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Fibering over the Circle within a Bordism Class

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In [1] Burdick considers the problem: which bordism classes in Ω_* contain representatives which fibre over the circle S^1? An obvious necessary condition is that the signature be zero and in this note we show that this condition is also sufficient. Modulo 2-torsion in Ω_* this was shown by Burdick and Conner in [1] and [2] using rather different methods.

Theorem. *If M is a closed oriented differentiable manifold, then M is oriented cobordant to a manifold which fibre over S^1 if and only if* $\operatorname{sign}(M)=0$.

Proof. Let M_i, $i=0,1,2$, be three compact oriented manifolds and $\varphi_i:\partial M_i\to\partial M_{i+1}$ (indices modulo 3) be orientation preserving diffeomorphisms of their boundaries. We consider closed oriented manifolds of the form

$$M=M_0\underset{\varphi_0}{\cup}-M_1+M_1\underset{\varphi_1}{\cup}-M_2+M_2\underset{\varphi_2}{\cup}-M_0$$

where $-M_i$ is M_i with reversed orientation and $+$ denotes disjoint union. Let $I_n\subset\Omega_n$ be the subgroup generated by all cobordism classes containing a manifold of this type. By theorem 1(a) of Jänich [3], a cobordism invariant ϱ vanishes on all manifolds of this type, and hence on I_n, if and only if ϱ is a multiple of signature. Hence I_n is simply the subgroup of Ω_n given by sign $=0$.

It thus only remains to show that the above generators of I_n are cobordant to manifolds which fibre over S^1. But by Theorem 1.1 of Burdick [1], $M_0\underset{\varphi_0}{\cup}-M_1+M_1\underset{\varphi_1}{\cup}-M_2+M_2\underset{\varphi_2}{\cup}-M_0$ is cobordant to the sum of $M_0\underset{id}{\cup}-M_0+M_1\underset{id}{\cup}-M_1+M_2\underset{id}{\cup}-M_2$ and a manifold which fibres over S^1. Since $M_i\underset{id}{\cup}-M_i$ bounds $M_i\times[0,1]$ after smoothing the latter, the theorem follows.

Note that the fibrations we obtain in the above proof have null-cobordant fibres, so one could add the condition "with null-cobordant fibre" in the above theorem. This is not as surprising as it may appear at first sight – if one has represented a bordism class more generally by a manifold M which fibres over S^k, then one can do it with null-cobordant fibre by adding $-F\times S^k$, where F is the fibre of M.

References

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