Topology Vol. 14, pp. 237–244. Pergamon Press 1975. Printed in Great Britain

MANIFOLD CUTTING AND PASTING GROUPS

WALTER D. NEUMANN†

(Received 24 July 1974; revised 6 November 1974)

SOME calculations are made of the group $SK_n(X)$ of closed oriented singular n-manifolds in X modulo cutting and pasting. In particular $SK_2(X)$ is completely calculated in terms of $\pi_1(X)$ and it is shown that the SK-groups of a space with trivial respectively finite fundamental group are trivial or torsion respectively for $n \neq 6$. Applications to existence of open book decompositions up to bordism and to signature of fibre bundles are discussed.

§1. INTRODUCTION

Given a closed oriented manifold M^n and a closed oriented submanifold $W^{n-1} \subset M^n$ of codimension 1, one can cut M open along W to obtain a manifold M' whose boundary is the disjoint union $W + (-W)$ of two (oppositely oriented) copies of W. By pasting these copies together in a new way (respecting orientation) one obtains a new closed oriented manifold N^n, which is said to have been obtained by cutting and pasting or briefly by SK (= "Schneiden und Kleben") from M.

More generally one can cut and paste singular manifolds in a space X (that is pairs (M, f) consisting of a closed oriented manifold M and a map $f: M \to X$) by allowing a homotopy of the map f after cutting M open and before pasting together in the new way. For instance for $X = BG$, a classifying space, this is equivalent via the classification of bundles with the natural definition of cutting and pasting of bundles with a fixed fibre and structure group G over closed oriented manifolds.

In the following we always assume for convenience that X is path connected. Let $SK_n(X)$ be the set of equivalence classes of non-empty oriented singular n-manifolds in X modulo the relation generated by cutting and pasting. In [8] it is shown that $SK_n(X)$ is a group (with operation induced by disjoint union) and it is calculated in many cases. In particular for n odd (except maybe $n = 5$) $SK_n(X)$ is always zero,‡ and the groups $SK_n = SK_n(*)$ of the trivial space are

$$SK_{4k+1} = 0 = SK_{4k+3};$$

$$SK_{4k+2} = \mathbb{Z}, \qquad \text{basis } [S^{4k+2}];$$

$$SK_{4k+4} = \mathbb{Z} \oplus \mathbb{Z}, \qquad \text{basis } [S^{4k+4}], [P\mathbb{C}^{2k+2}].$$

In fact euler characteristic and Hirzebruch signature give injective homomorphisms $SK_{4k+2} \to \mathbb{Z}$, $SK_{4k+4} \to \mathbb{Z} \oplus \mathbb{Z}$ onto subgroups of index 2 (to make all this work also in dimension zero it is convenient to allow the single point to exist with two orientations).

Define a reduced SK-group $\widetilde{SK}_n(X)$ as the kernel of the map $SK_n(X) \to SK_n$ induced by the map $X \to *$. Any map $* \to X$ induces a natural splitting (since X is path connected)

$$SK_n(X) = \widetilde{SK}_n(X) \oplus SK_n.$$

Since we know the group SK_n, we just need to investigate the reduced group. Our main results are:

THEOREM 1. *If X is simply connected than $\widetilde{SK}_n(X) = 0$ for $n \neq 6$. If X has finite fundamental group then $\widetilde{SK}_n(X)$ is torsion for $n \neq 6$.*

This probably also holds for $n = 6$. On the other hand the following theorem and corollary show that $\widetilde{SK}_n(X)$ can be large.

†Partially supported by the Sonderforschungsbereich Theoretische Mathematik 40 in Bonn.

‡This depended on an existence statement for open book decompositions in the odd-dimensional non-simply connected case made in a preprint, but not in the published version of [15]. Though the statement on open books is probably true, one can use [16] instead for this.

THEOREM 2. *$\widetilde{SK}_2(X) = H_2(\pi_1(X);\mathbb{Z})/I$, where I is the subgroup generated by all images*

$$\mathrm{Im}\,(H_2(A;\mathbb{Z})\to H_2(\pi_1(X);\mathbb{Z}))$$

where $A \subset \pi_1(X)$ either runs through all abelian subgroups of $\pi_1(X)$ or just through all two generator abelian subgroups.

COROLLARY 3. *Every finitely generated abelian group occurs as $\widetilde{SK}_2$ of a finite 2-dimensional complex.*

Theorems 1 and 2 will be proved in §3 and the above corollary is deduced in §4. These theorems led to the conjecture mentioned in [8] that $SK_n(X)$ always only depends on $\pi_1(X)$. We give examples disproving this conjecture in §5, where the SK-groups of certain products of surfaces are calculated in full using the fact that obstructions to multiplicativity of signature for fibre bundles lie in the SK-groups. In fact $SK_n(X)$ does only depend on the "n-homotopy type" of X (Theorem 9 below) but this is the best possible result by example 3 in §5.

