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MULTIPLICATIVITY OF SIGNATURE

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Dedicated to the memory of George Cooke

0. Introduction

Let $F \to E \to M$ be a fibration of closed oriented manifolds. Then one may ask under what conditions the multiplicativity formula

$$\operatorname{sign}(E) = \operatorname{sign}(M)\operatorname{sign}(F)$$

holds. This is false in general: counterexamples where first found by Atiyah [1] and Kodaira [11] and generalized variously by Meyer [16, 17], Hirzebruch [9] and Lusztig [12]. On the other hand we showed in [21] that for a large class of structure groups multiplicativity does in fact hold. In the present paper we improve this result as follows.

Let $\mathscr{C}$ be the class of all discrete groups G such that signature is multiplicative for any local coefficient system (hence for any fibration) with structure group G (see Section 1). Let $WU(G)$ be the Witt group of finite dimensional hermitian representations of G over $\mathbf{C}$. Using the signature theorem of Atiyah, Lusztig, Meyer [1, 12, 16] we show:

Theorem 0.1. *There is a natural ring homomorphism*

$$\tilde{\psi}_G : WU(G) \to \tilde{H}^{\mathrm{ev}}(G; \mathbf{Q})$$

such that $G \in \mathscr{C}$ if and only if $\tilde{\psi}_G = 0$.

Using properties of the Witt ring $WU(G)$ we deduce:

Theorem 0.2. *$\mathscr{C}$ has the following closure properties.*

(i) *$\mathscr{C}$ contains all G with $H^{\mathrm{ev}}(G; \mathbf{Q}) = 0$, for instance all finite groups, free groups, fundamental groups of non-orientable surfaces.*

(ii) *$\mathscr{C}$ is closed under formation of direct products, free products, direct limits.*

(iii) *$\mathscr{C}$ is closed under formation of finite extensions (not necessarily normal), quotients by subgroups of finite index, taking subgroups which possess normal complements. More generally if $H \in \mathscr{C}$ and $H \to G$ is a homomorphism which induces an injection $H^{\mathrm{ev}}(G; \mathbf{Q}) \to H^{\mathrm{ev}}(H; \mathbf{Q})$ then $G \in \mathscr{C}$.*

(iv) *If $H \subset G$ has finite index and $H \cdot Z_G(H) = G$ ($Z_G(H)$ = centralizer of H in G) and if $G \in \mathscr{C}$ then $H \in \mathscr{C}$.*

In [21] we had shown using other methods that a subclass $\mathscr{C}_0 \subseteq \mathscr{C}$ has (most of) these closure properties but not that $\mathscr{C}$ itself has them. A comparison of the methods is also of interest (Section 4, see also Section 5, especially Proposition 5.2 ff).

We obtain similar closure properties for the classes $\mathscr{C}_+$ and $\mathscr{C}_-$ of structure groups for which signature is always multiplicative for symmetric respectively skew-symmetric real bilinear coefficient systems (Theorem 6.1).

What appears to be a class of groups related to those discussed here has been described by Hirsch and Thurston [8], but this relationship may be superficial.

In a final section we describe applications to invariants of odd dimensional manifolds.

1. Preliminaries on signature

If M^{2n} is a compact connected oriented manifold and $\Gamma \to M$ a local coefficient system (= locally trivial sheaf) over M with fiber a complex hermitian vector space, then cup product plus the hermitian form combine to give a map

$$B : H^n(M, \partial M; \Gamma) \otimes H^n(M, \partial M; \Gamma) \to H^{2n}(M, \partial M; \Gamma \otimes_{\mathbf{R}} \Gamma)$$
$$\to H^{2n}(M, \partial M; \mathbf{C}) = \mathbf{C}$$

which is a hermitian form for n even and skew-hermitian for n odd. Define

$$\operatorname{sign}(M, \Gamma) = \operatorname{sign}(B),$$

where for n odd, $\operatorname{sign}(B)$ means signature of the hermitian form iB.

One can make a similar definition if Λ is a real $(-1)^n$-symmetric bilinear coefficient system, since then $B : H^n(M, \partial M; \Lambda) \otimes H^n(M, \partial M; \Lambda) \to \mathbf{R}$ is symmetric and has a signature ([16], but beware of the sign convention there). Tensoring with $\mathbf{C}$ and hermitianizing the form, this can be reduced to the hermitian case.

Theorem 1.1. *If $F^{2m} \to X^{2(m+n)} \to M^{2n}$, $m + n$ even, is a fibration of compact oriented manifolds, F closed, and Λ is the local coefficient system over M with fiber the cup product form on $H^m(F; \mathbf{R})$, then* $\operatorname{sign} X = (-1)^n \operatorname{sign}(M, \Lambda)$.

