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SIGNATURE RELATED INVARIANTS OF MANIFOLDS—I. MONODROMY AND γ-INVARIANTS

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IF X^{2n-1} IS A smooth closed oriented manifold and $\alpha: \pi_1(X) \to U(l)$ is a unitary representation, then Atiyah, Singer, and Patodi[2] defined an invariant $\gamma(X, \alpha)$ via the theory of spectral asymmetry (these denoted $\rho_\alpha(X)$; but their sign conventions differ from ours, see Section 1). In fact $\gamma(X, \alpha)$ measures the alteration or "defect" of the η-invariant of a Riemannian manifold X when "twisted" by such a representation α. This invariant had arisen also another way—as a signature defect—for instance in [12], [13], [7], in the case that a multiple qX bounds an oriented Y^{2n} admitting a representation $\bar{\alpha}: \pi_1(Y) \to U(l)$ extending α. Namely in this case

$$\gamma(X, \alpha) = \frac{1}{q}(\operatorname{sign}(Y, \bar{\alpha}) - l \operatorname{sign}(Y)),$$

where $\operatorname{sign}(Y, \bar{\alpha})$ is signature of Y with local coefficients.

Our main result is an intrinsic homotopy invariant computation of $\gamma(X, \alpha)$ in the case where α factors over a free abelian group; $\alpha: \pi_1(X) \to \mathbf{Z}^s \to U(l)$. The calculation is in terms of a certain linking form in homology of infinite cyclic covers of X, which we call "monodromy" of X. The invariant $\gamma(X, \alpha)$ is not a homotopy invariant in general, as computations for lens spaces easily show.

In Part II we will describe applications to a homotopy invariant calculation of α-invariants of certain group actions, signature defect of coverings, invariants of knots, etc.

The results of this paper were first announced at the Oberwolfach topology meeting of 1974 (see also [12–15]). The proofs have been considerably simplified since then, yielding also some improvement in the results. The research was supported in part by the National Science Foundation.

In the following, all manifolds are assumed smooth, compact and oriented. Smoothness is for convenience only and could be dispensed with. If X is not connected, $\pi_1(X)$ will mean the free product of the fundamental groups of the components of X.

This paper is organized as follows. In §1 we define the γ-invariant and recall some of its properties.

In §2 we define, for a closed manifold X^{2n-1} plus a given homotopy class $f \in [X, S^1]$, a homotopy invariant isometric structure $\mathcal{H}(X, f) = (H, b, t)$ consisting of a finite dimensional complex vector space H with a $(-1)^{n-1}$-hermitian form b and an isometry $t: H \to H$. We call $\mathcal{H}(X, f)$ the *monodromy* of (X, f), since if $f: X \to S^1$ is a fibration with fiber F say, then $\mathcal{H}(X, f) = (H_{n-1}(F; \mathbf{C}), b_F, t)$, where b_F is the $(-1)^{n-1}$-hermitian intersection form on $H_{n-1}(F; \mathbf{C})$ and $t: H_{n-1}(F; \mathbf{C}) \to H_{n-1}(F; \mathbf{C})$ is the monodromy of the fibration. Note that $[X, S^1] = \operatorname{Hom}(\pi_1(X), \mathbf{Z}) = H^1(X; \mathbf{Z})$, so we can consider f interchangeably also as a homomorphism $\pi_1(X) \to \mathbf{Z}$ or as a cohomology class in $H^1(X; \mathbf{Z})$.

The monodromy is in fact equivalent to the homology linking form on the $\mathbf{C}J$-torsion of $H_{n-1}(\tilde{X}; \mathbf{C})$, where $\tilde{X}$ is the infinite cyclic cover of X classified by f and $\mathbf{C}J$ is the group algebra over $\mathbf{C}$ of the infinite cyclic group. We use initially, however, a less abstract and hence more convenient definition of $\mathcal{H}(X, f)$ and postpone its description as a linking form to the final section (§11).

If $\beta: \pi_1(X) \to U(r)$ is a unitary representation, we can also define monodromy with coefficients in the corresponding local coefficient system, and we denote it $\mathscr{H}((X, \beta), f)$.

In §3 we define, to any isometric structure $\mathscr{H} = (H, b, t)$, an invariant $\lambda(\mathscr{H}) \in \mathbf{R}$ and state our main result. In the simplest case it is as follows. Note that a unitary representation $\tau: \mathbf{Z} \to U(l)$ determines an isometric structure $\mathscr{H}(\tau) = (\mathbf{C}^l, h, \tau(1))$ where h is the standard hermitian metric and *vice versa*. We use the notation $\mathscr{H} \otimes \tau$ for $\mathscr{H} \otimes \mathscr{H}(\tau) = (H \otimes \mathbf{C}^l, b \otimes h, t \otimes \tau(1))$.

THEOREM 1. (i) *If the representation* $\alpha: \pi_1(X) \to U(l)$ *factors as* $\alpha = \tau \circ f_*$ *with* $f_*: \pi_1(X) \to \mathbf{Z}$ *and* $\tau: \mathbf{Z} \to U(l)$, *then*

$$\gamma(X, \alpha) = \lambda(\mathscr{H}(X, f) \otimes \tau) - l \cdot \lambda(\mathscr{H}(X, f)).$$

(ii) *More generally, if* $\beta: \pi_1(X) \to U(r)$ *is a further representation, then with* $\mathscr{H} = \mathscr{H}((X, \beta), f)$

$$\gamma(X, \alpha \otimes \beta) - l \cdot \gamma(X, \beta) = \lambda(\mathscr{H} \otimes \tau) - l \cdot \lambda(\mathscr{H}).$$

Part (ii) of this theorem allows us to give an inductive computation of $\gamma(X, \alpha)$ if α is any unitary representation which factors over a free abelian group.

We have described the result for unitary representations $\alpha: \pi_1(X) \to U(l)$, but, in fact, the signature defect definition of $\gamma(X, \alpha)$ extends under suitable restrictions to indefinite hermitian representations $\alpha: \pi_1(X) \to U(l, m)$ and Theorem 1 holds in this more general situation. The precise statement is given in Theorem 3.2.

§§4–9 give the proof. §§4 and 5 are technical, giving an alternative definition of monodromy via "isometric relations". §6 collects some properties of Wall's non-additivity formula for signature[21] which are of interest in their own right. These are then used in §7 to give an initial computation of the γ-invariant in terms of monodromy in the bounding case. §8 completes the proof in the bounding case and finally in §9 the general case is deduced by a simple bordism argument.

In §8 we also prove the following result, which was, in fact, the original starting point of this research.

PROPOSITION 2. *If* $g: Y^{2n} \to S^1$ *is a fibration of the compact manifold-with-boundary* Y^{2n} *over* S^1, *then*

$$\operatorname{sign}(Y) = \lambda(\mathscr{H}(\partial Y, g|\partial y)).$$

This is true also with local coefficients (Proposition 8.5). Note that in this proposition $\mathscr{H} = \mathscr{H}(\partial Y, g|\partial y)$ is the usual middle dimensional monodromy of the fibration $\partial Y \to S^1$. The fact that this fibration bounds as a fibration implies that $\mathscr{H} = (H, b, t)$ is "null-bordant", that is, $\mathscr{H}$ has a "invariant kernel" $K \subset H$ with $K = K^\perp = tK$. For null-bordant $\mathscr{H}$ the invariant $\lambda(\mathscr{H})$ has a particularly simple description: $\lambda(\mathscr{H}) = -\operatorname{sign}(b'|H_1)$, where H_1 is the $(t-1)$-primary part of H and b' is the (maybe degenerate) form $b'(x, y) = b((t - t^{-1})x, y)$.

Throughout the paper we use monodromy with coefficients in $\mathbf{C}$, but it can be defined with any coefficients, and an easy universal coefficients argument shows that $\mathscr{H}(X, f)$ is the hermitianization $\mathscr{H}^{\mathbf{Q}}(X, f) \otimes \mathbf{C}$ of the rational monodromy of (X, f).

In §10 we describe how our results extend to compute certain torsion invariants of X in terms of rational monodromy. The rational monodromy is in fact a very rich invariant—in an appendix we show that every skew-symmetric isometric structure over $\mathbf{Q}$ occurs as $\mathscr{H}^{\mathbf{Q}}(X^3, f)$ for a suitable 3-manifold X.

Finally, in §11 we prove the promised description of monodromy (over any field F of coefficients) as a homology linking form on the FJ-torsion of $H_{n-1}(\bar{X}; F)$, where $\bar{X}$ is the infinite cyclic cover of X classified by f. This section is of interest in its own right and is in a sense a generalization of a duality theorem of Milnor[10] which applied to infinite cyclic covers $\bar{X}$ with $H_*(\bar{X}; F)$ finitely generated over F. The linking form we obtain is equivalent to the linking form discussed by Blanchfield[3], in that it determines and is determined by the latter.

It is worth mentioning that the homotopy invariance of $\gamma(X, \alpha)$ when α factors over a free abelian group, though somewhat surprising, is not hard to prove, at least in the bounding case, if one does not want an explicit formula. Namely, it suffices to show that in this case $\gamma(X, \alpha)$ can be defined in the category PD of Poincaré complexes. In the bounding case this just depends on the multiplicativity formula for twisted signature: $\text{sign}(Y, \alpha) = l \,\text{sign}(Y)$, where Y^{2n} is a closed Poincaré complex and $\alpha: \pi_1(Y) \to U(l)$. This formula is certainly false in general, but if α factors through a free abelian group $\alpha: \pi_1(Y) \xrightarrow{f} \mathbf{Z}^s \xrightarrow{\tau} U(l)$, then it is true. Indeed, we need only check it for generators of $\Omega_{2n}^{PD}(BZ^s)\otimes \mathbf{Q}$, since $\text{sign}(Y, \tau f) - l\,\text{sign}(Y)$ is a bordism invariant of (Y, f), but by the results of Quinn [19] and Farrel and Hsiang [6] such generators may be taken as manifolds, and for manifolds we know the desired multiplicativity formula. This argument also applies to α which factor over free groups, for example.

§1. DEFINITION OF γ

To define the γ-invariant also for indefinite hermitian representations, we must first introduce some terminology.

In the following U, V, W will always denote finite dimensional complex hermitian vector spaces. The hermitian form, which is assumed non-degenerate, will be denoted h_U, h_V or h_W. If G is a discrete group and $\tau: G \to \text{Aut}(U)$ is a hermitian representation of G, we shall use the notations $\text{sign}(\tau)$ and $\text{sign}(U)$ interchangeable to mean $\text{sign}(h_U)$.

If $\tau: G \to \text{Aut}(U)$ is a hermitian representation, then we call (G, τ) a *good structure pair* if a certain characteristic class $\text{ch}(\tau) \in \tilde{H}^*(G; \mathbf{Q})$ vanishes (see [13], where $\text{ch}(\tau)$ is denoted $\tilde{\psi}_G(\tau)$). $\text{ch}(\tau)$ is defined as follows: τ classifies a flat hermitian bundle $E \to BG$ which can be split as the sum $E = E^+ \oplus E^-$ of a positive definite and a negative definite bundle (no longer flat in general) and we put $\text{ch}(\tau) = \text{ch}(E^+) - \text{ch}(E^-)$, considered as an element of $\tilde{H}^*(BG; \mathbf{Q}) = \tilde{H}^*(G; \mathbf{Q})$.

It is shown in [13] that (G, τ) is good if μ is definite, and for arbitrary τ so long as G belongs to a large class $\mathscr{C}$ of groups defined in [13] which includes all finite groups, all abelian groups, and is closed under cartesian product free product direct limits finite extensions, and quotienting by finite normal subgroups, among other things. Furthermore, it follows easily from the definition that, if (G, τ) and (H, μ) are good, then so is $(G \times H, \tau \otimes \mu)$.

Now suppose we have a closed manifold X^{2n-1} and a hermitian representation $\alpha: \pi_1(X) \to \text{Aut}(U)$. Suppose further that some multiple $q(X, \alpha)$ bounds a $(Y^{2n}, \bar{\alpha})$ say (by this we mean, of course, that $\partial Y^{2n} = qX$ and that for each component X of ∂Y the composition $\pi_1(X) \to \pi_1(Y) \to \text{Aut}(U)$ equals α). Let $\Gamma \to Y$ be the local coefficient system classified by $\bar{\alpha}$. Then cup product, the hermitian form $\Gamma \otimes_{\mathbf{R}} \Gamma \to \mathbf{C}$ and evaluation on the fundamental class $[Y, \partial Y]$, together define a form

$$b_{Y,\Gamma}: H^n(Y, \partial Y; \Gamma) \otimes H^n(Y, \partial Y; \Gamma) \to H^{2n}(Y, \partial Y; \Gamma \otimes \Gamma) \to H^{2n}(Y, \partial Y; \mathbf{C}) \to \mathbf{C},$$

which is hermitian or skew hermitian according as n is even or odd. Denote by $\text{sign}(Y, \Gamma)$ or $\text{sign}(Y, \bar{\alpha})$ the signature of this form, where, if $b_{Y,\Gamma}$ is skew hermitian we mean signature of the hermitian form $ib_{Y,\Gamma}$.

The invariant we wish to define is

$$\gamma(X, \alpha) = \frac{1}{q}(\text{sign}(Y, \bar{\alpha}) - \text{sign}(\bar{\alpha})\,\text{sign}(Y))$$

If h_U is indefinite we need the following condition to ensure that $\gamma(X, \alpha)$ is well defined (independent of the choice of Y and $\bar{\alpha}$), see [13].

ASSUMPTION. *We assume that α factors through some good structure pair (G, τ); that is $\alpha = \tau \circ g$ for some $g: \pi_1(X) \to G$. Further, we assume that $\bar{\alpha}$ admits a similar factorization $\bar{\alpha} = \tau \circ \bar{g}$ extending the factorization of α. In particular, some multiple of (X, g) must bound for $\gamma(X, \alpha)$ to be defined.*

In general $\gamma(X, \alpha)$ will depend on the choice of factorization of α through a good structure pair (G, τ), but if we restrict α to be definite or G to be finite or abelian (or more generally G to be a central extension of a finite group) then $\gamma(X, \alpha)$ only depends on (X, α) ([13] Theorem 7.2). Since this includes the cases of most interest to us here, we will continue to suppress the factorization of α from our notation for the γ-invariant.

More generally, in the above situation suppose $\beta: \pi_1(X) \to \text{Aut}(V)$ is a further hermitian representation and suppose β also extends to a representation $\bar{\beta}: \pi_1(Y) \to \text{Aut}(V)$. Then we can define

$$\gamma((X, \beta), \alpha) = \frac{1}{q}(\text{sign}(Y, \bar{\alpha} \otimes \bar{\beta}) - \text{sign}(\bar{\alpha}) \text{sign}(Y, \bar{\beta}))$$

and this is well defined so long as α satisfies the assumption above; no condition on β. The proof is the same as for the previously mentioned special case, proved in [13].

LEMMA 1.1. *If β also factors through a good structure pair (H, μ) as $\beta = \mu \circ h$ say, then*

$$\gamma((X, \beta), \alpha) = \gamma(X, \alpha \otimes \beta) - \text{sign}(\alpha)\gamma(X, \beta)$$

if the right side is defined (that is, if some multiple of $(X, (g, h)\colon \pi_1(X) \to G \times H)$ bounds, a multiple of (X, h) then bounds, too).

Proof. The right side of the equation, if defined, is well defined, since $(G \times H, \tau \otimes \mu)$ is good. By definition the equations says

$$\text{sign}(Y, \bar{\alpha} \otimes \bar{\beta}) - \text{sign}(\bar{\alpha}) \text{sign}(Y, \bar{\beta}) = (\text{sign}(Y, \bar{\alpha} \otimes \bar{\beta}) - \text{sign}(\bar{\alpha} \otimes \bar{\beta}) \text{sign } Y)$$
$$- \text{sign } \bar{\alpha}(\text{sign}(Y, \bar{\beta}) - \text{sign}(\bar{\beta}) \text{sign } Y),$$

which holds, since $\text{sign}(\bar{\alpha} \otimes \bar{\beta}) = \text{sign}(\bar{\alpha}) \text{sign}(\bar{\beta})$.