Our results allow us to give many cases in which signature is multiplicative for fibre bundles (Theorem 12).

Maybe more interesting than the cases in which we have calculated $SK_*(X)$ are the cases in which we have not. For instance:

PROBLEM. *Does non-zero torsion actually occur in the second part of Theorem 1 with dimension $n > 2$?*

A positive answer would imply that the open book theorem[15] already fails in the corresponding dimension if one only requires finite, rather than trivial fundamental group (for otherwise the proof of the first part of Theorem 1 would extend). Note that for $n = 2$ such torsion does occur (by §4), so one can even ask more specifically: does this torsion maybe stay non-trivial when multiplied by powers of $P\mathbb{C}^2$?

§2. GENERAL SK-THEORY

In this section we collect some general facts about SK. Let $\overline{SK}_n(X)$ be $SK_n(X)$ factored by the bordism relations, that is

$$\overline{SK}_n(X) := SK_n(X)/I_n(X)$$

where $I_n(X)$ is the subgroup generated by all elements having a representative which bounds in X. The following lemma is Theorem 1.3b of [8] and is also clear from the calculation of SK_* mentioned above.

LEMMA 4. *$\overline{SK}_*$ is the polynomial algebra $\mathbb{Z}[x]$ on a single 4-dimensional generator (representable for instance by $P\mathbb{C}^2$). An isomorphism $\overline{SK}_{4k} \cong \mathbb{Z}$ is given by signature of manifolds.*

THEOREM 5. *$\widetilde{SK}_n(X)$ can be identified with the reduced $\overline{SK}$-group $\mathrm{Ker}(\overline{SK}_n(X)\to\overline{SK}_n)$ and hence is equal to*

$$\widetilde{SK}_n(X) = \{[M,f] \in \overline{SK}_n(X) \mid \mathrm{sign}\, M = 0\}.$$

Proof. The statement that the reduced SK-group is equal to the reduced $\overline{SK}$-group is equivalent to saying that the kernel $I_n(X)$ of $SK_n(X)\to\overline{SK}_n(X)$ is independent of X, and this is just Theorem 1.1 of [8] (in fact $I_n(X) = \mathbb{Z}$ or 0 according as n is even or odd, generated by the class $[S^n]$ of S^n mapping by the constant map to X). Since $\overline{SK}_n(X)\to\overline{SK}_n$ is the signature map by Lemma 4, the second statement follows.

THEOREM 6. *The kernel $F_n(X)$ of the natural epimorphism $\Omega_n(X)\to\overline{SK}_n(X)$ consists of those bordism classes containing a representative (M,f) such that M can be fibered over S^1. This is the same as those bordism classes containing a representative (M,f) such that M has an open book decomposition (with possibly empty binding; definition below).*

Proof. The first statement is just Theorem 1.2 of [8]. For the second recall that an open book decomposition[15] can be thought of as a smooth map $g: M\to D^2$ having zero as a regular value and such that $\bar{g}: M - g^{-1}(0)\to S^1$ defined by $\bar{g}(x) = g(x)/\|g(x)\|$ is a fibration. $g^{-1}(0)$ is the "binding" of the open book and the closures of the $\bar{g}^{-1}(t)$, $t \in S^1$, are the "pages". For such an

M it was shown in [8] Theorem 6.4 (by cutting M open along the union of two pages and then flattening the cut open M into one page of itself) that (M, f) is always in the kernel $F_n(X)$ for any $f: M \to X$. On the other hand every element of $F_n(X)$ can be represented by an (M, f) with M and open book (with empty binding) by the first statement of the theorem.

Another general fact that we shall need later is that product of singular manifolds induces pairings

$$SK_n(X) \otimes SK_m(Y) \to SK_{n+m}(X \times Y)$$
$$\overline{SK}_n(X) \otimes \overline{SK}_m(Y) \to \overline{SK}_{n+m}(X \times Y).$$

In particular $\overline{SK}_n(X)$ and hence also $\widetilde{SK}_n(X)$ are modules over the polynomial algebra $\overline{SK}_* = \mathbb{Z}[[P\mathbb{C}^2]]$.

Given a closed oriented manifold M^n, let $\theta(M) \in H_n(M)$ denote the fundamental homology class.

