197? Bonn notes of W. Neumann

Chapter II. Signature of hermitian coefficient systems; γ-invariant.

1. Signature of hermitian coefficient systems.

Let M^{2n} be a closed oriented manifold and $\Gamma \to M$ a local coefficient system (= locally trivial sheaf) over M with fiber a complex hermitian vector space (V,b). The hermitian structure can be interpreted as a map (also denoted by b) of sheaves

$$b\colon \Gamma \otimes_{\mathbb{R}} \Gamma \to \underline{\mathbb{C}} \quad \text{(the constant sheaf)}$$

whose restriction to a fiber is the given (not necessarily definite) hermitian form. This map induces a coefficient homomorphism

$$b^*\colon H^*(M;\, \Gamma \otimes_{\mathbb{R}} \Gamma) \to H^*(M;\mathbb{C})$$

and using the cup product map

$$H^n(M,\Gamma) \otimes H^n(M,\Gamma) \to H^{2n}(M,\Gamma \otimes_{\mathbb{R}} \Gamma)$$

we thus get a pairing

$$B\colon H^n(M;\Gamma) \otimes H^n(M,\Gamma) \to \mathbb{C}$$

defined by $B(x,y) = b_*(x \cup y)[M]$ ($[M]$ is the homology fundamental class) which is hermitian for n even and skew hermitian for n odd.

Definition. Define the signature of (M,Γ) as $\operatorname{sign}(M,\Gamma) := \operatorname{sign}(B)$.

Here if B is skew hermitian, $\operatorname{sign}(B)$ means the number of occurences of $-i$ minus the number of occurences of $+i$ in a diagonalization of B, that is signature of the hermitian form iB.

One can make the same definition in the case of non-empty boundary, using the cup product pairing on the relative group

$$B:\ H^n(M,\partial M;\Gamma) \otimes H^n(M,\partial M;\Gamma) \to \mathbb{C}.$$

This form is in general degenerate, with degeneracy (or "radical") equal to

$$\mathrm{rad}(B) := \{x \mid B(x,y)=0 \text{ for all } y\} = \mathrm{Ker}(H^n(M,\partial M;\Gamma) \to H^n(M;\Gamma)),$$

by Poincaré-Lefschetz duality (for duality with local coefficients see for instance [ ]). Thus one can also consider B as a non-degenerate form on $H^n(M,\partial M;\Gamma)/\mathrm{rad}(B) = \mathrm{Im}(H^n(M,\partial M;\Gamma) \to H^n(M;\Gamma))$. Either way $\mathrm{sign}(B)$ is defined and we call it $\mathrm{sign}(M,\Gamma)$.

If Γ is the trivial coefficient system $\underline{\mathbb{C}}$ with standard hermitian form $b(z,w) = z\bar{w}$, then $\mathrm{sign}(M,\underline{\mathbb{C}})$ is the usual signature $\mathrm{sign}(M)$, which is zero if n is odd.

2. Real bilinear coefficient systems.

One can make the same definition as above if we are given a local coefficient system Λ of real vector spaces with symmetric or antisymmetric bilinear form instead of a complex hermitian system. In order that the induced form

$$B: H^n(M,\partial M;\Lambda) \otimes H^n(M,\partial M;\Lambda) \to \mathbb{R}$$

be symmetric, so its signature $\operatorname{sign}(M,\Lambda)$ is defined, the form $b: \Lambda \otimes \Lambda \to \mathbb{R}$ must be $(-1)^n$-symmetric.

This can be reduced to the hermitian case as follows. Let $b_{\mathbb{C}}: (\Lambda \otimes \mathbb{C}) \otimes (\Lambda \otimes \mathbb{C}) \to \mathbb{C}$ be the $(-1)^n$-hermitian extension of b and take $\Gamma = \Lambda \otimes \mathbb{C}$ with hermitian form $b_{\mathbb{C}}$ if n is even and $-ib_{\mathbb{C}}$ if n is odd. The effect of this on the symmetric cup product form B is to hermitianize it, and then multiply it by $-i$ if n is odd (making it skew hermitian). This does not alter signature, so $\operatorname{sign}(M,\Gamma) = \operatorname{sign}(M,\Lambda)$.

There is thus no loss in just considering hermitian coefficient systems (at least for signature questions) so we shall do this. Note that real coefficient systems are what more usually arise "in nature", for instance by the following well known theorem.

Theorem 2.1. If $F^{2m} \to X^{2(m+n)} \to M^{2n}$, $m+n$ even, is a fibration of closed oriented manifolds and Λ is the corresponding local coefficient system over M with fiber the cup product from an $H^m(F;\mathbb{R})$, then $\operatorname{sign} X = (-1)^n \operatorname{sign}(M,\Lambda)$.

Proof. This is essentially Chern, Hirzebruch, Serre [ ] and follows by observing that $\operatorname{sign} X$ and $(-1)^n \operatorname{sign}(M,\Lambda)$ are respectively the signatures of the cup product forms on the E^∞ and E^2 terms of the Serre spectral sequence and are hence equal.

See also Meyer [ ].

With dimensions other than in the above theorem the signature of the total space X is always zero, by the same argument.

3. Basic properties of signature.

In this section we collect for easy reference the most important standard properties of signature of coefficient systems.

3.1. Elementary properties. Let $-\Gamma$ be Γ with negative hermitian form and $-M$ be M with orientation reversed. Then

$$\mathrm{sign}(M,\Gamma) = -\mathrm{sign}(-M,\Gamma) = -\mathrm{sign}(M,-\Gamma),$$

$$\mathrm{sign}(M,\Gamma_1 \oplus \Gamma_2) = \mathrm{sign}(M,\Gamma_2) + \mathrm{sign}(M,\Gamma_2).$$

These properties imply that for a trivial hermitian coefficient system $\underline{V}$ with fiber V

$$\mathrm{sign}(M,\Gamma \otimes \underline{V}) = \mathrm{sign}(M,\Gamma).\mathrm{sign}(V).$$

3.2. Bordism invariance. If $(M^{2n},\Gamma) = \partial(X^{2n+1},\Gamma')$ with X compact and oriented then $\mathrm{sign}(M,\Gamma) = 0$.

3.3. Novikov additivity. If $(M,\Gamma) = (M_1,\Gamma_1) \cup (M_2,\Gamma_2)$ pasted along boundary components of M_1 and M_2 then

$$\mathrm{sign}(M,\Gamma) = \mathrm{sign}(M_1,\Gamma_1) + \mathrm{sign}(M_2,\Gamma_2).$$

3.4. Product formula. If $\Gamma_1 \to M_1$ and $\Gamma_2 \to M_2$ are hermitian coefficient systems then

$$\mathrm{sign}(M_1 \times M_2,\ \Gamma_1 \times \Gamma_2) = \mathrm{sign}(M_1,\Gamma_1)\ \mathrm{sign}(M_2,\Gamma_2)$$

The standard proofs of these facts (see for instance [ ] for 3.2 and 3.4 and [ ] and [ ] for 3.3) carry over to the case with coefficient systems, see Meyer [ ] for details. 3.2 and 3.4 also follow from the signature theorem below. Note that 3.2 and 3.3

imply the stronger

3.2' Bordism invariance. If M has boundary then $\mathrm{sign}(M,\Gamma)$ is an invariant of bordism modulo boundary (i.e. keeping the boundary fixed).

Indeed, a bordism modulo boundary from say M to N can be re-interpreted as a zero-bordism of $M \cup (-N)$. In fact 3.2' also implies 3.2 and 3.3, since $(M \cup (-N)) \times [0,1]$ can be interpreted as a modulo boundary bordism from M to $N + (M \cup (-N))$.

To formulate the next theorem we need some preparation. If $\Gamma \to M$ is a hermitian coefficient system then Γ determines a hermitian vector bundle $\tilde{\Gamma} \to M$ (plus a flat structure on $\tilde{\Gamma}$ which we shall not need) by "putting the topology back into the fibres of Γ". This is possible by local triviality of Γ. This bundle can be split as the sum

$$\tilde{\Gamma} = \tilde{\Gamma}^+ \oplus \tilde{\Gamma}^-$$

of a positive definite and a negative definite hermitian bundle; – for instance, after choosing a metric $(\ ,\ )$ on $\tilde{\Gamma}$, the hermitian form b is given by $b(x,y) = (x,By)$ for some hermitian operator B and one takes $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ as the sum of the positive respectively negative eigenspaces of B. These bundles $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ are unique up to isomorphism, for given another such splitting $\Gamma = \tilde{\Gamma}_1^+ \oplus \tilde{\Gamma}_1^-$ then both $\tilde{\Gamma}^+$ and $\tilde{\Gamma}_1^+$ are complements of $\tilde{\Gamma}^-$ and hence isomorphic to $\tilde{\Gamma}/\tilde{\Gamma}^-$ and similarly for $\tilde{\Gamma}^-$ and $\tilde{\Gamma}_1^-$.

Theorem 3.5. ([ ], Signature theorem). If M^{2n} is a closed oriented manifold then $\mathrm{sign}(M,\Gamma) = \mathrm{ch}(\tilde{\Gamma}^+ - \tilde{\Gamma}^-)\,\mathcal{L}(M)[M]$. Here ch is chern character and $\mathcal{L}(M)$ is the unstable L-class given by $\prod_1^n x_i/\tanh(x_i/2)$ where $\prod_1^n (1+x_i)$ is the total Pontrjagin class $p(M)$ of M.

For real coefficient systems this theorem is also proved by Meyer [ ]. In this case the result can be reformulated as follows. If $\Lambda \to M^{2n}$ is a real coefficient system with $(-1)^n$-symmetric nondegenerate bilinear form b then b determines a splitting of the associated vector bundle $\Lambda = \tilde{\Lambda}^+ \oplus \tilde{\Lambda}^-$ for n even and a complex structure on $\tilde{\Lambda}$ for n odd as follows: with respect to a euclidian metric on $\tilde{\Lambda}$ one has $b(x,y) = (x,Ay)$ with A $(-1)^n$-symmetric, and one can normalize so that $A^2 = (-1)^n I$. Then the eigenspaces of A give the above splitting for n even and A determines a complex structure for n odd. An easy calculation (see Atiyah Singer [ ] pp.579-580) shows in the latter case that if $\Gamma = \Lambda \otimes \mathbb{C}$ with hermitian form $-ib_{\mathbb{C}}$ as in section 2, then $\tilde{\Gamma}^{\pm} \cong (\tilde{\Lambda}, \mp A)$. Thus the above theorem becomes

Corollary 3.6. $\operatorname{sign}(M,\Lambda) = ch(\tilde{\Lambda}^+ - \tilde{\Lambda}^-)\, \mathcal{L}(M)[M]$ (n even),

$= ch(\tilde{\Lambda},-A)-(\tilde{\Lambda},+A))\, \mathcal{L}(M)[M]$ (n odd).

This formula was proved directly for fiber bundles by Atiyah in [ ],which gives an alternative proof of 2.1.

Remark 1. Due to the sign $(-1)^n$ in Theorem 2.1 there is an apparant sign discrepancy with Atiyah [ ], which is however resolved by observing that the complex structure he uses is $(\tilde{\Lambda},-A)$ by [ ] p.574. Also our A is A^{-1} of Meyer [ ], introducing a sign discrepancy which is resolved by observing that his sign convention for $\operatorname{sign}(M,\Lambda)$ is different from ours if Λ is antisymmetric.

Remark 2. As pointed out by Meyer [ ], one can replace $\mathcal{L}$ by Hirzebruch's stable L-class $L = \prod_1^n x_i/\tanh(x_i)$ in the above if one simultaniously replaces ch by $ch^{(2)} = ch \circ \psi^2$ to re-absorb the powers of 2 which this introduces.

Remark 3. It is interesting to ask in what categories the signature theorem 3.5 is still valid. Putting $X = B(U(p,q)_{discrete})$ in the equality $\Omega_*^{SO}(X) \otimes \mathbb{Q} = \Omega_*^{BSTOP}(X) \otimes \mathbb{Q}$ shows that it is still valid in TOP. It is probably still valid for $\mathbb{Z}$-homology manifolds, though present knowledge (c.f. [ ], [ ]) of this category seems not yet quite enough to prove it. For $\mathbb{Q}$-homology manifolds it is not even known yet if signature is multiplicative for finite coverings, a fact which would follow from 3.5.

4. Multiplicativity of signature.

In this section M always denotes a closed connected oriented manifold. We are interested in the

Problem. Given a hermitian local coefficient system $\Gamma \to M$, under what conditions does the equality

(4.1) $$\operatorname{sign}(M,\Gamma) \overset{?}{=} \operatorname{sign}(M).\operatorname{sign}\Gamma$$

hold ? Here $\operatorname{sign}(\Gamma)$ means signature of a fiber of Γ.

This formula is false in general: for a coefficient system coming from a fibration $F \to X \to M$ it is the equation $\operatorname{sign}(X) = \operatorname{sign}(M).\operatorname{sign}(F)$, to which counterexamples were found by Atiyah [ ] and Kodaira [ ] and generalized variously by Hirzebruch [ ] and Meyer [ ]. Further counterexamples to 4.1 not necessarily coming from a fibration were given by Lusztig [ ] and Meyer [ ]. on the other hand we showed in [ ] that for a large class of structure groups 4.1 is in fact true. We shall prove this here in a stronger form, but we must first recall what a "structure group" for Γ is.

After choosing a base point $x \in M$, any local coefficient system $\Gamma \to M$ is classified by a homomorphism $\pi_1(M,x) \to \operatorname{Aut}(\Gamma_x)$ where Γ_x is the fiber over x.

Definition. If $\Gamma \to M$ is a hermitian local coefficient system with fiber V such that the classifying map $\pi_1(M) \to \operatorname{Aut}(V)$ can be written as a composition $\pi_1(M) \to G \overset{\rho}{\to} \operatorname{Aut}(V)$, we say Γ admits G as structure group with defining representation ρ. We call (G,ρ) a structure pair for Γ.

We plan to prove the following theorems

Theorem 4.2. Let $\mathcal{C}$ be the class of all groups G such that the multiplicativity formula 4.1 holds for any hermitian coefficient system $\Gamma \to M$ which admits G as structure group. Then

(i) $\mathcal{C}$ is closed under formation of direct products, free products, direct limits.

(ii) $\mathcal{C}$ contains all groups with $\tilde{H}^{ev}(BG;\mathbb{Q}) = 0$, in particular all finite groups and the infinite cyclic group are in $\mathcal{C}$. More generally:

(iii) If $H \in \mathcal{C}$ and $H \to G$ is a group homomorphism for which $H^*(BG;\mathbb{Q}) \to H^*(BH;\mathbb{Q})$ is injective in even dimensions then $G \in \mathcal{C}$. In particular $\mathcal{C}$ is closed under formation of finite (not necessarily normal) extensions and quotient by finite normal subgroups.

Theorem 4.3. For any $\Gamma \to M$ with structure group $G \in \mathcal{C}$ and for any further hermitian coefficient system $\Lambda \to M$ the following generalization of 4.1 holds: $\operatorname{sign}(M,\Lambda \otimes \Gamma) = \operatorname{sign}(M,\Lambda)$ $\operatorname{sign} \Gamma$.

We shall prove these theorems later, we first need some preparations.

For any group G, recall that RU(G) denotes the Grothendieck group of representations of G in finite dimensional non-degenerate hermitian vector spaces (Chapter I, section 4) with ring structure given by orthogonal sum and tensor product. Define a ring homomorphism

$$\Psi_G: RU(G) \to H^*(BG;\mathbb{Q})$$

as follows. Any hermitian representation $\rho: G \to \mathrm{Aut}(V)$ determines a hermitian coefficient system $\Gamma_\rho \to BG$ with fiber V. Let $\tilde{\Gamma}_\rho = \tilde{\Gamma}^+_\rho \oplus \tilde{\Gamma}^-_\rho$ be a splitting of the corresponding bundle into a positive and a negative definite summand and put

$$\Psi_G(\rho) = \mathrm{ch}(\tilde{\Gamma}^+_\rho - \tilde{\Gamma}^-_\rho) \in H^*(BG;\mathbb{Q}).$$

Lemma 4.4. If $\rho: G \to \mathrm{Aut}(V)$ is hyperbolic (i.e. there exists a G-invariant subspace $K \subset V$ with $K = K^\perp$, whence $\dim(K) = \dim(V)/2$) then $\Psi_G(\rho) = 0$.

Proof. Let $\tilde{\Gamma}_\rho = \tilde{\Gamma}^+_\rho \oplus \tilde{\Gamma}^-_\rho$ be the above splitting of the bundle $\tilde{\Gamma}_\rho \to BG$ defined by ρ. The G-invariant subspace $K \subset V$ defines a subbundle $\tilde{K} \subset \tilde{\Gamma}_\rho$, and since the hermitian form is zero on $\tilde{K}$, we must have $\tilde{K} \cap \tilde{\Gamma}^+_\rho = \tilde{K} \cap \tilde{\Gamma}^-_\rho = 0$. Thus $\dim(\tilde{\Gamma}^+_\rho) \leq \dim(\tilde{\Gamma}_\rho) - \dim \tilde{K} = \dim(V)/2$, and the same for $\dim(\tilde{\Gamma}^-_\rho)$. It follows that $\dim(\tilde{\Gamma}^\pm_\rho) = \dim(V)/2$, so $\tilde{\Gamma}^+_\rho$ and $\tilde{\Gamma}^-_\rho$ are both complements of $\tilde{K}$, hence both isomorphic to $\tilde{\Gamma}_\rho/\tilde{K}$. Thus $\Psi_G(\rho) = \mathrm{ch}(\tilde{\Gamma}_\rho/K - \tilde{\Gamma}_\rho/K) = 0$.

Corollary 4.5. Ψ_G induces a map (also called Ψ_G)

$$\Psi_G: WU(G) \to H^*(BG;\ \mathbb{Q}),$$

where $WU(G)$ is the Witt group obtained by factoring $RU(G)$ by the ideal generated by hyperbolic representations.