Now suppose it is not necessarily true that a multiple of (X, α) bounds, but that $\alpha: \pi_1(X) \to \text{Aut}(U)$ is a unitary representation, i.e. the form h_U is positive definite. Then $\gamma(X, \alpha)$ can be defined up to sign as the invariant $\rho_\alpha(X)$ of Atiyah, Patodi and Singer[2]. We choose our sign conventions so that $\gamma(X, \alpha)$ agrees with the previous definition if (X, α) bounds; this agrees with the sign conventions of [13], [14], [7] and via Theorem 3 of [14], relating γ- and α-invariants, it agrees with the usual sign conventions for the α-invariant. For our purposes, all we shall need about this invariant are the following properties.

THEOREM 1.2. (i) *If $\alpha_i: \pi_1(X_i) \to U(l)$ for $i = 1, 2$ then $\gamma(X_1 + X_2, \alpha_1 + \alpha_2) = \gamma(X_1, \alpha_1) + \gamma(X_2, \alpha_2)$, where $X_1 + X_2$ means disjoint union and $\alpha_1 + \alpha_2$ is the representation $\pi_1(X_1 + X_2) = \pi_1(X_1) * \pi_1(X_2) \to U(l)$ induced by α_1 on $\pi_1(X_1)$ and α_2 on $\pi_1(X_2)$.*

(ii) *$\gamma(-X, \alpha) = -\gamma(X, \alpha)$, where $-X$ is X with reversed orientation.*

(iii) *If a multiple of (X, α) bounds then $\gamma(X, \alpha)$ agrees with the previous definition as a signature defect.*

(iv) *$\gamma(X, \alpha_1 \oplus \alpha_2) = \gamma(X, \alpha_1) + \gamma(X, \alpha_2)$ for any two unitary representations α_1 and α_2 of $\pi_1(X)$.*

(v) *If N^{2m} is closed and $\delta: \pi_1(N) \to U(r)$ is a unitary representation then $\gamma(N^{2m} \times X^{2n-1}, \delta \otimes \alpha) = (-1)^{nm}. \text{sign}(N, \delta) \cdot \gamma(X, \alpha)$.*

(vi) *If $\tau: \pi_1(S^1) = \mathbf{Z} \to U(1)$ is the representation $\tau(1) = e^{2\pi i a}$) with $0 \leq a \leq 1$, then*

$$\gamma(S^1, \tau) = 0 \qquad a = 0$$
$$= 1 - 2a, \qquad 0 < a < 1.$$

Proof. With the sign conventions of [2], properties (i), (ii) and (iv) are trivial, (v) is an easy computation from the definition in [2] and (iii) and (vi) are proved in [2]. Thus up to sign the theorem is correct. That the signs are correct can be seen by comparing with the signature defect definition (there seems to be a sign confusion in the derivation of (vi) in [2]; our sign convention gives $\gamma(S^1, \tau)$ the value $-\rho_\tau(S^1)$, where $\rho_\tau(S^1)$ is computed to conform with the equation at the top of p. 412 rather than with equation (2.11) on p. 411 of [2]. This leads to the value claimed here). Note that in (v), $\text{sign}(N, \delta) = r \cdot \text{sign}(N)$, for instance by [9] or [13].

Remark. For unitary representations for which $\text{Im}(\alpha)$ is a central extension of a finite group (e.g. $\text{Im}(\alpha)$ finite or abelian) it is not hard to show that the above properties determine the γ-invariant uniquely (in fact (vi) can be weakened to: $\gamma(S^1, \tau)$ is continuous on some open set of $\tau \in \text{Hom}(\mathbf{Z}, U(1))$). One can also show by direct topological arguments that for such α an invariant exists satisfying the above properties, but this is rather harder. This was the method used in the original version of this work to define $\gamma(X, \alpha)$.

More generally, if α and β are two unitary representations of $\pi_1(X)$, we define $\gamma((X, \beta), \alpha)$ by the equation of Lemma 1.1: $\gamma((X, \beta), \alpha) = \gamma(X, \alpha \otimes \beta) - \text{sign}(\alpha) \cdot \gamma(X, \beta)$.

§2. MONODROMY

Suppose we have a closed manifold X^{2n-1} and a homomorphism $f_*: \pi_1(X) \to \mathbf{Z}$. Since S^1 is a $K(\mathbf{Z}, 1)$ we can represent f_* by a unique map $f: X \to S^1$ up to homotopy. We may assume f is smooth.

Let $\bar{X} \to X$ be the infinite cyclic covering classified by f_*. Equivalently $\bar{X}$ is the pullback

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \mathbf{R} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^1. \end{array}$$

If $p \in S^1$ is a regular value of f and $N = f^{-1}(p)$, then $\bar{X}$ can be constructed by cutting X open along N and pasting infinitely many copies of the resulting manifold with boundary together end to end (see §5).

Let $\hat{f} \in H_{2n-2}(\bar{X})$ be the homology class represented by one copy of N in $\bar{X}$. Equivalently $\hat{f}$ is the image of $1 \in \mathbf{Z}$ in the composition $\mathbf{Z} = H_c^1(\mathbf{R}) \to H_c^1(\bar{X}) \cong H_{2n-2}(\bar{X})$ induced by the proper map $\bar{f}: \bar{X} \to \mathbf{R}$ and Poincaré duality. Thus $\hat{f}$ only depends on the homotopy class of f, since a homotopy of f induces a proper homotopy of $\bar{f}$.

Define a hermitian or skew hermitian form

$$b_0: H^{n-1}(\bar{X}; \mathbf{C}) \times H^{n-1}(\bar{X}; \mathbf{C}) \to \mathbf{C},\ b_0(x, y) = \langle x \cup y, \hat{f} \rangle,$$

where we are using hermitian cup product $x \cup y = (-1)^{|x||y|}\overline{y \cup x}$. This form is degenerate in general, but it induces a non-degenerate form b on

$$H = H^{n-1}(\bar{X}; \mathbf{C})/\text{Rad}(b_0),$$

where

$$\text{Rad}(b_0) = \{x \in H^{n-1}(\bar{X}; \mathbf{C}) | b_0(x, y) = 0 \quad \text{for all } y\}.$$

LEMMA 2.1. *(H, b) is a finite dimensional vector space with $(-1)^{n-1}$-hermitian form. The covering transformation $T: \bar{X} \to \bar{X}$ induces an isometry $t: H \to H$.*

Definition. The $(-1)^{n-1}$-hermitian isometric structure $\mathcal{H}(X, f) = (H, b, t)$ will be called the (middle dimensional) *monodromy* of (X, f) over $\mathbf{C}$.

Proof of Lemma. The finite dimensionality of H is all that needs a proof. The inclusion $N \subset \bar{X}$ induces a map $i^*: H^{n-1}(\bar{X}) \to H^{n-1}(N)$ which is an isometry with respect to the form b_0 on $H^{n-1}(\bar{X})$ and the cup product form b_N on $H^{n-1}(N)$, since $\langle x \cup y, \hat{f} \rangle = \langle x \cup y, i_*[N] \rangle = \langle i^*x \cup i^*y, [N] \rangle$. Thus if we factor out degeneracy of these forms from each side of the map $i^*: H^{n-1}(\bar{X}) \to \text{Im}\, i^*$, the induced map $H = H^{n-1}(\bar{X})/\text{Rad}(b_0) \to \text{Im}\, i^*/\text{Rad}(b_N\, \text{Im}\, i^*)$, being a surjective isometry of non-degenerate (± 1)-hermitian spaces, is an isomorphism. Since $\text{Im}\, i^*$ is finite dimensional, this proves the lemma.

If $\Lambda \to X^{2n-1}$ is a hermitian coefficient system classified by a representation $\beta: \pi_1(X) \to \mathrm{Aut}(V)$ say, then Λ lifts to a coefficient system $\bar{\Lambda} \to \bar{X}$ and we can repeat the definition of $\mathcal{H}(X, f)$ using coefficients in $\bar{\Lambda}$ to define $\mathcal{H}((X, \Lambda), f)$. We also denote this $\mathcal{H}((X, \beta), f)$.

PROPOSITION 2.2. *If $\alpha: \pi_1(X) \to \mathrm{Aut}(U)$ factors as $\alpha = \tau \circ f_*$ with $f_*: \pi_1(X) \to \mathbf{Z}$ and $\tau: \mathbf{Z} \to \mathrm{Aut}(U)$, then $\mathcal{H}((X, \alpha), f) = \mathcal{H}(X, f) \otimes \tau$. More generally, if $\beta: \pi_1(X) \to \mathrm{Aut}(V)$ is a further hermitian representation, then*

$$\mathcal{H}((X, \alpha \otimes \beta), f) = \mathcal{H}((X, \beta), f) \otimes \tau.$$

Proof. Recall that $\mathcal{H} \otimes \tau$ means $\mathcal{H} \otimes (U, h_{\bar{U}}, \tau(1))$. If $\Gamma \to X$ is the coefficient system classified by α then Γ is the pullback under $f: X \to S^1$ of the coefficient system over S^1 classified by τ. Hence $\bar{\Gamma} \to \bar{X}$ is the pullback of a coefficient system over $\mathbf{R}$ and is thus trivial. Hence $H^{n-1}(\bar{X}, \bar{\Gamma} \otimes \bar{\Lambda}) = H^{n-1}(\bar{X}, \bar{\Lambda}) \otimes U$ and the proposition now follows directly from the definition of $\mathcal{H}((X, \alpha \otimes \beta), f)$.

We close this section with a brief digression. We can think of $f \in [X, S^1] = H^1(X, \mathbf{Z})$ as a cohomology class and form the first higher Novikov signature

$$\mathrm{sign}\,(f) = \langle L_{2n-2}(X) \cup f, [X] \rangle,$$

where $L_*(X)$ is the Hirzebruch L-class. If $\mathcal{H} = (H, b, t)$ is an isometric structure denote $\mathrm{sign}\,(\mathcal{H}) = \mathrm{sign}\,(b)$.

PROPOSITION 2.3. Sign $(\mathcal{H}(X, f)) = \mathrm{sign}\,(f) = \mathrm{sign}\,(N)$, *where $N^{2n-2} \subset X^{2n-1}$ is any submanifold dual to f, as above.*

Proof. The first equality is precisely how Novikov[17] proved the homotopy invariance of sign (f). It can be seen as follows (which is essentially Novikov's proof). It will follow from our discussion that if a multiple of (X, f) bounds then $\mathrm{sign}\,(\mathcal{H}(X, f)) = 0$. Thus $\mathrm{sign}\,(\mathcal{H}(X, f))$ is a bordism invariant of (X, f). So is sign (f), so one must only compare the values of these two invariants on generators of $\Omega_{2n-1}(S^1)$, which is a trivial calculation. The second equality follows the same way, or alternatively directly from the Hirzebruch index theorem.

§3. $\lambda(\mathcal{H})$ AND THE MAIN THEOREM

To any (± 1)-hermitian isometric structure $\mathcal{H} = (H, b, t)$ we shall define an invariant $\lambda(\mathcal{H}) \in \mathbf{R}$ with the following properties

$$\lambda(-\mathcal{H}) = -\lambda(\mathcal{H}), \quad \text{where} \quad -\mathcal{H} = (H, -b, t),$$

$$\lambda(\mathcal{H}_1 \oplus \mathcal{H}_2) = \lambda(\mathcal{H}_1) + \lambda(\mathcal{H}_2).$$

LEMMA 3.1. *For $z \in \mathbf{C}$ let $H_z \subset H$ be the $(t - z)$-primary part of H, that is $H_z = \{x \in H \mid (t - z)^k x = 0$ for some $k\}$. Then $H_z \perp H_w$ if $z \neq \bar{w}^{-1}$. In particular $H_z \perp H_w$ if $|z| = 1$ and $z \neq w$.*

Proof. Since $(t^{-1} - \bar{z}) | H_w$ is an isomorphism if $z \neq \bar{w}^{-1}$, this follows from the equation $0 = \beta((t - z)^k x, y) = b(x, (t^{-1} - \bar{z})^k y)$ for $x \in H_z$ and $y \in H_w$.

Since H splits as the direct sum of the H_z, the above lemma shows that $\mathcal{H}$ decomposes as the orthogonal sum

$$\mathcal{H} = \mathcal{H}_{(0)} \oplus \bigotimes_{|z|=1} \mathcal{H}_z,$$

where $\mathcal{H}_z = (H_z, b|H_z, t|H_z)$ for $|z| = 1$ is the part of $\mathcal{H}$ belonging to the eigenvalue z and $\mathcal{H}_{(0)}$ has no eigenvalues on the unit circle.

Definition. Let $\mathcal{H} = (H, b, t)$ be ϵ-hermitian ($\epsilon = \pm 1$) and define

$$\lambda(\mathcal{H}) = \sum_{|z|=1} \lambda(\mathcal{H}_z),$$

where

$$\lambda(\mathcal{H}_z) = \epsilon \cdot \operatorname{sign}(\mathcal{H}_z) \cdot (1-2a), \quad z = e^{2\pi i a}, \quad 0 < a < 1,$$
$$= -\operatorname{sign}(b'|H_1), \qquad z = 1,$$

where b' is the (usually degenerate) $(-\epsilon)$-hermitian form $b'(x, y) = b((t - t^{-1})x, y)$. (Recall that we define signature of a skew hermitian form h to be signature of the hermitian form ih.)

With the notation of §1, we can now state our main theorem, generalizing Theorem 1 of the introduction.

THEOREM 3.2. *Let X^{2n-1} be closed and f: $X^{2n-1} \to S^1$ be given. Suppose also we have two hermitian representations τ: $\mathbf{Z} \to \operatorname{Aut}(U)$ and β: $\pi_1(X) \to \operatorname{Aut}(V)$ and let $\alpha = \tau \circ f_\#$: $\pi_1(X) \to \operatorname{Aut}(U)$. Then with $\mathcal{H} = \mathcal{H}((X, \beta), f)$,*

$$\gamma((X, \beta), \alpha) = \lambda(\mathcal{H} \otimes \tau) - \operatorname{sign}(U) \cdot \lambda(\mathcal{H}),$$

if the left side of this equation is defined.

In particular, if β: $\pi_1(X) \to U(1)$ is the trivial representation, this becomes the equation $\gamma(X, \alpha) = \lambda(\mathcal{H}(X, f) \otimes \tau) - \operatorname{sign}(\tau) \cdot \lambda(\mathcal{H}(X, f))$, which can be considered to be an extension of the definition of $\gamma(X, \alpha)$ to the non-unitary non-bounding case for α which factor over a cyclic group (if α factors over a finite cyclic group, $\gamma(X, \alpha)$ is defined, since a multiple of (X, α) bounds). It is plausible that the definition can be further extended to any α which factors over an abelian group in such a way that Theorem 3.2 remains true if one interprets $\gamma((X, \beta), \alpha)$ by the equation of Lemma 1.1.

Before we start on the proof of 3.2 we describe how this Theorem (or rather the special case: Theorem 1 of the introduction) permits calculation of $\gamma(X, \alpha)$ for any α: $\pi_1(X) \to U(l)$ which factors over a free abelian group.

