AN INVARIANT OF PLUMBED HOMOLOGY SPHERES

Walter D. Neumann

If A is a ring, a 3-dimensional A-<u>sphere</u> will mean a closed oriented 3-manifold M^3 with the same A-homology as S^3. An A-<u>homology bordism</u> between two A-spheres M_1^3 and M_2^3 is a compact oriented 4-manifold W^4 with $\partial W = M_1 + (-M_2)$, such that the inclusions $M_i \hookrightarrow W$ induce isomorphisms in A-homology. Here, and in the following, $+$ means disjoint union, $-$ means reversed orientation, and all manifolds will be assumed smooth, compact, and oriented. Diffeomorphisms should preserve orientation.

The set of A-homology bordism classes of 3-dimensional A-spheres forms a group under connected sum, which we denote Θ_3^A. We are most interested in $A = \mathbb{Z}/2$ or $\mathbb{Z}$. Every $\mathbb{Z}$-sphere is a $\mathbb{Z}/2$-sphere and every $\mathbb{Z}/2$-sphere is a rational sphere.

If M^3 is a $\mathbb{Z}/2$-sphere, then its μ-invariant (see for instance [3], where the opposite sign convention is used) can be defined as

$$\mu(M^3) = \operatorname{sign}(W^4) \pmod{16},$$

where W^4 is any simply connected parallelizable compact manifold with $\partial W^4 = M^3$. This invariant is a $\mathbb{Z}/2$-homology bordism invariant and its range of values is summed up in the commutative diagram

$$\begin{array}{ccc} \Theta_3^{\mathbb{Z}} & \xrightarrow{\ \mu\ } & 8\mathbb{Z}/16\mathbb{Z} \\ \downarrow & & \downarrow \\ \Theta_3^{\mathbb{Z}/2} & \xrightarrow{\ \mu\ } & 2\mathbb{Z}/16\mathbb{Z} \end{array}.$$

The groups Θ_3^A are of interest in their own right, but also because of applications. The most important application is the result of Galewski and Stern [2] and Matumoto [5] that for any

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$n \geq 5$, all closed TOP n-manifolds are simplicially triangulable if and only if $\Theta_3^{\mathbb{Z}}$ contains a subgroup $\mathbb{Z}/2$ on which μ is an isomorphism.

Another application is to the classical knot concordance group C_3 . Given a knot (S^3,K) one can do a p/q-Dehn surgery on it to get a homology lens space $M^3_{p/q}(S^3,K) = M^3_{p/q}$ with $H_1(M^3_{p/q};\mathbb{Z}) = \mathbb{Z}/p$. Thus if p is invertible in A then $M^3_{p/q}$ is an A-sphere. The map $C_3 \longrightarrow \Theta_3^A$ given by $(S^3,K) \longmapsto [M^3_{p/q}(S^3,K)] - [L(p,p-q)]$ is well defined, though possibly not a homomorphism. Thus information about Θ_3^A could yield new knowledge of C_3 . Other similar constructions can be used to the same purpose.

Our main results are the following time-dependent theorem and the non time-dependent consequences, theorems 6.1 to 6.3, of section 6 below.

THEOREM. There exists an invariant $\bar{\mu}(M^3) \in \mathbb{Z}$ with the following properties:

(i). $\bar{\mu}(M^3)$ is defined for $\mathbb{Z}/2$-spheres M^3 which are plumbed manifolds (= graph manifolds in Waldhausen's sense [10]) and is an oriented diffeomorphism type invariant.

(ii). $\bar{\mu}(M^3)$ (mod 16) $= \mu(M^3)$.

(iii). For all known (May 1979) $\mathbb{Z}/2$-homology bordisms of such $\mathbb{Z}/2$-spheres, $\bar{\mu}$ is invariant.

If $\bar{\mu}$ is in fact a $\mathbb{Z}/2$-homology bordism invariant and if it can be defined for any $\mathbb{Z}/2$-sphere M^3 , then of course we have a negative answer to the triangulation problem. If on the other hand $\bar{\mu}$ turns out not to be a $\mathbb{Z}/2$-homology bordism invariant, it will hopefully be due to some new idea in constructing such bordisms which then might lead to an example which solves the triangulation problem positively. Moreover, $\bar{\mu}$ might be invariant under some stronger bordism relation of geometric significance; we discuss this briefly in the last section.

Some consequences of our results are: plumbed $\mathbb{Z}/2$-spheres with non-zero μ-invariant admit no orientation reversing homeomorphisms; the Kummer surface cannot be pasted together from two simply connected plumbed 4-manifolds. The latter (see 6.2) depends on a generalization of $\bar{\mu}$ described in section 4.

2. The invariant.

For the purpose of this paper we define a <u>connected plumbing graph</u> Γ to be a connected graph with no cycles, each of whose vertices, which we label $i = 1,2, \dots ,s$, carries an integer weight e_i . The <u>oriented 4-manifold</u> $P(\Gamma)$ <u>obtained by plumbing according to</u> Γ is then defined in the usual way, namely, for each vertex i we let E_i be the D^2-bundle over S^2 of euler number e_i and then plumb these together according to the edges of Γ (see for instance [3]). We call $M(\Gamma) = \partial P(\Gamma)$ the <u>oriented 3-manifold obtained by plumbing according to</u> Γ .

More generally, we allow disconnected plumbing graphs Γ whose components are as above by making the convention that if $\Gamma = \Gamma_1 + \Gamma_2$ is the disjoint union of Γ_1 and Γ_2 , then $P(\Gamma) = P(\Gamma_1) \natural P(\Gamma_2)$ (boundary connected sum), so $M(\Gamma) = M(\Gamma_1) \# M(\Gamma_2)$ (connected sum). We allow also the empty graph by defining $P(\emptyset) = D^4$ and $M(\emptyset) = S^3$.