This is proved in Meyer [16]; see also [5].

Suppose now $\Gamma \to M$ is a hermitian coefficient system as above. Then there is a corresponding vector bundle $\tilde{\Gamma} \to M$ (by "putting the topology back into the fibers of Γ") which can be split as a sum $\tilde{\Gamma} = \tilde{\Gamma}^+ \oplus \tilde{\Gamma}^-$ of vector bundles on which the hermitian form is respectively positive and negative definite. Lusztig [12] proved

Theorem 1.2. *If M is closed then* $\operatorname{sign}(M,\Gamma)=\operatorname{ch}(\tilde{\Gamma}^+-\tilde{\Gamma}^-)\mathscr{L}(M)[M]$, *where* ch *is chern character and $\mathscr{L}(M)$ is the unstable L-class $\Pi x_i/\tanh(x_i/2)$, where $\Pi(1+x_i)$ is the total Pontryagin class $p(M)$.*

For real coefficient systems the corresponding result was also proved by Meyer [16]. For fiber bundles it was proved by Atiyah [1].

2. Structure groups and Theorem 0.1

After choosing a base point $x \in M$, any local coefficient system $\Gamma \to M$ is classified by a homomorphism $\pi_1(M,x)\to \operatorname{Aut}(\Gamma_x)$ where Γ_x is the fiber over x.

Definition 2.1. If $\Gamma \to M$ is a hermitian local coefficient system with fiber V such that the classifying map $\pi_1(M)\to\operatorname{Aut}(V)$ can be written as a composition $\pi_1(M)\to G \xrightarrow{\rho} \operatorname{Aut}(V)$, we say admits G as *structure group with defining representation* ρ. We call (G,ρ) a *structure pair* for Γ.

For any group G, let $RU(G)$ denote the Grothendieck group of representations of G in finite dimensional non-degenerate hermitian vector spaces (not necessarily definite) with ring structure given by orthogonal sum and tensor product. Define a ring homomorphism

$$\psi_G : RU(G)\to H^*(BG;\mathbf{Q}) = H^*(G;\mathbf{Q})$$

as follows. Any hermitian representation $\rho: G\to \operatorname{Aut}(V)$ determines a hermitian coefficient system $\Gamma_\rho \to BG$ with fiber V. Let $\tilde{\Gamma}_\rho=\tilde{\Gamma}^+_\rho\oplus\tilde{\Gamma}^-_\rho$ be a splitting of the corresponding bundle into a positive and a negative definite summand and put

$$\psi_G(\rho)=\operatorname{ch}(\tilde{\Gamma}^+_\rho-\tilde{\Gamma}^-_\rho)\in H^*(BG;\mathbf{Q}).$$

Lemma 2.2. *If $\rho: G\to\operatorname{Aut}(V)$ is hyperbolic (i.e. there exists a G-invariant subspace $K\subset V$ with $K=K^\perp$, whence $\dim(K)=\frac{1}{2}\dim(V)$) then $\psi_G(\rho)=0$.*

Proof. Let $\tilde{\Gamma}_\rho=\Gamma^+_\rho\oplus\Gamma^-_\rho$ be the above splitting of the bundle $\tilde{\Gamma}_\rho\to BG$ defined by ρ. The G-invariant subspace $K\subset V$ defines a subbundle $\tilde{K}\subset\tilde{\Gamma}_\rho$, and since the hermitian form is zero on $\tilde{K}$, we must have $\tilde{K}\cap\tilde{\Gamma}^+_\rho=\tilde{K}\cap\tilde{\Gamma}^-_\rho=0$. Thus $\dim(\tilde{\Gamma}^+_\rho)\leq \dim(\tilde{\Gamma}_\rho)-\dim K=\frac{1}{2}\dim(V)$, and the same for $\dim(\tilde{\Gamma}^-_\rho)$. It follows that $\dim(\tilde{\Gamma}^\pm_\rho)=\frac{1}{2}\dim(V)$, so $\tilde{\Gamma}^+_\rho$ and $\tilde{\Gamma}^-_\rho$ are both complements of $\tilde{K}$, hence both isomorphic to $\tilde{\Gamma}_\rho/K$. Thus $\psi_G(\rho)=\operatorname{ch}(\tilde{\Gamma}_\rho/K-\tilde{\Gamma}_\rho/K)=0$.

Corollary 2.3. *ψ_G induces a map (also called ψ_G)*

$$\psi_G: WU(G)\to H^*(G;\mathbf{Q}),$$

where $WU(G)$ is the Witt group obtained by factoring $RU(G)$ by the ideal generated by hyperbolic representations.