LEMMA 7. *Suppose (M_i, f_i), $i \in I$, are singular manifolds in X such that the elements $(f_i)_*\theta(M_i)$ form a modulo torsion generating set of $H_*(X)$ (that is $H_*(X)$ factored by the generated subgroup is torsion). Then the $[M_i, f_i]$ form a modulo torsion $\overline{SK}_*$-module generating set of $\overline{SK}_*(X)$. The same holds without the "modulo torsion" qualifications if $H_*(X)$ is free.*

Proof. With $\overline{SK}_*(X)$ and $\overline{SK}_*$ replaced by $\Omega_*(X)$ and Ω_* this is shown by a standard argument using the modulo torsion triviality of the bordism spectral sequence (see [4] Theorem 18.1). But since $\overline{SK}_*(X)$ is a quotient of $\Omega_*(X)$, the lemma follows.

We similarly have a weak sort of Künneth formula:

LEMMA 8. *$\overline{SK}_*(X) \otimes \overline{SK}_*(Y) \to \overline{SK}_*(X \times Y)$ is modulo torsion surjective and is genuinely surjective if one of X and Y has free homology.*

Indeed, the corresponding statement again holds with $\overline{SK}_*$ replaced by Ω_* (actually with surjective replaced by isomorphic, see [4]), and hence certainly holds for $\overline{SK}_*$.

The next theorem was promised in the introduction. Recall that an n-equivalence $X \to Y$ of connected spaces is a map which induces isomorphisms $\pi_i(X) \to \pi_i(Y)$ for $0 \le i \le n-1$ and an epimorphism for $i = n$.

THEOREM 9. *Any n-equivalence $X \to Y$ induces isomorphisms $SK_q(X) \to SK_q(Y)$ for $q \le n$. (Note that this is one dimension better than one might first expect.)*

Proof. Since any mapping of a q-complex to Y lifts to X for $q \le n$ [13; Cor. 7.6.23], this holds in particular for q-manifolds, so $SK_q(X) \to SK_q(Y)$ is surjective. It is injective because the homotopy involved in cutting and pasting a singular manifold in Y is a homotopy of a q-manifold having (without loss of generality) no connected component with empty boundary. Thus it is a homotopy of a homotopy-$(q-1)$-complex, so it lifts (up to homotopy) to X.

§3. PROOFS OF THEOREMS 1 AND 2

First suppose X is simply connected. By Theorem 5 $\widetilde{SK}_n(X)$ can be considered to be the subgroup of $\overline{SK}_n(X)$ represented by manifolds of zero signature, so choose any $[M, f] \in \overline{SK}_n(X)$ with $\operatorname{sign}(M) = 0$. Since $\pi_1(X) = \{1\}$ we can do surgery on M ($n > 1$) to kill the fundamental group, so without loss of generality $\pi_1(M) = \{1\}$ with respect to any base point if $n > 1$. For $n > 1$ we can clearly also then assume M is connected, by taking connected sum of the components if necessary. But now M has an open book decomposition for $n \ge 7$ by [15], for $n = 5$ by [1], for $n = 3$ by [2], and for $n = 2$ and 1 trivially, so in these cases $[M, f] = 0$ by Theorem 6. If $n = 4$ or 6 it is not known to date if M has an open book decomposition, but for $n = 4$ we can get around this. Namely we can first assume (in addition to $\pi_1(M) = \pi_0(M) = \{1\}$) that M has odd intersection form, since otherwise we achieve this by taking the connected sum with the total space Σ of the non-trivial S^2 bundle over S^2, which has intersection form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Here Σ of course should be mapped by a constant map to X so the class in $\overline{SK}_n(X)$ is unchanged. Since odd unimodular forms of signature zero are classified by their rank alone (see for instance [7] Theorem 4.6), the intersection form of M is equivalent to the intersection form of $\Sigma \# (S^2 \times S^2) \# \cdots \# (S^2 \times S^2)$ (k summands) for some k. On the other hand, by [14], simply connected 4-manifolds are classified up to connected sum with copies of $S^2 \times S^2$ by their intersection forms alone, so by taking connected sum with further copies of $S^2 \times S^2$ if necessary, we can actually assume M itself is the above k-fold connected sum. But both Σ and S^2+S^2 fibre over S^2 and hence have open book decompositions by pulling back the open book decomposition of S^2. Thus the above connected sum has an open book decomposition so $[M, f]=0$ also in this case. Theorem 1 is thus proved for $\pi_1(X)=\{1\}$.

To prove the second part of Theorem 1 let $Y \to X$ be any finite covering of spaces. Then $H_*(Y) \to H_*(X)$ is modulo torsion surjective, so by Lemma 7, $\widetilde{SK}(Y) \to \widetilde{SK}(X)$ is modulo torsion surjective. In particular if X has finite fundamental group and Y is its universal cover, then since $\widetilde{SK}_n(Y)=0$ for $n \neq 6$ it follows that $\widetilde{SK}_n(X)$ is zero modulo torsion for $n \neq 6$, as was to be shown.