THEOREM 3.3. *If α: $\pi_1(X) \to U(l)$ factors over $\mathbf{Z}^s$, say $\alpha = \tau \circ f_\#$ with $f_\#$: $\pi_1(X) \to \mathbf{Z}^s$ and τ: $\mathbf{Z}^s \to U(l)$, then $\gamma(X, \alpha)$ is a homotopy invariant, computable via monodromy of X.*

Proof. It suffices to prove this if τ is irreducible, since a unitary representation is a sum of irreducibles and $\gamma(X, \alpha_1 \oplus \alpha_2) = \gamma(X, \alpha_1) + \gamma(X, \alpha_2)$. But any irreducible unitary representation τ of $\mathbf{Z}^s$ is an exterior tensor product $\tau = \tau_1 \otimes \cdots \otimes \tau_s$, where τ_i: $\mathbf{Z} \to U(1)$ is an irreducible representation of the ith factor $\mathbf{Z}$ of $\mathbf{Z}^s$. Put $\alpha_i = \tau_i \circ f_\#$, so $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_s$. Then

$$\gamma(X, \alpha_1 \otimes \cdots \otimes \alpha_s) = \sum_{i=1}^{s} (\gamma(X, \alpha_1 \otimes \cdots \otimes \alpha_i) - \gamma(X, \alpha_1 \otimes \cdots \otimes \alpha_{i-1}))$$

$$= \sum_{i=1}^{s} \gamma((X, \alpha_1 \otimes \cdots \otimes \alpha_{i-1}), \alpha_i).$$

Since each term on the right is computable via monodromy by Theorem 3.2 (or Theorem 1), we are done.

An alternative approach to the above theorem might be to use the fact that representations τ: $\mathbf{Z}^s \to U(l)$ which factor through $\mathbf{Z}$ are dense in all such representations and use a suitable continuity property of the invariant $\gamma(X, \alpha)$. Such an approach would allow one to deal also with the indefinite case. However $\gamma(X, \alpha)$ does not have very nice continuity properties—the best we know is the following theorem, which allows one to use this approach on an open dense set of representations.

THEOREM 3.4. *If $f_\#$: $\pi_1(X) \to \mathbf{Z}^s$ is given, such that a multiple of $(X, f_\#)$ bounds, then $\gamma(X, \tau \circ f_\#)$ is locally constant on an open dense set of $\tau \in \operatorname{Hom}(\mathbf{Z}^s, \operatorname{Aut}(U))$.*

In fact $\mathbf{Z}^s$ can be replaced in this theorem by any group in the class $\mathscr{C}$ mentioned in §1. We just sketch the proof. Assume for convenience that $(X, f_\#)$ itself, rather than just a multiple of $(X, f_\#)$, bounds $(Y^{2n}, g_\#)$. Then $\gamma(X, \tau \circ f_\#) = \operatorname{sign}(Y,$

$\tau\circ g_*) - \operatorname{sign}(\tau)\operatorname{sign}(Y)$ is the signature of a hermitian form which varies algebraically with $\tau \in \operatorname{Hom}(Z^t, \operatorname{Aut}(U))$, by first triangulating Y and then using Ranicki and Sullivan's "semi-local intersection matrix"[20] (which generalizes to manifolds with boundary and local coefficients) to compute the relevant signatures. The theorem follows from this.

Note that in the non-bounding case Theorem 3.4 fails already for $X = S^1$, by Theorem (1.2)(vi).

§4. ISOMETRIC RELATIONS ON HERMITIAN SPACES

This section contains technical results on relations on hermitian vector spaces which we will use to give an alternate description of the monodromy $\mathcal{H}(X^{2n-1}, f)$. We are interested in *additive relations* between two vector spaces V and W, that is subspaces $R \subset V \times W$. In analogy with composition of functions we define the composition of relations $R_1 \subset V \times W$ and $R_2 \subset U \times V$ in the usual backward way as

$$R_1 \circ R_2 = \{(x, z) \in U \times W \mid \exists\, y \in V \quad \text{with} \quad (x, y) \in R_2 \quad \text{and} \quad (y, z) \in R_1\}.$$

For $R \subset V \times W$ we also make the usual definitions ($A \subset V$):

$$R^{-1} = \{(y, x) \in W \times V \mid (x, y) \in R\},$$

$$RA = \{(y \in W \mid \exists\, x \in A \quad \text{with} \quad (x, y) \in R\}.$$

The following lemma lets one interpret an additive relation $R \subset V \times W$ as the graph of an isomorphism from a subquotient of V to a subquotient of W.

LEMMA 4.1. *$R \subset V \times W$ is an additive relation if and only if there exist subspaces $A \subset A' \subset V$ and $B \subset B' \subset W$ and an isomorphism ϕ: $A'/A \xrightarrow{\cong} B'/B$ such that*

$$R = \{(x, y) \subset A' \times B' \mid \phi[x] = [y]\}.$$

In particular $A = R^{-1}\{0\}$, $A' = R^{-1}W$, $B = R\{0\}$, $B' = RV$; these are called respectively the kernel, domain, indeterminacy and image of R.

Proof. If R is an additive relation and we define A, A', B, B' as in the lemma then R can clearly be interpreted as the graph of a linear map $A \to B/B'$ with kernel A'. The converse is trivial.

Definition. If (W, b) is a finite dimensional non-degenerate (± 1)-hermitian space then an *isometric relation* on W will mean an additive relation $R \subset W \times (-W)$ satisfying $R = R^\perp$ (equivalently $R \subset R^\perp$ and $\dim R = \dim W$). Here $-W$ is W with hermitian form $-b$.

For example the graph $R(f) = \{(x, f(x)) \mid x \in W\}$ of a linear map f: $W \to W$ is an isometric relation if and only if f is an isometry. More generally

LEMMA 4.2. *$R \subset W \times (-W)$ is an isometric relation if and only if in Lemma 4.1 we have $A' = A^\perp$, $B' = B^\perp$ and ϕ is an isometry $A^\perp/A \xrightarrow{\cong} B^\perp/B$ (since $A = \operatorname{Rad}(b|A^\perp)$, b induces a non-degenerate form on $A^\perp/A$ and similarly on $B^\perp/B$).*

Proof. If A, B and ϕ: $A^\perp/A \to B^\perp/B$ are as in the lemma and $R = \{(x, y) \in A^\perp \times B^\perp \mid \phi[x] = [y]\}$, then a trivial calculations shows $R \subset R^\perp$ and $\dim R = \dim W$, so R is an isometric relation. Conversely if R is an isometric relation then orthogonal complement of the equation $A \times \{0\} = (W \times \{0\}) \cap R$ gives $A^\perp \times W = (W + \{0\})^\perp + R^\perp = (\{0\} \times W) + R = A' \times W$, whence $A^\perp = A'$. Similarly $B^\perp = B'$ and an easy calculation shows that ϕ is an isometry.

LEMMA 4.3. *If R_1 and R_2 are isometric relations on W then so is $R_1 \circ R_2$.*

Proof. Certainly $R_1 \circ R_2 \subset (R_1 \circ R_2)^\perp$, so we must show that $\dim(R_1 \circ R_2) = \dim W$. Let $K = \{(x, y, z) \in W \times W \times W \mid (x, y) \in R_2 \text{ and } (y, z) \in R_1\}$. There are two short exact sequences

$$0 \to R_1^{-1}\{0\} \cap R_2\{0\} \xrightarrow{\alpha} K \xrightarrow{\beta} R_1 \circ R_2 \to 0,$$

given by $\alpha(y) = (0, y, 0)$ and $\beta(x, y, z) = (x, z)$ and

$$0 \to K \xrightarrow{\gamma} R_2 \times R_1 \xrightarrow{\delta} R_1^{-1}W + R_2W \to 0,$$

with maps $\gamma(x, y, z) = ((x, y), (y, z))$ and $\delta((x, y), (w, z)) = y - w$. These give dimension equations which combine to give

$$\dim(R_1 \circ R_2) = 2 \dim W - \dim(R_1^{-1}\{0\} \cap R_2\{0\}) - \dim(R_1^{-1}W + R_2W).$$

But $(R_1^{-1}W + R_2W)^\perp = (R_1^{-1})^\perp \cap (R_2W)^\perp = R_1^{-1}\{0\} \cap R_2\{0\}$ by Lemma 4.2, so the dimension equation becomes $\dim(R_1 \circ R_2) = 2 \dim W - \dim W = \dim W$, as required.

Remark. Lemmas 4.2 and 4.3 hold more generally for relations between two hermitian spaces, $R \subset V \times (-W)$, with $R = R^\perp$. We shall not need this and in any case the proofs are the same. Similarly everything holds equally well for non-degenerate finite-dimensional bilinear or sesquilinear spaces over any field.

The only application we shall make of Lemma 4.3 is the following. Let $R^\infty\{0\}$ and $R^\infty W$ be the limits of the sequences

$$\{0\} \subseteq R\{0\} \subseteq R^2\{0\} \subseteq \cdots \subseteq R^j\{0\} \subseteq \cdots,$$

$$W \supseteq RW \supseteq R^2W \supseteq \cdots \supseteq R^jW \supseteq \cdots.$$

Since W is finite dimensional these sequences stabilize after a certain time, that is $R^j\{0\} = R^\infty\{0\}$ and $R^jW = R^\infty W$ for j sufficiently large.

LEMMA 4.4. *If R is an isometric relation then $R^j\{0\} = (R^jW)^\perp$ and $R^{-j}\{0\} = (R^{-j}W)^\perp$ for $j = 0, 1, 2, \ldots, \infty$.*

Proof. This is immediate by 4.2, since R^j and R^{-j} are isometric relations by 4.3.

Taking the graph of an isometry allows one to consider an isometric structure as an isometric relation. We now describe conversely how to derive an isometric structure from an arbitrary isometric relation.

LEMMA AND DEFINITION 4.5. *Let R be an isometric relation and put $B = R^\infty\{0\}$, so $B^\perp = R^\infty W$ and there is an induced non-degenerate form (also denoted by b) on $H = B^\perp/B$. Then*

$$S = [(B \times B + R) \cap (B^\perp \times B^\perp)]/B \times B \subset H \times (-H)$$

is the graph of an isometry $t: H \to H$. We denote the isometric structure (H, b, t) by $\mathcal{H}(R)$.

Proof. Note that the modular law

$$X \subset Z \Rightarrow (X + Y) \cap Z = X + (Y \cap Z),$$

holds for subspaces of a vector space, so in such a situation we can and will omit parentheses. In particular for subspaces of a hermitian space it follows that

$$X \subset X^\perp \Rightarrow (X + Y \cap X^\perp)^\perp = X + Y^\perp \cap X^\perp.$$

Applying this to

$$S_0 = (B \times B) + R \cap (B^\perp \times B^\perp),$$

considered as a subspace of $W \times (-W)$, shows $S_0 = S_0^\perp$, whence also $S = S^\perp$, so S is an isometric relation.

Next we observe that if $(x, y) \in S_0$ then certainly $(x, y) \in (B \times B) + R$, so if $x \in B$ then $y \in RB$. But $RB = RR^\infty\{0\} = R^\infty\{0\} = B$, so we have shown $S_0B \subset B$. Hence $S\{0\} = \{0\}$. Thus by Lemma 4.2, $SH = \{0\}^\perp = H$ and S is the graph of an isometry from a subquotient of H onto the whole of H. For dimensional reasons it follows that S is the graph of an isometry $t: H \to H$.

The next lemma, which gives a more symmetric description of $\mathcal{H}(R)$, will be needed to determine the topological meaning of this isometric structure.

LEMMA 4.6. *Let R be an isometric relation on (W, b) and put $A = R^{-\infty}\{0\}$ and $B = R^{\infty}\{0\}$ and $C = A^{\perp} \cap B^{\perp}$ and $D = \operatorname{Rad}(b|C) = C \cap C^{\perp}$. Put $H_1 = C/D$ and $S_1 = [(D \times D) + R \cap (C \times C)]/D \times D \subset H_1 \times (-H_1)$. Then the pair (H_1, S_1) is isomorphic to the pair (H, S) of Lemma 4.5 and thus also defines $\mathcal{H}(R)$.*

Proof. We first show that the inclusion $i: C \to B^{\perp}$ induces an epimorphism $C \xrightarrow{\pi i} B^{\perp}/B$, in other words that $C + B = B^{\perp}$. Choose q such that $R^q W = R^{\infty} W = B^{\perp}$ and $R^{-q} W = R^{-\infty} W = A^{\perp}$, whence also $R^q\{0\} = B$ and $R^{-q}\{0\} = A$. Given any $y \in B^{\perp} = R^{\infty} W$, we can find $x \in R^{\infty} W$ with $y \in R^q\{x\}$. Then $x \in R^{-q}\{y\} \subset R^{-q} W = R^{-\infty} W$, so we can find $z \in R^q\{x\}$ with $z \in R^{-\infty} W$. By construction $z - y \in R^q\{x - x\} = B$ and $z \in R^q\{x\} \cap R^{-\infty} W \subset B^{\perp} \cap A^{\perp} = C$, so $y \in B + C$. Thus we have shown $B^{\perp} \subset B + C$. The other inclusion is trivial.

Now the epimorphism $C \xrightarrow{\pi i} B^{\perp}/B$ preserves hermitian forms, and since $B^{\perp}/B$ is non-degenerate, πi induces an isometry

$$\phi: C/\operatorname{Rad}(b|C) \xrightarrow{\cong} B^{\perp}/B.$$

It remains to show that $(\phi \times \phi)(S_1) = S$. First note that if $x \in C$ then there exists $y \in C$ with $(x, y) \in R$. Hence for $[x] \in H_1$ there exists $[y] \in H_1$ with $([x], [y]) \in S_1$, so $\dim S_1 \geq \dim H_1 = \dim H$. On the other hand $\dim S = \dim H$, since S is the graph of a map $H \to H$, so $\dim S_1 \geq \dim S$. Thus it suffices to prove $(\phi \times \phi)(S_1) \subset S$, which is trivial.

§5. MONODROMY VIA ISOMETRIC RELATIONS

Let X^{2n-1} be a closed manifold and $f: X \to S^1$ a map, given up to homotopy. For some smooth representative t and some regular value $p \in S^1$ of this f put $N = f^{-1}(p)$ (equivalently $N^{2n-2} \subset X^{2n-1}$ is any submanifold representing the Poincaré dual of $f \in [X, S^1] = H^1(X; \mathbf{Z})$). Let X' be X cut open along N, so the boundary of X' consists of two copies N^+ and N^- of N.

The infinite cyclic cover $\bar{X}$ of X, used in the definition of monodromy, can be constructed by taking $\mathbf{Z}$ copies $\ldots, X'_{-1}, X'_0, X'_1, \ldots$ of X' and pasting them together by pasting N_i^+ to N_{i+1}^- for each $i \in \mathbf{Z}$ (Fig. 1). The covering transformation $T: \bar{X} \to \bar{X}$ is the map which moves the picture one step to the right.

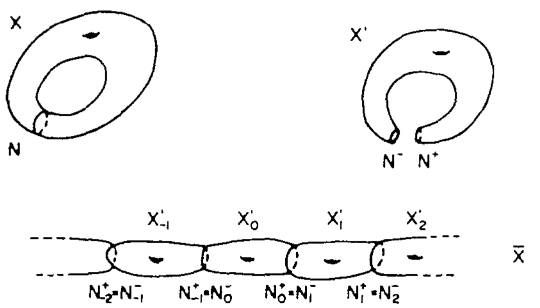

Fig. 1.

Now let $\Lambda \to X$ be any hermitian coefficient system over X. Then we have induced coefficient systems over X', N and $\bar{X}$, and in this section homology and cohomology are to be taken with coefficients in the corresponding local system, and cup product and intersection forms are the induced (± 1)-hermitian forms on these (co)-homology groups. To simplify notation, we will not write out the coefficients explicitly, and therefore also write $\mathcal{H}(X, f)$ for $\mathcal{H}((X, \Lambda), f)$ and so on.