Now let $\tilde{\psi}_G$ be the composition $WU(G) \xrightarrow{\psi_G} H^*(G;\mathbf{Q}) \to \tilde{H}^*(G;\mathbf{Q})$. Note that the image is only in even cohomology.

Theorem 2.4. *Given a hermitian representation $\rho: G \to \mathrm{Aut}(V)$, the following statements are equivalent.*

(i) $\tilde{\psi}_G(\rho) = 0$.

(ii) *For any hermitian coefficient system $\Gamma \to M$ over a closed manifold admitting (G, ρ) as structure pair and any further hermitian system $\Lambda \to M$ we have* $\mathrm{sign}(M, \Lambda \otimes \Gamma) = \mathrm{sign}(M, \Lambda)\,\mathrm{sign}(\Gamma)$.

(iii) *For any $\Gamma \to M$ as in* (ii) *we have* $\mathrm{sign}(M, \Gamma) = \mathrm{sign}(M) \cdot \mathrm{sign}(\Gamma)$.

Proof. (i) $\Longrightarrow$ (ii). Suppose Γ is as in (ii) and $\pi_1(M) \to G$ is a classifying map. This induces a map $f: M \to BG$ and then $\Gamma \cong f^*\Gamma_\rho$, where $\Gamma_\rho \to BG$ is the coefficient system determined by ρ. Thus if (i) holds then $\mathrm{ch}(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = f^*\psi_G(\rho)$ is contained in $H^0(M;\mathbf{Q})$. But for any vector bundle E the zero-dimensional component of $\mathrm{ch}(E)$ is $(\dim E) \cdot 1 \in H^0(M;\mathbf{Q})$, so $\mathrm{ch}(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = (\dim \tilde{\Gamma}^+ - \dim \tilde{\Gamma}^-) \cdot 1 = (\mathrm{sign}\, \Gamma) \cdot 1$. Thus $\mathrm{ch}((\tilde{\Lambda} \otimes \tilde{\Gamma})^+ - (\tilde{\Lambda} \otimes \tilde{\Gamma})^-) = \mathrm{ch}(\tilde{\Lambda}^+ - \tilde{\Lambda}^-) \cdot \mathrm{ch}(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = \mathrm{sign}(\Gamma) \cdot \mathrm{ch}(\tilde{\Lambda}^+ - \tilde{\Lambda}^-)$, so the multiplicativity formula follows from the signature Theorem 1.2.

(ii) $\Longrightarrow$ (iii) is trivial by taking $\Lambda = \mathbf{C}$, the trivial coefficient system.

(iii) $\Longrightarrow$ (i). Suppose (i) is false, so $\psi_G(\rho) = (\mathrm{sign}\, \Gamma_\rho) \cdot 1 + \alpha + \beta$ with $0 \neq \alpha \in H^{2n}(BG;\mathbf{Q})$, $n > 0$, and β a possibly zero sum of terms of higher degree. Choose a closed oriented singular manifold $f: M^{2n} \to BG$ such that $\alpha(f_*[M]) \neq 0$. This is possible by Steenrod representability of rational homology, see for instance Conner and Floyd [6, Theorem 15,3]. Now for $\Gamma = f^*\Gamma_\rho$ we have

$$\mathrm{ch}(\tilde{\Gamma}^+ - \tilde{\Gamma}^-) = (\mathrm{sign}\, \Gamma) \cdot 1 + f^*(\alpha),$$

since $H^*(M)$ is zero in degrees $> 2n$. Also

$$\mathscr{L}(M) = 2^n \cdot 1 + \cdots + \mathscr{L}_{2n}(M),$$

so by Theorem 1.2,

$$\begin{aligned} \mathrm{sign}(M, \Gamma) &= ((\mathrm{sign}\, \Gamma) \cdot 1 + f^*(\alpha))\mathscr{L}(M)[M] \\ &= \mathrm{sign}\, \Gamma \cdot \mathscr{L}_{2n}(M)[M] + 2^n f^*(\alpha)[M] \\ &= \mathrm{sign}\, \Gamma \cdot \mathrm{sign}(M) + 2^n \cdot \alpha(f_*[M]) \\ &\neq \mathrm{sign}\, \Gamma \cdot \mathrm{sign}(M). \end{aligned}$$

Theorem 0.1 of the introduction is an immediate corollary of the above theorem. Another corollary is:

Corollary 2.5. *If $\Gamma \to M$ is a hermitian coefficient system with definite hermitian form on the fiber, then the multiplicativity statement of Theorem* 2.4 (iii) *holds.*

Proof. Assume Γ is positive definite. Then $\tilde{\Gamma}^- = 0$ and $\tilde{\Gamma}^+ = \Gamma$ is a flat bundle with flat hermitian metric, so by the Chern Weil description of rational chern classes (see for instance [18, Appendix C, Corollary 2], $\operatorname{ch}(\tilde{\Gamma}) \in H^0(M;\mathbf{Q})$, as was to be proved.