The motivation for introducing this natural transformation Ψ is the following theorem.

Theorem 4.6. Given a hermitian representation $\rho: G \to Aut(V)$, the following statements are equivalent.

(i) $\Psi_G(\rho) \in H^\circ(BG;Q)$

(ii) For any hermitian coefficient system $\Gamma \to M$ over a closed manifold admitting (G,ρ) as structure pair and any further hermitian system $\Lambda \to M$ we have $sign(M,\Lambda \otimes \Gamma) = sign(M,\Lambda) sign(\Gamma)$.

(iii) For any $\Gamma \to M$ as in (ii) we have $sign(M,\Gamma) = sign(M) \cdot sign($

Proof. (i) $\Rightarrow$ (ii). Suppose Γ is as in (ii) and $\pi_1(M) \to G$ is a classifying map. This induces a map $f: M \to BG$ and then $\Gamma \cong f^*\Gamma_\rho$, where $\Gamma_\rho \to BG$ is the coefficient system determined by ρ. Thus if (i) holds then $ch(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = f^*\Psi_G(\rho)$ is contained in $H^\circ(M;\mathbb{Q})$. But for any vector bundle E the zero-dimensional component of $ch(E)$ is $(\dim E).1 \in H^0(M;\mathbb{Q})$, so $ch(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = (\dim\tilde{\Gamma}^+ - \dim\tilde{\Gamma}^-).1 = (sign\Gamma).1$. Thus $ch((\tilde{\Lambda} \otimes \tilde{\Gamma})^+ - (\tilde{\Lambda} \otimes \tilde{\Gamma})^-) = ch(\tilde{\Lambda}^+ - \tilde{\Lambda}^-).ch(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = sign(\Gamma).\, ch(\tilde{\Lambda}^+ - \tilde{\Lambda}^-)$, so the multiplicativity formula follows from the signature theorem 3.5.

(ii) => (iii) is trivial by taking $\Lambda = \underline{\mathbb{C}}$, the trivial coefficient system.

(iii) => (i). Suppose (i) is false, so $\Psi_G(\rho) = (sign\Gamma_\rho).1 + \alpha + \beta$ with $0 \neq \alpha \in H^{2n}(BG;Q)$, $n > 0$, and β a possibly zero sum of terms of higher degree. Choose a closed oriented singular manifold $f: M^{2n} \to BG$ such that $\alpha(f_*[M]) \neq 0$. This is possible by Steenrod representability of rational homology, see for instance Conner and Floyd [,theorem 15,3]. Now for $\Gamma = f^*\Gamma_\rho$ we have

$$ch(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = (sign\Gamma).1 + f^*(\alpha),$$

since $H^*(M)$ is zero in degrees $>2n$. Also

$$\mathcal{L}(M) = 2^n.1 + \ldots + \mathcal{L}_{2n}(M) ,$$

so by theorem 3.5

$$\begin{aligned} \operatorname{sign}(M,\Gamma) &= ((\operatorname{sign}\Gamma).1 + f^*(\alpha))\,\mathcal{L}(M)[M] \\ &= \operatorname{sign}\Gamma.\mathcal{L}_{2n}(M)[M] + 2^n f^*(\alpha)[M] \\ &= \operatorname{sign}\Gamma \,.\, \operatorname{sign}(M) + 2^n.\,\alpha(f_*[M]) \\ &\neq \operatorname{sign}\Gamma.\operatorname{sign}(M). \end{aligned}$$

Q.E.D.

<u>Corollary 4.7</u>. The class $\mathcal{C}$ of theorem 4.2 is just the class of all G for which $\operatorname{Im}\Psi_G \subset H^0(BG;\mathbb{Q})$.

Note that theorem 4.3 is a consequence of the equivalence (ii) <=> (iii) in theorem 4.6. We are also now ready to prove 4.2.

To prove closure of $\mathcal{C}$ under free products consider the commutative diagram

$$\begin{array}{ccc} WU(G * H) & \xrightarrow{\ \Psi\ } & H^*(B(G * H);\mathbb{Q}) \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ WU(G) \oplus WU(H) & \xrightarrow{\ \Psi \oplus \Psi\ } & H^*(BG;\mathbb{Q}) \oplus H^*(BH;\mathbb{Q}) , \end{array}$$

where α and β are induced by the inclusions $G \to G * H$ and $H \to G * H$. The maps $BG \to B(G * H)$ and $BH \to B(G * H)$ induce a homotopy equivalence $BG \vee BH \to B(G * H)$, so β is an isomorphism in positive dimensions. Hence if $G, H \in \mathcal{C}$ then $\operatorname{Im}\Psi_{G*H} \subset \beta^{-1}(\operatorname{Im}\Psi_G \oplus \operatorname{Im}\Psi_H) \subset H^0(B(G * H);\mathbb{Q})$, so $G * H \in \mathcal{C}$.

To see closure under direct products we use the commutative diagram

$$\begin{array}{ccc} WU(G) \otimes WU(H) & \xrightarrow{\Psi \otimes \Psi} & H^*(BG;\mathbb{Q}) \otimes H^*(BH;\mathbb{Q}) \\ \downarrow t & & \downarrow k \\ WU(G \times H) & \xrightarrow{\Psi} & H^*(B(G \times H);\mathbb{Q}) \end{array}$$

where t is the map induced by tensor product of representations and k is the Künneth map given by $BG \times BH = B(G \times H)$. Closure under direct products follows if t is surjective, but t is actually an isomorphism by theorem 4.4 of chapter I.

Finally for direct limits we use the diagram

$$\begin{array}{ccc} WU(\varinjlim G_i) & \xrightarrow{\Psi} & H^*(B(\varinjlim G_i);\mathbb{Q}) \\ \downarrow & & \downarrow \\ \varprojlim WU(G_i) & \xrightarrow{\varprojlim \Psi} & \varprojlim H^*(BG_i;\mathbb{Q}) \end{array}$$

and observe that the right vertical arrow is an isomorphism since our coefficients are a field, so closure of $\mathcal{C}$ under $\varinjlim$ follows.

To prove (iii) note that in the diagram induced by $H \to G$

$$\begin{array}{ccc} WU(G) & \xrightarrow{\Psi} & H^{ev}(BG;\mathbb{Q}) \\ \downarrow & & \downarrow \\ WU(H) & \xrightarrow{\Psi} & H^{ev}(BG;\mathbb{Q}), \end{array}$$

if the right vertical arrow is injective then $G \in \mathcal{C}$ follows from $H \in \mathcal{C}$. In particular, if $H \to G$ has finite kernel and image of finite index this is so, since then $BH \to BG$ is rationally a finite covering (the fiber is $B(\ker) \times H/\mathrm{Im}G$ which has the rational homology type of $H/\mathrm{Im}G$). As remarked in the theorem, (ii) is a special case of (iii) (take $H = 1$).

Another corollary of 4.6 is

<u>Corollary 4.8</u>. If $\Gamma \to M$ is a hermitian coefficient system with definite hermitian form on the fiber, then the multiplicativity statement 4.6 iii) holds.

Proof. Assume Γ is positive definite. Then $\tilde{\Gamma}^- = 0$ and $\tilde{\Gamma}^+ = \Gamma$ is a flat bundle with flat hermitian metric, so by the Chern Weil description of rational chern classes (see for instance [, appendix C, corollary 2]), $ch(\tilde{\Gamma}) \in H^0(M;\mathbb{Q})$, as was to be proved.

Before discussing some examples to theorem 4.2, let us consider the relationship to the analogous result in [ ]. In that paper and in [ ] a graded group $\overline{SK}_*(X)$ is defined as the group of singular manifolds in X modulo bordism and "cutting and pasting". Equivalently (by [ ]), $\overline{SK}_n(X) = \Omega_n(X)/F_n(X)$, where $F_n(X)$ is the subgroup of elements in $\Omega_n(X)$ representable by an (M,f) for which M can be fibered over S^1. This group $\overline{SK}_*(X)$ is a module over the ring $\overline{SK}_* = \overline{SK}_*(\text{point})$, which is a polynomial ring $\mathbb{Z}[P]$ in one 4-dimensional generator P, representable by any 4-manifold of signature 1.

Definition. Let $\mathcal{C}_0$ be the class of groups G for which the reduced SK-group $\widetilde{SK}_*(BG) = \mathrm{Ker}(\overline{SK}_*(BG) \to \overline{SK}_*)$ does not contain a free $\overline{SK}_*$-submodule. That is $\widetilde{SK}_*(BG)$ is a $\overline{SK}_*$-torsion module.

In [ ] it was shown by a simple geometric argument that signature is multiplicative for structure groups in $\mathcal{C}_0$, so $\mathcal{C}_0 \subset \mathcal{C}$. In fact

Proposition 4.9. $\mathcal{C}_0 \subset \mathcal{C}$ and $\mathcal{C}_0$ satisfies properties (i), (ii), (iii) of theorem 4.2.

Except for closure under direct limits, (i) and (ii) were proved in [ ]. Closure under direct limits and (iii) are proved similarly using in particular lemma 7 of [ ].

This proposition suggests the immediate

Problem. Is $\mathcal{C}_0 = \mathcal{C}$?

Examples 4.10. By 4.8, the class $\mathcal{C}_{\text{def}}$ of all groups G for which WU(G) is generated by definite hermitian representations (equivalently: irreducible hermitian representations are definite, by 2.3 in chapter I) is a subclass of $\mathcal{C}$. This class $\mathcal{C}_{\text{def}}$ contains all finite and abelian groups and is closed under direct products, formation of arbitrary quotient groups but not under free products.

Examples 4.11. Some groups obviously in $\mathcal{C}$ are free groups, abelian groups, $SL(2,\mathbb{Z})$ (since $\tilde{H}^*(SL(2,\mathbb{Z});\mathbb{Q}) = 0$), etc. Some groups not in $\mathcal{C}$ are fundamental groups of surfaces of genus >1, $SL(2,\mathbb{Z}[\frac{1}{2}])$, etc., since these groups occur as structure groups with non-multiplicative signature (see [ ] and example 5.6).

Less obvious is the following example

Example 4.12. If G has a finitely generated free abelian normal subgroup with infinite cyclic quotient, then $\widetilde{SK}_*(BG) = 0$, so $G \in \mathcal{C}_0$, so $G \in \mathcal{C}$.

The proof follows from the following two facts and is left to the reader: BG can be taken as a torus bundle over S^1;

a set of singular manifolds in BG which represents a generating set of $H_*(BG;\mathbb{Q})$ represents a $\overline{SK}_*$-module generating set of $\overline{SK}_*(BG)$ up to torsion (lemma 7 of [ ]). A direct algebraic proof that these groups are in $\mathcal{C}$ would be of interest.

Remark. This last example can be used to show that the closure properties (i), (ii), (iii), of $\mathcal{C}_0$ and $\mathcal{C}$ in theorem 4.2 are not enough to generate $\mathcal{C}_0$ or $\mathcal{C}$ out of the trivial group (and hence not enough to prove $\mathcal{C}_0 = \mathcal{C}$).

5. Definition of γ-invariants.

In this scetion we define certain "γ-invariants" of odd dimensional manifolds. Later we will show how they are related to the α-invariant for group actions and also give a useful homotopy invariant description in some cases.

We have seen (4.2, 4.6, 4.8) that under certain conditions signature of a hermitian coefficient system is multiplicative. These multiplicativity results fail for manifolds with boundary, however the error to multiplicativity turns out to be an invariant of the boundary. This is the invariant we wish to define and study. Suppose theorefore that we have the following data.

Data 5.1. Let X^{2n-1} be a closed oriented manifold and $f: \pi_1(X) \to G$ a homomorphism to a group G. This homomorphism is induced by a (unique up to homotopy preserving base points) map $X \to BG$, which we also denote by f. We assume some multiple of (X, f) bounds in BG.

Definition 5.2. Given the above data and any hermitian representation ρ of G. Choose (M,g) with $\partial(M,g) = q(X,f)$, the union of q copies of (X,f), and put

$$\gamma_\rho(X,f) := \frac{1}{q}(\mathrm{sign}(M,g^*\Gamma_\rho) - \mathrm{sign}(M).\mathrm{sign}(\rho))$$

where $\Gamma_\rho \to BG$ is the hermitian coefficient system defined by ρ. In general $\gamma_\rho(X,f)$ will not be well defined, but in the situation of theorem 4.6 it is:

Theorem 5.3. If $\Psi_G(\rho) = 0$, for instance if $G \in \mathcal{C}$ or ρ is a definite representation, then $\gamma_\rho(X,f)$ is well defined (independant of the choice of (M,g)).

Proof. This is a standard argument. Namely suppose $r(X,f)$ bounds a different pair (M_1,g_1). Then $r(M,g)$ and $q(M_1,g_1)$ have the same boundary, namely $qr(X,f)$, so we can paste to get a closed singular manifold in BG,

$$N = (rM \cup q(-M_1)) \xrightarrow{h = g \cup g_1} BG.$$

Multiplicativity 4.6 tells us that

$$\text{sign}(N, h^* \Gamma_\rho) = \text{sign}(N).\,\text{sign}(\rho).$$

On the other hand we can rewrite this equation using Novikov additivity 3.3 as

$$r.\text{sign}(M, g^*\Gamma_\rho) - q.\text{sign}(M_1, g_1^*\Gamma_\rho) = (r.\text{sign}\,M - q.\text{sign}\,M_1)\text{sign}\,\rho.$$

Dividing by qr and rearranging gives the required result:

$$\frac{1}{q}(\text{sign}(M, g^*\Gamma_\rho) - \text{sign}(M).\text{sign}(\rho)) = \frac{1}{r}(\text{sign}(M_1, g_1^*\Gamma_\rho) - \text{sign}(M_1).\text{sign}(\rho)).$$

One might expect that in the above situation $\gamma_\rho(X^{2n-1},f)$ only depends on the representation ρf of $\pi_1(X)$ and not on G. Though this is true for the cases of most interest to us, as the following theorem shows, we shall later show it is false in general.

Theorem 5.4. If one of conditions a), b) below holds then $\gamma_\rho(x,f)$ only depends in the representation ρf of $\pi_1(x)$ and not on G. We then often write $\gamma(x,\rho f)$ for $\gamma_\rho(x,f)$.

a) ρ is a definite representation.

b) The center of G has finite index in G (e.g. G abelian or finite).

Proof. a) is clear, since in this case in the proof of 5.3 we use the multiplicativity result 4.8, which does not depend on G. For b) we shall use the following lemma.

Lemma 5.5. Given a closed oriented manifold X^{2n-1} and homomorphisms $\pi_1(X) \xrightarrow{f} G \xrightarrow{\varphi} H \xrightarrow{\tau} \mathrm{Aut}(V)$, with $G, H \in \mathcal{C}$, then

(i) If $\gamma_{\tau\varphi}(X,f)$ is defined, then so is $\gamma_\tau(X,\varphi f)$, and they are equal.

(ii) If $\varphi: G \to H$ is injective in rational homology and $\gamma_\tau(X,\varphi f)$ is defined, then so is $\gamma_{\tau\varphi}(X,f)$, and they are equal.

Proof. (i) If $\gamma_{\tau\varphi}(X,f)$ is defined, that is $q(X,f) = \partial(M,g)$ for some $q > 0$ and $g: M \to BG$, then $q(X,\varphi f) = \partial(M,\varphi g)$, so $\gamma_\tau(X,\varphi f)$ is defined and the equality $\gamma_{\tau\varphi}(X,f) = \gamma_\tau(X,\varphi f)$ is clear from the definition.

(ii) The modulo torsion triviality of the bordism spectral sequence implies that the condition $\varphi_*: H_*(G;\mathbb{Q}) = H_*(BG;\mathbb{Q}) \to H_*(BH;\mathbb{Q}) = H_*(H;\mathbb{Q})$ injective is equivalent to $\Omega_*(BG) \otimes \mathbb{Q} \to \Omega_*(BH) \otimes \mathbb{Q}$ injective. Thus if $\gamma_\tau(X,\varphi f)$ is defined, that is $[X,\varphi f] = 0$ in $\Omega_*(BH) \otimes \mathbb{Q}$, then $[X,f] = 0$ in $\Omega_*(BG) \otimes \mathbb{Q}$, so $\gamma_{\tau\varphi}(X,f)$ is defined. It is equal to $\gamma_\tau(X,\varphi f)$ by (i).

To return to the proof of 5.4, suppose we have $\pi_1(X) \xrightarrow{f} G \xrightarrow{\rho} \mathrm{Aut}(V)$ such that the center of G has finite index in G (whence in particular $G \in \mathcal{C}$), and suppose some multiple of (X,f) bounds, so $\gamma_\rho(X,f)$ is defined. Put $H = \mathrm{Im}(\rho)$ and $K = \mathrm{Im}(\varphi f) \subset H \subset \mathrm{Aut}(V)$, so we have a diagram

$$\begin{array}{ccc}
\pi_1(X) & \xrightarrow{\psi} & K \\
\downarrow f & & \downarrow i \\
G & \xrightarrow{\varphi} & H \\
& \overset{\rho}{\searrow} & \downarrow j \\
& & \mathrm{Aut}(V)
\end{array}$$

where i and j are the inclusions and ψ and φ are just ρf and ρ with their ranges restricted.

Suppose we know that $i_*: H_*(K;\mathbb{Q}) \to H_*(H,\mathbb{Q})$ is injective. Then we can apply lemma 5.5 parts (i) and (ii) successively to show that $\gamma_j(X,\varphi f)$ is defined and equals $\gamma_\rho(X,f)$ and that $\gamma_{ji}(X,\psi)$ is defined and equals $\gamma_j(X,\varphi f)$. Thus $\gamma_\rho(X,f) = \gamma_{ji}(X,\psi)$ and since ji and ψ only depend on ρf and not on G, the theorem is proved. Thus we must just show the injectivity of i_*.