As oriented manifolds $\partial X' = N^- + (-N^+) \cong N + (-N)$, where $-N$ is N with reversed orientation. Let $W = H_{n-1}(N)$ with (± 1)-hermitian intersection form. Let

$$R = \operatorname{Ker}(H_{n-1}(\partial X') \to H_{n-1}(X')) \subset H_{n-1}(\partial X') = W \oplus (-W).$$

The Poincaré duality diagram

$$\begin{array}{ccccc} H_n(X',\partial X') & \longrightarrow & H_{n-1}(\partial X') & \xrightarrow{i_*} & H_{n-1}(X') \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ (H_{n-1}(X'))^* & \xrightarrow{(i_*)^*} & (H_{n-1}(\partial X'))^* & \longrightarrow & (H_n(X',\partial X'))^*, \end{array}$$

where vertical arrows are given by intersection forms, shows that $R = \operatorname{Ker} i_* = R^\perp$, so R is an isometric relation on W.

PROPOSITION 5.1. *In the notation of Lemma* 4.5, *the monodromy* $\mathcal{H}(X, f)$ *satisfies* $\mathcal{H}(X, f) = \mathcal{H}(R)$.

Proof. Let $i: N \subset \bar{X}$ be the inclusion of one copy of N in $\bar{X}$, and $i^*: H^{n-1}(\bar{X}) \to H^{n-1}(N)$ the induced map in cohomology. We need the following lemma.

LEMMA 5.2. *If* $\mathcal{H}(X, f) = (H, b, t)$, *then* $H \cong \operatorname{Im}(i^*)/\operatorname{Rad}(b_N|\operatorname{Im}(i^*))$, *where* b_N *is the cup product form on* $H^{n-1}(N)$. *Under this isomorphism b is the form induced by* b_N *and t is the unique homomorphism such that*

$$\begin{array}{ccc} H^{n-1}(\bar{X}) & \xrightarrow{\pi i^*} & \operatorname{Im}(i^*)/\operatorname{Rad} \\ \downarrow T^* & & \downarrow t \\ H^{n-1}(\bar{X}) & \xrightarrow{\pi i^*} & \operatorname{Im}(i^*)/\operatorname{Rad} \end{array}$$

commutes, where $\operatorname{Rad} = \operatorname{Rad}(b_N|\operatorname{Im}(i^*))$, $\pi: \operatorname{Im}(i^*) \to \operatorname{Im}(i^*)/\operatorname{Rad}$ *is the projection, and* T^* *is the map in cohomology induced by the covering transformation.*

Proof. Except for the statement about t, this is precisely the proof of Lemma 2.1. The commutative square involving t is by definition; it defines t uniquely by surjectivity of the horizontal arrows.

We shall translate this lemma by Poincaré duality into homology, which then easily gives the proposition.

Let $H_*^{cl}(\bar{X})$ denote homology with closed supports defined by allowing infinite but locally finite singular chains. Alternatively we can use Borel–Moore homology[4], which agrees with the above for "good" spaces (e.g. manifolds) by Olk[18]. Closed but not necessarily compact submanifolds of $\bar{X}$ represent cycles in such a theory. There is a Poincaré duality isomorphism

$$\cap[\bar{X}]: H^q(\bar{X}) \to H_{2n-1-q}^{cl}(\bar{X}).$$

The map $i^*: H^{n-1}(N) \to H^{n-1}(\bar{X})$ is Poincaré dual to the map

$$\delta: H_n^{cl}(\bar{X}) \to H_{n-1}(N)$$

defined by intersecting cycles in $\bar{X}$ with N. The dualized version of Lemma 5.2 is as follows.

LEMMA 5.3. $\mathcal{H}(X, f) \cong (\operatorname{Im}\delta/\operatorname{Rad}(b'_N|\operatorname{Im}\delta),\ b'_N,\ t')$, *where* b'_N *denotes both the homology intersection form on* $H_{n-1}(N)$ *and the induced form on* $\operatorname{Im}\delta/\operatorname{Rad}(b'_N|\operatorname{Im}\delta)$ *and* t' *is defined as follows*: $t'(\xi) = \eta$ *if and only if* ξ *and* η *can be represented by cycles x and y in N such that cycles* $i_\# x$ *and* $T_\# i_\# y$ *are homologous in the portion of* $\bar{X}$ *lying between N and TN.*

Proof. Except for the characterization of t', this is just the dualized version of Lemma 5.2. To prove the statement about t' first note that the equation $T^*a \cap [\bar{X}] = T_*^{-1}(a \cap T_*[\bar{X}]) = T_*^{-1}(a \cap [\bar{X}])$ shows that T^* is Poincaré dual to $T_*^{-1}: H_*^{cl}(\bar{X}) \to H_*^{cl}(\bar{X})$. Thus by Lemma 5.2 the map t' corresponding to t is characterized by commutativity of the diagram

$$\begin{array}{ccc} H_n^{cl}(\bar{X}) & \xrightarrow{\pi\delta} & \operatorname{Im}\delta/\operatorname{Rad}(b'_N|\operatorname{Im}\delta) \\ T_*^{-1}\downarrow & & \downarrow t' \\ H_n^{cl}(\bar{X}) & \xrightarrow{\pi\delta} & \operatorname{Im}\delta/\operatorname{Rad}(b'_N|\operatorname{Im}\delta). \end{array}$$

where $\pi: \operatorname{Im}\delta \to \operatorname{Im}\delta/\operatorname{Rad}(b'_N|\operatorname{Im}\delta)$ is the projection. Now suppose $\xi, \eta \in \operatorname{Im}\delta/\operatorname{Rad}(b'_N|\operatorname{Im}\delta)$ can be represented by cycles x and y as described in the lemma. Let c_1 and $c_2 \in Z_n^{cl}(\bar{X})$ be cycles with closed support in $\bar{X}$ intersecting N in i_*x and i_*y (possible, since $[x], [y] \in \operatorname{Im}\delta$). Let c_1^- be the part of c_1 "to the left of N in $\bar{X}$" and c_2^+ be the part of c_2 "to the right of N in $\bar{X}$" and let d be a homology from i_*x to T_*i_*y. Then $c = c_1^- \cup d \cup T_*c_2^+$ is a cycle in $Z_n^{cl}(\bar{X})$ with $\delta[c] = [x]$ and $\delta T_*^{-1}[c] = [y]$, whence $t'(\xi) = t'\pi[x] = t'\pi\delta[c] = \pi\delta T_*^{-1}[c] = \pi[y] = \eta$, as was to be shown. Conversely, given ξ and η with $t'\xi = \eta$, one can find a cycle $c \in Z_n^{cl}(\bar{X})$ with $\pi\delta[c] = \xi$ and $\pi\delta T_*^{-1}[c] = \eta$ and one can further assume that c intersects $N \subset \bar{X}$ and $TN \subset \bar{X}$ in cycles i_*x and T_*i_*y. Then x and y represent ξ and η and i_*x is homologous to T_*i_*y by the portion of c lying between N and TN in $\bar{X}$. This completes the proof.

We can now complete the proof of Proposition 5.1. Using the picture of $\bar{X}$ of Fig. 1, observe that $\operatorname{Im}\delta$ is represented by those classes in $N = N_0^{\frac{1}{2}}$ say which bound infinitely far to the left and right in $\bar{X}$. Observe also that $(x, y) \in R$ if and only if x, as a class in the left boundary N^- of X', is homologous in X' to y considered as a class in the right boundary N^+ of X'. Thus $\operatorname{Im}\delta$ is simply $R^\infty W \cap R^{-\infty}W$, which is C in the notation of Lemma 4.6, and Proposition 5.1 follows directly from a comparison of Lemma 5.3 and Lemma 4.6.

§6. WALL NON-ADDITIVITY

We shall need some properties of an invariant introduced by Wall[21]. We work with complex hermitian spaces rather than bilinear spaces as in [21], but this makes no significant difference.

Let W be a non-degenerate ϵ-hermitian space, $\epsilon = \pm 1$ and let A_1, A_2, A_3 be three kernels, that is $A_i \subset W$ and $A_i = A_i^\perp$. On the subspace

$$A_1 \cap (A_2 + A_3) = \{x_1 \in A_1 | \exists x_2 \in A_2 \quad \text{and} \quad x_3 \in A_3 \quad \text{with} \quad x_1 + x_2 + x_3 = 0\}$$

define a sesquilinear form w by

$$w(x_1, y_1) = h_W(x_1, y_2), \quad y_1 + y_2 + y_3 = 0, \quad y_i \in A_i.$$

It is easily verified that w is well defined and $(-\epsilon)$-hermitian. Also $\operatorname{Rad}(w) = (A_1 \cap A_2) + (A_1 \cap A_3)$, so w induces a non-degenerate form, also denoted w, on $(A_1 \cap (A_2 + A_3))/((A_1 \cap A_2) + (A_1 \cap A_3))$.

Definition. $\operatorname{Sign}(W; A_1, A_2, A_3) = \operatorname{sign} w$. Recall that if w is skew hermitian this means $\operatorname{sign}(iw)$.

It is not hard to see that $(A_1 \cap (A_2 + A_3))/((A_1 \cap A_2) + (A_1 \cap A_3))$ is unaltered up to isometry by even permutations of A_1, A_2, A_3 while odd permutations reverse the sign of w (see Wall[21]). Thus $\operatorname{sign}(W; A_1, A_2, A_3)$ is an alternating function of its last three arguments.

The reason for introducing this invariant is the following theorem.

THEOREM 6.1 (Wall[21]). *Let the oriented manifold Y^{2n} be the union of two pieces Y_+ and Y_-, pasted along a common zero-codimensional submanifold M_0 of their boundaries. Put $M_+ = \partial Y_+ - \operatorname{Int}(M_0)$, $M_- = \partial Y_- - \operatorname{Int}(M_0)$ and let N be the common boundary $N = \partial M_- = \partial M_+ = \partial M_0$, oriented as boundary of M_+. Let $\Lambda \to Y$ be a hermitian local coefficient system and denote the restriction to any subspace of Y also by Λ. Put*

$$\begin{aligned} W &= H_{n-1}(N; \Lambda) \text{ with } (\pm 1)\text{-hermitian intersection form,}\\ A_1 &= \ker(H_{n-1}(N; \Lambda) \to H_{n-1}(M_-; \Lambda)),\\ A_2 &= \ker(H_{n-1}(N; \Lambda) \to H_{n-1}(M_0; \Lambda)),\\ A_3 &= \ker(H_{n-1}(N; \Lambda) \to H_{n-1}(M_+; \Lambda)). \end{aligned}$$

Then $A_i = A_i^\perp$ for each i and

$$\operatorname{sign}(Y, \Lambda) = \operatorname{sign}(Y_+, \Lambda) + \operatorname{sign}(Y_-, \Lambda) - \operatorname{sign}(W; A_1, A_2, A_3).$$

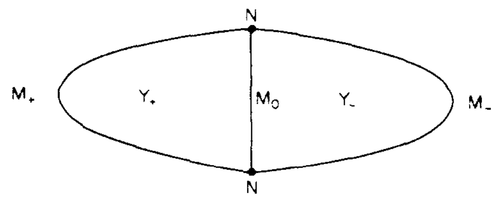

Fig. 1a.

Proof. Wall's proof for trivial local coefficients extends word for word to the present situation, see Meyer[9].

We shall need the following properties of Wall's invariant.

LEMMA 6.2(*i*). *If* $D \subset A_1 \cap A_2 \cap A_3$ *then*

$$\operatorname{sign}(W; A_1, A_2, A_3) = \operatorname{sign}(D^\perp/D; A_1/D, A_2/D, A_3/D).$$

(ii) *If* $D \subset A_1 \cap A_2$, *then*

$$\operatorname{sign}(W; A_1, A_2, A_3) = \operatorname{sign}(W; A_1, A_2, D + A_3 \cap D^\perp)$$

(*see the remark at the beginning of the proof of Lemma* 4.5).

Proof. (i). If $D \subset A_1 \cap A_2 \cap A_3$ then $A_i^\perp \subset D^\perp$, so $A_i \subset D^\perp$ for each i. The projection $D^\perp \to D^\perp/D$ restricts to an epimorphism

$$A_1 \cap (A_2 + A_3) \to A_1/D \cap (A_2/D + A_3/D),$$

which preserves Wall's form, so after factoring radicals on both sides it becomes an isometry, proving (i).

(ii). Since $D \subset A_i$ for $i = 1, 2$, $A_i \subset D^\perp$ for $i = 1, 2$, so $A_1 \cap (A_2 + A_3) = A_1 \cap (A_2 + A_3) \cap D^\perp = A_1 \cap (A_2 + (A_3 \cap D^\perp)) = A_1 \cap (A_2 + (D + A_3 \cap D^\perp))$. If $y_1 + y_2 + y_3 = 0$ with $y_i \in A_i$, then $y_1 + y_2 \in D^\perp$, so $y_3 \in D^\perp$, so $y_3 \in D + A_3 \cap D^\perp$. Thus Wall's form on the above group is the same whether defined using A_1, A_2, A_3 or $A_1, A_2, D + A_3 \cap D^\perp$, proving (ii).

Another property we shall need is the following pleasing "cocycle property".

PROPOSITION 6.3. *If* A_i, $i = 0, 1, 2, 3$, *are four kernels in* W *then*

$\operatorname{sign}(W; A_1, A_2, A_3) - \operatorname{sign}(W: A_0, A_2, A_3) + \operatorname{sign}(W; A_0, A_1, A_3) - \operatorname{sign}(W; A_0, A_1, A_2) = 0.$

Proof. Assume W is hermitian by multiplying h_W by i if necessary. Since W contains a kernel, it has zero signature, so $W \cong E \oplus (-E)$ where $E \cong \mathbf{C}^n$ with standard hermitian metric for some n. Any $A \subset E \oplus (-E)$ with $A = A^\perp$ is the graph $A = R(f)$ of an isometry $f \in \operatorname{Aut}(E) = U(n)$. Let $A_i = R(f_i)$, $i = 0, 1, 2, 3$. Consider four copies of the annulus $Y_i = \{x \in \mathbf{R}^2 | 1 \leq |x| \leq 2\}$, $i = 0, 1, 2, 3$. On Y_i consider the local coefficient system $\Lambda_i \to Y_i$ with fiber E defined by the following picture (Fig. 2, indices modulo 4).

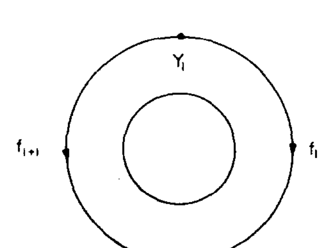

Fig. 2.

That is, Λ_i is classified by the representation $\pi_1(Y_i) = \mathbf{Z} \to \operatorname{Aut}(E) = U(n)$ which takes $1 \in \mathbf{Z}$ to $f_i^{-1} f_{i+1} \in U(n)$. Now we can paste these four annuli together as in Fig. 3 to get a four-punctured sphere Y with a coefficient system $\Lambda \to Y$. Denote Λ restricted to

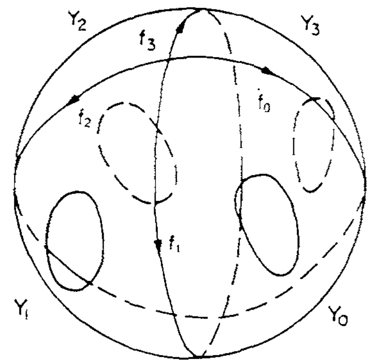

Fig. 3.

any subset of Y now also by Λ. Since Y is obtained by pasting $Y_0 \cup Y_1$ to $Y_2 \cup Y_3$ along a closed boundary component, Novikov additivity (or the case $N = \emptyset$ of Theorem 6.1) gives

$$\operatorname{sign}(Y, \Lambda) = \operatorname{sign}(Y_0 \cup Y_1, \Lambda) + \operatorname{sign}(Y_2 \cup Y_3, \Lambda).$$

Similarly

$$\operatorname{sign}(Y, \Lambda) = \operatorname{sign}(Y_1 \cup Y_2, \Lambda) + \operatorname{sign}(Y_3 \cup Y_0, \Lambda),$$

and subtracting these equations gives

$$\operatorname{sign}(Y_1 \cup Y_2, \Lambda) - \operatorname{sign}(Y_1 \cup Y_3, \Lambda) + \operatorname{sign}(Y_3 \cup Y_0, \Lambda) - \operatorname{sign}(Y_0 \cup Y_1, \Lambda) = 0.$$

On the other hand Theorem 6.1 gives (indices modulo 4)

$$\operatorname{sign}(Y_i \cup Y_{i+1}, \Lambda) = \operatorname{sign}(Y_i, \Lambda) + \operatorname{sign}(Y_{i+1}, \Lambda) - \operatorname{sign}(W; A_i, A_{i+1}, A_{i+2})$$

and inserting this into the previous equation proves the proposition.