It is more customary, when plumbing, to allow graphs with cycles and bundles over surfaces of higher genus, but, since such plumbing could never yield homology spheres, our more restricted plumbing graphs include all we want. They are precisely the plumbing graphs for which $P(\Gamma)$ is simply connected.

If Γ is a plumbing graph in our sense, recall that the matrix

$$A(\Gamma) = (a_{ij})_{i,j=1, \dots ,s}$$

$$a_{ij} = \begin{cases} e_i & \text{if } i = j \\ 1 & \text{if } i \text{ is connected to } j \text{ by an edge} \\ 0 & \text{otherwise} \end{cases},$$

is the intersection matrix for the natural basis of $H_2(P(\Gamma);\mathbb{Z})$, namely the basis represented by the zero-sections of the plumbed bundles.

<u>THEOREM 2.1</u>. <u>If</u> Γ <u>is a plumbing graph as above then</u> $M(\Gamma)$ <u>is a</u> $\mathbb{Z}/2$<u>-sphere if and only if</u> $\det A(\Gamma)$ <u>is odd. In this case there exists a unique subset</u> $S \subset \mathrm{Vert}(\Gamma) = \{1, \dots ,s\}$ <u>such that the following condition holds for any</u> $j = 1, \dots ,s$:

$$(*) \qquad \sum_{i\in S} a_{ij} \equiv a_{jj} \pmod 2 .$$

The integer

$$\bar{\mu}(M(\Gamma)) = \operatorname{sign}A(\Gamma) - \sum_{j\in S} a_{jj} \quad (\text{recall } a_{jj} = e_j)$$

then only depends on $M(\Gamma)$ and not on Γ. Its modulo 16 reduction is the usual μ-invariant $\mu(M(\Gamma))$.

Proof. Everything except the oriented diffeomorphism type invariance of $\bar{\mu}(M(\Gamma))$ was proved in [9]. We describe the main ingredient however, since we need it later.

Suppose M^3 is a $\mathbb{Z}/2$-sphere and $M^3 = \partial W^4$ where W^4 is oriented. Recall that an integral Wu class for W^4 is a class $d \in H_2(W;\mathbb{Z})$ such that (dot represents intersection number):

$$d.x \equiv x.x \pmod 2 \quad \text{for all} \quad x \in H_2(W;\mathbb{Z}) .$$

We assume that such a class d exists and moreover that it can be chosen to be spherical, that is, it is representable by a smoothly embedded sphere in W. Then the μ-invariant can be computed as

$$\mu(M^3) = \operatorname{sign}(W) - d.d \pmod{16} .$$

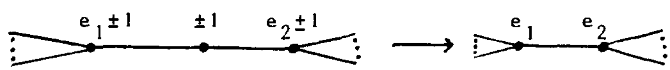
It is easily seen that d does exist if $H_1(W;\mathbb{Z})$ has no even torsion and is unique up to even multiples of elements of $H_2(W;\mathbb{Z})$ if $H_1(W;\mathbb{Z}) = 0$. In particular, if Γ is as in the theorem, then there is a unique subset $S \subset \operatorname{Vert}(\Gamma)$ such that $d = \sum_{j\in S} \alpha_j$ is a Wu class for $P(\Gamma)$, where $\alpha_1, \ldots, \alpha_s$ is the natural basis of $H_2(P(\Gamma);\mathbb{Z})$. The defining property of d translates to condition $(*)$ of the theorem, so this set is the set S of the theorem. Condition $(*)$ implies that no two adjacent vertices of Γ can both be in S. It follows that d is spherical and $d.d = \sum_{j\in S} a_{jj}$, so $\bar{\mu}(M(\Gamma)) = \operatorname{sign}P(\Gamma) - d.d \equiv \mu(M(\Gamma))$ modulo 16.

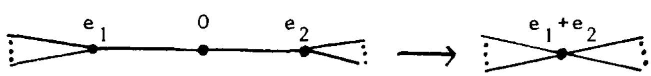
To see $\bar{\mu}(M(\Gamma))$ only depends on $M(\Gamma)$ we need the following proposition.

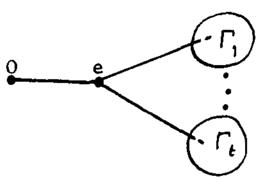
PROPOSITION 2.2. If Γ_1 and Γ_2 are two plumbing graphs as defined above, then $M(\Gamma_1) \cong M(\Gamma_2)$ if and only if Γ_1 can be obtained from Γ_2 by a sequence of the following moves and their inverses.

1a. Delete a component of Γ_1 consisting of an isolated vertex with weight ± 1 .

1b. 

1c. 

2. 

3. 

$\longrightarrow \Gamma_1 + \dots + \Gamma_t$ (disjoint union).

Move 1 is called blowing down, and its inverse blowing up.

This proposition is proved in [8 , theorem 3.2] and I understand a forthcoming paper of Bonahon and Siebenmann will also contain a proof.

To complete the proof of theorem 2.1 we need only verify that $\bar{\mu}(M(\Gamma))$ is invariant under the above moves. This is a trivial computation, so we leave it to the reader.

COROLLARY 2.3. The formulae of [9 , theorem 6.2] for the μ-invariants of Seifert manifolds which are $\mathbb{Z}/2$-spheres are actually formulae for $\bar{\mu}$, if one does not reduce modulo 16.

3. Examples.

A quite general method of generating homology spheres which are homology null-bordant is given by the following simple lemma.