Now both K and H have centers of finite index, since this property is inherited by subgroups and quotient groups. We first show that in the commutative diagram

$$\begin{array}{ccc}
H_*(Z(K) \cap Z(H);\mathbb{Q}) & \to & H_*(Z(H);\mathbb{Q}) \\
\downarrow & & \downarrow \\
H_*(K,\mathbb{Q}) & \to & H_*(H;\mathbb{Q})
\end{array}$$

the vertical arrows are isomorphisms. Indeed, in the Lyndon spectral sequence $E_2^{pq} = H^p(H/Z(H);\ H^q(Z(H);\mathbb{Q})) \Rightarrow H^{p+q}(H;\mathbb{Q})$ (see e.g. MacLane [ ] p.351) we have $E_2^{p,q} = 0$ for $p > 0$, since $H/Z(H)$ is finite and acts trivially on $H^q(Z(H);\mathbb{Q})$. Hence $H^*(H;\mathbb{Q}) \to H^*(Z(H);\mathbb{Q})$ is an isomorphism, so the homology map also is. The same argument holds for $Z(K) \cap Z(H) \subset K$.

Now an injection of abelian groups induces an injection in $\mathbb{Q}$-homology (this is clear for finitely generated abelian groups and homology commutes with direct limits), so the top map of the square is injective, so the bottom map is too, as was to be shown.

The following example shows that some condition is necessary for theorem 5.4.

Example 5.6. We shall give an example with $X = S^1$ and G a free group and $\gamma_\rho(X,f) \neq 0$ although ρf is trivial. Let $\Gamma \to F$ be a hermitian coefficient system with non-zero signature over a closed surface F (exists by [ ],[ ],[ ]), and let $m = F - D^2$ for some embedding $D^2 \subset F$. Then $\mathrm{sign}(M,\Gamma|M)) = \mathrm{sign}(F,\Gamma)$ by Novikov additivity. If we put $G = \pi_1(M)$ then G is free so $G \in \mathcal{C}$. Let $\rho\colon G \to \mathrm{Aut}(V)$ be a classifying homomorphism for $\Gamma|M$ and let $f\colon \pi_1(S^1) \to G$ be induced by $S^1 = \partial M \subset M$. Then we can use M to calculate $\gamma_\rho(S^1,f)$, so by definition $\gamma_\rho(S^1,f) = \mathrm{sign}(M,\Gamma|M) - \mathrm{sign}\, M\, \mathrm{sign}\,\Gamma = \mathrm{sign}(M,\Gamma|M) = \mathrm{sign}(F,\Gamma) \neq 0$. But ρf is clearly trivial, since it classifies $\Gamma|\partial M = \Gamma|\partial D^2$.

To give a numerical example we can use the example of Meyer [,p.55] which gives:

$$G = \langle a_1, a_2, b_1, b_2 \rangle \quad \text{(free on 4 generators);}$$

$f\colon \pi_1(S^1) \to G$ defined by

$$f(1) = [a_1,b_1]\,[a_2,b_2] \quad ([a,b] = aba^{-1}b^{-1});$$

$\rho\colon G \to \mathrm{Aut}(\mathbb{C}^2, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix})$ defined by

$$\rho(a_1) = \begin{pmatrix} -5 & 1 \\ -27/2 & 5/2 \end{pmatrix},\ \rho(a_2) = \begin{pmatrix} -4 & -1 \\ 33 & 8 \end{pmatrix},\ \rho(b_1) = \begin{pmatrix} -4 & 1 \\ -33 & 8 \end{pmatrix},$$
$$\rho(b_2) = \begin{pmatrix} -5 & -1 \\ 27/2 & 5/2 \end{pmatrix}.$$

Then ρf is trivial but $\gamma_\rho(S^1,f) = 4$.

To close this section we mention for later reference some simple properties of the γ-invariants.

Suppose we have $G \in \mathcal{C}$ and $X^{2n-1} \xrightarrow{f} BG$ representing torsion in $\Omega_*(BG)$, so $\gamma_\rho(X,f)$ is defined for any hermitian representation ρ of G. Then:

<u>5.7. Additivity</u>. $\gamma_{\rho \oplus \tau}(X,f) = \gamma_\rho(X,f) + \gamma_\tau(X,f)$.

<u>5.8. Multiplicativity</u>. Given $H \in \mathcal{C}$ with hermitian representation τ and $M^{2m} \xrightarrow{g} BH$, then $\gamma_{\rho \otimes \tau}(X \times M,\ f \times g) =$ $\mathrm{sign}(M, g^*\Gamma_\tau) . \gamma_\rho(X,f)$. In particular (taking $H = \{1\}$): $\gamma_\rho(X \times M,\ f \circ pr_1) = \mathrm{sign}(M) . \gamma_\rho(X,f)$.

<u>5.9. Bordism invariance mod $\mathbb{Z}$</u>. $(\gamma_\rho(X,f)$ modulo $\mathbb{Z}) \in \mathbb{Q}/\mathbb{Z}$ is a bordism invariant of (X,f) in $\Omega_*(BG)$.

The proofs are trivial from the definition of γ; for 5.8 one must use the multiplicativity of signature 3.4.

6. Extending the definition of γ.

For $G \in \mathcal{C}$ and [1] $f: X^{2n-1} \to BG$ we have so far only defined the invariant $\gamma_\rho(X,f)$ if $[X,f] \in \Omega_{2n-1}(BG)$ is torsion. If G is finite this is an empty condition and in some other cases (Theorem 5.4) we can weaken it somewhat, but in general it is a genuine restriction. We shall show that in some cases the definition of γ admits a natural extension. The basic principal is as follows.

Definition 6.1. Suppose we have a $(\Omega_* \otimes \mathbb{Q})$-module basis $\{[Y_i,g_i] \mid i \in I\}$ of $\Omega_{odd}(BG) \otimes \mathbb{Q}$, where I is some index set. Suppose further some number $\bar{\gamma}_\rho(Y_i,g_i)$ is given for each i. Then for any (X^{2n-1},f), some multiple $q(X,f)$ is bordant modulo torsion to a disjoint union $\sum_i M_i \times (Y_i,g_i)$ (only finitely many $M_i \neq 0$) and we define

$$\bar{\gamma}_\rho(X,f) = \frac{1}{q}\{\gamma_\rho[(X,f) + \sum_i(-M_i) \times (Y_i,g_i)] + \sum_i \operatorname{sign}(M_i)\bar{\gamma}_\rho(Y_i,g_i)\}.$$

Here the disjoint union $(X,f) + \sum_i(-M_i) \times (Y_i,g_i)$ represents torsion in $\Omega_*(BG)$, so γ_ρ of it is defined.

In other words we define $\bar{\gamma}$ by requiring that it extend γ_ρ and still be additive with respect to disjoint union and have the multiplicative property $\bar{\gamma}_\rho(M \times (X,f)) = \operatorname{sign}(M) \cdot \bar{\gamma}_\rho(X,f)$. We must check that the above $\bar{\gamma}_\rho(X,f)$ is well defined, that is, if some multiple $q'(X,f)$ is bordant modulo torsion to a different disjoint union $\sum_i M'_i \times (Y_i,g_i)$, then using this union gives the same

---

1) Recall that we use the map $X \to BG$ and the homomorphism $\pi_1(X) \to G$, which induce each other, completely interchangably.

value for $\bar{\gamma}_\rho(X,f)$. But we can assume $q = q'$, and then M_i is bordant modulo torsion to M_i' for each i (since the $[Y_i, g_i]$ are a $(\Omega_* \otimes \mathbb{Q})$-module basis), so $\operatorname{sign}(M_i) = \operatorname{sign}(M_i')$ and the well-definition boils down to showing $\gamma_\rho((M_i + (-M_i')) \times (Y_i, g_i)) = 0$. But more generally: $\gamma_\rho(M \times (Y,g)) = 0$ if M is a boundary modulo torsion follows directly from the definition of γ_ρ and the multiplicativity formula 3.4.

From now on we always assume our representation ρ is unitary (= positive definite hermitian). We intend to give a "good" definition of $\bar{\gamma}_\rho(X,f)$ whenever G has center of finite index, but we first look at the following simple case, which in fact is the only case we will need later.

<u>Special Case, $G = \mathbb{Z}$.</u> Then $BG = S^1$ and $\Omega_{odd}(BG) \otimes \mathbb{Q}$ is the free $(\Omega_* \otimes \mathbb{Q})$-module on the one generator $[S^1, id]$. In this case we have the suggestive Proposition which will be proved in chapter III.

<u>Proposition 6.2.</u> If $\tau_\theta : \mathbb{Z} \to U(1)$ is the unitary representation $\tau_\theta(1) = (e^{i\theta}) \in U(1)$, then

$$\gamma(S^1, \tau_\theta) = 1 - \frac{\theta}{\pi}, \quad 0 < \theta < 2\pi$$
$$= 0, \qquad \theta = 0,$$

where θ/π must be rational for $\gamma(S^1, \tau_\theta)$ to be defined.

We are here using the notation $\gamma_\rho(X,f) = \gamma(X, \rho f)$, allowable by theorem 5.4. The footnote on the previous page is relevant here.

In view of this proposition, the following definition is natural.

Definition 6.3. We define for ρ irreducible

$$\bar{\gamma}_\rho(S^1,\mathrm{id}) = 1 - \frac{\theta}{\pi} \quad \text{if} \quad \rho = \tau_\theta \qquad 0 < \theta < 2\pi$$
$$= 0 \quad \text{if} \quad \rho = \tau_0 .$$

More generally any unitary representation $\rho\colon \mathbb{Z} \to U(n)$ is a sum of representations of type τ_θ and we define $\bar{\gamma}_\rho(S^1,\mathrm{id})$ to be the corresponding sum of numbers $1 - \theta/\pi$ or 0.

Theorem 6.4. Use definitions 6.1 and 6.3 to define $\bar{\gamma}_\rho(X^{2n-1},f)$ for any $f\colon X \to B\mathbb{Z} = S^1$ and any unitary representation ρ of $\mathbb{Z}$. That is

$$\bar{\gamma}_\rho(X,f) = \gamma_\rho[(X,f) + (-M) \times (S^1,\mathrm{id})] + \mathrm{sign}(M) . \bar{\gamma}_\rho(S^1,\mathrm{id}),$$

where M is such that $M \times (S^1,\mathrm{id})$ is bordant modulo torsion to (X,f). Then $\bar{\gamma}_\rho(X,f) = \gamma(X,\rho f)$ whenever the latter is defined (so we also write $\bar{\gamma}(X,\rho f)$).

This follows directly from the definitions since $\gamma(X,\rho f)$ is defined only if $\mathrm{Im}(\rho f)$ is finite or if $\gamma_\rho(X,f)$ was already defined.

We now look at the more general case.

G has center of finite index.

Lemma 6.5. In this case $\Omega_*(BG) \otimes \mathbb{Q}$ has a $(\Omega_* \otimes \mathbb{Q})$-module basis of elements of the form $[T^k,g]$, where $g\colon T^k = B(\mathbb{Z}^k) \to BG$ is the map induced by an inclusion $\mathbb{Z}^k \to Z(G) \subset G$.

Proof. By the triviality of the bordism special sequence tensored with $\mathbb{Q}$ it suffices to show that $H_*(BG) \otimes \mathbb{Q}$ has a basis of elements of the form $\mu[T^k, g]$ where $\mu: \Omega_*(BG) \to H_*(BG)$ is the natural map (namely $\mu[T^k, g] = g_*[T^k]$ where $[T^k]$ is the fundamental homology class). This is certainly true for G finitely generated abelian, hence also for a direct limit of such groups, that is arbitrary abelian. Hence it is also true if G has center of finite index, since then the inclusion $Z(G) \subset G$ induces an isomorphism in rational homology by the proof of 5.5.

Thus to carry out Definition 6.1 we must define $\bar{\gamma}(T^k, \rho g)$ for unitary representations ρg of $\mathbb{Z}^k$. For $k = 1$ this has been done above. In general we make the following definition.

Definition 6.6. Define $\bar{\gamma}_\rho(T^k, g) = 0$ for $k > 1$ and $\bar{\gamma}_\rho(S^1, g)$ to be $\bar{\gamma}(S^1, \rho g)$ as defined above.

This is reasonable in view of the following lemma.

Lemma 6.7. If $\gamma(T^k, \rho)$ is defined, where ρ is a unitary representation of $\mathbb{Z}^k$ and $k > 1$, then $\gamma(T^k, \rho) = 0$.

Proof. If $\gamma(T^k, \rho)$ is defined it is because $\mathrm{Im}\, \rho$ is finite. Without loss of generality ρ is irreducible, hence 1-dimensional, so $\mathrm{Im}\, \rho$ is cyclic, so after a change of coordinates in T^k the pair (T^k, ρ) has the form $T^{k-1} \times (S^1, \rho)$. Since T^{k-1} bounds, this has zero γ-invariant by the multiplicativity formula 3.4.

Theorem 6.8. If we choose a basis for $\Omega_{odd}(BG)\otimes\mathbb{Q}$ by lemma 6.5 and then use Definitions 6.1 and 6.6 to define $\bar{\gamma}_\rho(X,f)$ for arbitrary $f: X^{2n-1}\to BG$ and any unitary representation of G then $\bar{\gamma}_\rho(X,f)$ is independant of the choice of basis of $\Omega_{odd}(BG)\otimes\mathbb{Q}$ and agrees with $\gamma(X,\rho f)$ whenever the latter is defined.

Proof. We shall just sketch the proof. First observe that the fact that $\bar{\gamma}_\rho(X,f) = \gamma(X,\rho f)$ if the latter is defined follows directly from the definitions. The independence of $\bar{\gamma}_\rho(X,f)$ from the choice of basis in Lemma 6.5 boils down to showing that if the disjoint unions

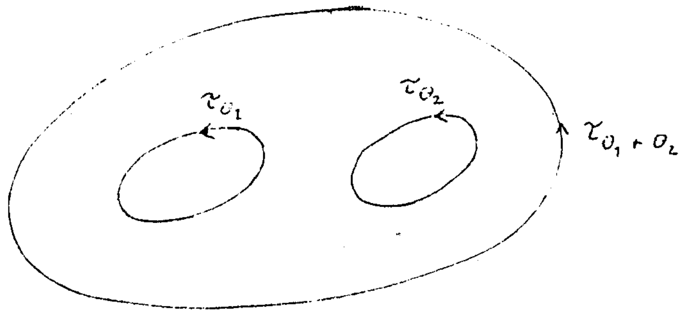
$$A = \sum_{i\in I} M_i \times (T^{k_i}, g_i) \quad \text{and} \quad B = \sum_{j\in J} N_j \times (T^{k_j}, g_j)$$

are representations, with respect to different bases, of the same element in $\Omega_{2n-1}(BG)\otimes\mathbb{Q}$, then

$$(6.9)\qquad \gamma_\rho(A+(-B)) = \sum_{i\in I}\operatorname{sign}(M_i)\bar{\gamma}_\rho(T^{k_i},g_i) - \sum_{j\in J}\operatorname{sign}(N_j)\bar{\gamma}_\rho(T^{k_j},g_j).$$

First observe that this equation is defined in a finitely generated subgroup of $Z(G)$, so we can assume without loss of generality that G is finitely generated abelian. We can also assume that ρ is irreducible.

We first prove equality (6.9) for $G=\mathbb{Z}$ and $A=(S^1,\tau_{\theta_1}) + (S^1,\tau_{\theta_2})$ and $B=(S^1,\tau_{\theta_1+\theta_2})$. For the left side of (6.9) we must calculate the signature of the local coefficient system over $D(3)$ (3-fold punctured S^2) indicated in the following picture



$0 \leq \theta_1 < 2\pi$,

$0 \leq \theta_2 < 2\pi$.

This can be calculated as in Meyer [ ] and turns out to be 0 if either θ_1 or θ_2 is zero and otherwise +1 if $\theta_1 + \theta_2 < 2\pi$, 0 if $\theta_1 + \theta_2 = 2\pi$, -1 if $\theta_1 + \theta_2 > 2\pi$. This agrees with the right side of (6.9) and thus proves this case.

Next, we prove the equality for $A = q(T^k, g)$ and $B = (T^k, g\varphi)$, where $\varphi\colon T^k \to T^k$ is a q-fold covering in one S^1-factor and the identity on the remaining factor T^{k-1}. This is done by using $D(q+1) \times T^{k-1}$, where $D(q+1)$ is the (q+1)-fold punctured S^2, as an explicit bordism between A and B. If $k > 1$ there is nothing to prove, as everything involved is zero. If $k = 1$ the calculation can be carried out by chopping $D(q+1)$ along circles into q-1 D(3)'s and arguing inductively. This case thus reduces to the case already considered.

Since any covering $\varphi\colon T^k \to T^k$ is a composition of coverings of the type just considered, equality (6.9) follows inductively for $A = q(T^k, g)$ and $B = (T^k, g\varphi)$, where $\varphi\colon T^k \to T^k$ is any q-fold covering. This allows us, in the general case of finitely generated abelian G, to go down to any subgroup of finite index in G and hence to assume G is free, say $G = \mathbb{Z}^r$.

We now have two ways of completing the proof. We can analyze the bordisms involved in changing a basis of $\Omega_{odd}(B(\mathbb{Z}^r)) \otimes \mathbb{Q}$;

by putting any basis change together out of elementary basis changes one can again reduce calculations to the case of local coefficient systems over $D(3)$, which has already been dealt with. An alternative method is to observe that the set of irreducible unitary representations of $G = \mathbb{Z}^r$ has a natural topology as $(U(1))^r = (S^1)^r$ and we already know equation (6.9) holds for ρ in the dense subset $(\mathbb{Q}/\mathbb{Z})^r \subset (S^1)^r$. But the right side of (6.9) has the continuity property (as a function of $\rho \in (U(1))^r$) that its value at a point of discontinuity is the "average over nearby values", since $\bar{\gamma}(S^1, \tau_\theta)$ has this property. The same continuity property follows for the left side of (6.9) from our later homotopy invariant calculation of the γ-invariant for G free abelian, so (6.9) holds in general.