The following consequence of the cocycle formula, involving a special case of Wall's invariant, is basic for later calculations.

Definition. If $\mathcal{H} = (H, b, t)$ is a (± 1)-hermitian isometric structure and $K \subset H$ satisfies $K = K^\perp$ (so $\operatorname{sign}(\mathcal{H}) = 0$), denote

$$\operatorname{sign}(\mathcal{H}, K) := \operatorname{sign}(H \oplus (-H), \Delta(H), K \oplus K, R(t)),$$

where $\Delta(H) = \{(x, x) \in H \oplus (-H)\}$ and $R(t) = \{(x, tx) \in H \oplus -H\}$ is the graph of t.

LEMMA 6.5. *Let $\mathcal{H} = (H, b, t)$ and $\mathcal{H}' = (H, b, t')$ be two isometric structures on the same hermitian space (H, b). Let $K_i \subset G$, $i = 1, 2$ satisfy $K_i = K_i^\perp$ and $t't^{-1}K_i = K_i$. Then*

$$\operatorname{sign}(\mathcal{H}, K_1) - \operatorname{sign}(\mathcal{H}', K_1) = \operatorname{sign}(\mathcal{H}, K_2) - \operatorname{sign}(\mathcal{H}', K_2).$$

Proof. The equation to be proved can be rewritten

$$\operatorname{sign}(\mathcal{H}, K_1) - \operatorname{sign}(\mathcal{H}, K_2) = \operatorname{sign}(\mathcal{H}', K_1) - \operatorname{sign}(\mathcal{H}', K_2)$$

or in other words

$$\operatorname{sign}(H \oplus (-H); \Delta H, K_1 \oplus K_1, R(t)) - \operatorname{sign}(H \oplus (-H); \Delta H, K_2 \oplus K_2, R(t))$$
$$= \operatorname{sign}(H \oplus (-H); \Delta H, K_1 \oplus K_1, R(t')) - \operatorname{sign}(H \oplus (-H); \Delta H, K_2 \oplus K_2, R(t')).$$

Using the cocycle formula (6.3) with $A_0 = \Delta H$, $A_1 = K_1 \oplus K_1$, $A_2 = K_2 \oplus K_2$, $A_3 = R(t)$, the left side of this equation can be rewritten

$$\operatorname{sign}(H \oplus (-H); K_1 \oplus K_1, K_2 \oplus K_2, R(t)) - \operatorname{sign}(H \oplus (-H); \Delta H, K_1 \oplus K_1, K_2 \oplus K_2).$$

Similarly the right side can be rewritten as the same thing with t replaced by t', so the equation to be proved becomes

$$\text{sign}\,(H \oplus (-H);\, K_1 \oplus K_1,\, K_2 \oplus K_2,\, R(t)) = \text{sign}\,(H \oplus (-H);\, K_1 \oplus K_1,\, K_2 \oplus K_2,\, R(t')).$$

But $id \oplus (t't^{-1})$: $H \oplus (-H)$ $H \oplus (-H)$ is an isometry which takes $K_1 \oplus K_1$ to $K_1 \oplus K_1$, $K_2 \oplus K_2$ to $K_2 \oplus K_2$ and $R(t)$ to $R(t')$, so this last equality is proved.

§7. COMPUTATION IN THE BOUNDING CASE

Let X^{2n-1} be a closed manifold and suppose $f: X \to S^1$ and $\beta: \pi_1(X) \to \text{Aut}\,(V)$ are given. In this section and the next we shall prove the bounding case of Theorem 3.2. It clearly suffices to do this if (X, β, f) itself bounds, rather than just a disjoint multiple, so suppose we have a Y^{2n} with $\partial Y = X$ and maps g: $Y^{2n} \to S^1$ and $\bar{\beta}$: $\pi_1(Y) \to \text{Aut}\,(V)$ extending f and β.

PROPOSITION 7.1 (i). *In the above situation if* $\mathcal{H} = \mathcal{H}((X, \beta), f) = (H, b, t)$ *then there exists a* $K \subset H$ *with* $K = K^{\perp}$.

(ii) *If* τ: $\mathbf{Z} \to \text{Aut}\,(U)$ *and* $\alpha = \tau \circ f_{\#}$: $\pi_1(X) \to \text{Aut}\,(U)$ *then for any* K *as in* (i) *above*

$$\gamma((X, \beta), \alpha) = \text{sign}\,(\mathcal{H} \otimes \theta, K \otimes U) - \text{sign}\,(\mathcal{H} \otimes \tau, K \otimes U),$$

where θ: $\mathbf{Z} \to \text{Aut}\,(U)$ *is the trivial representation.*

Proof. Let $p \in S^1$ be a regular value of g: $Y \to S^1$ and put $V = g^{-1}(p)$, $N = \partial V = f^{-1}(p)$. Let Y' be Y cut open along V, X' be X cut open along N.

Let $\Lambda \to Y$ be some hermitian coefficient system, to be specified later and in the following, homology is always to be taken with coefficients in Λ or in the induced system on Y', X', V, etc. As in §5 we suppress the coefficients in our notation, so we will just write $\text{sign}\,(Y)$ for $\text{sign}\,(Y, \Lambda)$, $\text{sign}\,(Y')$ for $\text{sign}\,(Y', \Lambda)$, $\mathcal{H}(X, f)$ for $\mathcal{H}((X, \Lambda), f)$, and so on.

As in §5, let $W = H_{n-1}(N)$, so $H_{n-1}(\partial X') = W \oplus (-W)$ and $R = \text{Ker}\,(H_{n-1}(\partial X') \to H_{n-1}(X')) \subset W \oplus (-W)$ is an isometric relation. Also put

$$L = \text{Ker}\,(H_{n-1}(N) \to H_{n-1}(V)) \subset W.$$

LEMMA 7.2. $\text{Sign}\,(Y) = \text{sign}\,(Y') - \text{sign}\,(W \oplus (-W);\, \Delta W,\, L \oplus L,\, R)$.

Proof. The boundary of Y' can be decomposed as $\partial Y' = V^- \cup X' \cup V^+$ where V^- and V^+ are copies of V. By thickening V slightly in Y we can write $Y = Y' \cup (V \times I)$, where we are pasting $V^+ \subset Y'$ to $V \times \{1\} \subset V \times I$ and $V^- \subset Y'$ to $V \times \{0\} \subset V \times I$ (Fig. 4). Wall's formula 6.1 gives $\text{sign}\,(Y) = \text{sign}\,(Y') + \text{sign}\,(V \times I) - \text{sign}\,(W \oplus W;\, \Delta W,\, L \oplus L,\, R)$, which proves the lemma since $\text{sign}\,(V \times I) = 0$.

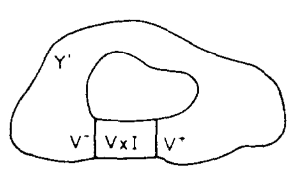

Fig. 4.

LEMMA 7.3. *By altering g in its homotopy class if necessary (and hence altering also V and Y'), we can arrange that* $R^{\infty}0 \subset L \subset R^{\infty}W$.

Proof. Adding a collar $\partial Y \times I$ to Y along its boundary does not alter the diffeomorphism type of Y, so we may work with $Y_1 = Y \cup_{\partial Y \times \{0\}} \partial Y \times I$ instead of Y. For some large integer k let

$$g': \partial Y \times I \to S^1,\, g'(x, r) = e^{2\pi i k r} g(x).$$

Then $g'|\partial Y \times \{0\} = g|\partial Y$, so g and g' fit together to give a map g_1: $Y_1 \to S^1$. Let $V_1 = g_1^{-1}(p)$. Then $V_1 \cong (X' \cup X' \cup \cdots \cup X') \cup V$ (k copies of X') where the copies of X' wind around $\partial Y \times I = X \times I$ parallel to the outer boundary and spiralling inwards.

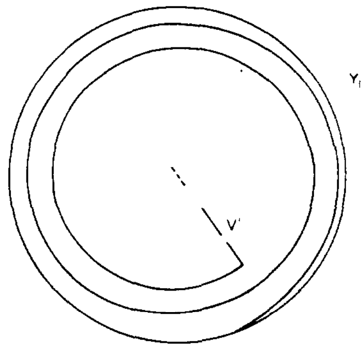

Fig. 5.

Let $L_1 = \text{Ker}(H_{n-1}(\partial V_1) \to H_{n-1}(V_1))$. Then clearly $L_1 = R^k L$, so $R^k 0 \subset L_1 \subset R^k W$. But for k sufficiently large $R^k 0 = R^\infty 0$ and $R^k W = R^\infty W$ by the remarks preceding Lemma 4.4, so Lemma 7.3 is proved.

For any subspace $B \subset W$ let $\Delta B = \{(x, x)\} \in W \oplus (-W) | x \in B\}$.

LEMMA 7.4. *If $B = R^\infty 0$, whence $B^\perp = R^\infty W$, then*

$$\Delta B + R \cap (\Delta B)^\perp = (B \oplus B) + R \cap (B \oplus B)^\perp.$$

Proof. By the remarks at the beginning of the proof of lemma 4.5, it is irrelevant how we bracket the above expressions. We shall show inductively that for any isometric relation R one has

$$(*) \quad \Delta R^k 0 + R \cap (\Delta R^k 0)^\perp = (R^k 0 \oplus R^k 0) + R \cap (R^k W \oplus R^k W).$$

For k large this is the formula of the lemma.

Let $R_1 = \Delta R0 + R \cap (\Delta R0)^\perp$. Since $0 \oplus R0 \subset R$, we have $\Delta R0 + R = \Delta R0 + (0 \oplus R0) + R = (R0 \oplus R0) + R$. Now $(\Delta R0)^\perp = \{(x, y) | x - y \in (R0)^\perp = RW\}$. Thus $R_1 = \{(x, y) \in (R0 \oplus R0) + R | x - y \in RW\}$ and this equals $\{(x, y) \in (R0 \oplus R0) + R | x \in RW\}$ since $(x, y) \in (R0 \oplus R0) + R$ implies $y \in RW$. On the other hand $(R0 \oplus R0) + R \cap (RW \oplus RW) = \{(x, y) \in (R0 \oplus R0) + R | x \in RW, \ y \in RW\} = \{(x, y) \in (R0 \oplus R0) + R | x \in RW\}$, so (*) is proved for $k = 1$.

Assume (*) is proved for k and denote the relation defined in (*) by R_k. Then $R_k 0 = \{y \in R^k W | \exists x \in R^k 0, \ (x, y) \in R\} = R^{k+1} 0 \cap R^k W = R^{k+1} 0$. Thus on the one hand $(R_k)_1 = \Delta R^{k+1} 0 + R_k \cap (\Delta R^{k+1} 0)^\perp = \Delta R^{k+1} 0 + \Delta R^k 0 + R \cap (\Delta R^k 0)^\perp \cap (\Delta R^{k+1} 0)^\perp = \Delta R^{k+1} 0 + R \cap (\Delta R^{k+1} 0)^\perp$ and on the other hand $(R_k)_1 = (R^{k+1} 0 \oplus R^{k+1} 0) + R_k \cap (R^{k+1} W \oplus R^{k+1} W) = (R^{k+1} 0 \oplus R^{k+1} 0) + R \cap (R^{k+1} W \oplus R^{k+1} W)$. Thus (*) follows for $k + 1$, completing the proof.

Now with $B = R^\infty 0$ as above, write $H = B^\perp / B$ with induced hermitian form denoted b and write $S = ((B \oplus B) + R \cap (B^\perp \oplus B^\perp))/(B \oplus B)$. By 5.1 and 4.5, S is the graph of an isometry $t: H \to H$ and $(H, b, t) = \mathcal{H}(X, f)$. By Lemma 7.3 we may assume $B \subset L$, so denote $K = L/B \subset H$. Then $K = K^\perp$, so (7.1)(i) is proved. We shall prove (7.1)(ii) first for this particular choice of K and then deduce it for arbitrary $K \subset H$ with $K = K^\perp$.

LEMMA 7.5 Sign $(W \oplus (-W); \Delta W, L \oplus L, R) = \text{sign}(\mathcal{H}, K)$ *with $\mathcal{H} = \mathcal{H}(X, f) = (H, b, t)$ and $K = L/B \subset H$ as above.*

Proof. Using first lemma (6.2)(ii) with $D = \Delta B$, then Lemma 7.4, then (6.2)(ii) with $D = B \oplus B$, then (6.2)(i) with $D = B \oplus B$, we get the sequence of equalities:

$$\begin{aligned}
&\text{sign}(W \oplus (-W); \Delta W, L \oplus L, R) = \text{sign}(W \oplus (-W); \Delta W, L \oplus L, \Delta B + R \cap (\Delta B)^\perp) \\
&\quad = \text{sign}(W \oplus (-W); \Delta W, L \oplus L, (B \oplus B) + R \cap (B^\perp \oplus B^\perp)) \\
&\quad = \text{sign}(W \oplus (-W); (B \oplus B) + \Delta W \cap (B^\perp \oplus B^\perp), L \oplus L, (B \oplus B) + R \cap (B^\perp \oplus B^\perp)) \\
&\quad = \text{sign}(H \oplus (-H); \Delta H, K \oplus K, S).
\end{aligned}$$

Since S is the graph of $t\colon H \to H$, the latter is by definition $\operatorname{sign}(\mathcal{H}, K)$, proving the lemma.

Putting (7.2), (7.3), (7.5) together we have shown:

COROLLARY 7.6. $\operatorname{Sign}(Y) = \operatorname{sign}(Y') - \operatorname{sign}(\mathcal{H}, K)$.

Now let us recall the local coefficients we have been suppressing. If Λ is the coefficient system classified by $\bar{\beta}\colon \pi_1(Y) \to \operatorname{Aut}(V)$, then (7.6) says

$$\operatorname{sign}(Y, \Lambda) = \operatorname{sign}(Y', \Lambda) - \operatorname{sign}(\mathcal{H}((X, \beta), f), K).$$

If $\tau\colon \mathbf{Z} \to \operatorname{Aut}(U)$ is as in Proposition 7.1 and $\Gamma \to Y$ is the coefficient system classified by $\alpha = \tau \circ g_*$, then using $\Lambda \otimes \Gamma$ in place of Λ replaces $\mathcal{H}((X, \beta), f)$ by $\mathcal{H}((X, \beta), f) \otimes \tau$ and replaces K by $K \otimes U$ (see Proposition 2.3). Thus with $\mathcal{H} = \mathcal{H}((X, \beta), f)$

$$\operatorname{sign}(Y, \Lambda \otimes \Gamma) = \operatorname{sign}(Y', \Lambda \otimes \Gamma) - \operatorname{sign}(\mathcal{H} \otimes \tau, K \otimes U).$$

Similarly, if $\theta\colon \mathbf{Z} \to \operatorname{Aut}(U)$ is the trivial representation, then using U also to denote the trivial coefficient system with fiber U

$$\operatorname{sign}(Y, \Lambda \otimes U) = \operatorname{sign}(Y', \Lambda \otimes U) - \operatorname{sign}(\mathcal{H} \otimes \theta, K \otimes U).$$

Now $\Gamma \to Y$ pulls back from a coefficient system over S^1 via the map $g\colon Y \to S^1$, so $\Gamma | Y'$ pulls back from a system over the interval and is hence trivial. Thus $\operatorname{sign}(Y', \Lambda \otimes \Gamma) = \operatorname{sign}(Y', \Gamma \otimes U)$, so subtracting the above two equations gives

$$\operatorname{sign}(Y, \Lambda \otimes \Gamma) - \operatorname{sign}(Y, \Lambda \otimes U) = \operatorname{sign}(\mathcal{H} \otimes \theta, K \otimes U) - \operatorname{sign}(\mathcal{H} \otimes \tau, K \otimes U).$$

But a trivial Künneth formula computation shows $\operatorname{sign}(Y, \Lambda \otimes U) = \operatorname{sign}(U) \operatorname{sign}(Y, \Lambda)$, so the left side above is by definition $\gamma((X, \beta), \alpha)$, so (7.1)(ii) is proved for this choice of K.

