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HOMOTOPY INVARIANCE OF ATIYAH INVARIANTS

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In the following, all manifolds are understood to be smooth, compact, and oriented. The invariants to be discussed are

(I). The α-invariants $\alpha(M^{2n-1}, g)$ of a smooth group action on a closed odd dimensional manifold M^{2n-1}. These were introduced by Atiyah and Singer [2] and have been extensively studied since then; see for instance Hirzebruch and Zagier [6] and the literature quoted there.

(II). Certain γ-invariants $\gamma(X^{2n-1}, \rho)$ associated to a representation $\rho\colon \pi_1(X) \to U(m)$ of the fundamental group of a closed odd dimensional manifold X^{2n-1}. These invariants arise for instance in Atiyah, Singer and Patodi [1] via the theory of spectral asymmetry. They came up in a different context in [8] and [11].

(III). A "monodromy" $\mathscr{H}$ associated to a manifold X^{2n-1} and a map $X \to S^1$; $\mathscr{H} = (H, S, t)$ is an isometric structure, that is it consists of a vector space H plus a symmetric or skew-symmetric bilinear form S on H and an isometry $t\colon (H, S) \to (H, S)$. If $X \to S^1$ is a fibration with fiber F^{2n-2}, then $(H, S) = H^{n-1}(F; Q)$ with the cup product form $S(x, y) = \langle x \cup y, [F]\rangle$ and t is the monodromy of the fibration.

The latter invariant will be, by its very definition, a homotopy invariant. On the other hand, if the representation $\rho\colon \pi_1(X) \to U(m)$ in (II) factors over a free abelian group ($\rho = \pi_1(X) \to Z^s \to U(m)$), then $\gamma(X, \rho)$ is calculable in terms of the monodromy invariants, and is hence also a homotopy invariant. Finally, in view of the intimate relationship between α- and γ-invariants, we obtain also homotopy invariant calculations of α-invariants in certain situations.

There seems to be a certain analogy of the homotopy invariance proved here with the homotopy invariance of higher Novikov signatures proved by Farrel and Hsiang [7] and Lusztig [9]. This analogy deepens a feeling the author has often had,

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that so-called "peripheral invariants", such as the α- and γ-invariants of odd dimensional manifolds, are connected in some deeper way than has yet been discovered to characteristic class and surgery type invariants.

The present paper is a fairly exact version of the talk given at Stanford; in particular no proofs are included. Some proofs appeared in the preliminary manuscript [11] (see also [13]). Complete proofs and further applications and examples will appear in the final version of [11].

1. α-invariants of group actions. Let N^{2n} be a closed manifold with a smooth G-action and $g \in G$. Then the Atiyah-Singer fixed point theorem [2] calculates the equivariant signature $\operatorname{sign}(N, g)$ as a polynomial in the characteristic classes of the fixed point set N^g and its normal bundle $\nu(N^g)$:

$$\operatorname{sign}(N, g) = f(N^g, \nu(N^g)).$$

If N has boundary M^{2n-1}, this equation is no longer valid, but a standard argument shows that if g has no fixed points in M, then the "error"

$$\alpha(M, g) = \operatorname{sign}(N, g) - f(N^g, \nu(N^g))$$

is an invariant of (M, g). More generally, given (M, G) and $g \in G$ acting without fixed points on M, some disjoint multiple $q(M, g)$ bounds, and one may define $\alpha(M, g) = \alpha(q(M, g))/q$. Denote by $\alpha(M, G)$ the (partially defined) map $G \to C$ whose value for $g \in G$ is $\alpha(M, g)$.

DEFINITION. Let G act effectively on a manifold M. We say an element $g \in G$ of finite order k is *S^1-induced* if there exists an equivariant map $(M, g) \to (S^1, e^{2\pi i q/k})$ for some q prime to k. In particular g acts freely.

We say G acts *h-injectively* on M if a dense set of elements of G are S^1-induced.

EXAMPLES. (i) One can show that a homologically injective action in the sense of Conner and Raymond [3] is h-injective.

(ii) If G is connected and acts on M then the following conditions are equivalent:

(a) (M, G) is h-injective;

(b) some nontrivial $g \in G$ is S^1-induced;

(c) some finite covering of G has the form $H \times S^1$ and the induced S^1-action on M is homologically injective.

(iii) If G is finite, any free orientation preserving action on a surface is h-injective. Very many free actions on Seifert spaces are (in a sense that can be made precise).