To prove Theorem 2 observe that for any space K the bordism spectral sequence shows that $\Omega_2(K)=H_2(K)$, so Theorems 6 and 5 give an exact sequence

$$0 \to F_2(K) \to H_2(K) \to \widetilde{SK}_2(K) \to 0,$$

where $F_2(K)$ is now the subgroup of $H_2(K)$ generated by all classes representable by the torus $T^2=S^1 \times S^1$.

Now if X is any path connected space, let $K=K(\pi, 1)$ be an Eilenberg Maclane space with $\pi=\pi_1(X)$ and let $\varphi: X \to K$ be a map inducing an isomorphism of the fundamental groups. φ is certainly a 2-equivalence, so it induces an isomorphism $\widetilde{SK}_2(X) \to \widetilde{SK}_2(K)$ by Theorem 9. It thus just remains to show that the subgroup $F_2(K)$ of $H_2(K)=H_2(\pi; \mathbb{Z})$ is the subgroup described in each of the two ways of Theorem 2. $F_2(K)$ is generated by the elements x which can be represented by a map $f: T^2 \to K$. Since T^2 is a $K(\mathbb{Z} \oplus \mathbb{Z}, 1)$, this map f is represented by a homomorphism $g: \mathbb{Z} \oplus \mathbb{Z} \to \pi$, so x is in the image of $g_*: H_2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}) \to H_2(\pi; \mathbb{Z})$ and is hence certainly in the image of $H_2(A; \mathbb{Z}) \to H_2(\pi; \mathbb{Z})$ where A is the 2-generator abelian subgroup $g(\mathbb{Z} \oplus \mathbb{Z})$ of π. Conversely let x be in the image of $i_*: H_2(A; \mathbb{Z}) \to H_2(\pi; \mathbb{Z})$, where A is any abelian subgroup (not necessarily 2-generated) of π. We must show that x is in the subgroup of $H_2(\pi; \mathbb{Z})=H_2(K)$ generated by mappings of T^2 to K. It clearly suffices to show that $H_2(A; \mathbb{Z})$ itself is generated by maps $T^2 \to K(A, 1)$. If A is finitely generated, and hence a product of cyclic groups, this follows from the Künneth formula. The general case follows by a direct limit argument.

§4. SOME EXAMPLES

We must give examples to prove Corollary 3. First note the following:

LEMMA 10. *$\widetilde{SK}_*(X \vee Y)=\widetilde{SK}_*(X) \oplus \widetilde{SK}_*(Y)$ for well pointed spaces X and Y (for instance CW complexes).*

Proof. The map $X+Y \to X \vee Y$ (where $+$ denotes disjoint union) induces

$$\widetilde{SK}_*(X) \oplus \widetilde{SK}_*(Y) \to \widetilde{SK}_*(X \vee Y)$$

which is injective because the maps $X \vee Y \to X$ and $X \vee Y \to Y$ induce a left inverse, and is surjective because

$$\widetilde{\Omega}_*(X) \oplus \widetilde{\Omega}_*(Y) \to \widetilde{\Omega}_*(X \vee Y)$$

is surjective (actually an isomorphism, by the Mayer Vietoris sequence).

In view of this lemma, to prove Corollary 3 it suffices to find a finite 2-complex K with

$$\widetilde{SK}_2(K)=\mathbb{Z}/m$$

for each cyclic group $\mathbb{Z}/m$ of infinite ($m=0$) or finite order (prime power orders would suffice). This is done in the following examples.

Example 1 ($m=0$). Let K be an orientable surface of genus at least 2. Then K is a $K(\pi, 1)$ and $\pi=\pi_1(K)$ contains no noncyclic abelian subgroup, so by Theorems 2 and 5 (see also [8; Theorem 2.9])

$$\widetilde{SK}_2(K)=\overline{SK}_2(K)=H_2(K; \mathbb{Z})=\mathbb{Z}.$$

Example 2. For any m we shall construct a finite group G with $\overline{SK}_2(K(G,1)) = \mathbb{Z}/m$. As our 2-complex K we can then take the 2-skeleton of $K(G,1)$, which still has fundamental group G. I am grateful to P. M. Neumann for suggesting the group G used here.