Now $\mathcal{H} \otimes \theta = (H \otimes U,\ b \otimes h_U,\ t \otimes id)$ and $\mathcal{H} \otimes \tau = (H \otimes U,\ b \otimes h_U,\ t \otimes \tau(1))$ and $(t \otimes \tau(1))(t \otimes id)^{-1} = id \otimes \tau(1)$ maps $K \otimes U$ to itself for any $K \subset H$. Lemma (6.4) is thus applicable and shows that $\operatorname{sign}(\mathcal{H} \otimes \theta, K \otimes U) - \operatorname{sign}(\mathcal{H} \otimes \tau, K \otimes U)$ is independent of the choice of $K \subset H$ with $K = K^{\perp}$.

§8. COMPLETION OF PROOF IN THE BOUNDING CASE

To complete the proof of (3.2) in the bounding case we must show, in view of Proposition 7.1:

PROPOSITION 8.1. *If $\mathcal{H} = (H, b, t)$ is an isometric structure with* $\operatorname{sign}(\mathcal{H}) = 0$, *so $K \subset H$ exists with $K = K^{\perp}$ then for any representation $\tau\colon \mathbf{Z} \to \operatorname{Aut}(U)$ and for some such K (and hence also for any such K)*

$$\operatorname{sign}(\mathcal{H} \otimes \theta, K \otimes U) - \operatorname{sign}(\mathcal{H} \otimes \tau, K \otimes U) = \lambda(\mathcal{H} \otimes \tau) - \operatorname{sign}(U)\lambda(\mathcal{H}).$$

LEMMA 8.2. Let *$\mathcal{H} = (H, b, t)$ be ϵ-hermitian and let $K \subset H$ satisfy $K = K^{\perp}$. Then*

$$\operatorname{sign}(\mathcal{H}, K) = \operatorname{sign}(b' | (1 - t)^{-1} K),$$

where $b'\colon H \times H \to \mathbf{C}$ is the (possibly degenerate) $(-\epsilon)$-hermitian form $b'(x, y) = b((t - t^{-1})x, y)$.

Proof. $\operatorname{sign}(\mathcal{H}, K) = \operatorname{sign}(H \oplus (-H);\ \Delta H,\ K \oplus K,\ R(t))$. It is convenient to calculate this as $\operatorname{sign}(H \oplus (-H); A_1, A_2, A_3)$ with $A_1 = R(t)$, $A_2 = \Delta H$, $A_3 = K \oplus K$. Then

$$\begin{aligned} A_1 \cap (A_2 + A_3) &= \{(x, tx) | x = a + b,\ tx = a + c,\ b \text{ and } c \in K\} \\ &= \{(x, tx) | x - tx \in K\} \\ &\cong (1 - t)^{-1} K. \end{aligned}$$

If $(y, ty) \in A_1 \cap (A_2 + A_3)$ then $(y, ty) + (-y, -y) + (0, y - ty) = 0$ with $(y, ty) \in A_1$, $(-y, -y) \in A_2$, $(0, t - ty) \in A_3$. Thus denoting the form on $H \oplus (-H)$ also by b, Wall's form on $A_1 \cap (A_2 + A_3)$ is given by $w((x, tx), (y, ty)) = b((x, tx), (-y, -y)) =$

$b(x, -y) - b(tx, -y) = b(tx - x, y)$. So, interpreted as a form on $(1-t)^{-1}K$, w is given by $w(x, y) = b((t-1)x, y)$. For $x, y \in (1-t)^{-1}K$ we have $b((1-t)x, (1-t)y) = 0$, so $b((1-t)x, y) = b((1-t)x, ty)$, so $b'(x, y) = b((t-t^{-1})x, y) = b((t-1)x, y) + b((1-t^{-1})x, y) = w(x, y) + b((t-1)x, ty) = w(x, y) + b((t-1)x, y) = 2w(x, y)$. Thus $b'|(1-t)^{-1}K = 2w$, so $\operatorname{sign}(\mathcal{H}, K) = \operatorname{sign}(w) = \operatorname{sign}(b'|(1-t)^{-1}K)$.

We shall need the following well known lemma.

LEMMA 8.3. *If (W, h) is a possibly degenerate hermitian space and $A \subset W$ satisfies $A^\perp \subset A$, then* $\operatorname{sign}(h) = \operatorname{sign}(h|A)$.

Proof. Since $\operatorname{Rad}(h) = W^\perp \subset A^\perp$, we can factor throughout by $\operatorname{Rad}(h)$ and assume h is non-degenerate. Let $A' = A/A^\perp$ with induced non-degenerate form h' and $\pi: A \to A'$ the projection. $L = \{(a, \pi a) \in W \oplus (-A') | a \in A\}$ satisfies $L \subset L^\perp$ and $\dim L = \frac{1}{2}\dim(W \oplus (-A'))$ so $L = L^\perp$, so $\operatorname{sign}(W \oplus (-A')) = 0$. That is, $\operatorname{sign}(h) - \operatorname{sign}(h') = 0$, but $\operatorname{sign}(h') = \operatorname{sign}(h|A)$, so the lemma is proven.

LEMMA 8.4. *If $\mathcal{H} = (H, b, t)$ has an invariant kernel, that is, there exists $K \subset H$ with $K = K^\perp = tK$, then for any such K,* $\operatorname{sign}(\mathcal{H}, K) = -\lambda(\mathcal{H})$.

Proof. Let
$$\mathcal{H} = \mathcal{H}_{(0)} \oplus \bigoplus_{|z|=1} \mathcal{H}_z$$
be the decomposition of $\mathcal{H}$ according to eigenvalues as in §3. An easy computation (see for instance [16]) shows that K decomposes correspondingly as $K_{(0)} \oplus \bigoplus_{|z|=1} K_z$, where each summand is an invariant kernel of the corresponding summand of $\mathcal{H}$.

For $z \neq 1$, $(1-t)$ is an isomorphism on H_z, so $(1-t)^{-1}K_z = K_z$ and since $(t - t^{-1})K_z \subseteq K_z$, it follows that $b'|(1-t)^{-1}K_z = 0$, so $\operatorname{sign}(\mathcal{H}_z, K_z) = 0$ by 8.2. The same argument shows $\operatorname{sign}(\mathcal{H}_{(0)}, K_{(0)}) = 0$. We have thus shown
$$\operatorname{sign}(\mathcal{H}, K) = \operatorname{sign}(\mathcal{H}_1, K_1).$$

On the other hand, since $\mathcal{H}_z$ has a kernel for each z, $\operatorname{sign}(\mathcal{H}_z) = 0$, so the definition of $\lambda(\mathcal{H})$ shows that
$$\lambda(\mathcal{H}) = \lambda(\mathcal{H}_1).$$

We must thus show that $\operatorname{sign}(\mathcal{H}_1, K_1) = -\lambda(\mathcal{H}_1)$, in other words that
$$\operatorname{sign}(b'|(1-t)^{-1}K_1) = \operatorname{sign}(b'|H_1).$$

We shall work in H_1, and for any $A \subset H_1$ we use the notation $A^{\perp\prime}$ or $A^\perp$ for orthogonal complement of A in H_1 with respect to the form b' or b respectively. By Lemma 8.3 it suffices to show
$$((t-1)^{-1}K_1)^{\perp\prime} \subset (t-1)^{-1}K_1.$$

Now
$$\begin{aligned}((t-1)^{-1}K_1)^{\perp\prime} &= \{x \in H_1 | b((t-t^{-1})y, x) = 0 \quad \text{for all} \quad x \in (t-1)^{-1}K_1\} \\ &= ((t-t^{-1})(t-1)^{-1}K_1)^\perp.\end{aligned}$$

But $t - t^{-1} = t^{-1}(t+1)(t-1)$, so $(t-t^{-1})(t-1)^{-1}K_1 = t^{-1}(t+1)(t-1)(t-1)^{-1}K_1 = t^{-1}(t+1)(K_1 \cap \operatorname{Im}(t-1)) = K_1 \cap \operatorname{Im}(t-1)$, since $t^{-1}(t+1)$ is an isomorphism on H_1 and maps K_1 to K_1 and $\operatorname{Im}(t-1)$ to $\operatorname{Im}(t-1)$. On the other hand $(\operatorname{Im}(t-1))^\perp = \{x \in H_1 | b((t-1)y, x) = 0 \text{ for all } y \in H_1\} = \{x \in H_1 | b(y, (t^{-1}-1)x) = 0 \text{ for all } y\} = \{x | (t^{-1}-1)x = 0\} = \operatorname{Ker}(t^{-1}-1) = \operatorname{Ker}(t-1)$. Thus
$$\begin{aligned}((t-1)^{-1}K_1)^{\perp\prime} &= (K_1 \cap \operatorname{Im}(t-1))^\perp = K_1^\perp + \operatorname{Im}(t-1)^\perp \\ &= K_1 + \operatorname{Ker}(t-1) \subset (t-1)^{-1}K_1,\end{aligned}$$
as we wished to prove.

We are now ready to prove Proposition 8.1. First observe that it suffices to prove it for hermitian $\mathcal{H}$, for if $\mathcal{H} = (H, b, t)$ is skew hermitian then replacing $\mathcal{H}$ by $(H, -ib, t)$ multiplies the hermitian form b' of Lemma 8.2 by $-i$ and hence does not change its signature, so the left side of equation (8.1) stays unchanged, while the right side stays unchanged by definition of $\lambda(\mathcal{H})$.

Observe next that the equation to be proved can be written

$$\alpha(\mathcal{H}, \tau) = 0,$$

where

$$\alpha(\mathcal{H}, \tau) = \operatorname{sign}(\mathcal{H} \otimes \theta, K \otimes U) - \operatorname{sign}(\mathcal{H} \otimes \tau, K \otimes U) - \lambda(\mathcal{H} \otimes \tau) + \lambda(\mathcal{H} \otimes \theta).$$

Indeed, $\lambda(\mathcal{H} \otimes \theta) = \operatorname{sign}(U)\lambda(\mathcal{H})$, since $\mathcal{H} \otimes \theta$ is just isomorphic to the sum of suitably many copies of $\mathcal{H}$ and $-\mathcal{H}$.

Let $WU(\mathbf{Z})$ denote the Witt group of hermitian representations of $\mathbf{Z}$, that is the semigroup of hermitian representations of $\mathbf{Z}$ with direct sum as addition, factored by the sub-semigroup of hermitian representations τ: $\mathbf{Z} \to \operatorname{Aut}(U)$ having an invariant kernel (that is, a subspace $L \subset U$ with $L = L^{\perp}$ and $\tau(\mathbf{Z})L = L$). Since hermitian representations of $\mathbf{Z}$ correspond one to one with hermitian isometric structures, $WU(\mathbf{Z})$ is also the Witt group of hermitian isometric structures. Let $WU^{\sim}(\mathbf{Z})$ be the "reduced Witt group", that is, the subgroup represented by isometric structures $\mathcal{H}$ with $\operatorname{sign}(\mathcal{H}) = 0$.

If $\mathcal{H}$ has an invariant kernel K, then $K \otimes U$ is an invariant kernel for both $\mathcal{H} \otimes \tau$ and $\mathcal{H} \otimes \theta$, so $\alpha(\mathcal{H}, \tau) = 0$ follows directly from Lemma 8.4. If τ: $\mathbf{Z} \to \operatorname{Aut}(U)$ has an invariant kernel L, then $\operatorname{sign}(\mathcal{H} \otimes \theta, K \otimes U) - \operatorname{sign}(\mathcal{H} \otimes \tau, K \otimes U) = \operatorname{sign}(\mathcal{H} \otimes \theta, H \otimes L) - \operatorname{sign}(\mathcal{H} \otimes \tau, H \otimes L)$ by Lemma (6.4), so again $\alpha(\mathcal{H}, \tau) = 0$ by Lemma 8.4. Thus α can be considered as a homomorphism

$$\alpha: WU^{\sim}(\mathbf{Z}) \times WU(\mathbf{Z}) \to \mathbf{R}$$

and to show this homorphism is trivial we need only check it on generators.

By [16], $WU(\mathbf{Z})$ is (freely) generated by irreducible representations τ: $\mathbf{Z} \to U(1)$. Hence $WU^{\sim}(\mathbf{Z})$, as a Witt group of isometric structures, is generated by isometric structures of the form

$$\mathcal{H} = \left(\mathbf{C}^2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\right), \ z \in S^1 - \{1\}.$$

It thus remains to prove $\alpha(\mathcal{H}, \tau) = 0$ for $\mathcal{H}$ and τ as above. If τ is the trivial representation θ then the statement is trivial, so assume $\tau(1) = w = e^{2\pi i a}$ and $z = e^{2\pi i b}$ with $0 < a, b < 1$.

Now $\mathcal{H} \otimes \theta = \mathcal{H}$ and $\mathcal{H} \otimes \tau = \left(\mathbf{C}^2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} wz & 0 \\ 0 & w \end{pmatrix}\right)$ so using $K = \{(x, x) \in \mathbf{C}^2\}$, direct computation using Lemma 8.2 shows

$$\begin{aligned} \operatorname{sign}(\mathcal{H} \otimes \theta, K) &= 0 \\ \operatorname{sign}(\mathcal{H} \otimes \tau, K) &= +1, \quad a + b < 1, \\ &= 0, \quad a + b = 1, \\ &= -1, \quad a + b > 1. \end{aligned}$$

On the other hand the definition of λ gives

$$\begin{aligned} \lambda(\mathcal{H} \otimes \theta) &= 1 - 2b \\ \lambda(\mathcal{H} \otimes \tau) &= (1 - 2(a + b)) - (1 - 2a), \quad && a + b < 1, \\ &= -(1 - 2a), && a + b = 1, \\ &= (1 - 2(a + b - 1)) - (1 - 2a), && a + b > 1. \end{aligned}$$

A trivial computation thus shows $\alpha(\mathcal{H}, \tau) = 0$, completing the proof of (8.1) and hence also of Theorem 3.2 in the bounding case.

Before completing the proof of 3.2 in the non-bounding case we note a further consequence of our computations, which was announced in the introduction.

PROPOSITION 8.5. *Let Y^{2n} be a compact manifold with $\partial Y = X$ and suppose g: $Y \to S^1$ is a fibration. Then for any coefficient system $\Lambda \to Y$ we have* $\operatorname{sign}(Y, \Lambda) = \lambda(\mathcal{H}((X, \Lambda), g|X))$. *In particular,* $\operatorname{sign}(Y, \Lambda)$ *only depends on* $\mathcal{H}((X, \Lambda|X), g|X)$.

Proof. In the notation of §7 we can choose V as a fiber of g: $Y \to S^1$ and then $Y' = V \times I$, so Lemmas 7.2 and 7.5 show

$$\operatorname{sign}(Y, \Lambda) = \operatorname{sign}(V \times I, \Lambda|V \times I) - \operatorname{sign}(\mathcal{H}, K)$$
$$= -\operatorname{sign}(\mathcal{H}, K),$$

with $\mathcal{H} = \mathcal{H}((X, \Lambda), g|X)$. But K is clearly an invariant kernel of $\mathcal{H}$, so $-\operatorname{sign}(\mathcal{H}, K) = \lambda(\mathcal{H})$ by Lemma 8.4.