This completes the sketch of proof. This proof is unsatisfactory in being more complicated that I feel should be necessary. The best proof would presumable be to identify $\bar{\gamma}_\rho$ directly with certain analytic invariants which Atiyah has recently defined.

Chapter III. Equivariant signature and the α-invariant

1. Witt invariant and equivariant signature

Let M^{2n} be a compact oriented manifold and G a group (finite or not) acting on $(M,\delta M)$ by orientation preserving diffeomorphisms, homeomorphisms, homotopy equivalences, or even just homology equivalences. Then G acts on

$$H = \mathrm{Im}(H^n(M,\delta M;\mathbb{Z}) \to H^n(M;\mathbb{Z}))/\mathrm{Torsion}$$

preserving the cup product form, which is non-degenerate on H. We thus have a non-degenerate $(-1)^n$-symmetric bilinear representation of G over $\mathbb{Z}$ which defines an element $w(M,G)$ in the Witt group of such representations, that is

$$w(M,G) \in W_\varepsilon(G;\mathbb{Q},\mathbb{Z}) \quad , \quad \varepsilon = (-1)^n ,$$

in the notation of I.7.4. Actually G acts from the right (i.e. contravariantly) on H and we must convert this to a left action by convening that $g \in G$ act by g^{-1} on H. This is equivalent to transposing the action via the form on H, and hence also equivalent via Poincaré-Lefschetz duality to the usual left action on homology with intersection form.

The properties of the Witt invariant are as follows.

1.1. If M is closed. Then the cup product form is non-singular, so $w(M,G)$ is in the subgroup $W_\varepsilon(G;\mathbb{Z}) \subset W_\varepsilon(G;\mathbb{Q},\mathbb{Z})$.

1.2. Bordism invariance. If M is a G-equivariant oriented boundary then $w(M,G) = 0$.

<u>1.3. Novikov additivity</u>. If $M = M_1 \cup M_2$ pasted G-equivariantly along boundary components then $w(M,G) = w(M_1,G) + w(M_2,G)$.

<u>1.4. Product formulas</u>. If G acts on M_1 and M_2 then with the diagonal action on $M_1 \times M_2$ we have $w(M_1 \times M_2, G) = w(M_1,G) \cdot w(M_2,G)$. If G_1 acts on M_1 and G_2 on M_2, then $w(M_1 \times M_2, G_1 \times G_2) = t(w(M_1,G_1) \otimes w(M_2,G_2))$, where $t\colon W_*(G_1;\mathbb{Q}) \otimes W_*(G_2;\mathbb{Q}) \to W_*(G_1 \times G_2;\mathbb{Q})$ is the natural map.

The usual proofs of 1.2 and 1.3 for signature also work here. In 1.4 is an easy application of the Künneth formula for a product. These properties show that the invariant w defines a ring homomorphism from the "equivariant bordism group" Ω_*^G to $W_*(G;\mathbb{Z})$ (and also from the equivariant "cutting and pasting group" SK_*^G to $W_*(G;\mathbb{Z})$).

<u>Digression 1.5</u>. The class of $w(M,G)$ in $W_*(G;\mathbb{Q},\mathbb{Z})/W_*(G;\mathbb{Z})$ is a Novikov additive invariant which vanishes for closed manifolds. It follows that it is an invariant of the boundary; call it $\ell(\delta M,G)$. In the proof of theorem 6.5 of chapter I, a natural injection $W_*(G;\mathbb{Q},\mathbb{Z})/W_*(G;\mathbb{Z}) \to W_*(G;\mathbb{Z}\text{-torsion})$ was constructed, so we can think of $\ell(\delta M,G)$ as an element of $W_*(G;\mathbb{Z}\text{-torsion})$. For trivial G, Alexander Hamrick and Wick [ ] have identified $-\ell(\delta M)$ as the class of the linking form on the torsion of $H_*(\delta M;\mathbb{Z})$, so in particular $\ell(N)$ is also defined for N^{2n-1} which do not bound, and is a homotopy invariant. Their proof also works in the present situation.

Note that the class of $\ell(N,G)$ in $\mathrm{Cok}(W_*(G;\mathbb{Q},\mathbb{Z}) \to W_*(G;\mathbb{Z}\text{-tor}))$ is a homotopy invariant bordism invariant of (N^{2n-1},G). This may be quite an interesting invariant.

To return to the more mundane invariants we are interested in at the moment, recall the character map $\chi\colon W_*(G;\mathbb{Q}) \to \mathbb{C}^G$ of section 8 in chapter I.

Definition. Given M^{2n} with G-action as above define the equivariant signature as

$$\mathrm{sign}(M,G) := \chi w(M,G)$$

and for $g \in G$

$$\mathrm{sign}(M,g) := \chi w(M,G)(g)$$

This can be described directly as follows: Let $(H \otimes \mathbb{C}, b)$ be the hermitianized, symmetrized (by multiplication by $+i$ if n is odd) cup product pairing for M. Then g induces a $\mathbb{Z}$-action on $(H \otimes \mathbb{C}, b)$, and up to hyperbolic hermitian $\mathbb{Z}$-modules, $H \otimes \mathbb{C}$ splits as a sum $V^+ \oplus V^-$ of a positive and a negative definite hermitian $\mathbb{Z}$-module. Then

$$\mathrm{sign}(M,g) = \mathrm{trace}(g|V^+) - \mathrm{trace}(g|V^-).$$

This makes it clear that if g is contained in a compact Lie group of transformations of M (equivalently g preserves some Riemannian metric on M), then $\mathrm{sign}(M,g)$ agrees with Atiyah and Singer's definition in [ ]. Note that $\mathrm{sign}(M,g)$ is real or pure imaginary according as n is even or odd.

For finite or abelian G, all we are doing by looking at $\mathrm{sign}(M,G)$ rather than $w(M,G)$ is throwing away torsion information (theorem I.8.3). However, example I.8.8 can be thought of as giving examples of a linear action of a free group G on $S^1 \times S^1$ for which $\mathrm{sign}(S^1 \times S^1, G) = 0$ although $w(S^1 \times S^1, G)$ is of infinite order.

2. The α-invariant

In this section we return completely to the "classical" case by assuming that $g: M \to M$ always preserves some riemannian metric. Atiyah and Singer [ ] showed

Theorem 2.1. Suppose M is closed. Then with g as above

$$\mathrm{Sign}(M,g) = AS(M^g, \nu M^g)$$

where $AS(M^g, \nu M^g)$ is a complicated expression in characteristic classes of the fixpoint set M^g and its normal bundle. In particular if $M^g = \emptyset$ then $\mathrm{sign}(M,g) = 0$.

That g preserve a riemannian metric is of absolute importance here, as we shall see in section 3.

Now $\mathrm{sign}(M,g) - AS(M^g, \nu M^g)$ is a Novikov additive invariant for manifolds with fixpoint free boundary which vanishes for closed manifolds, so by the usual argument (see proof of II.5.3) it is an invariant of the boundary $(\delta M, g)$. This invariant is called $\alpha(\delta M, g)$. More generally:

Definition. If (N^{2n-1}, g) is fixpoint free and some multiple $m(N,g)$ bounds, say $m(N,g) = \delta(M^{2n}, g)$ then we put

$$\alpha(N,g) = \frac{1}{m}(\mathrm{sign}(M,g) - AS(M^g, \nu M^g))$$

which is real or pure imaginary according as n is even or odd.

A multiple of (N,g) bounds for instance if g is contained in a finite group acting freely on N or in a compact connected Lie group acting without fixed points (see [ ],[ ]).

The following example will be needed in sections 3 and 4.

Example 2.2. Let S^1 act on the sphere $S^{2n-1} = \{z \in \mathbb{C}^n \mid \|z\| = 1\}$ by $\lambda(z_1, \dots, z_n) = (\lambda^{a_1} z_1, \dots, \lambda^{a_n} z_n)$, where the a_i are coprime integers. Then as M^{2n} we can take the disc D^{2n}. Then $\mathrm{sign}(D^{2n}, \lambda) = 0$, so

$$\alpha(S^{2n-1}, \lambda) = -AS \text{ (isolated fixed point with rotation numbers } a_1, \dots, a_n)$$

$$= - \prod_{i=1}^{n} \frac{\lambda^{a_i} + 1}{\lambda^{a_i} - 1} .$$

See for instance , .

3. Bordism of diffeomorphisms and examples

This section is something of a digression. We use the Witt invariant and equivariant signature to deduce some facts about the bordism group $\mathcal{O}_*(\mathbb{Z})$ of smooth $\mathbb{Z}$-actions on manifolds. This is of course nothing more than the bordism group of self-diffeomorphisms of oriented manifolds, denoted Δ_* in [ ].

In odd dimensions the group $\mathcal{O}_m(\mathbb{Z})$, $m \neq 3$, has recently been determined by M.Kreck [ ]. The result is that the map

$$\mathcal{O}_m(\mathbb{Z}) \rightarrow \Omega_m \oplus \hat{\Omega}_{m+1}$$

$$[M,f] \longrightarrow [M] \oplus [\text{Mapping torus of } f],$$

is an isomorphism, where $\hat{\Omega}_{m+1}$ is the kernel of signature on Ω_{m+1}. For even m this map is still epic, but the kernel is infinitely generated, by Winkelnkemper [ ] and Medrano [ ]. This also follows from the following result.

Theorem 3.1. The map $w: \mathcal{O}_{2n}(\mathbb{Z}) \rightarrow W_\varepsilon(\mathbb{Z};\mathbb{Z})$, $\varepsilon = (-1)^n$, is surjective for each $n > 0$.

Proof. For $n = 1$ this follows from the fact (Nielsen theorem plus for example Magnus, Karras, Solitar [, section 3.7]) that the map

$$\text{Diff}_+(F_g)/\text{Diff}_o(F_g) \rightarrow Sp(2g;\mathbb{Z})$$

is surjective. Here F_g is the orientable surface of genus g, $\text{Diff}_+(F_g)$ its orientable diffeomorphism group and $\text{Diff}_o(F_g)$ the component of unity. $Sp(2g;\mathbb{Z})$ is the automorphism group of $H_1(F_g;\mathbb{Z})$ with intersection form.

For $n = 2$, suppose we have a symmetric non-singular isometric structure (H,b,t) over $\mathbb{Z}$. By adding the hyperbolic structure $(\mathbb{Z}^2, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, id)$ if necessary we can assume the bilinear space (H,b) is indefinite and odd, so it is classified by its rank and signature alone (Serre [ ]), so it is isomorphic to the cup product form on some manifold of the form $M^4 = \mathbb{C}P^2 \# \ldots \# \mathbb{C}P^2 \# -\mathbb{C}P^2 \# \ldots \# -\mathbb{C}P^2 \# (S^2 \times S^2)$. By Wall [ ], any isometry of (H,b) is realizable by a diffeomorphism of M^4.

For $n > 2$ use the fact $w(N \times M, id \times f) = sign(N).w(M,f)$, which follows from 1.4 above. This completes the proof.

Corollary 3.2. $\mathcal{O}_{2n}(\mathbb{Z}) \otimes \mathbb{Q}$ is infinitely generated for any $n > 0$ and this fact is detected by equivariant signature.

Indeed $\mathcal{O}_{2n}(\mathbb{Z}) \otimes \mathbb{Q}$ surjects onto $W_\varepsilon(\mathbb{Z};\mathbb{Z}) \otimes \mathbb{Q}$ which is calculated in Theorem I.7.16. By section I.8, equivariant signature detects all of $W_\varepsilon(\mathbb{Z};\mathbb{Z}) \otimes \mathbb{Q}$.

Next, we give examples to show that, in contrast to theorem 2.1, a fixed point free diffeomorphism $f: M \to M$ may have $sign(M,f) \neq 0$.

Consider the S^1 action on S^{2n-1} of Example 2.2, given by nonzero coprime integers $a_1, a_2, \ldots, a_n$ and choose integers p_o and p_1 prime to all the a_i. For $t \in [0,1] = I$ let $\lambda_t = \exp(2\pi i(t/p_o + (1-t)/p_1)) \in S^1$ and consider the diffeomorphism

$$g: S^{2n-1} \times I \to S^{2n-1} \times I, \qquad (x,t) \longrightarrow (\lambda_t \cdot x,\ (t^2+t)/2).$$

Then $\lambda_i = g|S^{2n-1} \times \{i\}$ generates a free $\mathbb{Z}/p_i$ action for $i = 0,1$. Choose $m > 0$ such that the disjoint union $m(S^{2n-1}, \mathbb{Z}/p_i)$ bounds a free $(\mathbb{Z}/p_i)$-manifold M_i for $i = 0,1$ (possible since $\tilde{\Omega}_* B(\mathbb{Z}/p_i)$ is torsion) and let $g_i: M_i \to M_i$ denote the generator of this action. We can paste along boundaries to get

$$M^{2n} = M_o \cup m(S^{2n-1} \times I) \cup -M_1$$

with diffeomorphism $f = g_o \cup g \cup g_1 : M \to M$.

Example 3.3. Then $f: M^{2n} \to M^{2n}$ above has the properties:

(i) f is fixed point free;

(ii) If $n > 1$ then $f^{2p_op_1}$ is isotopic to the identity.

(iii) If $\lambda_j = \exp(2\pi i/p_j)$ for $j = 0,1$ then

$$\operatorname{sign}(M,f) = m\left(\prod_1^n \frac{\lambda_1^{a_i}+1}{\lambda_1^{a_i}-1} - \prod_1^n \frac{\lambda_o^{a_i}+1}{\lambda_o^{a_i}-1}\right)$$

Proof. We leave (i) and (ii) as exercises; for (ii) one must only show $g^{2p_op_1}: S^{2n-1} \times I \to S^{2n-1} \times I$ is isotopic to the identity keeping boundary fixed, which follows from $\pi_1(SO(2n)) = \mathbb{Z}/2$.

To prove (iii) note that $\operatorname{sign}(S^{2n-1} \times I, g) = 0$, so by Novikov additivity $\operatorname{sign}(M,t) = \operatorname{sign}(M_o, g_o) - \operatorname{sign}(M_1, g_1)$. But $\operatorname{sign}(M_j, g_j) = m(\alpha(S^{2n-1}, \lambda_j))$ by definition of the α-invariant, so apply example 2.2.

Remark. If one chooses M_o and M_1 above with zero euler characteristics (which is possible if $n > 1$) then one can modify the above f isotopically along a suitable vector field so that no nonzero power of f has a fixed point ! Another technique to get examples is to choose a manifold M^{2n} with periodic diffeomorphism f such that both M and every component of the fixed point set have zero euler characteristic. Then one can again alter f isotopically along a suitable vector field so that no non-zero power of f has a fixed point.

Finally we give examples to show

Proposition 3.4. $\mathcal{O}_{2n}(\mathbb{Z})$ contains infinite $(\mathbb{Z}/2)$-torsion for each $n > 0$.

Proof. For any $A \in Sp(2,\mathbb{Z}) = SL(2,\mathbb{Z})$ let $[A] \in W_{-}(\mathbb{Z};\mathbb{Z})$ denote the class of A as an antisymmetric isometric structure Denote by $f_A\colon T^2 \to T^2$ the corresponding linear automorphism of the torus. Then clearly $w(T^2, f_A) = [A]$.

Now if $a = \operatorname{tr} A$ and $D = a^2-4$ then the minimal polynomial of A is $m(t) = t^2 - at + 1$ which has roots $(a \pm \sqrt{D})/2$. Thus the field $\mathbb{Q}[t]/m(t)$ is $\mathbb{Q}(\sqrt{D})$ and by Corollary I.7.13 and the remark following it we have

Lemma 3.5. If $A \in Sp(2,\mathbb{Z})$ and $a = \operatorname{tr} A$ and $D = a^2 - 4$ then the order of $w(T^2, f_A) = [A]$ in $W_{-}(\mathbb{Z};\mathbb{Z})$ is:

∞ if $D < 0$,

1 if $D = 0$,

2 if $D > 0$ and -1 is a norm of $\mathbb{Q}(\sqrt{D})$,

4 if $D > 0$ and -1 is not a norm of $\mathbb{Q}(\sqrt{D})$.

Remark. Elementary number theory implies that -1 is not a norm of $\mathbb{Q}(\sqrt{a^2-4})$, $|a| > 2$, if the prime decomposition of $a+2$ or of $a-2$ contains a prime of the form $4k+3$ to an odd power. Otherwise, namely for $|a| = 3,6,7,11,15,18,27,34,38,39,43,47,\ldots,$ -1 is a norm.

Now put $B_r = \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix}$ and $A_r = B_r^2 = \begin{pmatrix} r^2+1 & r \\ r & 1 \end{pmatrix}$ for $r = 1,2,3,\ldots$. Then $[T^2, f_{A_r}] \in \mathcal{O}_2(\mathbb{Z})$ has order at least 2, since $[A_r]$ does, by the lemma. But B_r induces an f_{A_r}-equivariant orientation reversing diffeomorphism of T^2, so $[T^2, f_{A_r}] = -[T^2, f_{A_r}]$, so $[T^2, f_{A_r}]$ has order 2. Further, the $[A_r]$ are independent elements of $W_-(\mathbb{Z};\mathbb{Z})$ by Theorem I.7.2, since they have different minimal polynomials, so the $[T^2, f_{A_r}]$ are independent in $\mathcal{O}_2(\mathbb{Z})$.

To deal with $\mathcal{O}_4(\mathbb{Z})$ we need a lemma.