Let X be the abelian group $(\mathbb{Z}/m)^4$ with generators $x_1, \ldots, x_4$, say. Then the exterior square $\Lambda^2 X$ is isomorphic to $(\mathbb{Z}/m)^6$ with generators $x_i \wedge x_j$, $4 \geq i > j \geq 1$. Define a multiplicative group structure on

$$H = X \times \Lambda^2 X$$

by

$$\left(\sum a_i x_i, y\right)\left(\sum b_i x_i, z\right) = \left(\sum (a_i + b_i)x_i,\ y + z + \sum_{i>j} a_i b_j x_i \wedge x_j\right).$$

The group laws are easily checked and one calculates that the commutator of two elements $h = (x, y)$ and $k = (w, z)$ in H is

$$[h, k] = h^{-1}k^{-1}hk = (0, x \wedge w).$$

Thus the commutator subgroup of H is

$$H' = [H, H] = \{0\} \times \Lambda^2 X \subset H.$$

Observe that H' is in the center of H; in fact H' actually is the center, but we shall not need this.

Let us abbreviate the element $(0, x_i \wedge x_j) \in H'$ by y_{ij}. Then since H' is central, the cyclic subgroup generated by $y_{43}y_{21}$ is normal in H so we can define

$$G = H/\langle y_{43}y_{21}\rangle = X \times (\Lambda^2 X/(x_4 \wedge x_3 + x_2 \wedge x_1)).$$

This has commutator subgroup

$$G' = H'/y_{43}y_{21} = \Lambda^2 X/(x_4 \wedge x_3 + x_2 \wedge x_1) \cong (\mathbb{Z}/m)^5,$$

with quotient $G/G' = X$. Let B be the quotient $X/(x_3, x_4)$ and

$$\pi: G \to B \cong \mathbb{Z}/m \oplus \mathbb{Z}/m$$

the natural projection. We plan to show that the induced map $H_2(G) \to H_2(B) \cong \mathbb{Z}/m$ can be identified with the map $H_2(G) \to \overline{SK}_2(K(G,1))$. Here and in the following coefficients for homology are always $\mathbb{Z}$.

CLAIM 1. *For any abelian $A \subset G$ the image $\pi(A) \subset B$ has order dividing m.*

Proof. Since $\pi(A)$ is generated by two elements, we lose no generality in supposing that A is generated by two elements $h = (x, [y])$ and $k = (w, [z])$ say in $G = X \times \Lambda^2 X/(x_4 \wedge x_3 + x_2 \wedge x_1)$. By assumption, their commutator $(0, [x \wedge w])$ is zero, and since $x_4 \wedge x_3 + x_2 \wedge x_1$ and its non-zero multiples are indecomposible, $x \wedge w$ must be zero already in $\Lambda^2 X$. Thus the image of $x \wedge w$ in $\Lambda^2 B = \mathbb{Z}/m$ is zero, but this image is $\pi(h) \wedge \pi(k)$, so it follows that $\pi(h)$ and $\pi(k)$ generate a subgroup of B of index divisible by m, hence of order dividing m. Every subgroup of order dividing m actually occurs, but we do not need this, so we omit the (easy) proof.

CLAIM 2. *If $I \subset H_2(G)$ is as in Theorem 2 then $I \subset \mathrm{Ker}\,(\pi_*)$, that is, $\pi_*(I) = 0$, where π_*: $H_2(G) \to H_2(B)$ is the map induced by π.*

Proof. Since I is generated by all images $\mathrm{Im}\,(H_2(A) \to H_2(G))$ with $A \subset G$ abelian, $\pi_*(I)$ is generated by all images $\mathrm{Im}\,(H_2(\pi(A)) \to H_2(B))$ with $\pi(A)$ as in Claim 1. Since for abelian groups the homology functor H_2 and exterior square Λ^2 coincide, we are just looking at $\mathrm{Im}\,(\Lambda^2\pi(A) \to \Lambda^2 B) = \mathrm{Im}\,(\Lambda^2\pi: \Lambda^2 A \to \Lambda^2 B)$, which we have seen above is always zero.

By Claim 2 π_* induces a homomorphism

$$\pi': \overline{SK}_2(K(G,1)) = H_2(G)/I \to H_2(B) \cong \mathbb{Z}/m.$$

CLAIM 3. *This π' is an isomorphism.*

Proof. We must show that π_* is surjective and that $\mathrm{Ker}(\pi_*) \subset I$ (we already know $I \subset \mathrm{Ker}(\pi_*)$). For this we look at the exact sequence $0 \to G' \to G \to X \to 0$, and consider its Lyndon spectral sequence $\{E^2_{p,q} = H_p(X; H_q(G')) \Rightarrow H_{p+q}(G)\}$ (see for instance [10]). Since the extension is central, X acts trivially on $H_*(G')$, so the E^2 term is

	p = 0	p = 1	p = 2	...
q = 2	$H_2(G') = \Lambda^2 G'$	...	...	...
q = 1	$H_1(G') = G'$	$H_1(X) \otimes H_1(G')$	...	...
q = 0	$\mathbb{Z}$	$H_1(X) = X$	$H_2(X) = \Lambda^2 X$	...