3. Proof of Theorem 0.2

We use the description of $\mathscr{C}$ given by Theorem 0.1. Part (i) of Theorem 0.1 is thus trivial.

To prove closure of $\mathscr{C}$ under free products consider the commutative diagram

$$\begin{array}{ccc} WU(G*H) & \xrightarrow{\psi} & H^*(B(G*H);\mathbf{Q}) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ WU(G)\oplus WU(H) & \xrightarrow{\psi\oplus\psi} & H^*(BG;\mathbf{Q})\oplus H^*(BH;\mathbf{Q}), \end{array}$$

where α and β are induced by the inclusions $G \to G*H$ and $H \to G*H$. The maps $BG \to B(G*H)$ and $BH \to B(G*H)$ induce a homotopy equivalence $BG \vee BH \to B(G*H)$, so β is an isomorphism in positive dimensions. Hence if $G, H \in \mathscr{C}$ then

$$\operatorname{Im}\psi_{G*H} \subset \beta^{-1}(\operatorname{Im}\psi_G \oplus \operatorname{Im}\psi_H) \subset H^0(B(G*H):\mathbf{Q}), \quad \text{so} \quad G*H \in \mathscr{C}.$$

To see closure under direct products we use the commutative diagram

$$\begin{array}{ccc} WU(G)\otimes WU(H) & \xrightarrow{\psi\otimes\psi} & H^*(BG;\mathbf{Q})\otimes H^*(BH;\mathbf{Q}) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle k} \\ WU(G\times H) & \xrightarrow{\psi} & H^*(B(G\times H);\mathbf{Q}) \end{array}$$

where t is the map induced by tensor product of representations and k is the Künneth map given by $BG \times BH = B(G \times H)$. Closure under direct products follows if t is surjective, but t is actually an isomorphism by [22, Theorem 4.4].

Finally for direct limits we use the diagram

$$\begin{array}{ccc} WU(\varinjlim G_i) & \xrightarrow{\psi} & H^*(B(\varinjlim G_i);\mathbf{Q}) \\ \downarrow & & \downarrow \\ \varprojlim WU(G_i) & \xrightarrow{\lim\psi} & \varprojlim H^*(BG_i;\mathbf{Q}) \end{array}$$

and observe that the right vertical arrow is an isomorphism since our coefficients are a field, so closure of $\mathscr{C}$ under $\varinjlim$ follows.

To prove (iii) note that in the diagram induced by $H \to G$

$$\begin{array}{ccc} WU(G) & \xrightarrow{\psi} & H^{\mathrm{ev}}(BG;\mathbf{Q}) \\ \downarrow & & \downarrow \\ WU(H) & \xrightarrow{\psi} & H^{\mathrm{ev}}(BH;\mathbf{Q}), \end{array}$$

if the right vertical arrow is injective then $G \in \mathscr{C}$ follows from $H \in \mathscr{C}$. In particular, if $H \to G$ has finite kernel and image of finite index this is so, since then $BH \to BG$ is rationally a finite covering (the fiber is $B(\ker) \times H/\operatorname{Im} G$ which has the rational homology type of $H/\operatorname{Im}(G)$). If $G \subset H$ has a normal complement, then there exists a homomorphism $H \to G$ such that the composition $G \subset M \to G$ is the identity, so the induced map in cohomology is injective.

Finally to prove (iv), observe that the condition in (iv) is equivalent to saying that $H \subset G$ is normal and the action of G on H by conjugation is by inner automorphisms of H.

If $H \subset G$ is any subgroup of finite index then we have an induction map $WU(H) \xrightarrow{\mathrm{ind}} WU(G)$, by mapping a hermitian representation $\rho : H \to \operatorname{Aut}(V, b)$ to its induced representation $\operatorname{ind}(\rho) : G \to \operatorname{Aut}((V, b)^{G/H})$. Precisely, $\operatorname{ind}(\rho)$ is defined by choosing a set $r_1, \ldots, r_n$ of left coset representatives of H in G and putting $W = r_1 V \oplus \cdots \oplus r_n V$ (orthogonal sum), where each $r_i V$ is just a copy of V, with suggestive notation. If we denote the elements of $r_i V$ by $r_i v$, $v \in V$, then G acts on W by $g(r_i v) = r_j(hv)$, where $gr_i = r_j h$ with $h \in H$. This action preserves the form on W and is independent (up to isomorphism) on the choice of coset representatives.

