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Four-Manifolds Constructed via Plumbing

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The following procedure seemed to present a way of constructing interesting 4-manifolds: plumb disc bundles with even euler numbers over 2-spheres together in such a way that the boundary of the resulting 4-manifold M_0^4 is simply connected, hence S^3 by von Randow [7], Scharf [6], and Montesinos [5]. Then $M = M_0^4 \cup D^4$ is a 4-manifold with even intersection form.

We shall show that the 4-manifolds obtained this way are diffeomorphic to a connected sum of copies of $S^2 \times S^2$. More generally:

Theorem. *If $M^4 = M_0^4 \cup D^4$, where M_0^4 is a 4-manifold obtained by plumbing with $\partial M_0 = S^3$, then either $M = (S^2 \times S^2) \# \ldots \# (S^2 \times S^2)$ or $M = CP^2 \# \ldots \# CP^2 \# -CP^2 \# \ldots \# -CP^2$, according as M has even or odd intersection form.*

[As remarked above, we may replace the condition $\partial M_0 = S^3$ by the condition $\pi_1(\partial M_0) = 1$.]

Scharf, in his correction in [6] of von Randow's proof [5] of the Poincaré conjecture for graph manifolds showed that if M_0 is obtained by plumbing according to a weighted graph Γ and $\partial M_0 = S^3$, then Γ can be reduced by a series of reduction moves to a trivial graph. We prove the above theorem by analyzing the effect of these reduction moves on the plumbed 4-manifold (Theorem 1.3 below). A simply connected manifold satisfying the conclusion of the above theorem is called *completely reducible*. The theorem can thus be restated: M^4 is completely reducible if (and only if) it has a plumbed spine. We do not know if this helps in showing, for instance, complete reducibility of $X \# CP^2$, where X is a non-singular complex surface, proved in many cases by Mandelbaum and Moishezon (see Moishezon [4]).

We also observe that a completely reducible M^4 has a handle decomposition with only 0-, 2-, and 4-handles. However, a consequence of our theorem is that 4-manifolds with non-zero index and even intersection form, even if they have only 0-, 2-, and 4-handles, can never have a plumbed spine. In this regard, we note that Kirby [3] has shown the Kummer surface (and hence the Kummer surface connected sum with arbitrarily many copies of $S^2 \times S^2$) has such a handle decomposition.

Despite the above theorem, one can obtain interesting manifolds of the form $M_0 \cup M_1$ pasted along the boundary, where M_0 and M_1 are plumbed manifolds. For example, if $f(z_1, z_2, z_3)=0$ is a weighted homogeneous polynomial equation with isolated singularity at the origin and if $g(w_1, w_2, w_3, w_4)=w_4^d f(w_1/w_4, w_2/w_4, w_3/w_4)$ with $d=\deg(f)$, then the variety in CP^3 defined by $g=0$, once its singularities have been well resolved, is easily seen to have this form. This was pointed out to us by Dolgacev.

1. Reduction Theorem

If Γ is a weighted graph for plumbing, denote by $P\Gamma$ the 4-manifold with boundary obtained by plumbing according to Γ. We are mainly interested in the case $\partial P\Gamma = S^3$, in which case Γ must clearly be a tree, and the bundles being plumbed are D^2-bundles over S^2.

It is convenient to extend the definition of plumbing slightly. If $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_s$ is a disjoint union of plumbing graphs, let $P\Gamma$ denote $P\Gamma = P\Gamma_1 \natural \ldots \natural P\Gamma_s$, (boundary connected sum), so $\partial P\Gamma = \partial P\Gamma_1 \# \ldots \# \partial P\Gamma_s$. Until further notice, we assume Γ is a disjoint union of trees.

Before describing von Randow's and Scharf's reduction procedures we need a lemma.