LEMMA 3.1. *Let* A *be a principal ideal domain and let* V^4 *be a connected oriented* 4-*manifold with connected boundary* $\partial V^4 = N^3$, *such that*

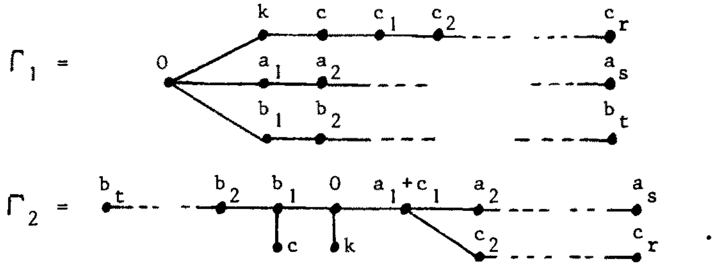
$$H_1(V;A) = A^s \quad , \quad H_2(V;A) = H_3(V;A) = 0 \quad .$$

Let M^3 *be an* A-*sphere obtained by performing* s *index* 2 *surgeries on* N. *Then* M *represents zero in* Θ_3^A.

Proof. Let $\alpha_i : S^1 \times D^2 \longrightarrow N$, $i = 1, \ldots, s$, be the embeddings on which the surgeries are performed. Then $\alpha_i S^1$, $i = 1, \ldots, s$, clearly represent an A-basis of $H_1(N;A) = H_1(V;A) = A^s$, the first equality here being by the exact homology sequence for the pair (V,N). Let W^4 be the result of adding s 2-handles to V along the α_i, so $\partial W = M^3$. Then W is clearly A-acyclic, proving the lemma.

Applied to $V^4 = S^1 \times D^3$, this lemma shows that any A-sphere M^3 which results by a single index 2 surgery on $S^1 \times S^2$ represents zero in Θ_3^A. Such manifolds are called *Mazur manifolds*.

PROPOSITION 3.2. *Let* Γ_1 *and* Γ_2 *be the following two plumbing graphs:*

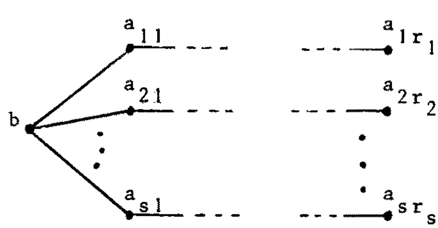


Then $M(\Gamma_1)$ *results by doing a single index* 2 *surgery on* $M(\Gamma_2)$ *and vice versa. Thus if one of them is* $S^1 \times S^2$, *then the other, if an* A-*sphere, is a Mazur manifold and represents zero in* Θ_3^A.

PROPOSITION 3.3. *If M is a plumbed $\mathbb{Z}/2$-sphere which is shown to be a Mazur manifold by proposition* 3.2 *then* $\bar{\mu}(M) = 0$.

Before giving proofs we make some remarks. Firstly it is not hard to write down many propositions of the form of 3.2. The above was the most productive one I found. Secondly it is a very easy matter to check if a plumbing graph Γ represents $S^1 \times S^2$: it follows from [8] that this is so if and only if Γ can be reduced by the moves of proposition 2.2 (without using the inverse moves) to an isolated vertex with weight 0 . For Seifert manifold plumbing graphs it is even easier.

Recall from [9] that the plumbing graph



yields the Seifert manifold

$$M(0;(1,-b),(\alpha_1,\beta_1), \ldots ,(\alpha_s,\beta_s))$$

(unnormalized Seifert invariants, as in [6], [9]) with

$$\alpha_i/\beta_i = [a_{i1}, \ldots ,a_{ir_i}] \quad , \quad i = 1, \ldots ,s .$$

We are using the continued fraction notation

$$[x_1, \ldots ,x_r] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_r}}} ,$$

and we assume $\alpha_i \neq 0$ for all i .

This Seifert manifold is $S^1 \times S^2$ if and only if: $\alpha_i = \pm 1$ for at least $s-2$ of the indices i , and $b - \sum \beta_i/\alpha_i = 0$.

It is thus a simple but tedious matter to list all Seifert manifolds which are shown to be Mazur manifolds by proposition

3.2. One obtains all the examples of Casson and Harer [1] plus the following three parameter family of Seifert manifolds:

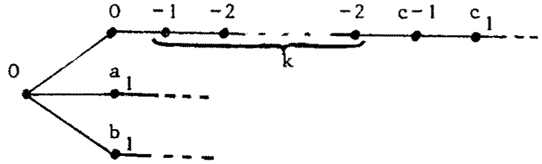
$$M(0;(1\ ,\ 1),(pq-q+1,q),(qr-r+1,r),(rp-p+1,p))\ .$$

Here p,q,r are arbitrary integers whose product is not -1 and $H_1(M;\mathbb{Z})$ has order $(pqr+1)^2$ (by convention a Seifert pair (α,β) with α negative is taken to mean $(-\alpha,-\beta)$).

It is remarkable that, although there are about 100 cases to check, each of which yields a 3-parameter family of examples, they all seem to give one of Casson and Harer's four families or the above family. The calculations are exceedingly tedious however, and I have not checked every case. One way of obtaining the family mentioned above is to take $c = -1$ and $r = 1$, so Γ_2 reduces to a star shaped Seifert manifold graph, and then choose the weights so that $[a_1+c_1,a_2, \dots ,a_s]$ is the reciprocal of an integer and $M(\Gamma_2) = S^1 \times S^2$; as the reader can check, with patience.

We shall just prove 3.2 for $k > 0$. The case $k < 0$ then follows by reversing orientations and the case $k = 0$ is given by suitably interpreting the proof, though it can be seen much more easily directly.