The automorphism group of H acts in the obvious fashion on $WU(H)$ and inner automorphisms act trivially. Hence if H is normal in G, then $K = G/H$ acts on $WU(H)$. By the above description of ind it is clear that the composition

$$f : WU(H) \xrightarrow{\mathrm{ind}} WU(G) \longrightarrow WU(H)$$

is the map

$$f(x) = \sum_{k \in K} k(x).$$

Thus under the condition of Theorem 0.2 (iv), K acts trivially on $WU(H)$, so f is just multiplication by n. Hence

$$WU(G) \otimes \mathbf{Q} \to WU(H) \otimes \mathbf{Q}$$

is surjective, so the conclusion of the theorem follows from the diagram

$$\begin{array}{ccc} WU(G) \otimes \mathbf{Q} & \xrightarrow{\tilde\psi_G} & \tilde H^*(G;\mathbf{Q}) \\ \downarrow & & \downarrow \\ WU(H) \otimes \mathbf{Q} & \xrightarrow{\tilde\psi_H} & \tilde H^*(H;\mathbf{Q}). \end{array}$$

4. Cutting and pasting

Before discussing some examples to Theorem 0.1, let us consider the relationship to the analogous result in [21]. In that paper and in [10] a graded group $\overline{SK}_*(X)$ is defined as the group of singular manifolds in X modulo bordism and "cutting and pasting". Equivalently, by [10], $\overline{SK}_*(X) = \Omega_n(X)/F_n(X)$, where $F_n(X)$ is the subgroup of elements in $\Omega_n(X)$ representable by an (M, f) for which M can be fibered over S^1. This group $SK_*(X)$ is a module over the ring $\overline{SK}_* = \overline{SK}_*(\text{point})$, which is a polynomial ring $\mathbf{Z}[P]$ in one 4-dimensional generator P, representable by any 4-manifold of signature 1.

Definition 4.1. Let $\mathscr{C}_0$ be the class of groups G for which the reduced SK-group $\widetilde{SK}_*(BG) = \operatorname{Ker}(\overline{SK}_*(BG) \to \overline{SK}_*)$ does not contain a free $\overline{SK}_*$-submodule. That is $\widetilde{SK}_*(BG)$ is a $\overline{SK}_*$-torsion module.

In [21] it was shown by a simple geometric argument that signature is multiplicative for structure groups in $\mathscr{C}_0$, so $\mathscr{C}_0 \subset \mathscr{C}$. In fact

Proposition 4.2. *$\mathscr{C}_0 \subseteq \mathscr{C}$ and $\mathscr{C}_0$ satisfies the closure properties* (i) *to* (iv) *of Theorem* 0.2.

Proof. Most of these properties were proved in [21]. The others follow easily by the same methods.

Problems. Are $\mathscr{C}_0$ and $\mathscr{C}$ equal?

5. Examples

Some groups not in $\mathscr{C}$:

(i) If F is an orientable surface of genus > 1, then $\pi_1(F) \not\in \mathscr{C}$.

(ii) If $\mathrm{SP}(2, \mathbf{Z}[1/2]) \subseteq G \subseteq U(1,1)$ or $\mathrm{Sp}(2n, \mathbf{Z}) \subseteq G \subseteq U(n, n)$, $n > 2$, then $G \not\in \mathscr{C}$.

These are by Meyer [16, 17] who constructed examples of nonmultiplicativity for such groups.

In contract, by Theorem 0.2 (i), if N is a nonorientable surface then $\pi_1(N) \in \mathscr{C}$ and also $\mathrm{Sp}(2, \mathbf{Z}) \in \mathscr{C}$. Observe that if F is an orientable surface then $\pi_1(F) \subset \pi_1(N)$ of index 2 for some nonorientable surface.

Corollary 5.1. *Being in $\mathscr{C}$ is not inherited by normal subgroups of finite index.*

Less obvious is the following example.

Proposition 5.2. *If G has a finitely generated free abelian normal subgroup with infinite cyclic quotient, then $\widetilde{SK}_*(BG) = 0$, so $G \in \mathscr{C}_0$, so $G \in \mathscr{C}$.*

The proof follows from the following facts: BG can be taken as a torus bundle over S^1; a set of singular manifolds in BG which represents a generating set of $H_*(BG; Q)$ represents a $(\overline{SK}_* \otimes Q)$-module generating set of $\widetilde{SK}_*(BG) \otimes \mathbf{Q}$ [21, Lemma 7]; such a generating set of singular manifolds can be chosen as a set of torus bundles over S^1, so they represent zero in $\widetilde{SK}_*(BG)$.

The G of this proposition is not in the smallest class of groups satisfying Theorem 0.1, so those closure properties do not characterize $\mathscr{C}$ or $\mathscr{C}_0$.