Now d_2: $E^2_{2,0} \to E^2_{0,1}$ is (up to sign) the natural map $H_2(X) = \Lambda^2 X \to H_1(G') = G' = \Lambda^2 X/(x_4 \wedge x_3 + x_2 \wedge x_1)$ (this is maybe most easily seen by using naturality and comparing with the spectral sequence of $1 \to F_2' \to F_2 \to \mathbb{Z}^2 \to 1$, where F_2 is the free group on two generators).

It follows that the E^∞ term has the form

	p = 0	p = 1	p = 2	...
q = 2	$\Lambda^2 G'/(?)$	...	...	...
q = 1	0	$(X \otimes G')/(?)$	...	...
q = 0	$\mathbb{Z}$	X	$(x_4 \wedge x_3 + x_2 \wedge x_1)$	...

Using the map of exact sequences

$$\begin{array}{ccccccc} 0 \to & G' & \to & G & \to & X & \to 0 \\ & \downarrow & & \downarrow \pi & & \downarrow & \\ 0 \to & 0 & \to & B & \to & B & \to 0 \end{array}$$

to compare the above spectral sequence with the (trivial) spectral sequence of $0 \to 0 \to B \to B \to 0$ we see that π_*: $H_2(G) \to H_2(B)$ is surjective and has kernel given by the terms $E^\infty_{0,2} = \Lambda^2 G'/(?)$ and $E^\infty_{1,1} = (X \otimes G')/(?)$. The former is just $\mathrm{Im}\,(H_2(A) \to H_2(G))$ with $A = G'$ and the latter is generated by all $\mathrm{Im}\, H_2(A) \to H_2(G))$ with $A = (x_i, y_{jk}) \subset G$, so $\mathrm{Ker}\,(\pi_*) \subset I$, as was to be shown.

The following remarks may give more insight into the above example. First one can check that the group G above has a presentation

$$G = \langle x_1, x_2, x_3, x_4 | x_i^m = [x_i, x_j]^m = [x_i, [x_j, x_k]] = [x_4, x_3][x_2, x_1] = 1\rangle.$$

The relations $[x_i, x_j]^m = 1$ are actually redundant, since modulo relations of the form $[x, [y, z]]$ they are equivalent to $[x_i^m, x_j] = 1$ (see for instance [5; p. 150].

On the other hand, given a free presentation $1 \to R \to F \to G \to 1$ of any group G one has the Hopf description

$$H_2(G) = R \cap [F, F]/[R, F]$$

of $H_2(G)$. If G is abelian on two generators one can choose $F = \langle x, y\rangle$ free on two generators and then $H_2(G)$ is generated by the class of the relation $[x, y] \in R$. It follows by the naturality of the above equation that for arbitrary G the subgroup $I \subset H_2(G)$ of Theorem 2 is the subgroup of $R \cap [F, F]/[R, F]$ generated by those relations which are commutators; that is

$$I = \langle R \cap \{[x, y] | x, y \in F\}\rangle/[R, F],$$

where $\langle \ldots \rangle$ here means "group generated by". We hence have

PROPOSITION. $\overline{SK}_2(K(G, 1)) = R \cap [F, F]/\langle R \cap \{[x, y]\}\rangle$.

For instance in the above example $\overline{SK}_2(K(G, 1))$ is generated by the relation $[x_4, x_3][x_2, x_1]$. One might expect the relation $[x_i, x_j]^m$ also to give a non-trivial element of $\overline{SK}_2(BG)$, but the argument above showing that this relation is redundant also shows that it is in $\langle R \cap \{[x, y]\}\rangle$, hence represents zero in $\overline{SK}_2(BG)$.

§5. MULTIPLICATIVITY OF SIGNATURE

Let $F^m \to E^{4k} \to M^n$ be a fibration of closed oriented manifolds $(m - n = 4k)$. It is known that under certain conditions, for instance if $\pi_1(M)$ is finite, then

$$\operatorname{sign} E = \operatorname{sign} F . \operatorname{sign} M,$$

however this equality does not always hold[3, 6, 9, 11].

In discussing the validity of the above equality no generality is lost by assuming sign $M = 0$ above (since otherwise we can instead consider the disjoint union $E + (F \times -M)$ as a bundle over $M' = M + -M$). Let BG be any classifying space for the bundle and $f: M \to BG$ a classifying map, so $E = f^*(EG \times_G F)$.