First note that by blowing up (-1)-vertices directly to the right of the k-weighted vertex one obtains the equivalent graph to Γ_1 (after k such blow-ups)



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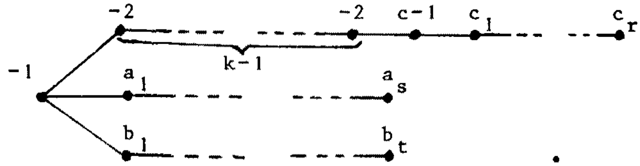
which is equivalent, by move 2 of 2.2, to the graph

$\Gamma_1' =$ 

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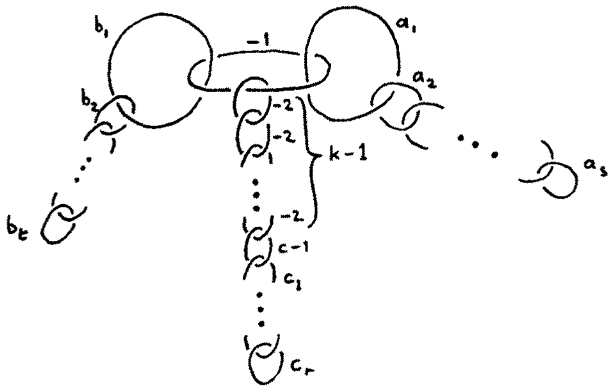
This can be represented in Kirby's link calculus [4] by the framed link



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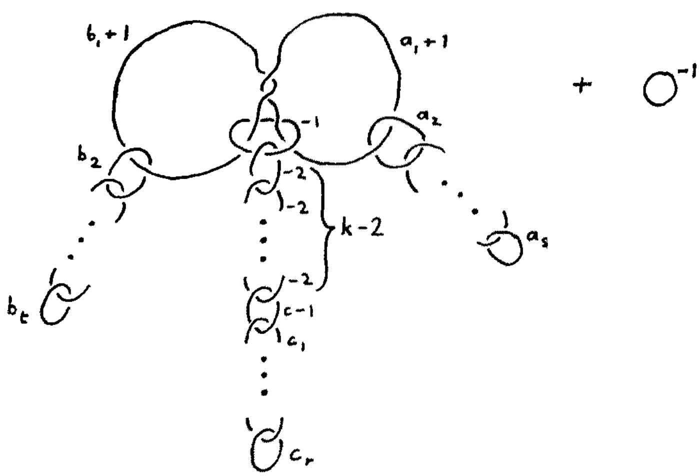
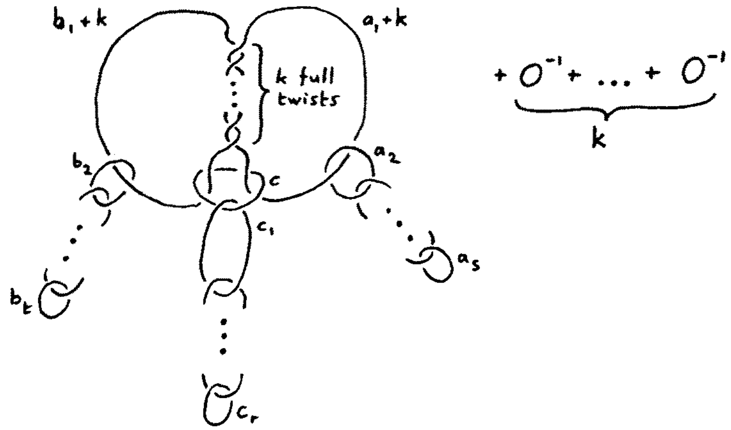
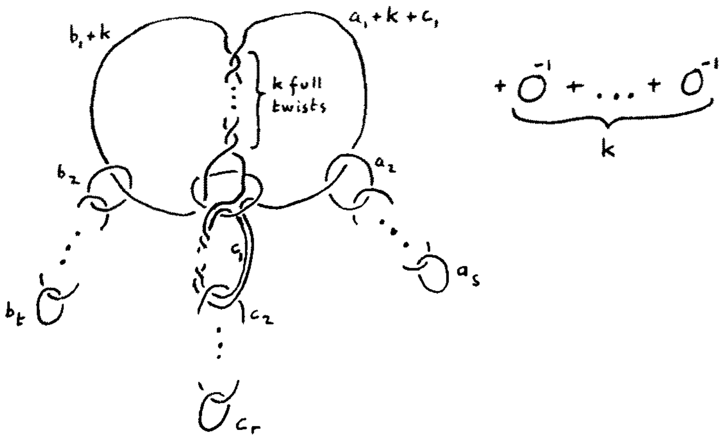
By sliding the handles which are linked with the (-1)-framed handle over this handle we obtain

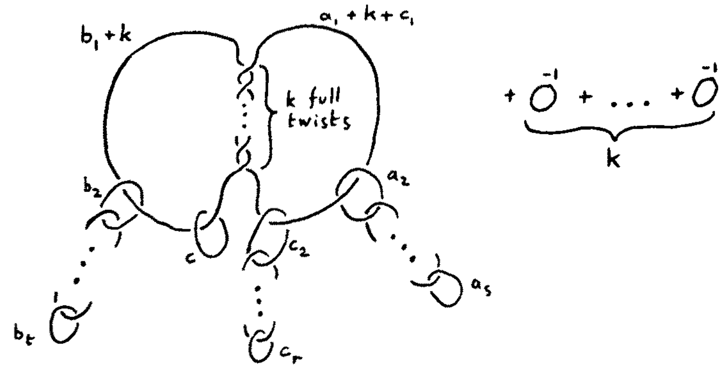


and by iterating this procedure, we eventually obtain the first link on the next page. We then slide the (a_1+k)-framed handle over the c_1-framed handle to obtain the second link on the next page.

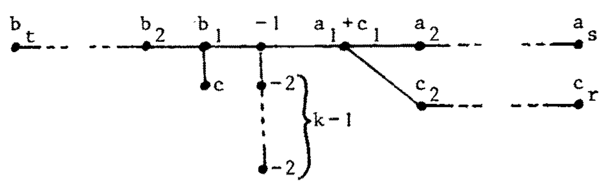




We now perform a single index 2 surgery by deleting the handle with framing c_1, giving us the next link (next page). Then by reversing our initial procedure which produced the k twists we can unwind them again, to end up with a framed link represented by the graph on the next page, called Γ_2'. Finally, the same argument which showed that Γ_1 is equivalent to Γ_1' shows that Γ_2' is equivalent to Γ_2, completing the proof.



