \sum_{w \in \mathcal{R}} \sum_{v \in \mathcal{V}_w} \Big(s(m_v, a; m_w) - a \cdot s(m_v, 1; m_w) \Big).$$

5.16. Corollary. [Conjectured by Brieskorn, Durfee and Zagier, first proved by Neumann].
The correspondence $(a \mapsto \sigma(g + z^a))$ is a sum of a periodic function and a linear function.
5.17. Example. If $g = x^2 + y^3$ then $\mathcal{R}$ has only one element $\{w\}$ with $m_w = 6$ and $\{m_v\}_{v \in \mathcal{V}_w} = \{1, 2, 3\}$.
5.18. Example.* [Némethi] [37]

Consider $g : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$, $g(z_1, z_2) = z_1^{a_1} + z_2^{a_2}$, and set $a = a_3$. For simplicity, we will assume that $g.c.d.(a_1, a_2) = 1$. Fix integers b_1 and b_2 such that $b_1 a_1 + b_2 a_2 = 1$. Then $\mathcal{W}$ has only one element $\{w\}$ with $m_w = a_1 a_2$ and the multiplicities $\{m_v\}_{v \in \mathcal{V}_w}$ modulo m_w are given by $\{-a_1 b_1, -a_2 b_2, 1\}$. Now, using the following properties of the Dedekind sums:

$$s(b, c; a) = (a, b, c) \cdot s(\frac{b}{(a, b)}, \frac{c}{(a, b, c)}; \frac{a}{(a, b)}), \text{ and} \tag{5.19}$$

$$s(b, c; a) = s(kb, kc; a) \text{ if } (k, a) = 1; \tag{5.20}$$

and the famous Dedekind Reciprocity Law:

$$s(b, 1; a) + s(a, 1; b) = -\frac{1}{4} + \frac{a^2 + b^2 + (a, b)^2}{12ab},$$

one can deduce that:

$$\sigma(z_1^{a_1} + z_2^{a_2} + z_3^{a_3}) = -a_3 \frac{(a_1^2 - 1)(a_2^2 - 1)}{3a_1 a_2} + 4 \cdot \sum_{v \in \mathcal{V}_w} s(m_v, a_3; a_1 a_2).$$

(Recall that the set $\{m_v\}$ is given above.)

This formula is equivalent (via 5.7) with Mordell's formula which gives the number of lattice points in the open tetrahedron $T(a_1, a_2, a_3)$ (in Mordell's case the a_i's are pairwise relatively prime numbers). If we drop the condition $g.c.d.(a_1, a_2) = 1$, then by a similar computation, we reobtain a recent result of Pommersheim, which counts the number of lattice points in the open tetrahedron (with no restriction about the a_i's).
5.21. Remark. Notice that the formula (5.12) or (5.15) is not valid if $h_{nil}(g) \neq 0$. Take,

for example, A'Campo's germ $g=(x^2+y^3)(x^2+y^3)$ (cf. E.3.12 and 4.5). Set $f=g+z^2$ i.e. $a=2$. Then, even without computation, using the symmetry of the graph G_g, the right hand side of (5.15) gives an odd number. Notice that $\mu(f)=11$, and one has, in general, the relation $\mu(f)+\sigma(f)\equiv \dim\ker(,)_f$ $(mod\ 2)$. Since in our case $\dim\ker(,)_f=1$ (cf E.3.2 or E.4.2), $\sigma(f)$ is even. (Actually, $\sigma(f)=-8$, see E.5.5.).

The signature $\sigma(g+z^a)$, without any restriction about g, is given in:

5.22. Theorem.* [Némethi] [36] *For any isolated plane curve singularity g and positive integer a, one has:*

$$\sigma(g+z^a)=-(r_g-1)(a-1)+\sum_{e\in\mathcal{E}}\Big(g.c.d.(a,d_e)-1\Big)-\sum_{w\in\mathcal{W}}\Big(g.c.d.(a,d_w)-1\Big)+$$

$$+4\sum_{w\in\mathcal{R}}\sum_{v\in\mathcal{V}_w}\Big(s(m_v,a;m_w)-a\cdot s(m_v,1;m_w)\Big).$$

Above, similarly as in (4.4): for any $e=(w_1,w_2)\in\mathcal{E}$ we let $d_e:=g.c.d.(m_{w_1},m_{w_2})$, and for any $w\in\mathcal{W}$, we let $d_w:=g.c.d.(m_v|v\in\mathcal{V}_w\cup\{w\})$.

5.23. Remark.

(a) In [Némethi] [32, 33] the (equivariant) signature of $f=g+z^a$ is clarified for arbitrary n.