§9. COMPLETION OF THE PROOF IN GENERAL

It remains to prove Theorem 3.2 in the case that (X^{2n-1}, β, f) does not bound. For the γ-invariant to be defined, all representatives in question must be unitary, so Theorem 3.2 becomes Theorem 1 of the introduction.

Since a unitary representation of $\mathbf{Z}$ decomposes as a sum of one-dimensional representations, it suffices to prove Theorem 1 when τ is an irreducible representation τ: $\mathbf{Z} \to U(1)$. We first check two special cases.

(i) If f: $X \to S^1$ is homotopically trivial then $\mathcal{H}((X, \beta), f) = 0$ and $\alpha = \tau \circ f_\#$ is trivial, so Theorem 1 becomes trivial.

(ii) Let $X = N^{2n-2} \times S^1$ and let π: $X \to N$ and p: $X \to S^1$ be the projections. Assume $\beta = \mu \circ \pi_\#$ for some representation μ: $\pi_1(N) \to U(r)$ and let $\alpha = \tau \circ p_\#$ with τ as above. Then if $\mathcal{H} = \mathcal{H}((X, \beta), p)$ we have

$$\gamma(N \times S^1, \alpha \otimes \beta) = (-1)^{n-1} \operatorname{sign}(N, \mu)\gamma(S^1, \tau) = \lambda(\mathcal{H} \otimes \tau),$$

the first equation by Theorem (1.2)(v) and the second by direct computation from the definition of $\lambda(\mathcal{H} \oplus \tau)$ and Theorem (1.2)(vi). Since this equation also holds for trivial τ, Theorem 1 follows in this case.

Finally suppose (X, β, f) is arbitrary. Assume f: $X \to S^1$ is smooth and let $q \in S^1$ be a regular value and denote $f^{-1}(q) = N$. We shall construct a bordism of (X, β, f) to something simpler.

Let $g_0 = f \times id$: $X \times [0, 2] \to S^1 \times [0, 2]$. Then $(q, 1)$ is a regular value of g_0, so for a sufficiently small disc D about $(q, 1) \in S^1 \times [0, 2]$ we have $g_0^{-1}(D) \cong N \times D$. Let $M = X \times [0, 2] - g_0^{-1}(\operatorname{Int} D)$. Let g_1: $(S^1 \times [0, 2]) - \operatorname{Int}(D) \to S^1$ be a map with $g_1|S^1 \times \{0\} = id_{S^1}$; $g_1|S^1 \times \{2\} = c$, a constant map; $g_1|\partial D$: $\partial D \to S^1$ an isomorphism. Let g: $M \to S^1$ be $g = g_1 \circ (g_0|M)$. Then the boundary of (M, g) is the disjoint union

$$\partial(M, g) = (X, f) + (-X, c) + (N \times S^1, p).$$

Furthermore the representation β: $\pi_1(X) \to U(r)$ induces representations on the fundamental groups of $X \times [0, 2]$, hence M, hence on each component of ∂M. We denote these representations also by β.

Now Theorem 1 is true for $\partial(M, \beta, g) = (X, \beta, f) + (-X, \beta, c) + (N \times S^1, \beta, p)$, since this is the bounding case already proven. It is true for $(-X, \beta, c)$ and $(N \times S^1, \beta, p)$ since these are cases (i) and (ii) discussed above. It follows that it is true for (X, β, f), as was to be proven.

§10. INVARIANTS OF RATIONAL MONODROMY

One can define monodromy equally well using other coefficients instead of $\mathbf{C}$. Using rational coefficients the monodromy $\mathcal{H}^{\mathbf{Q}}(X^{2n-1}, f)$ is a $(-1)^{n-1}$-symmetric isometric structure over $\mathbf{Q}$. A simple universal coefficient argument shows that the complex monodromy $\mathcal{H}(X, f)$ is the hermitianization $\mathcal{H}^{\mathbf{Q}}(X, f) \otimes \mathbf{C}$ of the rational monodromy. This of course restricts the possibilities for $\mathcal{H}(X, f)$, but as we show in an appendix, it is the only restriction, at least for n even: every skew-symmetric isometric structure over $\mathbf{Q}$ occurs as monodromy.

Most of the discussion of this paper holds with rational (or other) coefficients if one replaces signature of forms by "Witt invariant" of forms throughout. This leads to torsion invariants analogous to γ-invariants and computations of these via rational monodromy. We describe this in the interesting special case that the representation α: $\pi_1(X)\to \mathrm{Aut}\,(U)$ involved, is a non-singular integral bilinear representation.

Let $W_\pm(\mathbf{Z})$ and $W_\pm(\mathbf{Q})$ denote the Witt groups of non-singular (± 1)-symmetric bilinear spaces over $\mathbf{Z}$ and $\mathbf{Q}$. Let $W_\pm(\mathbf{Q}/\mathbf{Z})$ denote the Witt group of non-singular (± 1)-symmetric bilinear forms $T\times T\to \mathbf{Q}/\mathbf{Z}$ on finite abelian groups T. There is a split exact sequence due to Knebusch and Milnor ([8], [11], see also [16] or [1] for an exposition closest to the present one).

$$0\to W_+(\mathbf{Z})\to W_+(\mathbf{Q})\xrightarrow{\delta} W_+(\mathbf{Q}/\mathbf{Z})\to 0.$$

Further, $W_+(\mathbf{Z})\cong \mathbf{Z}$ by signature of forms. Also the natural maps $W_+(\mathbf{F}_p)\to W_+(\mathbf{Q}/\mathbf{Z})$, where $\mathbf{F}_p$ is the finite prime field, induce an isomorphism

$$\bigoplus_p W_+(\mathbf{F}_p)\equiv W_+(\mathbf{Q}/\mathbf{Z}).$$

By Witt[27], $W_+(\mathbf{F}_p)$ is $\mathbf{Z}/2$, $\mathbf{Z}/2\oplus\mathbf{Z}/2$, or $\mathbf{Z}/4$, according as $p=2$, $p=4k+1$, $p=4k-1$.

The groups $W_-(\mathbf{Z})$ and $W_-(\mathbf{Q})$ are trivial, while $W_-(\mathbf{Q}/\mathbf{Z})$ is $\mathbf{Z}/2$, given by the non-trivial form on $T=\mathbf{Z}/2$. Note that if W_* denotes $W_+\oplus W_-$, then $W_*(\mathbf{Q})$ and $W_*(\mathbf{Q}/\mathbf{Z})$ are modules over $W_*(\mathbf{Z})$.

Now let X^{2n-1} be a closed oriented manifold and α: $\pi_1(X)\to \mathrm{Aut}\,(A)$ a representation, where A is a $(-1)^n$-symmetric non-singular bilinear space over $\mathbf{Z}$. Then if $\Lambda\to X$ is the corresponding coefficient system, a linking form can be defined on the torsion subgroup of $H_{n-1}(X;\Lambda)$ in just the same way as for trivial coefficients. We denote the Witt class of this linking form by $l(X,\alpha)\in W_*(\mathbf{Q}/\mathbf{Z})$.

If, in the above situation, $X^{2n-1}=Y^{2n}$ and α extends to $\bar\alpha$: $\pi_1(Y)\to \mathrm{Aut}\,(A)$, then the intersection form $S_{Y,\bar\alpha}$ is a symmetric form over $\mathbf{Q}$, so it defines an element $W(Y,\alpha)\in W_+(\mathbf{Q})$. Alexander, Hamrick and Vick[1] showed (for trivial α, but their proof extends to local coefficients with no change):

THEOREM 10.1. $\delta W(Y,\alpha)=-l(X,\alpha)\in W_+(\mathbf{Q}/\mathbf{Z})$.

Thus if we denote

$$\gamma^{\mathbf{Q}}(X,\alpha)=W(Y,\bar\alpha)-W(A)\cdot W(Y)\in W_+(\mathbf{Q}),$$

then the free part of $\gamma^{\mathbf{Q}}$, being given by signature, is just our previous γ-invariant, while the torsion part $\delta\gamma^{\mathbf{Q}}(X,\alpha)$ is given by

$$\delta\gamma^{\mathbf{Q}}(X,\alpha)=W(A)\cdot l(X)-l(X,\alpha),$$

and is thus defined even if (X,α) does not bound.

The analogue of Proposition 7.1 holds (with the same proof as before) for $\gamma^{\mathbf{Q}}$, by replacing signature by Witt invariant throughout. This gives a computation of $\gamma^{\mathbf{Q}}(X,\alpha)$ in terms of monodromy in the bounding case. For the torsion part of $\gamma^{\mathbf{Q}}$ this computation can be extended to the non-bounding case by the argument of §9. This is an easy calculation and yields the result:

THEOREM 10.2. (i) *Suppose α above factors through $\mathbf{Z}$, say $\alpha=\tau\circ f_\#$ with $f_\#$: $\pi_1(X)\to\mathbf{Z}$ and τ: $\mathbf{Z}\to \mathrm{Aut}\,(A)$. Let $\mathcal{H}=\mathcal{H}^{\mathbf{Q}}(X,f)$, but if n is odd add sufficiently many copies of the trivial isometric structure* ($\mathbf{Q}$, (± 1), *id*) *to $\mathcal{H}$ to make* $\mathrm{sign}\,(\mathcal{H})=0$, *so we can find $K\subset H$ with $K=K^\perp$. Then for any such K*

$$l(X,\alpha)-W(A)\cdot l(X)=\delta(W(\mathcal{H}\otimes\tau,K\otimes A)-W(\mathcal{H}\otimes\theta,K\otimes A))+\mathrm{sign}\,(\mathcal{H})\cdot l(\tau),$$

where: θ: $\mathbf{Z}\to\mathrm{Aut}\,(A)$ *is the trivial representation*; $W(\mathcal{H},K)$ *is defined just like* sign $(\mathcal{H}; K)$ *but using Witt invariant rather than signature*; *and* $l(\tau)$ *is the class in* $W_+(\mathbf{Q}/\mathbf{Z})$ *of the following form* ϕ *on the torsion of* $A/(\tau(1)-1)A$:

$$\phi(a, b) = \frac{1}{q} h_A(x, b) \pmod 1 \text{ if } q \cdot a = (\tau(1)-1)x,\ q \in \mathbf{Z}.$$

Here h_A *is the given form on* A.

(ii) *If* $\beta\colon \pi_1(X) \to \mathrm{Aut}(B)$ *is a further non-singular integral bilinear representation, then the same formula gives* $l(X, \alpha \otimes \beta) - W(A) \cdot l(X, \beta)$ *on replacing* $\mathcal{H}^{\mathbf{Q}}(X, f)$ *by* $\mathcal{H}^{\mathbf{Q}}((X, \beta), f)$ *above.*

It would be more satisfactory to have a formula more like the one of Theorem 3.2, but we have not been able to find a suitable substitute for the algebraic invariant $\lambda(\mathcal{H})$ of Theorem 3.2.

§11. MONODROMY AS A LINKING FORM

Let X^m be a closed oriented manifold and $\bar{X} \to X$ an infinite cyclic covering classified by an element $f \in \mathrm{Hom}(\pi_1(X), \mathbf{Z}) = H^1(X, \mathbf{Z})$. For any field F of coefficients, $H_*(\bar{X}; F)$ is a finitely generated module over the group ring FJ of the (multiplicative) infinite cyclic group J. Let $\mathrm{Tor}\, H_*(\bar{X})$ denote the FJ-torsion submodule of $H_*(\bar{X})$. FJ is a principal ideal domain, so

$$\mathrm{Tor}\, H_*(\bar{X}) \cong FJ/(\phi_1) \oplus \cdots \oplus FJ/(\phi_k),$$

for some $\phi_1, \ldots, \phi_k \in FJ - \{0\}$. Each $FJ/(\phi_i)$ is finite dimensional over F, so $\mathrm{Tor}\, H_*(\bar{X})$ is also.

THEOREM 11.1. (i) *There exists a natural non-degenerate F-bilinear graded-symmetric linking form*

$$S\colon \mathrm{Tor}\, H_q(\bar{X}) \times \mathrm{Tor}\, H_{m-1-q}(\bar{X}) \to F.$$

(ii) *The action of* J *on* $\mathrm{Tor}\, H_*(\bar{X})$ *is by isometries of this form. If* $m = 2n-1$ *is odd then* $(\mathrm{Tor}\, H_{n-1}(\bar{X}), S, t)$ *is the monodromy* $\mathcal{H}^F(X, f)$ *over* F, *where* $t \in J$ *is the generator.*

Milnor[10] proved an analogous statement to (i) above in cohomology in case $H_*(\bar{X}) = \mathrm{Tor}\, H_*(\bar{X})$. In this case there is in fact a duality isomorphism $H_q(\bar{X}) \cong H^{m-1-q}(\bar{X})$ (see Milnor *loc. cit.*: this follows from the long exact sequence in our proof of (i) below, the analogous one in cohomology and Poincaré duality) and our statement is "dual" to his. In the more general situation of the above theorem one can also translate to cohomology, but not at all so pleasantly.

Before giving proofs we describe the linking pairing S. If P is an integral domain, let QP denote its quotient field. For any space X the short exact sequence $0 \to P \to QP \to QP/P \to 0$ induces a long exact sequence

$$\cdots \to H_q(X; QP/P) \xrightarrow{\delta} H_{q-1}(X; P) \to H_{q-1}(X; QP) \to \cdots$$

and it is easy to see that

$$\mathrm{Im}\,\delta = \mathrm{Tor}\, H_{q-1}(X; P) = \{x \in H_{q-1}(X; P) \mid px = 0 \text{ for some } 0 \neq p \in P\}.$$

If X^m is a closed oriented manifold one therefore obtains a linking form

$$L\colon \mathrm{Tor}\, H_q(X; P) \times \mathrm{Tor}\, H_{m-q-1}(X; P) \to QP/P,$$

by $L(x, y) = x \cdot \delta^{-1} y$, where the dot denotes the intersection pairing $H_q(X; P) \times H_{m-q}(X; QP/P) \to QP/P$ (defined in the usual way either via intersection of cycles or as Poincaré dual of the cup product pairing $H^{m-q}(X; P) \times H^q(X; QP/P) \to QP/P$). For $P = \mathbf{Z}$ this is a well known description of the classical linking form on $\mathrm{Tor}\, H_*(X; \mathbf{Z})$.

Now if $\bar{X} \to X$ is the above infinite cyclic covering then we may take the corresponding local coefficient system FJ over X as our coefficients P. On $P = FJ$ and on any other ring or module constructed from J we have an involution called *conjugation induced by the automorphism* $t \mapsto t^{-1}$ on J. If we base the intersection form $H_q(X; P) \times H_{m-q}(X, QP/P) \to QP/P$ on the hermitian coefficient map $P \times QP/P \to QP/P$, $(a, b) \to (a\bar{b})$, then the resulting linking form

$$L: \operatorname{Tor} H_q(X; P) \times \operatorname{Tor} H_{m-q-1}(X; P) \to QP/P$$

is graded hermitian. Since $P = FJ$ and $H_*(X; FJ) = H_*(\bar{X}; F)$, this linking form can be written

$$L: \operatorname{Tor} H_q(\bar{X}; F) \times \operatorname{Tor} H_{m-q-1}(\bar{X}; F) \to QFJ/FJ.$$

This pairing is discussed in greater generality by Blanchfield[3].