A direct algebraic proof that such G are in $\mathscr{C}$ would be of interest, since presumably this example can be greatly generalized.

Problem. Are polycyclic groups always in $\mathscr{C}$? Nilpotent groups*? Maybe even solvable groups?

It is not generally true that an extension of a $\mathscr{C}$-group by a $\mathscr{C}$-group is in $\mathscr{C}$, for instance surface groups are free extensions of free groups.

6. Real coefficient systems

One can define the classes $\mathscr{C}_-$ and $\mathscr{C}_+$ of all structure groups for which signature of symmetric respectively antisymmetric coefficient systems is always multiplicative. Let $W_+(G)$ and $W_-(G)$ denote respectively the Witt groups of real symmetric or antisymmetric bilinear representations of G. One proves exactly as for Theorem 0.1

Theorem 6.1. *There are natural maps $\tilde{\psi}_G^{\pm}: W_{\pm}(G) \to \tilde{H}(G; \mathbf{Q})$ such that $\mathscr{C}_{\pm} = \{G : \tilde{\psi}_G^{\pm} = 0\}$.*

In fact $\tilde{\psi}_G^{\pm}$ is the composition $W_{\pm}(G) \to WU(G) \xrightarrow{\tilde{\psi}_G} \tilde{H}*(G; \mathbf{Q})$, where the first map is hermitianization.

Theorem 6.2. *$\mathscr{C}_-$ and $\mathscr{C}_+$ both satisfy all the closure properties of Theorem 0.1 except for product closure of $\mathscr{C}_+$. We have*

$$G \in \mathscr{C}_- \quad \text{and} \quad H \in \mathscr{C}_- \Longleftrightarrow G \times H \in \mathscr{C}_-$$

$$G \in \mathscr{C}_+ \quad \text{and} \quad H \in \mathscr{C}_+ \quad \text{and} \quad (G \text{ or } H \in \mathscr{C}_-) \Longleftrightarrow G \times H \in \mathscr{C}_+.$$

Further $\mathscr{C} = \mathscr{C}_+ \cap \mathscr{C}_-$.

* J. Roitberg has shown that signature is multiplicative if G acts nilpotently on the fiber of Γ ("The signature of quasi-nilpotent fiber bundles", preprint).

Proof. That $\mathscr{C} = \mathscr{C}_+ \cap \mathscr{C}_-$ follows from 6.1 and the fact [22] that the ring homomorphism

$$W_+(G) \oplus W_-(G) \to WU(G)$$

given by hermitianization is a modulo torsion isomorphism. The closure properties are proved exactly as for $\mathscr{C}$. For the statements about products we use the diagrams

$$\begin{array}{ccc} (W_+(G) \otimes W_-(H)) \oplus (W_-(G) \otimes W_+(H)) & \to & H^*(G;\mathbf{Q}) \otimes H^*(H;\mathbf{Q}) \\ \downarrow & & \downarrow \\ W_-(G \times H) & \longrightarrow & H^*(G \times H;\mathbf{Q}) \end{array}$$

and

$$\begin{array}{ccc} (W_+(G) \otimes W_+(H)) \oplus (W_-(G) \otimes W_-(H)) & \longrightarrow & H^*(G;\mathbf{Q}) \otimes H^*(H;\mathbf{Q}) \\ \downarrow & & \downarrow \\ W_+(G \times H) & \longrightarrow & H^*(G \times H;\mathbf{Q}) \end{array}$$

where the left vertical arrows are isomorphisms modulo torsion by the modulo torsion isomorphism above and the fact that $WU(G \times H) \cong WU(G) \otimes WU(H)$. If one observes also that ψ_G^- is zero in dimension zero while ψ_G^+ is always non-zero in dimension zero, the statements on products follow easily.

7. Manifolds with boundary; γ-invariants

The multiplicativity results for signature fail for coefficient systems over compact manifolds with boundary, but this failure leads in a standard way to interesting invariants of the boundary.

Data. Let (G, ρ) be a structure pair consisting of a group G and a hermitian representation $\rho : G \to \mathrm{Aut}(V, b)$ which is "good" for multiplicativity of signature, that is $\tilde{\psi}_G([\rho]) = 0$ in $\tilde{H}^*(G;\mathbf{Q})$ (Theorem 2.3).

If M^{2n} is a compact oriented manifold with boundary X^{2n-1} and $f : \pi_1(M) \to G$ a homomorphism, then f classifies a hermitian coefficient system $\Gamma \to M$.