Lemma 3.6. If $A, B \in Sp(2,\mathbb{Z})$ with $\operatorname{tr} A = a \neq \pm 2$ and $\operatorname{tr} B = b \neq \pm 2$ and either a or b nonzero then $[A \otimes B] \neq 0$ in $W_+(\mathbb{Z};\mathbb{Z})$.

Proof. If the minimal polynomials of A and B are $t^2 - at + 1 = (t-\lambda)(t-\lambda^{-1})$ and $t^2 - bt + 1 = (t-\mu)(t-\mu^{-1})$, then our assumptions imply that either $\lambda\mu \notin \mathbb{Q}$ or $\lambda\mu^{-1} \notin \mathbb{Q}$, say $\lambda\mu \notin \mathbb{Q}$. The minimal polynomial $m(t)$ for $A \otimes B$ is either irreducible, in which case the Remark following I.7.13 shows $[A \otimes B] \neq 0$, or its prime decomposition is

$$[(t-\lambda\mu)(t-\lambda^{-1}\mu^{-1})].[(t-\lambda\mu^{-1})(t-\lambda^{-1}\mu)]$$

or

$$[(t-\lambda\mu)(t-\lambda^{-1}\mu^{-1})].[t-\lambda\mu^{-1}].[t-\lambda^{-1}\mu],$$

in which case the first factor is the minimal polynomial of a 2-dimensional summand C of $A \otimes B$ which satisfies $[C] \neq 0$ by the Remark already mentioned and is independent of the other summand(s) by Theorem I.7.2.

Now if $[T^2, f_A]$ has order 2, then for any B with $\operatorname{tr} B \neq \pm 2$ the element $[T^2 \times T^2, f_A \times f_B] \in \mathcal{O}_4(\mathbb{Z})$ has clearly order at most 2, hence order exactly 2 by Lemma 3.6. Thus using the previously found $[T^2, f_{A_r}] \in \mathcal{O}_2(\mathbb{Z})$ of order 2 we can easily find infinitely many elements in $\mathcal{O}_4(\mathbb{Z})$ of order 2.

Finally use the fact that $w(N \times M, \mathrm{id} \times f) = \operatorname{sign}(N) w(M,f)$ to show that $\mathcal{O}_n(\mathbb{Z})$ contains infinite 2-torsion also for any $n > 2$

4. Relation between the α-invariant and γ-invariant

Suppose we are given a finite normal covering

$$N^{2n-1} \longrightarrow X^{2n-1}$$

of closed oriented manifolds. Then if G is the group of covering transformations, we can think of the situation in two ways:

(i) G acts freely on N with orbit space X, so the α-invariants $\alpha(N,g)$ are defined for $g \in G-\{1\}$.

(ii) The covering is classified by a map $f:\pi_1(X) \to G$, so the γ-invariants $\gamma(X,\rho f)$ are defined for any hermitian representation φ of G.

Since G is finite, any hermitian representation is an orthogonal sum of irreducible representations, each of which then has definite hermitian form (see I.3.6). It hence suffices to calculate $\gamma(X,\rho f)$ just for irreducible unitary representations ρ of G. Let

$$\rho_j: G \to U(n_j) \qquad j = 0,\dots,r$$

be a complete set of representatives for the irreducible unitary representations of G, and let χ_j be the character χ_{ρ_j} (so $\chi_j(g) = \text{trace}\ \rho_i(g))$ for each j. Assume $\rho_0: G \to U(1)$ is the trivial representation (so $n_0 = 1$).

Theorem 4.1. In the above situation the α-invariants and γ-invariants determine each other as follows:

$$\alpha(N,g) = \sum_{j=1}^{r} \chi_j(g).\gamma(X,\rho_j f), \qquad (g \neq 1),$$

$$\gamma(X,\rho_i f) = \frac{1}{|G|} \sum_{g \neq 1} (\chi_i(g^{-1}) - n_i).\alpha(N,g).$$

<u>Proof</u>. By multiplying by a positive integer if necessary, we can assume N^{2n-1} equivariantly bounds a free G-manifold M^{2n}, so by definition

$$\alpha(N,g) = \mathrm{sign}(M,g)$$

Also, if $M/G = Y$ and $g\colon \pi_1(Y) \to G$ classifies the covering $M \to Y$, then by definition

$$\gamma(X,\rho_j f) = \mathrm{sign}(Y,\Gamma_j) - n_j.\mathrm{sign}(M),$$

where $\Gamma_j \to Y$ is the unitary local coefficient system classified by $\rho_j g$. Now if $\Gamma \to Y$ is the coefficient system classified by the regular representation $\rho\colon G \to U(|G|)$ of G, then we have a G-isomorphism

$$\mathrm{Im}(H^n(M,N;\mathbb{C}) \to H^n(M;\mathbb{C})) \cong \mathrm{Im}(H^n(Y,X;\Gamma) \to H^n(Y;\Gamma))$$

which preserves the hermitianized cup product forms (if n is odd we multiply the skew hermitian cup product by i to make it hermitian). But the regular representation ρ of G splits as the orthogonal sum

$$\rho \cong n_o\rho_o \oplus \dots \oplus n_r\rho_r$$

so the coefficient system Γ is the orthogonal sum

$$\Gamma \cong n_o\Gamma_o \oplus \dots \oplus n_r\Gamma_r .$$

Thus, setting $H_j = \mathrm{Im}(H^n(Y,X;\Gamma_j) \to H^n(Y;\Gamma_j))$ for $j = 0,\dots,r$ we have

$$\mathrm{Im}(H^n(M,N;\mathbb{C}) \to H^n(M;\mathbb{C})) \cong n_oH_o \oplus \dots \oplus n_rH_r .$$

Let $H_j = H_j^+ \oplus H_j^-$ be a G-invariant orthogonal splitting into positive and negative define subspaces. Observe that the irreducible components of the G-action on H_j are all of type ρ_j, since this is true on the chain level and taking homology does not alter this. Hence $tr(g|H_j^{\pm}) = \chi_j(g).(\dim H_j^{\pm})/n_j$, so

$$\alpha(N,g) = \mathrm{sign}(M,g) = tr(g|H^+) - tr(g|H^-)$$

$$= \sum_{j=o}^{r} n_j(tr(g|H_j^+) - tr(g|H_j^-))$$

$$= \sum_{j=o}^{r} \chi_j(g)(\dim H_j^+ - \dim H_j^-)$$

$$= \sum_{j=o}^{r} \chi_j(g).\mathrm{sign}(Y,\Gamma_j).$$

Now using the fact that $\sum_{j=o}^{r} \chi_j(g).n_j = 0$ for $g \neq 1$, this becomes

$$\alpha(N,g) = \sum_{j=o}^{r} \chi_j(g).(\mathrm{sign}(Y,\Gamma_j) - n_j.\mathrm{sign}(Y,\Gamma_o))$$

$$= \sum_{j=1}^{r} \chi_j(g).(\mathrm{sign}(Y,\Gamma_j) - n_j.\mathrm{sign}\ Y)$$

$$= \sum_{j=1}^{r} \chi_j(g).\gamma(X,\rho_j f),$$

as was to be proved. To prove the second formula, insert the above formula for $\alpha(N,g)$ into it and then apply the orthogonality relation $\sum_g \chi_i(g^{-1}).\chi_j(g) = \delta_{ij}|G|$ and its special case $\sum_g \chi_j(g) = \delta_{oj}|G|$.

Example 4.2. (Lens spaces). If we take the action of S^1 on S^{2n-1} with rotation numbers $a_1,\dots,a_n$, and if d is an integer prime to the a_i, then $\mathbb{Z}/d \subset S^1$ acts freely on S^{2n-1} with orbit space the lens space $L(d;a_1,\dots,a_r)$. Example 2.2 and Theorem 4.1 give

$$\gamma(L(d;a_1,\dots,a_n),\rho_j) = \frac{-1}{d} \sum_{\substack{\lambda^d=1\\ \lambda \neq 1}} (\lambda^{-j}-1)\left(\frac{\lambda^{a_1}+1}{\lambda^{a_1}-1}\right)\cdots\left(\frac{\lambda^{a_n}+1}{\lambda^{a_n}-1}\right),$$

where ρ_j is the representation which maps the generator of $\mathbb{Z}/d$ to $(e^{2\pi ij/d}) \in U(1)$. As a special case of this we have

Proposition 4.3. $\gamma(S',\tau_\theta) = 1-\frac{\theta}{\pi}$, $(0<\theta<\pi)$. Here $\tau_\theta\colon \pi_1(S^1)\to U(1)$ is the representation $\tau_\theta(1) = (e^{i\theta})$, and in order that $\gamma(S^1,\tau_\theta)$ be defined, we must have θ/π rational.

Proof. If $\theta = 2\pi j/d$, then $(S^1,\tau_\theta) = (L(d;1)\rho_j)$ in the above notation, so (with sums over all $\lambda \neq 1$ with $\lambda^d = 1$):

$$\gamma(S^1,\tau_\theta) = \frac{-1}{d}\sum(\lambda^{-j}-1)\left(\frac{\lambda+1}{\lambda-1}\right) = \frac{-1}{d}\sum(\lambda^{-j}-1)\left(\frac{1+\lambda^{-1}}{1-\lambda^{-1}}\right)$$

$$= \frac{1}{d}\sum(\lambda^{j}-1)\left(\frac{\lambda+1}{\lambda-1}\right)$$

$$= \frac{1}{d}\sum(\lambda^{j-1}+\lambda^{j-2}+\cdots+1)(\lambda+1)$$

$$= \frac{1}{d}\sum(\lambda^{j}+2\lambda^{j-1}+\cdots+2\lambda+1)$$

$$= \frac{1}{d}(-1-2-\cdots-2+(d-1)) = \frac{1}{d}(d-2j)$$

$$= 1-\frac{\theta}{\pi}.$$

Example 4.2 ctd. If we plug the above formula for $\gamma(S^1,\tau_\theta)$ into the formula 4.1 for $\alpha(S^1,\lambda)$, where λ is a d-th root of unity we get

$$-\left(\frac{\lambda+1}{\lambda-1}\right) = \sum_{k=1}^{d-1} \lambda^k\left(1-\frac{2k}{d}\right)$$

Inserting this into the above formula for $\gamma(L(d;a_1,\dots,a_n),\rho_j)$ gives, after a trivial simplification:

Proposition 4.4. $\gamma(L(d;a_1,\ldots,a_n),\rho_j) = \delta_o(d;a_1,\ldots,a_n) - \delta_j(d;a_1,\ldots,a_n)$, where $\delta_j(d;a_1,\ldots,a_n)$ is defined by

$$\delta_j(d;a_1,\ldots,a_n) = \sum_{\substack{0<k_1,\ldots,k_n<d \\ d|(a_1k_1+\ldots+a_nk_n-j)}} \left(\frac{2k_1}{d} - 1\right)\cdots\left(\frac{2k_n}{d} - 1\right).$$

These $\delta_j(d;a_1,\ldots,a_n)$ generalize the "generalized Dedekind sums" $\delta(d;a_1,\ldots,a_n)$ of Zagier [ ],[ ],[ ], which are the particular case $j = 0$. They have the obvious symmetry property

$$\delta_j(d;a_1,\ldots,a_n) = (-1)^n \delta_{d-j}(d;a_1,\ldots,a_n),$$

as can be seen by replacing each k_i by $d-k_i$ in the sum. In particular $\delta_o(d;a_1,\ldots,a_n) = 0$ if n is odd. By a similar argument

$$\sum_{j=0}^{d-1} \delta_j(d;a_1,\ldots,a_n) = 0 .$$

There is also a recursion formula

$$\delta_j(d;a_1,\ldots,a_n,a) = \delta_j(d;a_1,\ldots,a_n) + \sum_{k=0}^{d-1} \frac{2k}{d} \delta_{j-ak}(d;a_1,\ldots,a_n).$$

Zagier [ ] showed that the denominator of $\delta_o(d;a_1,\ldots,a_n)$ (n even) divides

$$\mu_k = \prod_{\substack{p \text{ prime} \\ p \text{ odd}}} p^{[\frac{n}{p-1}]} , \quad k = n/2$$

which is the denominator of the Hirzebruch L_k-polynomial. His argument was purely number-theoretic. At least for d an odd prime power (but presumably for all d) the denominator of $\delta_j(d;a_1,\ldots,a_n)$ divides

$$\text{g.c.d}\left(d.\prod_{p\text{ prime}} p^{\left[\frac{2n-1}{2p-2}\right]}, d^n\right)$$

This follows for d an odd prime power from Proposition 4.4, the modulo $\mathbb{Z}$ bordism invariance property II.5.9, and the fact that the above number is in this case the exponent of the bordism group $\tilde{\Omega}_{2n-1}(B(\mathbb{Z}/d))$ by [,§37].

For G finite acting freely on N with orbit space X the invariants $\gamma_p(X,f)(\text{mod }\mathbb{Z})$ and the invariants $\alpha(N,g)(\text{mod }\mathbb{Z}(e^{2\pi i/|G|}))$ are both bordism invariants in $\Omega_*(BG)$ and determine each other. They have been given an interpretation as K-theoretic characteristic numbers by Knapp [ ] (see also Wilson [ ]).

5. Signature defect for coverings

If $M' \to M$ is a d-fold covering (not necessarily normal) of closed oriented 2n-manifolds then $\operatorname{sign} M' = d.\operatorname{sign} M$. If M' and M have boundaries this is no longer true in general but the usual argument shows that the difference $\operatorname{sign} M' - d.\operatorname{sign} M$ is an invariant of $\delta M' \to \delta M$, which we call the signature defect. More generally:

Definition. Let $X' \to X$ be a d-fold covering (not necessarily normal) of closed oriented (2n-1)-manifolds. Then some multiple $q(X' \to X)$ bounds a covering $M' \to M$ of oriented 2n-manifolds and we define the signature defect

$$\operatorname{def}(X' \to X) = \tfrac{1}{q}(\operatorname{sign} M' - d.\operatorname{sign} M).$$

That some multiple of $X' \to X$ bounds, even for non-normal coverings, follows from Lemma 5.1 below. If n is odd then trivially $\operatorname{def}(X' \to X) = 0$, but the discussion of this section is still of interest.

If $X' \to X$ is a normal covering with covering transformation group G and classifying map $f\colon X \to BG$, then $\operatorname{def}(X' \to X) = \gamma_\rho(X,f)$, where $\rho\colon G \to U(|G|)$ is the regular representation, because we can assume $M' \to M$ is also normal and then $H^*(M',\delta M';\mathbb{C}) = H^*(M,\delta M;\Gamma)$, where Γ is the corresponding coefficient system. To generalize this to any covering, we need the following standard result of covering theory.

Lemma 5.1. Any finite covering $X' \to X$ has the form $N/H \to N/G$, where G is a finite group acting freely on N and $H \subset G$ is a subgroup. The covering is normal if and only if $H \subset G$ is a normal subgroup.

Proof. If $X' \to X$ is classified by the subgroup $A \subset \pi_1(X)$ of finite index, let B be the intersection of all conjugates of A, of which there are finitely many, so B still has finite index. B classifies a normal covering $N \to X$ with covering transformation group $G = \pi_1(X)/B$ and if we put $H = A/B$, then $X' = N/H$. The second statement is clear.

Theorem 5.2. Let $X' = N/H \to N/G = X^{2n-1}$ be as above. Let $f: X \to BG$ classify the covering $N \to X$. Let $\rho_j: G \to U(n_j)$, $j = 0,\ldots,r$, be the list of irreducible representations of G as in theorem 4.1 (ρ_o = trivial). Let $\mathrm{Fix}\rho_j|H \subset \mathbb{C}^{n_j}$ denote the trivial component of $\rho_j|H$. Then

$$\mathrm{def}(X' \to X) = \sum_{j=1}^{r} (\dim \mathrm{Fix}\, \rho_j|H).\gamma(x,\rho_j f).$$

Corollary 5.3. a) $\mathrm{def}(N \to N/G) = \sum_{j=1}^{r} n_j \gamma(x,\rho_j f)$,

b) $$\mathrm{def}(N/H \to N/G) = \frac{-1}{|H|} \sum_{g \in G-H} \alpha(N,g) ,$$

c) $$\mathrm{def}(N \to N/G) = - \sum_{g \in G-\{1\}} \alpha(N,g) .$$

Proof. In the corollary, a) is the special case $H = 1$ of 5.2, b) follows by applying 4.1 to 5.2, and c), which is a well known formula (see for instance [ ] ,[ ]), is a special case of b).

However, we shall prove 5.2 by first proving the corollary and then deducing 5.2 from it.

Note that in 5.2 and 5.3a) it is irrelevant whether we sum $j = 0$ or $j = 1$, since $\gamma(X,\rho_o f) = 0$. We have already observed that $\mathrm{def}(N \to N/G) = \gamma(X,\rho t)$ where ρ is the regular represensentation of G. But the regular representation splits as the orthogonal sum $n_o\rho_o \oplus \ldots \oplus n_r\rho_r$, which proves a). To prove c), evaluate its right side using the first formula of 4.1 and the relation $\sum_g \chi_j(g) = 0$, $j \neq 0$, mentioned in the proof of 4.1. Finally b) follows by taking formula c) for the coverings $N \to N/H$ and $N \to N/G$ and applying the following lemma.

Lemma 5.4. If $N \to X' \to X$ are finite coverings of degree h and d respectively, then

$$\operatorname{def}(X' \to X) = \frac{1}{h}(\operatorname{def}(N \to X) - \operatorname{def}(N \to X')).$$

This lemma follows by observing that for suitable $Y^{2n} \to M' \to M$, $\operatorname{def}(N \to X) = \frac{1}{4}(\operatorname{sign} Y - hd.\operatorname{sign} M)$, $\operatorname{def}(N \to X') = \frac{1}{4}(\operatorname{sign} Y - h.\operatorname{sign} M)$, and $\operatorname{def}(X' \to X) = \frac{1}{4}(\operatorname{sign} M' - d.\operatorname{sign} M)$.