(b) (Cf. 2.26-2.27) In [37] there is a very short proof of the negativity of the signature of $f=g+z^a$. Stronger inequalities are proved in [35, 36]. For a review about the signature of $f=g+z^a$, see [38].

(c) The semi–ring structure of "simple" variation structures (cf. 4.13) (i.e. the tensor product of indecomposable blocks, without the restriction $h_{nil}=0$) is given in [Némethi] [31].

Some more results about the signature*. Case $n=2$.

5.24. Assume that $f:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ defines an isolated singularity. We recall that in general the diffeomorphism type of its link K_f says very little about the smoothing invariants $\mu(f)$, or $\sigma(f)$ (cf. E.1.15.b). Nevertheless, there are some connections between these two (i.e. link and smoothing) type of invariants. Moreover, there is a big interest in particular cases, when K_f determines the smoothing invariants $\mu(f)$ resp. $\sigma(f)$.

First, we recall an important invariant of K_f. Let $\phi:(\mathcal{Y},E)\to(f^{-1}(0),0)$ be a resolution with $E:=\phi^{-1}(0)$. Then the **canonical class** (associated with ϕ) is an element $K\in H_2(\mathcal{Y},\mathbf{Q})$. We use the same notation $\{E_w\}_{w\in\mathcal{W}}$ for the 2–homology classes of the irreducible exceptional divisors $\{E_w\}_w$ (which is a base in $H_2(\mathcal{Y},\mathbf{Q})$). Then $K=\sum r_wE_w$. K satisfies the relations (adjunction formula):

$$-K\cdot E_w=E_w^2+\chi(E_w)\text{ for any }w\in\mathcal{W}.$$

Since the intersection matrix $(A_{\alpha\beta})$ (cf. 1.15) is negative definite, the above relations determine K. In fact, in our case, K associated with the graph of $(V_f,0)$ provided by an isolated hypersurface singularity, has integer coefficients $\{r_w\}_w$. The invariant K^2 is the auto–intersection of this class, i.e. $K^2 = K \cdot K = \sum_{i,j} r_i r_j E_i \cdot E_j$. Notice that K depends on the choice of the resolution ϕ, but the numerical invariant $K^2 + \#\mathcal{W}$ is independent of ϕ, and depends only on the link K_f.

5.25. Theorem.* [Durfee] [14] *Let $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ be an isolated singularity and define the numerical invariant $K^2 + \#\mathcal{W}$ from one of the resolutions of $(V_f, 0)$. As usual, K_f denotes the link of f. Then:*

$$3\sigma(f) + 2\mu(f) + 2 \cdot \dim H_1(K_f) + K^2 + \#\mathcal{W} = 0.$$

In particular, modulo the link K_f, the smoothing invariants $\mu(f)$ and $\sigma(f)$ are equivalent. Now, it is an interesting problem to find particular cases, when the three–dimensional manifold K_f determines both $\mu(f)$ and $\sigma(f)$. A very interesting result is the following.

Casson defined an invariant $\lambda(\Sigma)$ for any integer homology 3–manifold Σ (a fact generalized for rational homology spheres by Walker). Hence, if the link K_f is an integer homology sphere, then it has its well–defined Casson invariant $\lambda(K_f)$.

5.26. Theorem.* *Let f be as before, and assume that K_f is an integer homology sphere. Then $8 \cdot \lambda(K_f) = \sigma(f)$ in the following cases:*

(a) if f is a Brieskorn singularity; [Fintushel–Stern] [15]

(b) if $f(x, y, z) = g(x, y) + z^a$, [Neumann–Wahl] [45].

In particular, for these singularities, the smoothing invariants $\mu(f)$ and $\sigma(f)$ are determined completely by the link only !

Open problems and conjectures.

5.27. [Némethi] [36] For any isolated plane curve singularity g and positive integer a, show that

$$\sigma(g + z^a) \leq -\frac{a+1}{3a}\,\mu(g + z^a).$$

5.28. (cf. 2.28) For any isolated plane curve singularity g and positive integer a, show that:

$$\sigma(g + z^a) > \sigma(g + z^{a+1}).$$

5.29. (cf. 5.26) [Neumann–Wahl] [45] If the link of the isolated singularity $f : (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ is an integer homology sphere then $8 \cdot \lambda(K_f) = \sigma(f)$.

Exercises for the fifth section.

E.5.1. Prove (5.19) (Use E.4.7.b) and (5.20).