Theorem 7.1. $\mathrm{sign}(M, \Gamma) - \mathrm{sign}(M)\mathrm{sign}(b)$ *is an invariant of* $g = f \mid X$ *which we denote by* $\gamma_{(G,\rho)}(X, g)$. *By* $f \mid X$ *we mean the composition of* f *with the map* $\pi_1(X) \to \pi_1(M)$ *induced by the inclusion. Equivalently interpret* f *as a map* $M \to BG$ *and then* $g = f \mid X$ *in the usual sense.*

Proof. If (M', f') is another pair with the same boundary and $\Gamma' \to M'$ the corresponding coefficient system, then we can paste (M, Γ) to $(-M', \Gamma')$ along boundaries to get a closed manifold with coefficient system (Y, Δ). Novikov additivity gives

$$\mathrm{sign}(Y, \Delta) = \mathrm{sign}(M, \Gamma) - \mathrm{sign}(M', \Gamma').$$

But on the other hand multiplicativity of signature plus Novikov additivity gives

$$\begin{aligned}\operatorname{sign}(Y,\Delta) &= \operatorname{sign}(Y)\cdot\operatorname{sign}(b)\\ &= \operatorname{sign}(M)\cdot\operatorname{sign}(b)-\operatorname{sign}(M')\cdot\operatorname{sign}(b).\end{aligned}$$

Subtracting these equations gives the desired result.

More generally if (X,g) itself does not bound but some nonzero multiple $q(X,g)$ does, then we can define

$$\gamma_{(G,\rho)}(X,g)=\tfrac{1}{q}\gamma_{(G,\rho)}(q(X,g)).$$

One might expect that $\gamma_{(G,\rho)}(X,g)$ only depends on the composed representation $\rho g:\pi_1(X)\to \operatorname{Aut}(V,b)$, but this is false in general, as is shown by the following example. Let $\Gamma\to F$ be a coefficient system over an orientable surface such that $\operatorname{sign}(F,\Gamma)\neq 0$. Let $D^2\subset F$ be an embedded disc, $M=\operatorname{cl}(F-D^2)$. Let $\rho:\pi_1(M)\to\operatorname{Aut}(V,b)$ be the classifying map for $\Gamma\mid M$. Since $G=\pi_1(M)$ is free, $G\in\mathscr{C}$, so (G,ρ) is a good structure pair. If $g:\pi_1(S^1)\to G=\pi_1(M)$ is induced by the inclusion $S^1=\partial M\subset M$, then by definition:

$$\gamma_{(G,\rho)}(S^1,g)=\operatorname{sign}(M,\Gamma\mid M)=\operatorname{sign}(F,\Gamma)\neq 0,$$

the second equality being by Novikov additivity. But ρg is the trivial representation, since it extends over D^2, so $\gamma_{(G,\rho)}(S^1,g)$ does certainly not only depend on ρg.

On the other hand the following result was proved in [20].

Theorem 7.2. *In each of the following cases $\gamma_{(G,\rho)}(X,g)$ only depends on the representation ρg of $\pi_1(X)$.*

(a) *ρ is a definite representation.*

(b) *The center of G has finite index in G (e.g. G finite or abelian).*

We repeat the proof for completeness. We shall simply write $\gamma_\rho(X,g)$ for $\gamma_{(G,\rho)}(X,g)$ if G is understood.

Proof. (a) is clear, since in this case in the proof of 7.1 we use the multiplicativity result 2.4, which does not depend on G. For (b) we shall use the following lemma.

Lemma 7.3. *Given a closed oriented manifold X^{2n-1} and homomorphisms $\pi_1(X)\xrightarrow{f}G\xrightarrow{\varphi}H\xrightarrow{\tau}\operatorname{Aut}(V)$, with $G,H\in\mathscr{C}$, then:*

(i) *If $\gamma_{\tau\varphi}(X,f)$ is defined, then so is $\gamma_\tau(X,\varphi f)$ and they are equal.*

(ii) *If $\varphi:G\to H$ is injective in rational homology and $\gamma_\tau(X,\varphi f)$ is defined, then so is $\gamma_{\tau\varphi}(X,f)$, and they are equal.*

Proof. (i) If $\gamma_{\tau\varphi}(X,f)$ is defined, that is $q(X,f)=\partial(M,g)$ for some $q>0$ and $g:M\to BG$, then $q(X,\varphi f)=\partial(M,\varphi g)$, so $\gamma_\tau(X,\varphi f)$ is defined and the equality $\gamma_{\tau\varphi}(X,f)=\gamma_\tau(X,\varphi f)$ is clear from the definition.