To prove Theorem 5.2, we now insert the first formula of 4.1 into 5.3)b, giving

$$\operatorname{def}(N/H \to N/G) = \frac{-1}{|H|} \sum_{g\in G-H} \sum_{j=1}^{r} \chi_j(g).\gamma(N/G,\rho_j f).$$

Reversing the order of summation and using $\sum_{g\in G} \chi_j(g) = 0$ for $j \neq 0$ gives

$$\operatorname{def}(N/H \to N/G) = \frac{-1}{|H|} \sum_{j=1}^{r} \sum_{h\in H} \chi_j(h).\gamma(N/G,\rho_j f).$$

The fact that $\sum_{h\in H} \chi_j(h) = |H|.\dim \operatorname{Fix}(\rho_j|H)$ shows that the above equation is just what we wished to prove.

Example 5.5. $\operatorname{def}(S^{2n-1} \to L(d;a_1,\ldots,a_n)) = \delta(d;a_1,\ldots,a_n)$, where $\delta(d;a_1,\ldots,a_n) = \delta_o(d;a_1,\ldots,a_n)$ is Zagier's generalized Dedekind sum (see Proposition 4.4).

This follows by applying 5.2 to Proposition 4.4. Alternatively plugging 2.2 into 5.3c) gives one of Zagier's formulas [ ] for the Dedekind sum. The above example was much of Zagier's motivation for introducing the generalized Dedekind sums.

Corollary 5.6. If $X' \to X^{2n-1}$ is a normal covering with soluble covering transformation group G and n is even, then the denominator of $\operatorname{def}(X' \to X)$ divides the denominator μ_k, $k = n/2$ of the Hirzebruch L_k-polynomial.

Proof. The claim is true for $S^{2n-1} \to L(d;a_1,\ldots,a_n)$ by Zagier's number-theoretic determination [ ] of the denominator of generalized Dedekind sums (see §4). It is therefore true for any normal covering $X' \to X$ of prime degree d, since $\operatorname{def}(X' \to X)$ is a bordism invariant modulo integers and the relevant bordism group $\tilde{\Omega}_*(B(\mathbb{Z}/d))$ is generated as an Ω_*-modulo by lens spaces ([ ],§36). Now any normal covering $X' \to X$ with soluble covering transformation group is a composition $X' \to X_1 \to \ldots \to X$ of normal coverings of prime degree, so the result follows by repeated application of lemma 5.4.

The denominator μ_k arises purely number-theoretically in this proof. It would be interesting to understand topologically [1)] why just this denominator occurs. Such an understanding would hopefully also include a proof of the following conjecture.

Conjecture. Corollary 5.6 is true for arbitrary finite coverings.

---

1) A topological proof of Zagier's result on the denominator of generalized Dedekind sums has been found by Knapp [ ].

We can prove this conjecture for 3-manifolds by reinterpreting the signature defect $\operatorname{def}(X' \to X)$ as the covering defect for a certain invariant of Atiyah and Kreck (see [ ]). This goes as follows. If X^{2n-1} is a framed manifold with framing f say, then some multiple of (X,f) bounds a framed manifold $(M^{2n},\bar{f})$, say $q(X,f) = \delta(M,\bar{f})$, and Atiyah and Kreck define an invariant $\delta(X,f) = \frac{1}{q}\operatorname{sign}(M)$. Now we clearly have

Proposition 5.7. If $X' \to X$ is a d-fold covering and (X',f') is the pullback of a framing (X,f) of X, then

$$\operatorname{def}(X' \to X) = \delta(X',f') - d.\delta(X,f).$$

Thus the above conjecture follows for 3-manifolds by observing that every 3-manifold is stably parallelizable, i.e. can be framed, by [ ], and Kreck has shown [ ] that for framed $(4k-1)$-manifolds the denominator of $\delta(X,f)$ always divides μ_k.

Remark. $\operatorname{def}(X' \to X)$ is also the covering defect for the Atiyah-Patodi invariant of a riemanian manifold

$$\eta(\delta M,\omega) = \operatorname{sign} M - \int_M L(\bar{\omega}), \quad (\omega \text{ a metric on } \delta M),$$

where M is a riemannian $2n$-manifold with riemannian metric $\bar{\omega}$ equal to the product metric on a collar $\delta M \times [0,1] \subset M$ of the boundary, and $L(\bar{\omega})$ is the signature form associated with this metric (see [ ]).

Indeed, the integral behaves multiplicatively for coverings, since it is locally defined, so it cancels out when one takes the covering defect.

CHAPTER IV. HOMOTOPY INVARIANTS

1. Monodromy

Let X^{2n-1} be a closed oriented manifold and $\pi_1(X) \xrightarrow{a}$ any homomorphism. We intend to give a homotopy invariant description of the invariants $\gamma_\rho(X, a) = \gamma(X, \rho a)$ for hermitian representations ρ of $\mathbb{Z}$. This description will be in terms of a generalized "monodromy" of the map $X \longrightarrow B\mathbb{Z} = S^1$ induced by a. As usual we denote this map $X \longrightarrow S^1$, which is unique up to homotopy, also by a.

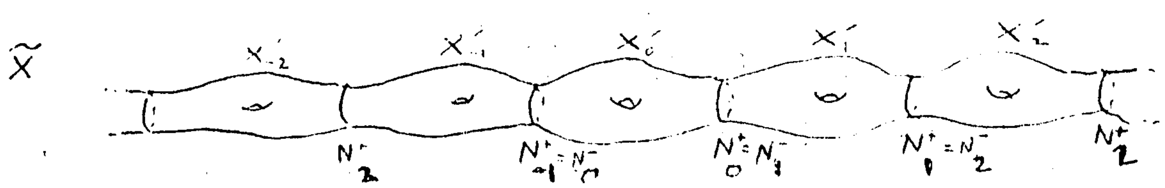
We must say what ordinary "monodromy" is. If $F^{2n-2} \longrightarrow X^{2n-1} \longrightarrow S^1$ is a fibration of closed oriented manifolds then translating the cohomology of F around the circle S^1 yields an automorphism $t: H^{n-1}(F) \longrightarrow H^{n-1}(F)$, which preserves the cup product form b on $H^{n-1}(F)$. The isometric structure $(H^{n-1}(F), b, t)$, or equivalently the representation $\pi_1(S^1) = \mathbb{Z} \longrightarrow Aut(H^{n-1}(F), b)$ it generates, is called the monodromy (in middle dimensional cohomology) of the fibration. More generally of course the monodromy can be defined for any fibration $F \longrightarrow X \longrightarrow B$ of closed manifolds, as a bilinear representation of $\pi_1(B)$. Our generalization below of monodromy to arbitrary maps $X \longrightarrow B$ works and is reasonable if B is a $K(\pi, 1)$-manifold, however the γ-invariants are only invariants of the monodromy if $\pi = \mathbb{Z}$, so we shall only consider the case $B = B\mathbb{Z} = S^1$. We shall work with coefficients in $\mathbb{Q}$, though other coefficients are of course possible (see section 2).

We plan to define for any closed oriented X^{2n-1} and any homotopy class $\alpha \in [X, S^1]$ a certain $(-1)^{n-1}$-symmetric isometric structure $\mathcal{H}_\alpha = (H_\alpha, b_\alpha, t_\alpha)$ as the monodromy of (X, α). Represent α by a map $a : X \longrightarrow S^1$ and let $\tilde{X}$ be the pullback of the universal cover of S^1:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{a}} & \mathbb{R} \\ \downarrow & & \downarrow \\ X & \xrightarrow{a} & S^1 \end{array}$$

We digress slightly by giving a cut and paste description of $\tilde{X}$, which we do not actually need now, but will be needed later and is useful to keep in mind to clarify things.

Identify $[X, S^1] = H^1(X; \mathbb{Z})$, so $\alpha \in H^1(X; \mathbb{Z})$. Represent the Poincaré dual $D\alpha \in H_{2n-2}(X; \mathbb{Z})$ by an oriented submanifold $N^{2n-2} \subset X^{2n-1}$. It is well known (and an easy exercise) that $N \subset X$ represents $D\alpha$ if and only if $N = a^{-1}(1)$ for some map $a : X \longrightarrow S^1$ transverse to the point $1 \in S^1$. Let X' be X cut open along N, so the boundary of X' consists of two copies N^+ and N^- of N. Then $\tilde{X}$ is obtained by taking $\mathbb{Z}$ copies $\ldots, X'_{-1}, X'_0, X'_1, \ldots$ of X' and pasting them together by pasting N_i^+ to N_{i+1}^- for each $i \in \mathbb{Z}$:



the orientation of $N = N_0^+ \subset X$ is such that, [illegible] followed by a normal toward $+\infty$

Shifting the picture of $\widetilde{X}$ one step to the right gives a generator $t : \widetilde{X} \longrightarrow \widetilde{X}$ of the group of covering transformations.

To return to the definition of $\mathcal{H}_\alpha$, let $\widehat{\alpha} \in H_{2n-2}(\widetilde{X})$ be the homology class represented by the submanifold $N = N_0^+ \subset \widetilde{X}$. Consider the bilinear pairing

$$\mathbf{b} : H^{n-1}(\widetilde{X}; \mathbb{Q}) \otimes H^{n-1}(\widetilde{X}; \mathbb{Q}) \longrightarrow \mathbb{Q} \quad , \quad \mathbf{b}(x,y) = (x \cup y)(\widehat{\alpha}) \, .$$

This is $(-1)^{n-1}$-symmetric, but is in general neither finite dimensional nor non-degenerate. The covering transformation $t : \widetilde{X} \longrightarrow \widetilde{X}$ maps $\widehat{\alpha}$ to a homologous class so it induces an isometry $t^* : H^{n-1}(\widetilde{X}; \mathbb{Q}) \longrightarrow H^{n-1}(\widetilde{X}, \mathbb{Q})$.

<u>DEFINITION 1.1</u> Let $\mathcal{H}_\alpha = (H_\alpha, b_\alpha, t_\alpha)$ be the result of quotienting out the degeneracy of $(H^{n-1}(\widetilde{X}; \mathbb{Q}), \mathbf{b}, t^*)$. That is $H_\alpha = H^{n-1}(\widetilde{X}; \mathbb{Q}) / \mathrm{Rad}(\mathbf{b})$, where $\mathrm{Rad}(b) = \{x \mid b(x,y) = 0$ for all $y\}$, and b_α and t_α are the bilinear form and isometry induced by b and t^*.

We must show

<u>PROPOSITION 1.2</u> $\mathcal{H}_\alpha$ is a homotopy invariant of (X^{2n-1}, α) and is a $(-1)^{n-1}$-symmetric finite dimensional non-degenerate isometric structure.

<u>Proof</u>. Since the covering $\widetilde{X} \longrightarrow X$ only depends on the homotopy class α, to show homotopy invariance of $\mathcal{H}_\alpha$

we need only show that the class $\hat{\alpha} \in H_{2n-2}(\tilde{X}, \mathbb{Q})$ is a homotopy invariant. This is proved by Novikov [ ]. The following proof is given for its simplicity. $\hat{\alpha}$ is just the image of $1 \in \mathbb{Z}$ under the composition

$$\mathbb{Z} = H^1_c(\mathbb{R}; \mathbb{Z}) \longrightarrow H^1_c(\tilde{X}; \mathbb{Z}) \xrightarrow{\cong} H_{2n-2}(\tilde{X}; \mathbb{Z}) \to H_{2n-2}(\tilde{X}; \mathbb{Q}),$$

where the subscript c means compact supports, the first arrow is induced by $\tilde{\alpha}: \tilde{X} \longrightarrow \mathbb{R}$ and is well defined since $\tilde{\alpha}$ is a proper map and unique up to proper homotopy, the second arrow is Poincaré duality, and the third one is the coefficient map. Any homotopy equivalence $X \simeq X'$ of compact manifolds mapping α to $\alpha' \in H^1(X'; \mathbb{Z}) = [X', \mathbb{Z}]$ induces a proper homotopy equivalence of the corresponding coverings $\tilde{X} \simeq \tilde{X}'$ and thus maps the above row of maps into the corresponding row for (X', α') and hence maps $\hat{\alpha}$ to $\hat{\alpha}'$.

That H_α is non-degenerate and $(-1)^{n-1}$-symmetric is clear. The finite dimensionality is immediate from the following lemma.

LEMMA 1.3. ~~If $N^{2n-2} \subset \tilde{X}$ be a submanifold representing~~ If $i^*: H^{n-1}(\tilde{X}; \mathbb{Q}) \longrightarrow H^{n-1}(N; \mathbb{Q})$ is induced by the inclusion $N \subset \tilde{X}$, then $H_\alpha \cong \mathrm{Im}(i^*)/\mathrm{Rad}(b_N|\mathrm{Im}(i^*))$, where b_N is the cup product form on $H^{n-1}(N; \mathbb{Q})$. Under this isomorphism b_α is the form induced by b_N and t_α is the unique homomorphism such that

$$\begin{array}{ccc} H^{n-1}(\tilde{X}; \mathbb{Q}) & \xrightarrow{\pi i^*} & \mathrm{Im}(i^*)/\mathrm{Rad}(b_N|\mathrm{Im}(i^*)) \\ \downarrow t^*_{\tilde{X}} & & \downarrow t_\alpha \\ H^{n-1}(\tilde{X}; \mathbb{Q}) & \xrightarrow{\pi i^*} & \mathrm{Im}(i^*)/\mathrm{Rad}(b_N|\mathrm{Im}(i^*)) \end{array}$$

commutes, where $\pi: \mathrm{Im}(i^*) \to \mathrm{Im}(i^*)/\mathrm{Rad}$ is the projection. ~~where t is induced by the covering transformation~~

Proof. If $i : N \to \tilde{X}$ is the inclusion then by definition $\hat{\alpha} = i_*[N]$. For $x, y \in H^{n-1}(\tilde{X})$ we have

$$\mathbf{b}(x,y) = (x \cup y)(\hat{\alpha}) = (x \cup y)(i_*[N]) = i^*(x \cup y)([N]) = (i^*x \cup i^*y)([N]) = b_N(i^*x, i^*y)$$

so $\operatorname{Ker} i^*$ is contained in the radical $\operatorname{Rad}(\mathbf{b})$ of the form $\mathbf{b}$ on $H^{n-1}(\tilde{X}; \mathbb{R})$ and we have a bilinear form preserving epimorphism

$$\operatorname{Im}(i^*) = H^{n-1}(\tilde{X}) / \operatorname{Ker}(i^*) \longrightarrow H^{n-1}(\tilde{X}) / \operatorname{Rad}(\mathbf{b}) = H_\alpha .$$

This induces an isometry

$$\operatorname{Im}(i^*) / \operatorname{Rad}(b_N | \operatorname{Im}(i^*)) \xrightarrow{\cong} H_\alpha .$$

The commutative square involving t_α is by definition. The uniqueness of t_α in this square is by surjectivity of the horizontal arrows.

The definition of H_α can be translated to homology using Poincaré duality. We shall do our later calculations in homology, as this is more easily visualizable, so we give the translation.

Recall that ordinary homology has compact supports since it is based on finite chains. Let $H^{cl}_*(\tilde{X})$ denote homology with closed supports. Such a theory seems to have been first defined by Borel and Moore [ ]. A singular theory of this type is given by allowing infinite but locally finite singular chains and agrees with the Borel-Moore theory for "good" spaces (e.g. manifolds) by Olk [ ]. Closed but not necessarily compact submanifolds of $\tilde{X}$ represent cycles in such a theory. There is a Poincaré duality isomorphism

$$\cap[\tilde{X}] : H^q(\tilde{X}) \xrightarrow{\cong} H^{cl}_{2n-1-q}(\tilde{X})$$

and via this isomorphism the form b on $H^{n-1}(\tilde{X};\mathbb{R})$ becomes the following intersection form on $H^{cl}_n(\tilde{X};\mathbb{R})$: given classes $[x],[y] \in H^{cl}_n(\tilde{X};\mathbb{R})$, then the cycles x, y and $N \subset \tilde{X}$ intersect "generically" in a finite set of points, which we can count (taking orientations into account) to get the intersection number $b([x],[y])$. The definition of H_α can thus be carried out in homology (but note that t^* is Poincaré dual to t_*^{-1}, not to t_*, see proof of 1.4 below).

Alternatively, intersecting cycles in $\tilde{X}$ with N induces a map

$$\partial : H^{cl}_n(\tilde{X};\mathbb{Q}) \longrightarrow H_{n-1}(N;\mathbb{R})$$

which is Poincaré dual to $i^* : H^{n-1}(N;\mathbb{R}) \longrightarrow H^{n-1}(\tilde{X};\mathbb{R})$, so we can translate Lemma 1.3. Since this is the version we shall use, we expand on it a bit.

<u>PROPOSITION 1.4</u> $H_\alpha \cong (\,\mathrm{Im}\,\partial/\mathrm{Rad}(b_N|\mathrm{Im}\,\partial),\ b_N,\ h)$ where b_N denotes both the intersection form on $H_{n-1}(N;\mathbb{R})$ and the induced form on $\mathrm{Im}\,\partial/\mathrm{Rad}(b_N|\mathrm{Im}\,\partial)$ and h is defined as follows: $h(\xi) = \eta$ if and only if ξ and η can be represented by cycles x and y in N such that the cycles $i_\# x$ and $t_\# i_\# y$ are homologous in $\tilde{X}$.