THEOREM 11. *If $F^m \to E \to M^n$ is a fibration as above with* sign $M = 0$ *and* sign $E \neq 0$ *then*

(i) *$[M, f]$ generates a free $\overline{SK}_*$-submodule of $\widetilde{SK}_*(BG)$;*

(ii) *$[M, \mathrm{id}]$ generates a free $\overline{SK}_*$-submodule of $\widetilde{SK}_*(M)$.*

Proof. If $q + m \equiv 0$ modulo 4 then $\varphi: \widetilde{SK}_q(BG) \to \mathbb{Z}$ defined by $\varphi[N^q, g] = \operatorname{sign} g^*(EG \times_G F)$ is a homomorphism which takes the value $\operatorname{sign}(PC^{2r} \times E) = \operatorname{sign} E \neq 0$ on the element $[PC^{2r}][M, f] \in \widetilde{SK}_{n+4r}(BG)$. This element hence has infinite order in $\widetilde{SK}_*(BG)$, proving (i). Statement (ii) is immediate from (i) and the fact that $[M, f]$ is the image $f_*[M, \mathrm{id}]$ under the map $f_*: \widetilde{SK}_n(M) \to \widetilde{SK}_n(BG)$.

Remark. We can strengthen the above theorem to replace G in (i) by its discrete quotient G/G_0 by considering the local coefficient system over M with fibre $H^*(F)$ (which has structure group G/G_0) instead of the actual bundle. Alternatively one can consider the structure group of this coefficient system to be $\pi_1(M)$ or the image of $\pi_1(M)$ in G/G_0 and can replace G in (i) by either of these groups. By calculating $\widetilde{SK}_*(BH)$ modulo torsion for the following discrete groups we obtain the consequence:

THEOREM 12. *If G/G_0 is in the class of discrete groups H generated from the class of groups for which $\widetilde{H}_*(BH; \mathbb{Z})$ is torsion in even dimensions* (e.g. *finite groups and the infinite cyclic group*) *by forming cartesian and free products and finite extension groups* (*not necessarily normal*), *then any fibration with structure group G has multiplicative signature. The same holds for a fibration $F \to E \to M$ with $\pi_1(M)$ or* $\operatorname{Im}(\pi_1(M) \to G/G_0)$ *in the above class.*

Proof. By Theorem 11 and the remarks preceding and succeeding it, it suffices to show that $\widetilde{SK}_*(BH)$ is torsion for all the groups in question. This follows immediately from the facts that the classifying space of a cartesian respectively free product of discrete groups is the cartesian product respectively wedge of the classifying spaces, together with Lemmas 7, 8, 10, and the covering space argument in the proof of Theorem 1. A similar theorem can also be obtained using Atiyah's signature formula for fibrations[3] (see [12] for a detailed discussion).

Another consequence of Theorem 11 is the following calculation of SK for certain products of surfaces, giving us examples promised in the introduction. This calculation can in fact easily be extended to any product of closed orientable surfaces, since factors of genus ≤ 1 contribute nothing to the calculation, by Lemma 8.

PROPOSITION. *Let $X = F_1 \times F_2 \times \ldots \times F_r$ be the product of r surfaces each having sufficiently large genus* (*see below*; *genus ≥ 2 is in fact sufficient*). *Then $\widetilde{SK}_*(X)$ is the free SK_*-module on the $2^r - 1$ generators $[F_I, j_I]$ where I runs through the non-empty subsets of $\{1, \ldots, r\}$, $F_I = \prod_{i \in I} F_i$, and j_I is the inclusion of F_I in X.*

Proof. That the above elements generate $\widetilde{SK}_*(X)$ follows for $r = 1$ from Lemma 7 and then for $r > 1$ by induction using Lemma 8. We must thus only show that they are linearly independant over $\overline{SK}_*$.

We assume the genus of each F_i is sufficiently large that a bundle $E_i \to F_i$ exists with E_i a compact manifold of non-zero signature (genus 2 suffices by [11], using a local coefficient system instead of a bundle). For each subset $I \subseteq \{1, \ldots, r\}$ let

$$E_I \to X = F_1 \times \ldots \times F_r$$

be the bundle with total space

$$E_I = \prod_{i \in I} E_i \times \prod_{i \notin I} F_i.$$

Let

$$\varphi_I : \widetilde{SK}_*(X) \to \mathbb{Z}$$

be the homomorphism $\varphi_I[M, f] = \operatorname{sign} f^*(E_I)$. If we make $\mathbb{Z}$ into an $\overline{SK}_*$-module via sign: $\overline{SK}_* \to \mathbb{Z}$, then φ_I is $\overline{SK}_*$-linear. If one orders the subsets of $\{1, \dots, r\}$ in some fashion such that the subsets of size one come first, then those of size two, etcetera, then the matrix $(\varphi_I[F_J, j_J] | I, J \subset \{1, \dots, r\})$ is lower triangular with non-zero diagonal terms so it follows that the elements $[F_J, j_J]$ are linearly independant over $\overline{SK}_*$, as was to be shown.