$\Gamma_2' =$



For the proof of 3.3 we need an extension of our invariant.

4. $\bar{\mu}$ for spin structures.

The μ-invariant is actually defined for any closed oriented 3-dimensional spin manifold. It is well defined for a $\mathbb{Z}/2$-sphere M^3 without spin structure because such an M admits a unique spin structure (in general the various spin structures on a spin manifold X are classified by $H^1(X;\mathbb{Z}/2)$).

Let Γ be a plumbing graph as in section 2 but suppose $M(\Gamma)$ is not necessarily a $\mathbb{Z}/2$-sphere. Then to any spin structure γ on $M(\Gamma)$ one has an obstruction in $H^2(P(\Gamma),M(\Gamma);\mathbb{Z}/2)$ to extending γ over $P(\Gamma)$. The Poincaré dual $w_\gamma \in H_2(P(\Gamma);\mathbb{Z}/2)$ of this obstruction will be called the homology Wu class. A class $w \in H_2(P(\Gamma);\mathbb{Z}/2)$ is the homology Wu class of some spin structure on $M(\Gamma)$ if and only if w satisfies

$$w.x \;=\; x.x \quad \text{for all} \quad x \in H_2(P(\Gamma);\mathbb{Z}/2)\ .$$

The Wu class w_γ is a linear combination of the natural basis $\alpha_1, \dots, \alpha_s$ of $H_2(P(\Gamma);\mathbb{Z}/2)$, so it can be written as

$$w_\gamma = \sum_{j \in S_\gamma} \alpha_j$$

for some well defined subset S_γ of the vertex set $\{1, \dots, s\}$ of Γ. We call S_γ the <u>Wu set for the spin structure</u> γ. A subset $S \subset \{1, \dots, s\}$ is the Wu set for some spin structure on $M(\Gamma)$ if and only if it satisfies condition (*) of theorem 2.1, so we call such a subset simply a <u>Wu set</u>.

In analogy to the definition in section 2 we define

$$\bar{\mu}(M(\Gamma),\gamma) = \operatorname{sign} A(\Gamma) - \sum_{j \in S_\gamma} e_j \quad .$$

A proof completely parallel to the proof of theorem 2.1 shows:

<u>THEOREM 4.1.</u> $\bar{\mu}(M(\Gamma),\gamma)$ <u>only depends on the oriented spin diffeomorphism type of</u> $(M(\Gamma),\gamma)$ <u>and not on</u> Γ. <u>Its modulo</u> 16 <u>reduction is the usual</u> μ<u>-invariant of the spin manifold</u> $(M(\Gamma),\gamma)$.

This involves strengthening proposition 2.2 to include the spin structure, but this is not hard to do by the same method as was used to prove it without spin structure in [8].

We shall need two simple lemmas on bilinear spaces, that is finite dimensional vector spaces equipped with bilinear forms.

<u>LEMMA 4.2.</u> <u>If</u> V <u>is a non-degenerate symmetric bilinear space over</u> $\mathbb{Z}$ <u>and</u> $V_1 \subset V$ <u>has codimension</u> 1, <u>then</u> V_1 <u>is degenerate (with induced form) if and only if</u> $\operatorname{sign} V = \operatorname{sign} V_1$.

<u>LEMMA 4.3.</u> <u>If</u> V <u>is a non-degenerate symmetric bilinear space over</u> $\mathbb{Z}/2$ <u>and</u> $V_1 \subset V$ <u>has codimension</u> 1, <u>then</u> V_1 <u>is degenerate if and only if the Wu class</u> w <u>of</u> V <u>is in</u> V_1. <u>The Wu class is the unique element with</u> $w.x = x.x$ <u>for all</u> $x \in V$.

<u>Proofs</u>. We prove 4.3 and leave 4.2 to the reader. V_1 degenerate means $V_1^\perp \subset V_1$ where $\perp$ is orthogonal complement. Now $V_1^\perp$ is 1-dimensional, so $V_1^\perp = \{0,x\}$ for some x. Since $w.x = x.x = 0$ we have $w \in (V_1^\perp)^\perp = V_1$. Conversely if V_1 is non-degenerate then $V = V_1 \oplus V_1^\perp$ and $w = w(V) = w(V_1) \oplus w(V_1^\perp)$ is not in

V_1 since $w(V_1^{\perp}) \neq 0$.

We can now prove 3.3. We use the notation of proposition 3.2 and its proof and we assume $M(\Gamma_1)$ is a $\mathbb{Z}/2$-sphere and $M(\Gamma_2) = S^1 \times S^2$. $P(\Gamma_1')$ is diffeomorphic to the 4-manifold V_1 obtained by adding 2-handles to D^4 along the framed link in $S^3 = \partial D^4$ first pictured in the proof of 3.2. $H_2(V_1;\mathbb{Z})$ has non-degenerate intersection form (in fact this is true modulo 2) since $\partial V_1 = M(\Gamma_1)$ is a $\mathbb{Z}/2$-sphere. As we alter the link in the proof of 3.2 we do not alter V_1 until the point where we delete a component of the link. This corresponds to removing a handle, and gives us a manifold V_2 diffeomorphic to $P(\Gamma_2')$. Thus $H_2(V_2;\mathbb{Z})$ is a codimension 1 subspace of $H_2(V_1;\mathbb{Z})$. But $H_2(V_2;\mathbb{Z})$ has degenerate intersection form, since $\partial V_2 = M(\Gamma_2) = S^1 \times S^2$. Thus sign V_1 = sign V_2 by lemma 4.2, that is, sign $A(\Gamma_1')$ = sign $A(\Gamma_2')$, whence also follows easily: sign $A(\Gamma_1)$ = sign $A(\Gamma_2)$.