(ii) The modulo torsion triviality of the bordism spectral sequence implies that the condition $\varphi_*: H_*(G;\mathbf{Q}) = H_*(BG;\mathbf{Q}) \to H_*(BG;\mathbf{Q}) = H_*(H;\mathbf{Q})$ injective is equivalent to $\Omega_*(BG)\otimes Q \to \Omega_*(BH)\otimes Q$ injective. Thus if $\gamma_\tau(X,\varphi f)$ is defined, that is $[X,\varphi f]=0$ in $\Omega_*(BH)\otimes\mathbf{Q}$, then $[X,f]=0$ in $\Omega_*(BG)\otimes Q$, so $\gamma_{\tau\varphi}(X,f)$ is defined. It is equal to $\gamma_\tau(X,\varphi f)$ by (i).

To return to the proof of Theorem 7.2, suppose we have $\pi_1(X)\xrightarrow{f} G \xrightarrow{\rho} \mathrm{Aut}(V)$ such that the center of G has finite index in G (whence in particular $G \in \mathscr{C}$), and suppose some multiple of (X,f) bounds, so $\gamma_\rho(X,f)$ is defined. Put $H = \mathrm{Im}(\rho)$ and $K = \mathrm{Im}(\varphi f) \subset H \subset \mathrm{Aut}(V)$, so we have a diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "\psi"] \arrow[d, "f"] & K \arrow[d, "i"] \\
G \arrow[r, "\varphi"] \arrow[dr, "\rho"'] & H \arrow[d, "j"] \\
& \mathrm{Aut}(V)
\end{tikzcd}$$

where i and j are the inclusions and ψ and φ are just ρf and ρ with their ranges restricted.

Suppose we know that $i_*: H_*(K;\mathbf{Q}) \to H_*(H;\mathbf{Q})$ is injective. Then we can apply Lemma 7.3 parts (i) and (ii) successively to show that $\gamma_j(X,\varphi f)$ is defined and equals $\gamma_\rho(X,f)$ and that $\gamma_{ji}(X,\psi)$ is defined and equals $\gamma_j(X,\varphi f)$. Thus $\gamma_\rho(X,f) = \gamma_{ji}(X,\psi)$ and since ji and ψ only depend on ρf and not on G, the theorem is proved. Thus we must just show the injectivity of i_*.

Now both K and H have centers of finite index, since this property is inherited by subgroups and quotient groups. We first show that in the commutative diagram

$$\begin{array}{ccc}
H_*(Z(K)\cap Z(H);\mathbf{Q}) & \to & H_*(Z(H);\mathbf{Q}) \\
\downarrow & & \downarrow \\
H_*(K,\mathbf{Q}) & \longrightarrow & H_*(H;\mathbf{Q})
\end{array}$$

the vertical arrows are isomorphisms. Indeed, in the Lyndon spectral sequence $E_2^{pq} = H^p(H/Z(H);\ H^q(Z(H);\mathbf{Q})) \Longrightarrow H^{p+q}(H;\mathbf{Q})$ (see e.g. MacLane [13, p. 351]) we have $E_2^{p,q}=0$ for $p>0$, since $H/Z(H)$ is finite and acts trivially on $H^q(Z(H);\mathbf{Q})$. Hence $H^*(H;\mathbf{Q}) \to H^*(Z(H);\mathbf{Q})$ is an isomorphism, so the homology map also is. The same argument holds for $Z(K)\cap Z(H) \subset K$.

Now an injection of abelian groups induces an injection in $\mathbf{Q}$-homology (this is clear for finitely generated abelian groups and homology commutes with direct limits), so the top map of the square is injective, so the bottom map is too, as was to be shown.

The γ-invariants have many interesting applications. For instance the calculations of Meyer [16, 17] of signature of coefficient systems over surfaces can be interpreted as giving connections between γ-invariants and classical dedekind sums. γ-invariants of lens spaces give generalized dedekind sums and enable one to prove topologically, number theoretic results about such sums [20].

For ρ definite, these invariants have come up analytically in recent work of

Atiyah, Patodi, and Singer. In particular $\gamma_\rho(X, g)$ can be defined even if the pair (X, g) does not bound. For G abelian we have been able to find a purely topological description of this.

In the forthcoming paper [20] we show that $\gamma_{(G,\rho)}(X, g)$ is a homotopy invariant of (X, g) if G is free abelian (this is definitely false if G is not free abelian, for instance lens spaces are classified by their γ-invariants). This result is reminiscent of the homotopy invariance of the higher Novikov signatures and our proof in fact has certain (seemingly superficial) similarities with Lusztig's proof of the latter fact [12].

In view of the intimate connection between γ-invariants and α-invariants of free finite group actions [20, 3], we also obtain homotopy invariant calculations of α-invariants in many interesting cases, namely for a class of (not necessarily free) actions which includes all homologically injective actions in the sense of Conner and Raymond [7]. The periodicity results for signature of coverings of manifolds with boundary announced in [19] are also a corollary of these calculations.

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