Now let FJ_+ denote the ring of formal Laurent series $\Sigma_{i\geq n} a_i t^i$ $(a_i \in F,\ n \in \mathbf{Z})$. Every element $x \neq 0$ of FJ is invertible in FJ_+, so there is a natural embedding i_+: $QFJ \to FJ_+$. For $x \in QFJ$ define $tr(x) = (i_+(x))_0 - (i_+(\bar{x}))_0$, where the subscript 0 means coefficient of t^0 in the Laurent series and $\bar{x}$ denotes conjugation. Clearly, $tr(x) = 0$ if $x \in FJ$, so tr induces a map

$$tr: QFJ/FJ \to F.$$

Define

$$S: \operatorname{Tor} H_q(\bar{X}; F) \times \operatorname{Tor} H_{m-q-1}(\bar{X}; F) \to F$$

by $S(x, y) = tr(L(x, y))$.

Remark. It is not hard to show that tr satisfies the equation $[x] = [i_+^{-1} \Sigma_{j\geq 0} tr(t^{-j}x) \cdot t^j]$ for all $[x] \in QFJ/FJ$ (and is in fact uniquely determined by this). In particular the form L can be recovered from S by $L(x, y) = [i_+^{-1} \Sigma_{j\geq 0} S(t^{-j}x, y) \cdot t^j]$.

Proof of Theorem 11.1. We shall first prove the theorem using a differently defined form S' and then show that $S' = S$.

We can construct the infinite cyclic covering $\bar{X}$ from its classifying element $f \in H^1(X; \mathbf{Z})$ by cutting X open along a submanifold $N^{m-1} \subset X$ dual to f to obtain a manifold X' with $\partial X' = N + (-N)$ and then pasting infinitely many copies X'_i, $i \in \mathbf{Z}$, of X' together end to end (see §5). Let $\bar{X}_+ = \cup_{i\geq 0} X'_i$ and $\bar{X}_- = \cup_{i<0} X'_i$, so $\bar{X} = \bar{X}_- \cup_N \bar{X}_+$.

Let $F[J]$ be the FJ-module of Laurent series $a = \Sigma_{i=-\infty}^{\infty} a_i t^i$, $a_i \in F$. Then $FJ_+ = \{a \in F[J] \mid a_i = 0$ for i sufficiently small$\}$ and $FJ_- = \{a \in F[J] \mid a_i = 0$ for i sufficiently large$\}$ are submodules and $FJ_- \cap FJ_+ = FJ$. There is a short exact sequence

$$0 \to FJ \to FJ_+ \oplus FJ_- \to F[J] \to 0$$

given by maps $x \to (x, -x)$ and $(x, y) \to x + y$. Considering these modules as local coefficient modules over X, we get a long exact homology sequence

$$\cdots \to H_{j+1}(X; FJ_+) \oplus H_{j+1}(X; FJ_-) \to H_{j+1}(X; F[J])$$

$$\xrightarrow{\delta} H_j(X; FJ) \to H_j(X; FJ_+) \otimes H_j(X; FJ_-) \to \cdots.$$

This sequence can be rewritten

$$\cdots \to H^+_{j+1}(\bar{X}) \oplus H^-_{j+1}(\bar{X}) \to H^{cl}_{j+1}(\bar{X}) \xrightarrow{\delta'} H_j(\bar{X}) \xrightarrow{\alpha} H^+_j(\bar{X}) \oplus H^-_j(\bar{X}) \to \cdots$$

where $H_*(\bar{X})$ is usual homology (that is with compact supports), $H^{cl}_*(\bar{X})$ is homology with closed supports (based on infinite but locally finite chains), $H_*^+(\bar{X})$ is homology with supports in $\{t^k\bar{X}_+ \mid k \in \mathbf{Z}\}$ (based on locally finite chains c which are supported in some $t^k\bar{X}_+$), and $H^-_*(\bar{X})$ is homology with supports in $\{t^k\bar{X}_- \mid k \in \mathbf{Z}\}$. Indeed, if we assume X triangulated and work simplicially, then the chain complexes defining $H_*(X; F[J])$ and $H^{cl}_*(\bar{X})$, $H_*(X; FJ_+)$ and $H^+_*(\bar{X})$ and so on, are identical, but we can also work in any other theory that allows closed supports—sheaf theoretic, singular, Čech-type, etc., see for instance Olk[18] for a comparison of these theories.

Looking at the connecting homomorphism δ' on the chain level we see that it is equal to the composition

$$\delta': H^{\mathrm{cl}}_{j+1}(\bar{X}) \to H_j(N) \to H_j(\bar{X}),$$

where the first map intersects cycles in $\bar{X}$ with N and the second is induced by the inclusion $N \subset \bar{X}$.

Define a pairing S_0: $H^{\mathrm{cl}}_{j+1}(\bar{X}) \times H^{\mathrm{cl}}_{m-j}(\bar{X}) \to F$ by $S_0(x, y) = \delta' x \cdot y$, where the dot represents the usual intersection pairing $H_j(\bar{X}) \times H^{\mathrm{cl}}_{m-j}(\bar{X}) \to F$. By the description of δ' above, S_0 is graded symmetric. Also, since the above intersection pairing is non-singular, the radical of S_0 is precisely Ker δ', so S_0 induces a non-singular form S' on $H^{\mathrm{cl}}_*(\bar{X})/\mathrm{Ker}\,\delta' = \mathrm{Im}\,\delta'$.

To complete the proof of part (i) of the theorem for S' we must show $\mathrm{Im}\,\delta' = \mathrm{Tor}\,H_*(\bar{X})$. By our description of δ' above, Im δ' has finite dimension over F, so $\mathrm{Im}\,\delta' \subset \mathrm{Tor}\,H_*(\bar{X})$. On the other hand suppose $x \in H_*(\bar{X})$ is a FJ-torsion element, say $a \cdot x = 0$ with $a \in FJ$. Let $\alpha_\pm$ be the map $H_*(\bar{X}) \to H^\pm_*(\bar{X})$. Since $H^+_*(\bar{X})$ is a module over FJ_+ and a is invertible in FJ_+, it follows from $a \cdot \alpha_+(x) = \alpha_+(a \cdot x) = 0$ that $\alpha_+(x) = 0$. Similarly $\alpha_-(x) = 0$, so $\alpha(x) = (\alpha_+(x), -\alpha_-(x)) = 0$, so $x \in \mathrm{Ker}\,\alpha = \mathrm{Im}\,\delta'$. Thus $\mathrm{Tor}\,H_*(\bar{X}) \subset \mathrm{Im}\,\delta'$, as was to be shown.

The first statement of part (ii) of the theorem is clear, while the second follows immediately on observing that the form S_0 above is the Poincaré dual of the form S_0 we used to define monodromy (see §§2 and 5). Thus the theorem is proved for S'.

Finally we show that $S' = S$. Note that S' can be described as the form

$$S': \mathrm{Tor}\,H_j(X; FJ) \times \mathrm{Tor}\,H_{m-j-1}(X; FJ) \to F$$

given by $S'(x, y) = x \cdot (\delta')^{-1} y$, where the dot is the intersection pairing $H_j(X;\ FJ) \times H_{m-j}(X,\ F[J]) \to F$ induced by the coefficient pairing ψ: $FJ \times F[J] \to F$ given by $\psi(\Sigma a_i t^i,\ \Sigma b_i t^i) = \Sigma a_i b_i$. Observe that $\psi = tr' \circ \phi$ where ϕ: $FJ \times F[J]$ is $\phi(a, b) = a\bar{b}$ and tr': $F[J] \to F$ is $tr'(c) = c_0 =$ coefficient of t^0 in c. Thus if we use the intersection pairing $H_j(X; FJ) \times H_{m-1}(X;\ F[J]) \to F[J]$ defined by the coefficient pairing ϕ, we obtain a form

$$L': \mathrm{Tor}\,H_j(X; FJ) \times \mathrm{Tor}\,H_{m-j-1}(X; FJ) \to F[J]$$

by the formula $L'(x, y) = x \cdot (\delta')^{-1}$, and $S' = tr' \circ L'$.

Now we have natural embeddings i_+: $QFJ \to FJ_+$ and i_-: $QFJ \to FJ_-$ (i_+ has been defined; i_- is defined analogously or alternatively by $i_-(x) = i_+(\bar{x})$). The map $i_+ \oplus (-i_-)$: $QFJ \to FJ_+ \oplus FJ_-$ induces an inclusion of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & FJ & \to & QFJ & \to & QFJ/FJ & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & FJ & \to & FJ_+ \oplus FJ_- & \to & F[J] & \to & 0 \end{array}$$

which takes the definition of L to the definition of L' and takes tr: $QFJ/FJ \to F$ to tr': $F[J] \to F$ and hence shows $S = tr \circ L = tr' \circ L' = S'$, as desired.

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APPENDIX. REALIZING RATIONAL MONODROMY

Let $\mathcal{H} = (H, b, t)$ be an arbitrary skew-symmetric isometric structure over $\mathbf{Q}$. We shall construct a 3-manifold X^3 and a map $f\colon X^3 \to S^1$ such that $\mathcal{H}^{\mathbf{Q}}(X^3, f) = \mathcal{H}$. In fact the corresponding infinite cyclic cover $\bar{X}$ will have finitely generated homology over $\mathbf{Q}$, so $\mathcal{H}^{\mathbf{Q}}(X^3, f)$ can be identified either with the linking form of §11 on $H_1(\bar{X}; \mathbf{Q})$ or with Milnor's cup product form on $H^1(\bar{X}, \mathbf{Q})$ [10].

By taking the cartesian product of the example with $P_{2k}\mathbf{C}$, we obtain examples in any dimension $4k+3$. A similar realizibility theorem in dimensions $4k+1$ seems plausible, but would be much harder to prove.

Recall that every skew-symmetric bilinear space over $\mathbf{Q}$ is isomorphic to $(\mathbf{Q}^{2n}, b)$ for some n, where b is the form given by the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We shall in fact show the following, where the last statement is included to expedite the proof.

PROPOSITION. *If $\mathcal{H} = (\mathbf{Q}^{2n}, b, t)$ is a skew-symmetric isometric structure then there exists a closed oriented X^3 and $f \in [X^3, S^1]$ such that $\mathcal{H}^{\mathbf{Q}}(X, f) = \mathcal{H}$. Further, this can be done such that there exists an embedded surface $N \subset X$ of genus n dual to f.*

Proof. We consider the set $S \subset \operatorname{Aut}(\mathbf{Q}^{2n}, b) = Sp(2n; \mathbf{Q})$ of all t which can be realized as in the theorem. We first show that S is a subgroup. Indeed suppose t_1 and t_2 are realized respectively by (X_1, f_1) and (X_2, f_2) and let $N \subset X_1$, $N \subset X_2$ be embeddings of the surface of genus g dual to f_1 and f_2 respectively. Let X_i' be X_i cut open along N, so $\partial X_i' = N_i^- \cup (-N_i^+)$ is the union of two copies N_i^- and N_i^+ of N for $i = 1, 2$. Let $X = X_1' \cup (-X_2')$ pasted by pasting N_1^+ to N_2^+ and N_1^- and let f: $X \to S^1$ be the obvious map. It then follows easily from Proposition 5.1 that (X, f) realizes $t_1t_2^{-1}$, so $t_1t_2^{-1} \in S$, so S is a subgroup.

We now list some realizable matrices in $Sp(2n; \mathbf{Q})$. Note that a matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is in $Sp(2n; \mathbf{Q})$ if and only if $A^{-1}B = (A^{-1}B)^t$ and $D = (A^t)^{-1}$.

Case 1. $t \in Sp(2n, \mathbf{Q})$ is integral, that is $t \in Sp(2n, \mathbf{Z}) = \operatorname{Aut}(H_1(N; \mathbf{Z}), S_N)$. In this case we can take $X \to S^1$ as a fibration with fiber N, since, as is well known, every $t \in Sp(2n, \mathbf{Z})$ is realizable by a diffeomorphism of N.

Case 2. $t = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$ with A diagonal. We need only realize the case

$$A = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & q & & \\ & & & & 1 & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}, \quad q \in \mathbf{Z} - \{0\},$$

since every diagonal matrix is a product of such matrices and their inverses. An example will suffice to show how this is done. Suppose therefore that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let Y be the solid pretzel of genus $n = 3$ and $j\colon Y \to Y$ the embedding indicated in the following picture:

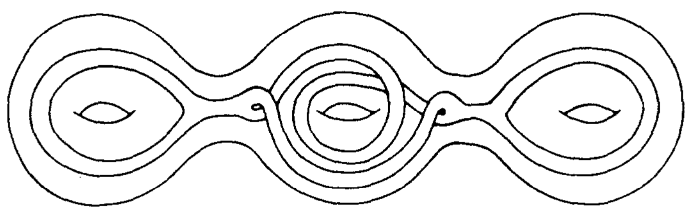

Fig. 6.

Let $X' = Y - j(\text{int } Y)$ and use $j|\partial Y$ to paste the two boundary components of X' together to get X. Choose $f[X, S^1]$ dual to the homology class of $N = \partial Y \subset X$.

Case 3. $t = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$ with A elementary; that is $A = I + (p/q)E_{ij}$, where E_{ij} is the matrix with a 1 in the (i, j) position and zeroes elsewhere. This A is a product of integral and diagonal matrices as follows, so case 3 follows from cases 1 and 2.

$$A = \begin{pmatrix} 1 & \cdots & & & 0 \\ \vdots & \ddots & & & \\ & & 1/q & & \\ & & & q & \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 \cdots & pq & \\ & & & 1 & \\ 0 & & & \ddots & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & & & 0 \\ \vdots & \ddots & & & \\ & & q & & \\ & & & 1/q & \\ 0 & & & \ddots & 1 \end{pmatrix}$$

Case 4. As Case 3 but $A \in GL(n, \mathbf{Q})$ arbitrary. Since $GL(n\ \mathbf{Q})$ is generated by elementary and diagonal matrices, this case follows from Cases 2 and 3.

Case 5. $t = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$. Then $B = B^t$. In the cases $B = bE_{ii}$ or $B = b(E_{ij} + E_{ji})$, t can be expressed as a product of diagonal and integral matrices similarly to case 3. Since $\begin{pmatrix} I & B_1 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & B_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & B_1 + B_2 \\ 0 & I \end{pmatrix}$, we can then generate any $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$.

Case 6. Any $t = \begin{pmatrix} A & B \\ 0 & (A^b)^{-1} \end{pmatrix} \in Sp(2n, \mathbf{Q})$ is now realizable as a product of matrices from Cases 4 and 5.

Conclusion. Let $T \subset Sp(2n, \mathbf{Q})$ be the subgroup of matrices as in case 6. We claim $Sp(2n, \mathbf{Z}) \cdot T = Sp(2n, \mathbf{Q})$, completing the proof, in view of cases 1 and 6.

Indeed, let $\mathcal{K} = \{K \subset \mathbf{Q}^{2n} | K = K^\perp\}$. Then $Sp(2n, \mathbf{Q})$ acts transitively on $\mathcal{K}$ with isotropy subgroup T (see below), so $\mathcal{K} \cong Sp(2n, \mathbf{Q})/T$. But also $Sp(2n, \mathbf{Z})$ acts transitively on $\mathcal{K}$ with subgroup $T \cap Sp(2n, \mathbf{Z})$, so the inclusion of $Sp(2n, \mathbf{Z}) \to Sp(2n, \mathbf{Q})$ induces a bijection $Sp(2n, \mathbf{Z})/T \cap Sp(2n, \mathbf{Z}) \to Sp(2n, \mathbf{Q})/T$, which proves our claim.

To see that $Sp(2n, \mathbf{Z})$ acts transitively on $\mathcal{K}$, note that $Sp(2n, \mathbf{Z})$ certainly acts transitively on the set of simplectic bases of $\mathbf{Z}^{2n}$ and $\mathcal{K}$ is an equivariant quotient of this set. Since $Sp(2n, \mathbf{Z})$ acts transitively, $Sp(2n, \mathbf{Q})$ does so too. The isotropy subgroups are evidently as claimed.