THEOREM 1. *If G acts h-injectively on M^{2n-1} then $\alpha(M, G)$ is a homotopy invariant of (M, G).*

Our results in fact give a reasonably calculable intrinsic description of $\alpha(M, G)$ for h-injective actions.

EXAMPLE. Given a cyclic action $(N^{2n-2}, Z/k)$ we can form the S^1-manifold $M = N \times_{Z/k} S^1$. The S^1-action on M is h-injective; in fact by Conner and Raymond [3], this construction gives all homologically injective S^1-actions. For $t \in S^1 = \{t \in C \mid \|t\| = 1\}$ our calculation gives the formula (n even)

$$\alpha(M, t) = -2 \sum_{q=1}^{[(k-1)/2]} a_q \frac{t^{q'} + t^{q'+1} + \cdots + t^{k'-q'-1}}{1 + t + \cdots + t^{k'-1}},$$

where q'/k' is q/k in lowest terms and a_q is the integer (!)

$$a_q = \frac{1}{k} \sum_{r=0}^{k-1} e^{-2\pi i q r/k} \operatorname{sign}(N, g^r),$$

where g is a generator of the Z/k-action on N. If one compares this with Ossa's calculation [14] of the α-invariant of S^1-actions on 3-manifolds, one obtains some interesting identities between rational functions.

A similar result holds if n is odd.

REMARK. For non-h-injective actions the α-invariant is in general not homotopy invariant, as can be seen for instance by the standard free cyclic actions on the 3-sphere; see the remark at the end of §2.

2. γ-invariants. Given a compact connected manifold Y^{2n} and a unitary representation $\rho\colon \pi_1(Y) \to U(m)$ of its fundamental group, there is an induced local coefficient system (= locally trivial sheaf) $\Gamma \to Y$ with fiber (C^m, h), where h is the standard hermitian metric on C^m. These metrics on the fibers of Γ fit together to give a bilinear map of sheaves $b\colon \Gamma \times \Gamma \to C$, where C also denotes the trivial sheaf over Y with fiber C. Define a cup product form on $H^n(Y, \partial Y; \Gamma)$ by

$$S_{Y,\Gamma}\colon H^n(Y, \partial Y; \Gamma) \otimes H^n(Y, \partial Y; \Gamma) \to H^{2n}(Y, \partial Y; \Gamma \otimes \Gamma) \to H^{2n}(Y, \partial Y; C) = C,$$

where the first map is cup product and the second is the coefficient map induced by b. This $S_{Y,\Gamma}$ is a (in general not nondegenerate) hermitian or skew-hermitian form, according as n is even or odd. Define $\operatorname{sign}(Y, \rho) = \operatorname{sign}(S_{Y,\Gamma})$, where, if $S_{Y,\Gamma}$ is skew-hermitian we mean signature of the hermitian form $+iS_{Y,\Gamma}$.

Let $X^{2n-1} = \partial Y$ and denote the composed representation $\pi_1(X) \to \pi_1(Y) \to U(m)$ also by ρ. Then

$$\gamma(X, \rho) = \operatorname{sign}(Y, \rho) - n \cdot \operatorname{sign}(Y)$$

is an invariant of (X, ρ). If (X, ρ) does not bound, but some disjoint multiple $q(X, \rho)$ does, we can define $\gamma(X, \rho) = \gamma(q(X, \rho))/q$.

This invariant also arises analytically via Atiyah, Patodi and Singer theory in [1]. In particular it can be defined even if no multiple of (X, ρ) bounds. If $\operatorname{Im}(\rho\colon \pi_1(X) \to U(m))$ is abelian I can give a purely topological description of the invariant also in this case, but in general this appears to be still an open problem.

One can also carry through the above definition permitting indefinite unitary representations $\rho\colon \pi_1(X) \to U(p, q)$, where $U(p, q)$ is the group of isometries of C^{p+q} with the indefinite hermitian form of type (p, q). $\gamma(X, \rho)$ is then only well defined under suitable additional assumptions, for instance if one restricts $\operatorname{Im}(\rho)$ to be abelian (or more generally to be a central extension of a finite group), see [13]. The results to be described hold also for this more general definition.

THEOREM 2. *If the representation $\rho\colon \pi_1(X) \to U(m)$ factors over a free abelian group, $\pi_1(X) \to Z^s \to U(m)$, then $\gamma(X, \rho)$ is a homotopy invariant.*

In view of the following relationship between α- and γ-invariants, Theorem 1 is in fact an easy consequence of this Theorem 2.