Similarly one can use lemma 4.3 to follow what happens to the Wu set $S \subset \mathrm{Vert}(\Gamma_1)$ as one goes through the proof of 3.2. In particular one sees that the vertices weighted a_1 and c_1 are either both in S or both not and that there is a Wu set $S_2 \subset \mathrm{Vert}(\Gamma_2)$ which contains precisely the vertices of Γ_2 which correspond to the vertices in $S \subset \mathrm{Vert}(\Gamma_1)$, with the possible exception of the vertices weighted 0 and k . But by considering condition (*) of theorem 2.1 at the 0-weighted vertex one then sees that the k-weighted vertices are also either in both Wu sets S and S_2 or in neither. Thus $\sum_{j\in S} e_j = \sum_{j\in S_2} e_j$, so

$$\bar{\mu}(M(\Gamma_1)) = \operatorname{sign} A(\Gamma_2) - \sum_{j\in S_2} e_j .$$

Thus $\bar{\mu}(M(\Gamma_1))$ equals the $\bar{\mu}$-invariant of some spin structure on $M(\Gamma_2) = S^1 \times S^2$. But $S^1 \times S^2$ can be given by the graph consisting of a single point with weight 0 , so both spin structures on $S^1 \times S^2$ have $\bar{\mu} = 0$. Thus the proof of 3.3 is complete for this case. The case that $M(\Gamma_2)$ is a $\mathbb{Z}/2$-sphere and $M(\Gamma_1)$ is $S^1 \times S^2$ is completely analogous.

5. Seifert spheres.

We recall some facts from [9]. Let $\alpha_1, \ldots, \alpha_n$ be pairwise coprime integers, each $\alpha_i \geq 2$. Then there exists a unique Seifert manifold whose unnormalized Seifert invariants are $(0;(\alpha_1,\beta_1), \ldots, (\alpha_n,\beta_n))$ and satisfy

$$\alpha_1 \ldots \alpha_n (\sum \beta_i/\alpha_i) = 1 .$$

This manifold will be denoted $\Sigma(\alpha_1, \ldots, \alpha_n)$. It is a $\mathbb{Z}$--sphere and every Seifert manifold which is a $\mathbb{Z}$-sphere is diffeomorphic to such a manifold after reversing orientation if necessary. Various other descriptions of these manifolds are given in [7] and [9], for instance as links of certain complex surface singularities.

Note that

$$\Sigma(\alpha_1, \ldots, \alpha_n) = \Sigma(\alpha_1', \ldots, \alpha_n')$$

if and only if $(\alpha_1', \ldots, \alpha_n')$ is a permutation of $(\alpha_1, \ldots, \alpha_n)$. We define

$$\Sigma(\alpha_1, \ldots, \alpha_n, 1) = \Sigma(\alpha_1, \ldots, \alpha_n)$$

in order to permit $\alpha_i = 1$ in the definition.

No closed formula for $\bar{\mu}(\Sigma(\alpha_1, \ldots, \alpha_n))$ is known and in practice the formula of [9, theorem 6.2] seems the fastest way of computing it (see corollary 2.3 above). The only general formulae for it that I know are the following.

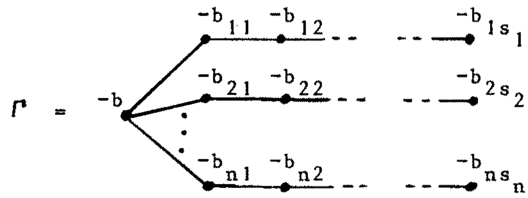
<u>THEOREM 5.1.</u> (i). <u>If</u> $\alpha_n \equiv \alpha_n' \mod 2\alpha_1 \ldots \alpha_{n-1}$ <u>then</u>

$$\bar{\mu}(\Sigma(\alpha_1, \ldots, \alpha_n)) = \bar{\mu}(\Sigma(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n')) .$$

(ii). <u>If</u> $\alpha_n = -\alpha_n' \mod 2\alpha_1 \ldots \alpha_{n-1}$ <u>then</u>

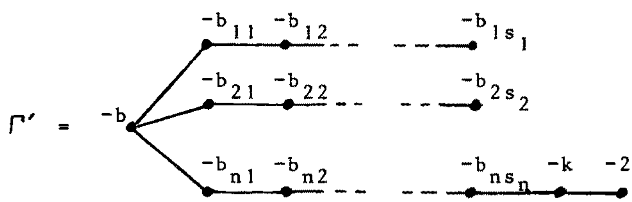
$$\bar{\mu}(\Sigma(\alpha_1, \ldots, \alpha_n)) = -\bar{\mu}(\Sigma(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n')) .$$

<u>Proof</u>. (i). It suffices to prove (i) for $\alpha_n' = \alpha_n + 2\alpha_1 \ldots \alpha_{n-1}$. Let



be the normal form plumbing graph in the sense of [8] (called the canonical graph in [9]) for $\Sigma = \Sigma(\alpha_1, \ldots, \alpha_n)$. It is characterized, among plumbing graphs for Σ , by the condition that all b_{ij} be $\geqslant 2$. As shown in [9], its intersection form $A(\Gamma)$ is negative definite.

A reasonably efficient algorithm for computing Γ is given in [9], and a simple computation with that algorithm shows that the normal form plumbing graph Γ' for $\Sigma' = \Sigma(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n')$ is



with $k = [\alpha_1 \ldots \alpha_{n-1}/\alpha_n] + 2$.