E.5.2. Prove that

$$-s(b,c;a)=\sum_{k=1}^{a-1}((\frac{kb}{a}))\cdot\{\frac{kc}{a}\}.$$

(Use: $\sum_{k=1}^{a-1}((\frac{kb}{a}))=0$, cf. E.4.7.b)
E.5.3. If $a>0$, prove that:

$$|s(b,c;a)|\leq\frac{(a-1)(a-2)}{12a}.$$

E.5.4. Compute the signature of the Milnor fiber of $f=x^3+y^5+z^a$ for any $a\geq 1$.
E.5.5. Consider the singularity $f=(x^2+y^3)(x^3+y^2)+z^2$ (cf. E.3.12).

(a) Using (5.22), prove that $\sigma(f)=-8$.

(b) Consider the minimal resolution of $(V_f,0)$ (as it is described in (E.3.12.c). Prove that $K=-(E_1+E_2)$, and $K^2=-2$. Verify the relation (5.25) for f.

E.5.6. Prove that:

(a) $\sigma(x^d+y^d+z^d)=-(d-1)(d^2+d-3)/3$.

(b) $\sigma(x^a+y^a+z^c)=a-1+4a\cdot s(c,1;a)-c(a^2-1)/3$.

(c) Consider again $f=x^d+y^d+z^d$. Then its minimal resolution graph has only one irreducible exceptional divisor E, with genus $(d-1)(d-2)/2$ and $E^2=-d$. Moreover, $K=-(d-2)E$. Recompute $\sigma(f)$ using (5.25).

E.5.7. Let $g^{-1}(0)$ be the union of three cusps of the form x^2+y^3 with distinct tangent lines, and take $f=g+z^2$. Then $\mu(f)=28$ and $\sigma(f)=-18$.
E.5.8. The signature of quasi–homogeneous singularities.* [Steenbrink] [55]

Let $f:(\mathbf{C}^{n+1},0)\to(\mathbf{C},0)$ be a quasi–homogeneous isolated singularity with weights $\{w_i\}_i$. Define the function $l(\alpha)$ for $\alpha\in\Lambda$ as in (E.4.4). Then:

$$\sigma(f)=\sum_{\substack{\alpha\in\Lambda\\ l(\alpha)\notin\mathbf{Z}}}(-1)^{[l(\alpha)]}.$$

Show, that in the case of Brieskorn singularities, this gives exactly the Brieskorn formula (5.5).
Remark 1. The above formula of Steenbrink follows easily from (E.4.4), but originally, the proof of (the author's real Seifert form formula) (E.4.4) is based on Steenbrink's result [55] on the Hodge numbers of a quasi-homogeneous singularity, which implies both relations.
Remark 2. Take $f=\sum_{j=1}^3 z_j^{a_j}$ (i.e. $n=2$), such that the numbers a_j are pairwise relatively prime. Then $S_0=0$ is equivalent to $l(\alpha)\in(1,2)$ for any $\alpha\in\Lambda$. Therefore, $S_0=(\mu+\sigma)/4$ is the measure of the non–rationality of the singularity $(f^{-1}(0),0)$ (cf. E.4.5). (S_0 is called the "geometric genus of the singularity", which can be defined for any isolated singularity f.)

Appendix.

I guess that the result of this appendix is known for specialists, but I never have seen it written down in this form. The idea and the method is already in the book of Laufer [19].

We start with an isolated plane curve singularity g. We give a recipe how one can obtain the resolution graph $G(V_f)$ of the normal surface singularity $(V_f, 0)$ (where $f = g + z^a$) from the embedded resolution graph G_g of g and from the integer a. Actually, we will consider the germ $z : (V_f, 0) \to (\mathbf{C}, 0)$ (induced by the projection $(x, y, z) \mapsto z$), and we will determine the resolution graph G_z of this germ z. Obviously, $G(V_f)$ is the graph G_z without arrowheads, so it is simpler, but computing G_z has an advantage. Sometimes, it is easier to compute multiplicities than Euler–numbers (characteristic classes). So, computing G_z, first we compute all the multiplicities of the germ z, then some of the Euler–numbers can be more easily computed using the relation (1.16). For the convenience of the reader, we recall (1.16):

$$(A0) \qquad e_w m_w + \sum_{v \in \mathcal{V}_w} m_v = 0 \quad \text{for any } w \in \mathcal{W}.$$

1. For any three positive integers u, v and a, with $(u, v, a) = 1$, we consider the congruence:

$$v + x \cdot \frac{u}{(u,a)} \equiv 0 \pmod{\frac{a}{(u,a)}}.$$

Let $0 \le x_1 < a/(u, a)$ be its solution, and take:

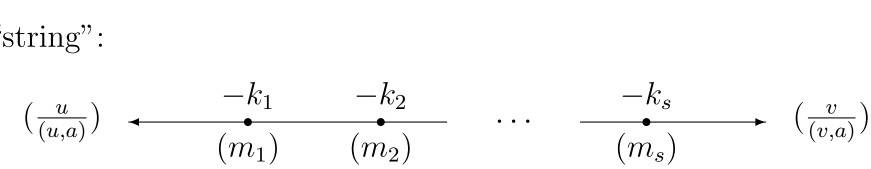
$$(A1) \qquad v + x_1 \cdot \frac{u}{(u,a)} = m_1 \cdot \frac{a}{(u,a)}.$$

If $x_1 \neq 0$, then consider the continuous fraction:

$$\frac{a/(u,a)}{x_1} = k_1 - \cfrac{1}{k_2 - \cfrac{1}{\ddots - \cfrac{1}{k_s}}}, \quad k_1, \dots, k_s \ge 2.$$

Consider the "string":

$G(u, v, a)$:



where all vertices have genus $g = 0$ (i.e. they represent rational irreducible exceptional divisors), their self intersection (Euler) numbers are $-k_1, \dots, -k_s$ respectively. The arrowheads have multiplicities $u/(u, a)$ and $v/(v, a)$, and the first vertex has multiplicity m_1

given by (A1). Therefore, $m_2, \ldots, m_s$ can be easily computed using (A0), namely

$$-k_1 m_1 + \frac{u}{(u,a)} + m_2 = 0, \text{ and } -k_i m_i + m_{i-1} + m_{i+1} = 0 \text{ for } i \geq 2.$$

If $x_1 = 0$, then the string $G(u, v, a)$ has no vertices, it is only an edge.

2. Now, consider the embedded resolution graph G_g of g. It is a tree. For the definition of d_e ($e \in \mathcal{E}$) and d_w ($w \in \mathcal{W}$) see (4.4) or (5.22). Recall also that $\delta_w = \#\mathcal{V}_w$.

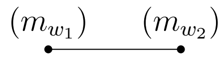
The graph G_z can be considered as a (branched) "covering" $q : G_z \to G_g$.

(a) Above $w \in \mathcal{W}(G_g)$ there are (d_w, a) vertices of G_z, each with multiplicity $m_w/(m_w, a)$ and genus $\tilde{g}$, where:

$$2 - 2\tilde{g} = \frac{(2 - \delta_w)(m_w, a) + \sum_{v \in \mathcal{V}_w} (m_w, m_v, a)}{(d_w, a)}.$$

The vertices in $q^{-1}(w)$ can be indexed by the group $\mathbf{Z}_{(d_w,a)}$.

(b) An edge $e = (w_1, w_2)$ of G_g



is covered by (d_e, a) copies of strings of type $G(m_{w_1}/(d_e, a), m_{w_2}/(d_e, a), a/(d_e, a))$. These strings can be indexed by the group $\mathbf{Z}_{(d_e,a)}$. The arrowheads of the strings are identified with the vertices $q^{-1}(w_1)$ respectively $q^{-1}(w_2)$ via the natural morphisms $\mathbf{Z}_{(d_e,a)} \to \mathbf{Z}_{(d_{w_1},a)}$, respectively $\mathbf{Z}_{(d_e,a)} \to \mathbf{Z}_{(d_{w_2},a)}$.

(c) An arrowhead of G_g

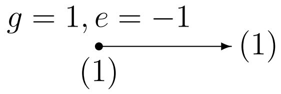
is covered by one string of type $G(m_w, 1, a)$, whose right arrowhead will remain an arrowhead of G_z with multiplicity 1, and its left arrowhead is identified with the unique vertex above w.

(d) In this way, we obtain all the vertices and edges of G_z, and all the multiplicities (which are the multiplicities of the germ z). Moreover, by the description of the strings (cf. part 1), one has all the auto–intersection numbers of the vertices which are situated on the new strings. The auto–intersection numbers of the vertices $q^{-1}(w)$ ($w \in \mathcal{W}(G_g)$) can be computed using (A0).

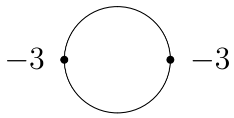
3. If we drop the arrowheads and multiplicities of G_z, we obtain $G(V_f)$. The graphs G_z and $G(V_f)$, in general, are not minimal. They can be simplified by blowing down the (-1)–rational curves.

Exercise.

(a) If $f = x^2 + y^3 + z^6$, then the minimal resolution graph G_z is:



(b) If $f = (x^2 + y^3)(x^3 + y^2) + z^2$, then the minimal resolution graph $G(V_f)$ is:



and $g_w = 0$ for both vertices.

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