THEOREM 3. *Let G be a finite group acting freely on M^{2n-1}. Then the covering*

$M^{2n-1} \to M/G = X^{2n-1}$ is classified by a homomorphism $f: \pi_1(X) \to G$. Let ρ_i: $G \to U(n_i)$, $i = 1, \cdots, r$, be all irreducible representations of G. Then

$$\alpha(M, g) = \sum_{j=1}^{r} \mathit{trace}(\rho_j(g))\gamma(X, \rho_j f),$$

$$\gamma(X, \rho_i f) = \frac{1}{|G|} \sum_{g \neq 1} (\mathit{trace}(\rho_i(g^{-1})) - n_i)\alpha(M, g).$$

Thus to calculate $\alpha(M^{2n-1}, G)$ for general G one can look at the $g \in G$ of finite order which act freely and compute $\alpha(M, g)$ for these via the γ-invariant. Then try to extend to arbitrary $g \in G$ using the continuity properties of $\alpha(M, G)$. If G acts freely or if G is connected this works and gives a complete calculation of $\alpha(M, G)$ in terms of γ-invariants.

Theorem 3 is a quite easy character computation and was proved in **[1]** and **[11]**.

REMARK. Some condition is necessary in Theorems 1 and 2 to conclude homotopy invariance. For example the lens spaces $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent. The γ-invariants of $L(7, 1)$ with respect to the six nontrivial irreducible representations $Z/7 \to U(1)$ are respectively: $-3/7$, $-13/7$, $-17/7$, $-17/7$, $-13/7$, $-3/7$, while for $L(7, 2)$ they are: $1/7$, $-3/7$, $-5/7$, $-5/7$, $-3/7$, $1/7$. In fact, 3-dimensional lens spaces are classified up to diffeomorphism by their γ-invariants; equivalently free linear cyclic actions on S^3 are classified up to equivariant diffeomorphism by their α-invariants.

γ-invariants of lens spaces in any dimension were completely calculated in **[11]**. They are generalized Dedekind sums (see also **[15]**).

3. Monodromy. Suppose we are given a closed manifold X^{2n-1} and a homomorphism $f_\#: \pi_1(X) \to Z$. Since S^1 is a $K(Z, 1)$, we can represent $f_\#$ by a unique map $f: X \to S^1$ up to homotopy. If this map f can be chosen as a fibration with fiber F^{2n-2} say, then one has the monodromy transformation $H^{n-1}(F) \to H^{n-1}(F)$ which preserves the cup product form. It is this monodromy that we wish to generalize to the case that $f: X \to S^1$ is not a fibration.

Let $\bar{X} \to X$ be the infinite cyclic covering classified by the homomorphism $f_\#$: $\pi_1(X) \to Z$. Equivalently $\bar{X}$ is the pullback

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & R \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^1 \end{array}$$

If $t \in S^1$ is a regular value of f and $N = f^{-1}(t)$, then $\bar{X}$ can be constructed by cutting X open along N and pasting infinitely many copies of the resulting manifold with boundary together end to end.

Let $\hat{f} \in H_{2n-2}(\bar{X})$ be the homology class represented by one copy of N in $\bar{X}$. Equivalently, $\hat{f}$ is the image of $1 \in Z$ in the composition $Z = H_c^1(R) \to H_c^1(\bar{X}) \cong H_{2n-2}(\bar{X})$ induced by the proper map $\bar{f}: \bar{X} \to R$ and Poincaré duality, so $\hat{f}$ only depends on the homotopy class of f.

Define a bilinear form

$$S_0: H^{n-1}(\bar{X}; Q) \otimes H^{n-1}(\bar{X}; Q) \to Q, \qquad S_0(x, y) = \langle x \cup y, \hat{f} \rangle.$$

This form is degenerate in general, but it induces a nondegenerate form S on

$H = H^{n-1}(X; Q)/\mathrm{Rad}\, S_0$, where $\mathrm{Rad}\, S_0 = \{x \in H^{n-1}(\bar{X}; Q) \mid S_0(x, y) = 0$ for all $y\}$.

LEMMA 4. *(H, S) is a finite dimensional vector space with nondegenerate $(-1)^{n-1}$-symmetric form. The covering transformation $\bar{X} \to \bar{X}$ induces an isometry $t: H \to H$.*

DEFINITION. The $(-1)^{n-1}$-symmetric isometric structure $\mathscr{H}(X, f) = (H, S, t)$ will be called the (middle-dimensional) *monodromy* of (X, f).