Now let S be the Wu set for Γ. Considering S as a subset of $\mathrm{Vert}(\Gamma')$ one sees that either S itself or S plus the (-2)-weighted vertex of Γ' is a Wu set S' for Γ' , and hence, by uniqueness, <u>the</u> Wu set. Since sign $A(\Gamma)$ and sign $A(\Gamma')$ differ by 2 and $\sum_{j\in S} e_j$ and $\sum_{j\in S'} e_j$ differ by at most 2 , we see that $\bar{\mu}(\Sigma)$ and $\bar{\mu}(\Sigma')$ differ by at most 4 . But they are both divisible by 8 , so they are equal.

It is enough to prove (ii) for $\alpha_n < 2\alpha_1 \ldots \alpha_{n-1}$ and $\alpha_n' = 2\alpha_1 \ldots \alpha_{n-1} - \alpha_n$, because then one can apply (i). The argument proceeds like case (i) after verifying that, if Γ is as above, then the above Γ' with minus signs deleted and k reduced by 1 is a plumbing graph for $\Sigma(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n')$.

The above proof is valid, with suitable interpretation, also for $\alpha_n = 1$. For example, $\Sigma(\alpha_1, \alpha_2, 1)$ is always the 3-sphere, so the theorem implies $\bar{\mu}(\Sigma(\alpha_1, \alpha_2, 2m\alpha_1\alpha_2 \pm 1)) = 0$ for all $m > 0$.

Until recently, almost all known examples of plumbed $\mathbb{Z}/2$--spheres which represent zero in $\Theta_3^{\mathbb{Z}/2}$ were included in Casson and Harer's lists, so their $\bar{\mu}$-invariants were zero by proposition 3.3, while the remaining examples could be checked by hand. Recently R. Stern has announced some additional examples. They are of the form $\Sigma(\alpha_1, \alpha_2, 2\alpha_1\alpha_2 \pm \alpha_3)$ for certain $\Sigma(\alpha_1, \alpha_2, \alpha_3)$ in the Casson Harer lists and hence have $\bar{\mu}$ equal to zero by the above theorem.

6. Final comments.

PROBLEM 1. Is $\bar{\mu}$ a $\mathbb{Z}/2$-homology bordism invariant? Equivalently, does every plumbed $\mathbb{Z}/2$-sphere M^3 which bounds a $\mathbb{Z}/2$-acyclic 4-manifold have $\bar{\mu}(M) = 0$?

Our results weigh both for and against a positive answer, for although they give many examples on the positive side, they also show, by giving a uniform construction for most of them, that the examples found to date are quite restricted in their construction. But there does seem to be a nontrivial obstruction to finding counterexamples: by analyzing the proof of 3.3 I found constructions which at first seemed certain to yield some, but they only gave rational homology spheres and no $\mathbb{Z}/2$-spheres.

A property of the examples found so far is that the $\mathbb{Z}/2$-acyclic manifold W with $\partial W = M$ can always be chosen such that $\pi_1(M) \longrightarrow \pi_1(W)$ is onto. If M is a $\mathbb{Z}$-sphere, W can even be chosen contractible. This again points out the probable restricted nature of these examples. However these conditions on W lead to stronger concepts of homology bordism which are still of interest and the answer to problem 1, if negative, might still be positive for such a relation. Non-triviality with respect to this stronger homology bordism relation can sometimes be detected by the Casson Gordon invariant, even when the μ-invariant and the only other known homology bordism invariant, the Witt class of the torsion linking form of M^3 , vanish.

If a counterexample to problem 1 exists, the new idea in its construction might be applicable to construct other sought for objects. One was mentioned in the introduction. Another might be a closed simply connected 4-manifold with nontrivial even definite intersection form.

In this context we mention the following problem, which seems unlikely to have a positive answer, though it would be nice if it did.

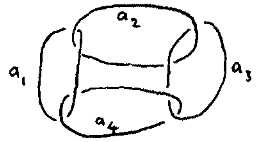
PROBLEM 2. If M is a plumbed $\mathbb{Z}/2$-sphere and $M = \partial V^4$ where V is simply connected and has even definite form, is then $\bar{\mu}(M)$ = sign V ? What if one weakens simply connected to $H_1(V;\mathbb{Z}/2) = 0$?

A positive answer to the second part would answer problem 1 for such M and suggest a faint hope for a general definition of $\bar{\mu}$.

PROBLEM 3. Can one extend the definition of $\bar{\mu}$ to arbitrary $\mathbb{Z}/2$-spheres, keeping its properties (whatever they are)?

If one just wants an oriented diffeomorphism invariant, additive for connected sum, satisfying $\bar{\mu}(-M) = -\bar{\mu}(M)$, whose mod 16 reduction is μ , then the answer is probably yes (it is equivalent to theorem 6.1 below holding for any $\mathbb{Z}/2$-sphere), although the answer to the same question for spin manifolds is no. Indeed the Lie-invariant spin structure on T^3 satisfies $\mu(T^3,\gamma) = 8 \pmod{16}$ and $(T^3,\gamma) \cong -(T^3,\gamma)$, so 6.1 does not extend to arbitrary spin manifolds.

One might hope that for arbitrary framed links representing Z/2-spheres there is a formula like the one of theorem 2.1, but this hope must fail. Indeed, the link



, all a_i odd,

if some a_i is -1 , represents the Seifert manifold $M = M(0;(1,-1),(a_{i+1}+2,1),(a_{i-1}+2,1),(a_{i+2}+1,1))$ (indices mod 4) , which has $\bar{\mu}(M) = -(a_1+a_2+a_3+a_4) - 4 + \text{sign } A$, where A is the intersection form determined by the link. But if $a_1 = a_3 = 1$ and $a_2 = a_4 = 3$ it gives $\Sigma(2,3,7)$, whose $\bar{\mu}$-invariant is +8 rather than -8 , which the above formula would give.