Proof. Except for the characterization of h, this is just the dualized version of Lemma 1.3. To prove the statement on h first note that the equation $(t^*a \cap [\tilde{X}]) = t_*^{-1}(a \cap t_*[\tilde{X}])$ $= t_*^{-1}(a \cap [\tilde{X}])$ for $a \in H^*(\tilde{X})$ shows that t^* is Poincaré dual to $t_*^{-1} : H_*^{cl}(\tilde{X}) \longrightarrow H_*^{cl}(\tilde{X})$. Thus by Lemma 1.3 the map h corresponding to t_* is characterized by commutativity of the diagram

$$\begin{array}{ccc} H_n^{cl}(\tilde{X};\mathbb{R}) & \xrightarrow{\pi\delta} & \operatorname{Im}\delta / \operatorname{Rad}(b_N | \operatorname{Im}\delta) \\ \downarrow t_*^{-1} & & \downarrow h \\ H_n^{cl}(\tilde{X};\mathbb{R}) & \xrightarrow{\pi\delta} & \operatorname{Im}\delta / \operatorname{Rad}(b_N | \operatorname{Im}\delta) \end{array},$$

where $\pi : \operatorname{Im}\delta \longrightarrow \operatorname{Im}\delta/\operatorname{Rad}(b_N|\operatorname{Im}\delta)$ is the projection. Now suppose $\xi, \eta \in \operatorname{Im}\delta/\operatorname{Rad}(b_N|\operatorname{Im}\delta)$ can be represented by cycles x and y as described in the lemma. Let c_1 and $c_2 \in Z_n^{cl}(\tilde{X})$ be cycles with closed support in $\tilde{X}$ intersecting N in $i_\# x$ and $i_\# y$ (possible, since $[x], [y] \in \operatorname{Im}\delta$). Let c_1^- be the part of c_1 "to the left of N in $\tilde{X}$" and c_2^+ be the part of c_2 "to the right of N in $\tilde{X}$" and let d be a homology from $i_\# x$ to $t_\# i_\# y$. Then $c = c_1^- \cup d \cup t_\# c_2^+$ is a cycle in $Z_n^{cl}(\tilde{X})$ with $\partial[c] = [x]$ and $\partial t_*^{-1}[c] = [y]$, whence $h(\xi) = h\pi[x] =$ $h\pi\partial[c] = \pi\partial t_*^{-1}[c] = \pi[y] = \eta$, as was to be shown. Conversely, given ξ and η with $h\xi = \eta$, one can find a cycle $c \in Z_n^{cl}(\tilde{X})$ with $\pi\partial[c] = \xi$ and $\pi\delta t_*^{-1}[c] = \eta$, and one can further assume that c intersects $N \subset \tilde{X}$ and $tN \subset \hat{X}$ in cycles $i_\# x$ and $t_\# i_\# y$. Then x and y represent ξ and η and $i_\# x$ is homologous to $t_\# i_\# y$ by the portion of c lying between N and tN in $\tilde{X}$. This completes the proof.

One can actually identify the radicals occurring in 1.3 and 1.4 explicitly. Though we shall not need it, we mention the ~~the result in homology. The cohomology result is Poincaré dual.~~

~~PROPOSITION 1.5 The radical $\mathrm{rad}(b_N | \mathrm{Im}\,\partial)$ of 1.4 is equal to $\mathrm{Ker}(i_* : H_{n-1}(N) \to H_{n-1}(\tilde{X}))$.~~

~~*Proof.* The form b on $H_n^{cl}(\tilde{X})$ can be expressed as the intersection number $b(x,y) = x \cdot i_* \partial y$, where the dot represents the intersection pairing $H_n^{cl}(\tilde{X}) \otimes H_{n-1}(\tilde{X}) \to \mathbb{R}$. But $H_n^{cl}(\tilde{X})$ and $H_{n-1}(\tilde{X})$ are dually paired by intersection numbers, so $\mathrm{Rad}\, b = \mathrm{Ker}(i_* \partial)$. The statement of the proposition follows.~~

result (in homology), since it involves a further and rather elegant description of H_ω. We leave it to the reader to check the details and to describe the dual statement in cohomology.

REMARK 1.5. (i) The map $\Phi = i_* \delta : H_n^{cl}(\tilde{X}) \to H_{n-1}(\tilde{X})$ is a homotopy invariant and does not depend on the choice of N.

(ii). The form b on $H_n^{cl}(\tilde{X})$ is given by $b(x,y) = x \cdot \Phi y$, where the dot is intersection numbers $H_n^{cl}(\tilde{X}) \otimes H_{n-1}(\tilde{X}) \to \mathbb{R}$.

(iii). Hence $\mathrm{Rad}(b) = \mathrm{Ker}\,\Phi$, so $H_\omega = H_n^{cl}(\tilde{X})/\mathrm{Rad}(b)$ is equal to $\mathrm{Im}(\Phi)$ with the induced form.

(iv). Hence in Proposition 1.4, $\mathrm{Rad}(b_N | \mathrm{Im}\,\delta) = \mathrm{Ker}\, i_* \cap \mathrm{Im}\,\delta$.

A homotopy invariant description of Φ is as the Bockstein of the following short exact sequence of local coefficient systems over X:

$$0 \to \mathbb{Z}J \to \widehat{\mathbb{Z}J}^+ \oplus \widehat{\mathbb{Z}J}^- \to \widehat{\mathbb{Z}J} \to 0 ,$$

where J is the infinite cyclic group with generator t say, $\mathbb{Z}J$ its group ring, $\widehat{\mathbb{Z}J} = \mathbb{Z}^J$, and $\widehat{\mathbb{Z}J}^+$ ($\widehat{\mathbb{Z}J}^-$) the subgroup of elements of $\widehat{\mathbb{Z}J} = \mathbb{Z}^J$ whose t^i-component is zero for i sufficiently small (large).

2. Realizability of isometric structures as monodromy

This section is a digression. Its purpose is to indicate the richness of the monodromy as an invariant, by showing that every antisymmetric isometric structure over $\mathbb{Q}$ is realizable as the monodromy of some (X^3, α). Presumably some similar statement holds in the symmetric case, but this is very much harder. The situation for other coefficients is as follows.

The monodromy H_α^A of (X^{2n-1}, α) is definable in the same way for any ring A of coefficients. For a field F of characteristic zero the universal coefficient theorem easily shows

$$H_\alpha^F = H_\alpha^{\mathbb{Q}} \otimes_{\mathbb{Q}} F \quad ,$$

so realizability depends on realizability for $\mathbb{Q}$. Universal coefficient arguments break down in trying to show $H_\alpha^{\mathbb{Q}} = H_\alpha^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and in fact this equation is clearly false in general, since not every rational antisymmetric isometric structure is definable over $\mathbb{Z}$. In fact the γ-invariants of (X^{2n-1}, α), which are invariants of the $(-1)^{n-1}$-hermitian monodromy $H_\alpha^{\mathbb{C}}$, are not invariants of $H_\alpha^{\mathbb{Z}}$. The relation (if any) between $H_\alpha^{\mathbb{Z}}$ and $H_\alpha^{\mathbb{Q}}$ is rather unclear, in fact it is not even clear if $H_\alpha^{\mathbb{Z}}$ is always non-singular rather than just non-degenerate.

To return to the result announced in the first paragraph, recall that every antisymmetric bilinear space over $\mathbb{Q}$ is isomorphic to $(\mathbb{Q}^{2n}, b)$ for some n, where b is the form given by the matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We shall show the following theorem, where the last statement is included only to ease the proof.

<u>THEOREM 2.1.</u> If $(\mathbb{Q}^{2n}, b, t)$ is any antisymmetric isometric structure over $\mathbb{Q}$, then there exists an oriented 3-manifold X^3 and $\alpha \in [X^3, S^1]$ such that $H_\alpha^{\mathbb{Q}} = (\mathbb{Q}^{2n}, b, t)$. Further this can be done so that the submanifold $N \subset X$ of section 1 representing $D\alpha \in H_2(X; \mathbb{Z})$ can be chosen as the surface of genus n.

<u>Proof.</u> We consider the set $S \subset Aut(\mathbb{Q}^{2n}, b) = Sp(2n; \mathbb{Q})$ of t which can be realized as described in the theorem. We first show S is a subgroup. Indeed suppose t_1 and t_2 are realized respectively by (X_1, α_1) and (X_2, α_2), and suppose $N \subset X_1$ and $N \subset X_2$ are embeddings of the surface of genus n dual to α_1 and α_2. Let X_1' and X_2' be X_1 and X_2 cut open along N, so the boundary of X_i' consists of two copies N_i^+ and N_i^- of N for each i.
~~Paste X_1' to $-X_2'$ by identifying N_1^+ with N_2^+ and N_1^- with N_2^-. Using Proposition 1.4 it is clear that the result realizes $t_1 t_2^{-1}$, so $t_1 t_2^{-1} \in S$.~~ Let $X = X_1' \cup (-X_2')$ pasted by identifying N_1^+ with N_2^+ and N_1^- with N_2^- and let $X \xrightarrow{\alpha} S^1$ be the obvious map. In the terminology of Proposition 1.4 it is clear for dimensional reasons that $\operatorname{Im}\delta = H_1(N; \mathbb{Q})$ for each of (X_1, α_1), (X_2, α_2). It follows that it also holds for (X, α) and Proposition 1.4 then implies that (X, α) realizes $t_1 t_2^{-1}$, so $t_1 t_2^{-1} \in S$, as was to be shown.

We now give a list of realizable matrices in $Sp(2n, \mathbb{Q})$. Note that a matrix $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is in $Sp(2n, \mathbb{Q})$ if and only if $A^{-1}B = (A^{-1}B)^t$ and $D = (A^t)^{-1}$.

Case 1. $t \in Sp(2n, \mathbb{Q})$ is integral, that is $t \in Sp(2n, \mathbb{Z})$. Then we can take $X \to S^1$ as a fibration with fiber N, see proof of theorem III.3.1.

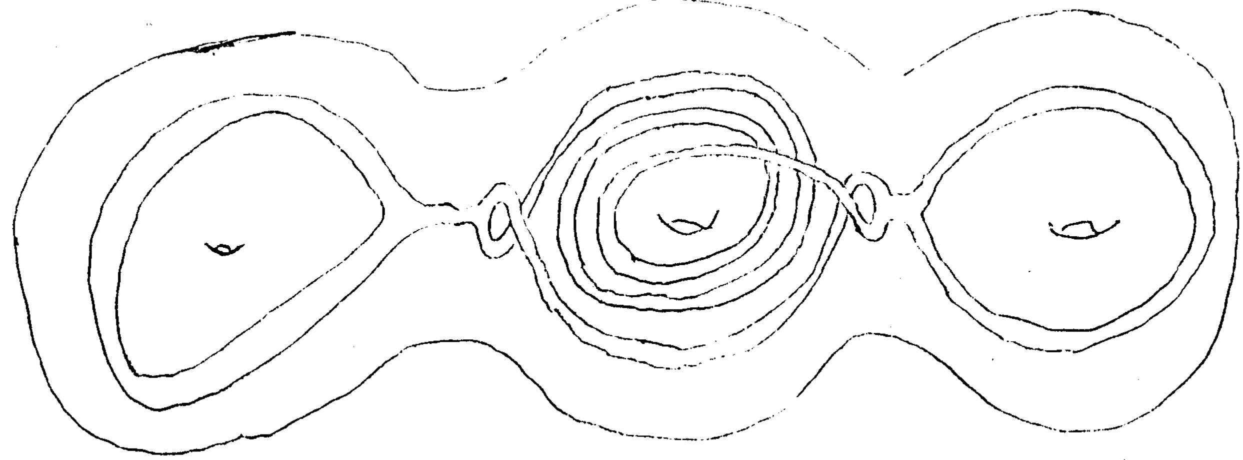
Case 2. $t = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$, with A diagonal. We need only realize the case

$$A = \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & q & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad q \in \mathbb{Z} - \{0\},$$

since any diagonal matrix is a product of such matrices and their inverses. An example will suffice to show how this is done. Suppose therefore that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let Y be the solid pretzel of genus $n=3$ and $j: Y \to Y$ the embedding indicated in the following picture.



Let $X' = Y - j(Y - \partial Y)$ and paste the two boundary comp of X' together using $j|\partial Y$, to get X. Then if $\alpha \in [X, S$ $H^1(X;\mathbb{Z})$ is dual to the homology class represented by $N = \partial$ then (X, α) does what is required.

Case 3. $t = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$ with A elementary, that is $A = I + \frac{p}{q} E_{ij}$ where E_{ij} is the matrix with a 1 in the (i,j)-position and zero elsewhere. This A is a product of integral and diagonal matrices as follows

$$A = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1/q & & \\ & & & q & \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & 0 \\ & 1 & \cdots & p & \\ & & & & 0 \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & 0 \\ & q & & & \\ & & \ddots & & \\ & & & 1/q & \\ 0 & & & & 1 \end{pmatrix},$$

so this case follows from cases 1 and 2.

Case 4. $t = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$ with $A \in GL(n, \mathbb{Q})$ arbitrary. Since $GL(n, \mathbb{Q})$ is generated by elementary and diagonal matrices, this follo from cases 2 and 3.

Case 5. $t = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$. Then $B = B^t$. In the cases

$$B = \begin{pmatrix} 0 & & 0 \\ & b & \\ 0 & & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & & b \\ & & \\ b & & 0 \end{pmatrix},$$

t can be expressed as a product of diagonal and integral matrices similarly to case ~~because~~ 3. But since $\begin{pmatrix} I & B_1 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & B_2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & B_1 + B_2 \\ 0 & I \end{pmatrix}$,

we can then generate any $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$.

Case 6. Any matrix of the form $\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \in Sp(2n, \mathbb{Q})$

is now realizable as a product of matrices from cases 4 and 5.

Case 7. Let $T \subset Sp(2n, \mathbb{Q})$ be the subgroup of matrices as in case 6. We claim $Sp(2n, \mathbb{Z}).T = Sp(2n, \mathbb{Q})$, which completes the proof.

Indeed, let $\mathcal{K} = \{ K \subset \mathbb{Q}^{2n} \mid K = K^\perp \}$. Then $Sp(2n, \mathbb{Q})$ acts transitively on $\mathcal{K}$ with isotropy subgroup T (see below), so

$$\mathcal{K} \cong Sp(2n, \mathbb{Q}) / T.$$

On the other hand $Sp(2n, \mathbb{Z})$ already acts transitively on $\mathcal{K}$ with isotropy subgroup $T \cap Sp(2n, \mathbb{Z})$, so the inclusion $Sp(2n, \mathbb{Z}) \longrightarrow Sp(2n, \mathbb{Q})$ induces a bijection

$$Sp(2n, \mathbb{Z}) / T \cap Sp(2n, \mathbb{Z}) \xrightarrow{\cong} Sp(2n, \mathbb{Q}) / T ,$$

which certainly implies our claim.

To see that $Sp(2n, \mathbb{Z})$ acts transitively on $\mathcal{K}$, note that $Sp(2n, \mathbb{Z})$ certainly acts transitively on the set Υ of simplectic bases of $\mathbb{Z}^{2n}$ and $\mathcal{K}$ is obviously an equivariant quotient of Υ.

3. Algebraic invariants of monodromy and the main theorem.

From now on we shall just consider complex (± 1)-hermitian isometric structures. In particular, by the monodromy $\mathcal{H}_\alpha$ of a homotopy class ~~[illegible]~~ $\alpha \in [X^{2n-1}, S^1]$ we mean the $(-1)^{n+1}$-hermitian monodromy $\mathcal{H}^c_\alpha$. If $\mathcal{H} = (H, b, t)$ is any (± 1)-hermitian isometric structure then by $\operatorname{sign}(\mathcal{H})$ we mean the signature of the form b (recall that this is defined as $\operatorname{sign}(ib)$ if b is antihermitian). We plan to define certain invariants $\gamma_p(\mathcal{H})$ of isometric structures with $\operatorname{sign}(\mathcal{H}) = 0$, which we shall then identify with the invariants $\gamma_p(X^{2n-1}, a)$ if $\mathcal{H}$ is the monodromy of (X^{2n-1}, a). The definition of $\gamma_p(\mathcal{H})$ is slightly involved ~~[illegible]~~; alternative descriptions and specific calculations will be given later.

<u>Notation.</u> It is clear what the tensor product $\mathcal{H} \otimes \mathcal{H}'$ of two isometric structure is. If $\mathcal{H}$ is an isometric structure and $\rho : \mathbb{Z} \longrightarrow \operatorname{Aut}(V, b)$ is a hermitian representation then $\mathcal{H} \otimes \rho$ means the isometric structure $\mathcal{H} \otimes (V, b, \rho(1))$.

<u>DEFINITION 3.1.</u> If $\mathcal{H} = (H, b, t)$ is an ε-hermitian isometric structure, $\varepsilon = \pm 1$, and $K \subset H$ is a subspace with $K = K^\perp$ (so $\operatorname{sign}(\mathcal{H}) = 0$) we define

$$\operatorname{sign}(\mathcal{H}, K) = \operatorname{sign}(\varphi \mid (1-t)^{-1}K)$$

where φ is the (possibly degenerate) $(-\varepsilon)$-hermitian form

$$\varphi(x,y) = b((t-\bar{t})x, y)$$

on H. Furthermore for any hermitian representation $\rho : \mathbb{Z} \to Aut(V)$ of $\mathbb{Z}$ we define

$$\gamma_\rho(\mathcal{H}, K) = sign(\mathcal{H}\otimes\rho, K\otimes V) - sign(\mathcal{H}\otimes triv, K\otimes V),$$

where $triv : \mathbb{Z} \longrightarrow Aut(V)$ is the trivial representation.

THEOREM 3.2. Given $\mathcal{H} = (H, b, t)$ with $sign(\mathcal{H}) = 0$, then for any ρ the number $\gamma_\rho(\mathcal{H}, K)$ defined above is independent of the choice of $K \subset H$ with $K = K^\perp$, so we denote it $\gamma_\rho(\mathcal{H})$.

THEOREM 3.3. Given $X^{2n-1} \xrightarrow{a} B\mathbb{Z} = S^1$ so that (X, a) represents zero in $\Omega_{2n-1}(S^1)\otimes\mathbb{Q}$ (e.g. n even), then $sign(\mathcal{H}_{[a]}) = 0$ and for any ρ

$$\gamma_\rho(X, a) = \gamma_\rho(\mathcal{H}_{[a]}).$$

In particular $\gamma_\rho(X, a)$ is a homotopy invariant.