We have defined the monodromy over Q. We could equally well have used other coefficients. If K is a field of characteristic 0 then using universal coefficient theorems it is easy to see $\mathscr{H}^K(X, f) \cong \mathscr{H}^Q(X, f) \otimes K$, where the superscript indicates coefficients, so $\mathscr{H}^Q(X, f)$ contains the most information (it is however false that $\mathscr{H}^Q(X, f)$ equals $\mathscr{H}^Z(X, f) \otimes Q$; in fact the precise relation between the $\mathscr{H}^R(X, f)$ for different coefficient rings R remains unclear in general).

$\mathscr{H}(X, f) = \mathscr{H}^Q(X, f)$ is a very rich invariant. Not only is the set of isometric structures over Q extremely abundant, but every isometric structure occurs as monodromy, at least in the skew-symmetric case. In fact:

THEOREM 5. *For any skew-symmetric isometric structure $\mathscr{H} = (H, S, t)$ over Q there exists a 3-manifold M^3 and a map $f: M^3 \to S^1$ such that $\mathscr{H}(M^3, f) \cong \mathscr{H}$.*

The relation between the monodromy and our previous invariants is given in the simplest case by the following theorem.

THEOREM 6. *Given $\rho: \pi_1(X) \to U(m)$ such that $\rho = \tau f_\#$ for some $f_\#: \pi_1(X) \to Z$ and $\tau: Z \to U(m)$, then $\gamma(X, \rho)$ only depends on $\mathscr{H}^R(X, f)$.*

Here is a precise description of the dependence in the antisymmetric case (that is n even); the result in the symmetric case is similar. Let S_q be the $(-1)^{q-1}$-symmetric bilinear form given by the $q \times q$ matrix

$$\begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ 0 & & & 1 & & \\ & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & 0 & & \\ (-1)^q & & & & & \end{pmatrix}$$

and let t_q be the isometry of S_q having matrix of the form

$$\begin{pmatrix} 1 & \frac{1}{2} & * & \cdots & & * \\ & 1 & \frac{1}{2} & & & \vdots \\ & & \ddots & \ddots & & \\ & 0 & & & 1 & \frac{1}{2} \\ & & & & & 1 \end{pmatrix},$$

(t_q is uniquely determined by this). Define

$$\begin{aligned} \mathscr{H}^{(q)}_{\pm 1} &= (R^q, S_q, \pm t_q), && q \text{ even}, \\ &= \left(R^{2q}, \begin{pmatrix} 0 & S_q \\ -S_q & 0 \end{pmatrix}, \pm(t_q \oplus t_q)\right), && q \text{ odd}, \\ \mathscr{H}^{(q)}_{\lambda} &= \left(R^{2q}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^q \otimes S_q, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \otimes t_q\right), && \lambda = e^{i\theta},\ 0 < \theta < \pi. \end{aligned}$$

For any isometric structure $\mathscr{H} = (H, S, t)$ define $-\mathscr{H}$ to mean $(H, -S, t)$.

By Milnor [10], any skew-symmetric isometric structure $\mathscr{H}$ over $\boldsymbol{R}$ is an orthogonal sum of an $\mathscr{H}_0 = (H_0, S_0, t_0)$ such that t_0 has no eigenvalue of unit length and a sum of isometric structures of the form $\pm \mathscr{H}_\lambda^{(q)}$, $\lambda = e^{i\theta}$, $0 \leqq \theta \leqq \pi$, defined as above.

For $A \in U(m)$ define an invariant $\gamma(\mathscr{H}, A)$ by requiring $\gamma(-\mathscr{H}, A) = -\gamma(\mathscr{H}, A)$ and $\gamma(\mathscr{H} \oplus \mathscr{H}', A) = \gamma(\mathscr{H}, A) + \gamma(\mathscr{H}' A)$, and putting

$$\begin{aligned}
\gamma(\mathscr{H}_0, A) &= 0, && \mathscr{H}_0 \text{ as above;}\\
\gamma(\mathscr{H}_1^{(q)}, A) &= -(-1)^{q/2}\cdot \operatorname{rank}(A - I), && q \text{ even,}\\
&= 0, && q \text{ odd;}\\
\gamma(\mathscr{H}_{-1}^{(q)}, A) &= (-1)^{q/2}\cdot \operatorname{corank}(A + I), && q \text{ even,}\\
&= 0, && q \text{ odd;}\\
\gamma(\mathscr{H}_\lambda^{(q)}, A) &= \operatorname{corank}(A - \lambda I) + \operatorname{corank}(\lambda A - I), && q \text{ even,}\\
&= 2 \sum_{-\theta \leqq \phi < 2\pi - \theta} d(\phi, \theta)\cdot \operatorname{corank}(A - e^{i\phi} I), && q \text{ odd;}
\end{aligned}$$

where

$$\begin{aligned}
d(\phi, \theta) &= 0 \quad \text{if } -\theta < \phi < \theta,\\
&= 1 \quad \text{if } \phi = \pm\theta,\\
&= 2 \quad \text{if } \theta < \phi < 2\pi - \theta.
\end{aligned}$$