PROBLEM 4. Is the Browder Livesay invariant of a free involution on a plumbed $\mathbb{Z}/2$-sphere M always equal to $\bar{\mu}(M)$?

The answer is yes for Seifert manifolds, at least if the orbit space is either sufficiently large or a Seifert manifold.

Despite the many questions about $\bar{\mu}$, the invariant does have applications.

THEOREM 6.1. *If M^3 is the boundary of a simply connected plumbed 4-manifold, then for any spin structure γ on M with $\mu(M,\gamma)$ nontrivial, (M,γ) admits no orientation reversing spin homeomorphism.*

Proof. Clearly $\bar{\mu}(M,\gamma) \neq 0$, so $\bar{\mu}(-(M,\gamma)) = -\bar{\mu}(M,\gamma) \neq \bar{\mu}(M,\gamma)$.

As already remarked, this theorem fails for more general 3-dimensional spin manifolds. However, at this conference, L. Siebenmann announced that it is true for all sufficiently large 3-dimensional $\mathbb{Z}$-spheres. In this case, indeed for $\mathbb{Z}/2$-spheres, the spin structure is unique and can thus be disregarded.

THEOREM 6.2. *If the closed oriented spin manifold W^4 can be written as the union $W_1 \cup (-W_2)$ of two simply connected plumbed 4-manifolds, pasted along their boundaries, then* $\operatorname{sign} W^4 = 0$.

Proof. Let $\partial W_1 = \partial W_2 = M$ and let γ be the restriction of the spin structure to M . Then $\operatorname{sign} W = \operatorname{sign} W_1 - \operatorname{sign} W_2 = \bar{\mu}(M,\gamma) - \bar{\mu}(M,\gamma) = 0$ by theorem 4.1.

Let $\Sigma = \Sigma(\alpha_1, \ldots, \alpha_n)$ be as in section 5 . We can write Σ as the boundary of a 4-manifold V with negative definite intersection form by considering Σ as the link of a complex surface singularity and taking the minimal resolution of this singularity. In [9] a list was given for $n = 3$ and low values of α_1 and α_2 for which this V has even form. It turned out that α_3 was then also restricted to be small. This is explained by the following theorem.

THEOREM 6.3. *The above* V *never has even form if* $\alpha_n > \alpha_1\alpha_2 \cdots \alpha_{n-1}$.

We just sketch the proof. Suppose $\alpha_n > \alpha_1 \dots \alpha_{n-1}$ and V has even form. Let $\bar{V}$ be constructed like V, but using the minimal good resolution. Then $\bar{V} \cong P(\Gamma)$ as smooth manifolds with Γ as in the proof of 5.1. Let V_m and $\bar{V}_m$ be the corresponding manifolds for $\Sigma_m = \Sigma(\alpha_1, \dots, \alpha_{n-1}, \alpha_n + 2m\alpha_1 \dots \alpha_{n-1})$ and let Γ_m be the corresponding graph. Γ_m is obtained from Γ by adding a chain of $2m$ (-2)-weighted vertices to Γ, as in the proof of 5.1 (note our assumption on α_n implies $k = 2$ in the proof of 5.1).

V is obtained from $\bar{V}$ by blowing down (in the sense of complex manifolds) some exceptional curves of $\bar{V}$, and it follows that the Wu set S for Γ consists only of vertices corresponding to exceptional curves which are blown down in this process. In particular, the vertex weighted $-b_{ns_n}$ is not in S, for if it were, then in passing from $\bar{V}_m$ to V_m, the whole chain of new (-2)-weighted exceptional curves would blow down, and this would contradict negative definiteness of the intersection form of V_m for m sufficiently large. The proof of 5.1 now shows that $\bar{\mu}(\Sigma_m) = \bar{\mu}(\Sigma) - 2m$, which is a contradiction to 5.1.

Added September 1979: Jonathon Wahl has obtained examples of rational homology spheres which bound rational homology balls as a by-product of work on smoothing singularities. Namely let p,q,r be positive integers and $\zeta = \exp(2\pi i/(pqr+1))$. Then $G = \mathbb{Z}/(pqr+1)$ acts on $\mathbb{C}^3$ by $\zeta(x,y,z) = (\zeta x, \zeta^{-p}y, \zeta^{pq}z)$ and this action is free on $\mathbb{C}^3 - \{0\}$ and leaves invariant the polynomial $f\colon \mathbb{C}^3 \to \mathbb{C}$ given by $f(x,y,z) = x^p y + y^q z + z^r x$. Thus f induces $g\colon \mathbb{C}^3/G \to \mathbb{C}$ and the fibers $g^{-1}(t)$, $t \neq 0$, are smoothings of the isolated surface singularity $(g^{-1}(0),0)$. Wahl showed that the "Milnor fiber" $W^4 = g^{-1}(t) \cap (D/G)$ (D the unit disc in $\mathbb{C}^3$ and $0 < |t|$ small) satisfies $H_1(W) = G$, $H_2(W) = H_3(W) = 0$, whence also $\partial W^4 = M^3$ satisfies $|H_1(M)| = (pqr+1)^2$, $H_2(M) = 0$. (This is because the Milnor fiber V^4 of f is homotopy equivalent to a wedge of (pqr) 2-spheres, and $W = V/G$.)

In fact, using the methods of [9] one can verify that M^3 is the Seifert manifold $M(0;(1,1),(pq-q+1,q),(qr-r+1,r),(rp-p+1,p))$ mentioned in section 3 . It is surprizing, given the difference of Wahl's construction from earlier ones, that his examples are not "new", but the same approach could well give many other examples.

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October: L.C. Siebenmann, in "On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres", these Proceedings, shows $M \cong -M \Rightarrow \mu(M) = 0$ for a very large class of $(\mathbb{Z}/2)$-spheres. He also gives a different definition of $\bar{\mu}$ and interesting examples.