The proofs of these two results will take the rest of this chapter. Before starting on them we note two corollaries. The first gives the homotopy invariance of the ~~[illegible]~~ Hirzebruch L_{4k}-class of a $(4k+1)$-manifold. If one filters our ~~[illegible]~~ proof of this corollary down to its basics it is essentially Novikovs original proof.

COROLLARY 3.4. (Novikov [ ]) If $\alpha \in [X^{4k+1}, S^1] = H^1(X^{4k+1}; \mathbb{Z})$ then $\mathrm{sign}(\mathcal{H}_\alpha) = (L_{4k}(X) \cup \alpha)[X]$, the "Novikov signature" of α. By letting α range through all of $H^1(X; \mathbb{Z})$ this gives a homotopy invariant calculation of $L_{4k}(X)$.

Proof. By the above theorem $\mathrm{sign}(\mathcal{H}_\alpha)$ vanishes if (X, α) represents zero in $\Omega_*(S^1)$, so $\mathrm{sign}(\mathcal{H}_\alpha)$ is a bordism invariant. So is $(L_{4k}(X) \cup \alpha)[X]$, since it is a characteristic number. But they agree on singular manifolds of the form $(N^{4k} \times S^1, pr_2)$ in S^1 and $\Omega_{4k+1}(S^1)$ is generated modulo torsion by such manifolds, so they agree in general.

Recall that in Theorem II.6.4 we extended the definition of γ_ρ to the case of nonbounding (X, a) when ρ is a definite hermitian representation.

COROLLARY 3.5 Given $X^{2n-1} \xrightarrow{a} S^1$ with $\mathrm{sign}(\mathcal{H}_{[a]}) = s$ (of course $s = 0$ if n is even) and ρ a definite hermitian representation, then

$$\bar{\gamma}_\rho(X, a) = \gamma_\rho(\mathcal{H}_{[a]} \oplus (\mathbb{R}^{|s|}, -\mathrm{sign}(s).e, \mathrm{id})) + s\,\bar{\gamma}(S^1, \rho),$$

where $\bar{\gamma}$ is the extension of γ of section II.6 and e is a euclidean metric on $\mathbb{R}^{|s|}$. In particular $\bar{\gamma}_\rho(X, a)$ is a homotopy invariant.

This is an immediate consequence of II.6.4.

The following is a summary of the proof of theorems 3.2 and 3.3. It is clearly sufficient to prove 3.3 in the case that (X,a) actually bounds, rather than just represents torsion in $\Omega_{2n-1}(S$ so we shall restrict attention to this case.

We first give yet another description (section 4) of the monodromy of (X,a), this time in terms of a certain "isometric relation" on $H_{n-1}(N)$, where $N^{2n-2} \subset X$ is dual to $[a]$. This description is used in section 5 to prove that $\gamma_p(X,a) = \gamma_p(\mathcal{H}_{[a]}, K)$ for a suitable $K \subset H_{[a]}$ with $K = K$. The main tool here is C.T.C. Wall's non-additivity formula for signature of a union of two manifolds pasted along portions of their boundaries.

It then only remains to prove theorem 3.2, that is that $\gamma_p(\mathcal{H}, K)$ is independant of K. Though this is a purely algebraic result, we know of no purely algebraic proof. Our proof, which is given in section 6, uses the result of section 5.

4. Relations on hermitian spaces

This section contains technical results on relations on hermitian vector spaces which will be needed for the proof of theorems 3.2 and 3.3. We are interested in <u>additive relations</u> between two vector spaces V and W, that is subspaces $R \subset V \times W$. In analogy with composition of functions we define the composition of relations $R_1 \subset V \times W$ and $R_2 \subset U \times V$ in the usual backward way as

$$R_1 \circ R_2 = \{(x,z) \in U \times W \mid \exists y \in V \text{ with } (x,y) \in R_2 \text{ and } (y,z) \in R_1\}.$$

For $R \subset V \times W$ we also make the usual definitions $(A \subset V)$:

$$R^{-1} = \{(y,x) \in W \times V \mid (x,y) \in R\},$$
$$RA = \{y \in W \mid \exists x \in A \text{ with } (x,y) \in R\}.$$

The following lemma tells one to think of an additive relation $R \subset V \times W$ as the graph of an isomorphism from a subquotient of V to a subquotient of W.

<u>LEMMA 4.1</u> $R \subset V \times W$ is an additive relation if and only if there exist subspaces $A \subset A' \subset V$ and $B \subset B' \subset W$ and an isomorphism $\varphi : A'/A \xrightarrow{\cong} B'/B$ such that

$$R = \{(x,y) \subset A' \times B' \mid \varphi[x] = [y]\}.$$

In particular $A = R^{-1}\{0\}$, $A' = R^{-1}W$, $B = R\{0\}$, $B' = RV$; these are sometimes called respectively the kernel, domain, indeterminacy, and image of R.

Proof. If R is an additive relation and we define A, A', B, B' as in the lemma then R can clearly be interpreted as the graph of a linear map $A \longrightarrow B/B'$ with kernel A'. The converse is trivial.

DEFINITION. If (W, b) is a finite dimensional non-degenerate (± 1)-hermitian space then an *isometric relation* on W will mean an additive relation $R \subset W \times (-W)$ satisfying $R = R^{\perp}$ (equivalently $R \subset R^{\perp}$ and $\dim R = \dim W$). Here $-W$ is W with hermitian form $-b$.

For example the graph $R(f) = \{(x, f(x)) \mid x \in W\}$ of a linear map $W \longrightarrow W$ is an isometric relation if and only if f is an isometry. More generally

LEMMA 4.2. $R \subset W \times (-W)$ is an isometric relation if and only if in Lemma 4.1 we have $A' = A^{\perp}$, $B' = B^{\perp}$ and φ is an isometry $A^{\perp}/A \xrightarrow{\cong} B^{\perp}/B$ (since $A = \mathrm{Rad}(b|A^{\perp})$, b induces a nondegenerate form on $A^{\perp}/A$, and similarly on $B^{\perp}/B$).

Proof. If A, B and $\varphi: A^{\perp}/A \longrightarrow B^{\perp}/B$ are as in the lemma and $R = \{(x, y) \in A^{\perp} \times B^{\perp} \mid \varphi[x] = [y]\}$, then a trivial calculation shows $R \subset R^{\perp}$ and $\dim R = \dim W$, so R is an isometric relation. Conversely if R is an isometric relation then orthogonal complement of the equation $A \times \{0\} = (W \times \{0\}) \cap R$ gives $A^{\perp} \times W = (W \times \{0\})^{\perp} + R^{\perp}$ $(\{0\} \times W) + R = A' \times W$, whence $A^{\perp} = A'$. Similarly $B^{\perp} = B'$, and an easy calculation shows that φ is an isometry.

LEMMA 4.3 If R_1 and R_2 are isometric relations on W then so is $R_1 \circ R_2$.

Proof. Certainly $R_1 \circ R_2 \subset (R_1 \circ R_2)^\perp$, so we must show that $\dim(R_1 \circ R_2) = \dim W$. Let $K = \{(x,y,z) \in W \times W \times W \mid (x,y) \in R_2 \text{ and } (y,z) \in R_1\}$. There are two short exact sequences

$$0 \to R_1^{-1}\{0\} \cap R_2\{0\} \xrightarrow{\alpha} K \xrightarrow{\beta} R_1 \circ R_2 \to 0$$

given by $\alpha(y) = (0, y, 0)$ and $\beta(x,y,z) = (x,z)$, and

$$0 \to K \xrightarrow{\gamma} R_2 \times R_1 \xrightarrow{\delta} R_1^{-1}W + R_2 W \to 0$$

with maps $\gamma(x,y,z) = ((x,y),(y,z))$ and $\delta((x,y),(w,z)) = y - w$. These give dimension equations which combine to give

$$\dim(R_1 \circ R_2) = 2\dim W - \dim(R_1^{-1}\{0\} \cap R_2\{0\}) - \dim(R_1^{-1}W + R_2 W).$$

But $(R_1^{-1}W + R_2W)^\perp = (R_1^{-1}W)^\perp \cap (R_2W)^\perp = R_1^{-1}\{0\} \cap R_2\{0\}$ by Lemma 4.2, so the dimension equation becomes $\dim(R_1 \circ R_2) = 2\dim W - \dim W = \dim W$, as required.

Remark. Lemmas 4.2 and 4.3 hold more generally for relations between two hermitian spaces, $R \subset \mathbf{W} \times (-W)$, with $R = R^\perp$. We shall not need this and in any case the proofs are the same. Similarly everything holds equally well for non-degenerate finite-dimensional bilinear or sesquilinear spaces over any field.

The only application we shall make of lemma 4.3 is the following. Let $R^\infty\{0\}$ and $R^\infty W$ be the limits of the sequences

$$\{0\} \subseteq R\{0\} \subseteq R^2\{0\} \subseteq \cdots \subseteq R^j\{0\} \subseteq \cdots ,$$
$$W \supseteq RW \supseteq R^2 W \supseteq \cdots \supseteq R^j W \supseteq \cdots .$$

Since W is finite dimensional these sequences stabilize after a certain time, that is $R^j\{0\} = R^\infty\{0\}$ and $R^j W = R^\infty W$ for j sufficiently large.

<u>LEMMA 4.4.</u> If R is an isometric relation then $R^j\{0\} = (R^j W)^\perp$ and $R^{-j}\{0\} = (R^{-j}W)^\perp$ for $j = 0, 1, 2, \cdots, \infty$.

<u>Proof</u>. This is immediate by 4.2, since R^j and R^{-j} are isometric relations by 4.3.

Taking the graph of an isometry allows one to consider an isometric structure as an isometric relation. We now describe how to go the other way and derive an isometric structure from an isometric relation.

<u>PROPOSITION 4.5</u> Let R be an isometric relation and put $B = R^\infty\{0\}$, so $B^\perp = R^\infty W$ and there is an induced non-degenerate form on $H = B^\perp / B$. Then

$$S = [(B \times B + R) \cap (B^\perp \times B^\perp)]/B \times B \subset H \times (H)$$

is the graph of an isometry $h : H \longrightarrow H$. We denote the isometric structure (H, h) by $\mathcal{H}(R)$.

Proof. Note that the modular law

$$X \subset Z \implies (X+Y) \cap Z = X + (Y \cap Z)$$

holds for subspaces of a vector space, so in such a situation we can and will omit brackets. In particular for subspaces of a hermitian space it follows that

$$X \subset X^{\perp} \implies (X + Y \cap X^{\perp})^{\perp} = X + Y^{\perp} \cap X^{\perp}.$$

Applying this to

$$S_0 = (B \times B) + R \cap (B^{\perp} \times B^{\perp}) \quad ,$$

considered as a subspace of $W \times (-W)$, shows $S_0 = S_0^{\perp}$, whence also $S = S^{\perp}$, so S is an isometric relation.

Next we observe that if $(x,y) \in S_0$ then certainly $(x,y) \in (B \times B) + R$; so if $x \in B$ then $y \in RB$. But $RB = RR^{\infty}\{0\} = R^{\infty}\{0\} = B$, so we have shown $S_0 B \subset B$. Hence $S\{0\} = \{0\}$. Thus by Lemma 4.2, $SH = \{0\}^{\perp} = H$ and S is the graph of an isometry from a subquotient of H onto the whole of H. For dimensional reasons it follows that S is the graph of an isometry $h : H \to H$.

The next lemma, which gives a more symmetric description of $\mathcal{H}(R)$, is needed to determine the topological meaning of this isometric structure.

LEMMA 4.6. Let R be an isometric relation on (W, b) and put $A = R^{-\infty}\{0\}$ and $B = R^{\infty}\{0\}$ and $\mathbb{C} = A^{\perp} \cap B^{\perp}$ and $\mathbb{D} = \mathrm{Rad}(b|\mathbb{C}) = \mathbb{C} \cap \mathbb{C}^{\perp}$. Put $H_1 = C/D$ and $S_1 = [(\mathbb{D} \times \mathbb{D}) + R \cap (\mathbb{C} \times \mathbb{C})]/\mathbb{D} \times \mathbb{D} \subset H_1 \times (-H_1)$. Then the pair (H_1, S_1) is isomorphic to the pair (H, S) of Lemma 4.5 and thus also defines $\mathcal{H}(R)$.

Proof. We first show that the inclusion $i: \mathbb{C} \to B^\perp$ induces an epimorphism $\mathbb{C} \xrightarrow{\pi i} B^\perp/B$, in other words that $\mathbb{C} + B = B^\perp$. Choose ε such that $R^\varepsilon W = R^\infty W = B^\perp$ and $R^{-\infty} W = R^{-\infty} W = A^\perp$, whence also $R^\varepsilon\{0\} = B$ and $R^{-\varepsilon}\{0\} = A$. Given any $y \in B^\perp = R^\infty W$, we can find $x \in R^\infty W$ with $y \in R^\varepsilon\{x\}$. Then $x \in R^{-\varepsilon}\{y\} \subset R^{-\varepsilon} W = R^{-\infty} W$, so we can find $z \in R^\varepsilon\{x\}$ with $z \in R^{-\infty} W$. By construction $z - y \in R^\varepsilon\{x - x\} = B$ and $z \in R^\varepsilon\{x\} \cap R^{-\infty} W \subset B^\perp \cap A^\perp = \mathbb{C}$, so $y \in B + \mathbb{C}$. Thus we have shown $B^\perp \subset B + \mathbb{C}$. The other inclusion is trivial.

Now the epimorphism $\mathbb{C} \xrightarrow{\pi i} B^\perp/B$ preserves hermitian forms, and since $B^\perp/B$ is non-degenerate, πi induces an isometry

$$\varphi : \mathbb{C}/\mathrm{Rad}(b|\mathbb{C}) \xrightarrow{\cong} B^\perp/B .$$

~~In particular it follows that $X \cap B = \mathrm{Rad}(b|X) = Y$, which we shall use in showing that $\varphi(S_1) = S$, which is all that remains to be done.~~

It remains to show that $(\varphi \times \varphi)(S_1) = S$. First note that if $x \in C$ then there exists $y \in C$ with $(x, y) \in R$. Hence for $[x] \in H_1$ there exists $[y] \in H_1$ with $([x], [y]) \in S_1$, so $\dim S_1 \geq \dim H_1 = \dim H$. On the other hand $\dim S = \dim H$, since S is the graph of a map $H \to H$, so it suffices to prove $(\varphi \times \varphi)(S_1) \subset S$, which is trivial.

The following example motivates this section. Let X^{2n-1} be a closed oriented manifold and let $\alpha \in [X, S^1]$. Choose a submanifold $N^{2n-2} \subset X^{2n-1}$ dual to α and let X' be X cut open along N, so $\partial X'$ is the union of two copies N^- and N^+ of N, as in section 1. As oriented manifolds $\partial X' = N^- + (-N^+) = N + (-N)$. Using $\mathbb{C}$ as coefficients, put

$$W = H_{n-1}(N),$$
$$R = \mathrm{Ker}(H_{n-1}(\partial X') \longrightarrow H_{n-1}(X')) \subset W \times (-W).$$

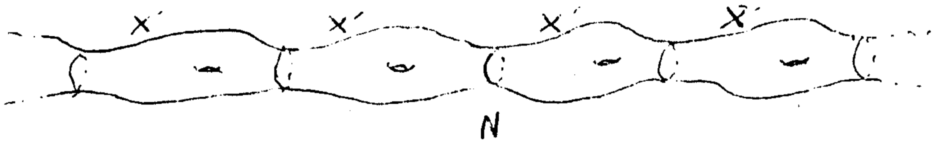
We use the $(-1)^{n-1}$-hermitian intersection form on $W \cong H_{n-1}(N;\mathbb{R}) \otimes \mathbb{C}$. The Poincaré duality diagram

$$\begin{array}{ccccc} H_n(X', \partial X') & \longrightarrow & H_{n-1}(\partial X') & \xrightarrow{i_*} & H_{n-1}(X') \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ (H_{n-1}(X'))^* & \xrightarrow{(i_*)^*} & (H_{n-1}(\partial X'))^* & \longrightarrow & (H_n(X', \partial X'))^* \end{array},$$

where the vertical arrows are given by intersection forms, shows that $R = \mathrm{Ker}(i_*) = R^{\perp}$, so R is an isometric relation.

<u>PROPOSITION 4.7.</u> In the above situation the monodromy $\mathcal{H}_\alpha$ of (X, α) is isomorphic to $\mathcal{H}(R)$.

<u>Proof</u>. We shall use the description of $\mathcal{H}_\alpha$ of Proposition 1.4, so we must calculate $\mathrm{Im}\,\delta$, where δ is the map $H_n^{cl}(\widetilde{X}) \longrightarrow H_{n-1}(N)$. If we consider $\widetilde{X}$ to be pasted together out of infinitely many copies of X' as in section 1,

$\widetilde{X} =$ 

then $\mathrm{Im}\,\delta$ is represented by those classes in $N \subset \widetilde{X}$ which bound infinitely far to the left and right in $\hat{X}$. Observe that $(x,y) \in R$ if and only if x, as a class in the left boundary N^- of X', is homologous ~~xxxx~~ in X' to y considered as a class in the right boundary N^+ of X'. Hence $\operatorname{Im} \delta$ is simply $R^{\infty} W \cap R^{-\infty} W$, which is G in the notation of Lemma 4.6, and the present proposition follows directly from 1.4 and 4.6.

To follow:

§5 – 6 Proof of theorems 3.2 & 3.3

CHAPTER V : APPLICATIONS & EXAMPLES

Contents:

Calculation of α - invariants of injective group actions.

Almost periodicity of signature for coverings.

" " " " for cyclic suspension of knots.

Numerical calculations

etc.