THEOREM $6\frac{1}{2}$. *In Theorem* 6 $\gamma(X, \rho) = \gamma(\mathscr{H}^{\boldsymbol{R}}(X, f), \rho(1))$.

THEOREM 7. *If* $\rho: \pi_1(X) \to U(m)$ *factors as* $\rho = \tau g_\#$, *where* $g_\#: \pi_1(X) \to \boldsymbol{Z}^s$ *and* $\tau: \boldsymbol{Z}^s \to U(m)$, *then* $\gamma(X, \rho)$ *can be calculated via a limiting process from the monodromies* $\mathscr{H}(X, hg)$ *where* h *runs through all maps* $h: \boldsymbol{Z}^s \to \boldsymbol{Z}$. *Alternatively one can give an explicit calculation in terms of finitely many monodromies of* X *calculated with suitable local coefficients on* X.

4. Application to signature defect. The α- and γ-invariants arise naturally as correction terms to multiplicativity of signature of branched coverings and coverings of bounded manifolds; see for instance Hirzebruch [5].

EXAMPLE. Let $N^{4k} \to Y^{4k}$ be a d-fold covering of oriented manifolds with boundaries $M^{4k-1} = \partial N \to X^{4k-1} = \partial Y$. Then the error to multiplicativity of signature, namely $\operatorname{sign}(N) - d\cdot\operatorname{sign}(Y)$, is an invariant of $M \to X$ which is denoted "signature defect":

$$\operatorname{def}(M \to X) = \operatorname{sign}(N) - d\cdot \operatorname{sign}(Y).$$

In fact if $\pi_1(M) \subset \pi_1(X)$ is the induced inclusion of fundamental groups and $H \subset \pi_1(M)$ is a normal subgroup of $\pi_1(X)$ of finite index (H exists), and if $\rho_i: \pi_1(X)/H \to U(n_i)$, $i = 1, \cdots, r$, are all the irreducible representations of $\pi_1(X)/H$ and m_i is the dimension of the trivial component of $\rho_i|(\pi_1(M)/H)$ for each i, then

$$\operatorname{def}(M \to X) = \sum_{i=1}^{r} m_i \gamma(X, \rho_i).$$

For a proof see [11, Chapter III].

One obtains similar results for branched coverings by cutting out the branch locus and considering the resulting unbranched covering of manifolds with boundary.

Our calculations lead to the following periodicity result for the "signature defect" $\operatorname{def}(M \to X)$.

THEOREM 8. *Let M_r^{4k-1}, $r = 1, 2, \cdots$, be a family of closed manifolds, and whenever r divides s let an (s/r)-fold cyclic covering $M_s \to M_r$ be given such that all the obvious diagrams commute. Then* $\operatorname{def}(M_s \to M_1)$ *is a linear plus an almost periodic function of s.*

By an *almost periodic function* is meant the restriction of a linear combination of periodic functions $\boldsymbol{R} \to \boldsymbol{R}$. If the periods in question are rationally dependent then this linear combination is itself periodic. In fact in the above theorem the coverings $M_s \to M_1$ can all be pulled back from the standard coverings of S^1 via a suitable map $M_1 \to S^1$ and the periods in question are the numbers $1/q$, where $e^{2\pi i q}$, $0 < q \leqq 1$, runs through the eigenvalues of unit length of the monodromy of this map.

As a corollary one obtains the periodicity statement for signature of cyclic suspension of knots announced in [12]. This result has also been shown subsequently (for fibered knots, but the proof works for arbitrary knots) by Durfee and Kauffman [4] and by Cappell and Shaneson (unpublished) using an alternative method.

REMARK. $\operatorname{def}(M \to X)$ also arises as the error to multiplicativity for coverings of the Atiyah-Patodi ν-invariant of a riemmanian manifold and of the Atiyah-Kreck δ-invariant of a framed manifold, so in particular we get similar "linear plus almost periodic" statements for these invariants, see [11].

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