We can now easily give an example to show that $SK_n(X)$ does not only depend on $\pi_1(X)$, in contrast to what was conjectured in [8].

EXAMPLE 3. Let X be as in the above proposition, $r > 1$, and let Y be X minus one point. Then $\widetilde{SK}_*(Y)$ is freely generated over SK_* by $2^r - 2$ elements, namely the same elements as for X, omitting the element $[F_I, j_I]$ with $I = \{1, \dots, r\}$. That these elements are generators follows from Lemma 7 and the fact that anything involving odd-dimensional homology of Y is zero in $\overline{SK}_*(Y)$, since circles always represent zero and the odd homology of Y all arises from products involving circles. They are independant since they are mapped to independant elements of $\widetilde{SK}_*(X)$ by the inclusion of Y in X.

One can even obtain a counterexample to the abovementioned conjecture in which Y is also a closed manifold as follows:

EXAMPLE 4. *For any finitely generated group π, we can find a manifold Y^n (any $n > 5$) having an open book decomposition and with fundamental group π.* Then $[Y, \mathrm{id}] = 0$ in $\widetilde{SK}_n(Y)$, so by Lemma 7 it follows that any element of $\widetilde{SK}_n(Y)$ is modulo torsion a linear combination (over $\overline{SK}_*$) of lower dimensional elements. This property distinguishes the SK groups of any such Y from those of the X in the proposition above. The manifold Y is constructed by taking a closed regular neighbourhood V^{n-1} of any 2-complex $K \subset \mathbb{R}^{n-1}$, $n > 5$, with $\pi_1(K) = \pi$, and putting $Y = (V \times S^1) \cup (\partial V \times D^2)$ pasted at the boundary.

Observe that the Example 3 actually shows that $SK_n(X)$ does not in general only depend on the $(n-1)$-homotopy type of X, so Theorem 9 is the best possible result.

REFERENCES

1. N. A'CAMPO: Feuilletages de codimension 1 sur les variété simplement connexes de dimension 5, *Comm. Math. Helv.* **47** (1972), 514–525.
2. J. W. ALEXANDER: A lemma on systems of knotted curves, *Proc. Nat. Acad. Sci. U.S.A.* **9** (1923), 93–95.
3. M. ATIYAH: The signature of fibre bundles, *Global Analysis*, Papers in Honor of K. Kodaira, Princeton (1969), 73–84.
4. P. E. CONNER and E. E. FLOYD: *Differentiable Periodic Maps*. Ergebnisse der Math. und Ihre Grezgebiete 33, Berlin–Heidelberg (1964).
5. M. HALL: *The Theory of Groups*. New York (1959).
6. F. HIRZEBRUCH: The signature of ramified coverings, *Global Analysis*, Papers in honor of K. Kodaira, Princeton (1969).
7. F. HIRZEBRUCH, W. D. NEUMANN and S. S. KOH: *Differentiable manifolds and quadratic forms*, Lecture Notes in Pure and Appl. Math., Vol. **4**. M. Dekker, New York (1971).
8. U. KARRAS, M. KRECK, W. D. NEUMANN and E. OSSA: *Cutting and pasting of manifolds; SK-groups*, Math. lecture Series 1. Publish or Perish, Boston (1973).
9. K. KODAIRA: A certain type of irregular algebraic surface, *J. d'Analyse math.* **19** (1967), 207–215.
10. S. MACLANE: *Homology*. Berlin–Heidelberg (1963).
11. W. MAYER: Die Signatur von lokalen Koeffizientensystemen und Faserbündeln, *Bonner Math. Schriften* **53** (1972).
12. W. D. NEUMANN: Signature related invariants of manifolds, (in preparation).
13. E. SPANIER: *Algebraic Topology*. McGraw-Hill (1966).
14. C. T. C. WALL: On simply connected 4-manifolds, *J. Lond. Math. Soc.* **39** (1964), 141–149.
15. H. E. WINKELNKEMPER: Manifolds as open books, *Bull A.M.S.* **79** (1973), 45–51.
16. H. E. WINKELNKEMPER: On equators of manifolds and the action of θ^n. Thesis. Princeton (1970).

Mathematisches Institut
Universität Bonn