For integers $a_1, \cdots, a_n$ define inductively

$$p(\ )=1, \qquad p(a_1)=a_1$$
$$p(a_1, \ldots, a_n)=a_n p(a_1, \ldots, a_{n-1})-p(a_1, \ldots, a_{n-2}).$$

$p(a_1, \ldots, a_n)$ is easily seen to be the determinant of the intersection form of $P\Gamma$, where

$$\Gamma = \overset{a_1}{\bullet}\!\!-\!\!-\!\!-\overset{a_2}{\bullet}\!\!-\!\!-\!\!-\cdots-\!\!-\!\!-\overset{a_n}{\bullet}.$$

$\partial P\Gamma$ is a lens space $L(p,q)$ in this case, with $p=|p(a_1, \ldots, a_n)|$, and so it is S^3 if and only if $p(a_1, \ldots, a_n)=\pm 1$ ([6, 1]).

Lemma 1.1. *If* $p(a_1, \ldots, a_n)=\pm 1$ *and* $n>0$ *then some* a_i *is* ± 1 *or* 0.

Proof ([7]). By a trivial induction, if all $|a_i| \geqq 2$ then $|p(a_1, \ldots, a_n)| > |p(a_1, \ldots, a_{n-1})|$, so $|p(a_1, \ldots, a_n)| \geqq n+1$.

Von Randow defined reduction moves RI and RII, which replace a graph Γ by a simpler graph Γ' with $\partial P\Gamma = \partial P\Gamma'$, as follows: Assume $p(a_i, \ldots, a_k)=\pm 1$.

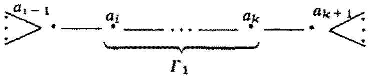
RI. *Replace*

$$\Gamma = \underbrace{\overset{a_i}{\bullet}\!\!-\!\!-\!\!-\cdots-\!\!-\!\!-\overset{a_k}{\bullet}}_{\Gamma_1}\!\!-\!\!-\!\!-\overset{a_{k+1}}{\bullet}\!\!<\!\vdots$$

by

$$\Gamma' = \overset{a'_{k+1}}{\bullet}\!\!<\!\vdots$$

RII. *Replace*

$\Gamma =$ 

by

$\Gamma' =$ 

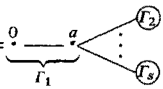
where

$$a'_{k+1} = a_{k+1} - p(a_i, \ldots, a_k)\, p(a_i, \ldots, a_{k-1}),$$
$$a'_{i-1} = a_{i-1} - p(a_i, \ldots, a_k)\, p(a_{i+1}, \ldots, a_k).$$

Note that by Lemma 1.1 it is sufficient to have reductions RI and RII for $\Gamma_1 = \overset{\pm 1}{\bullet}$ and $\Gamma_1 = \overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{a}{\bullet}$. The general case then follows by induction. This is basically how the general case was proved in [7] (in fact one can reduce to the cases RI and RII with $\Gamma_1 = \overset{\pm 1}{\bullet}$ and the inverse operations, so called "blowing down" and "blowing up").

Scharf defined the additional move

RIII. *Replace*

$\Gamma =$ 

by

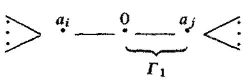
$$\Gamma' = \text{disjoint union of } \Gamma_2, \ldots, \Gamma_s,$$

and showed $\partial P\Gamma = \partial P\Gamma'$. He showed also (see Lemma 1.4).

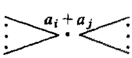
Theorem 1.2. *If $\partial P\Gamma$ is simply connected then Γ can be reduced to a disjoint union of trivial graphs $\Gamma_i = \overset{\pm 1}{\bullet}$ or $\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{a}{\bullet}$ by* RI, RII, RIII *above, so $\partial P\Gamma = S^3$.*

There is a further reduction move one can define which is not needed in the above theorem.

RIV. *Replace*

$\Gamma =$ 

by

$\Gamma' =$ 

Theorem 1.3. *In the situation of each of the reduction moves* RI, RII, RIII, RIV *above*

$$P\Gamma = P\Gamma_1 \natural P\Gamma'.$$

We first describe how this theorem implies the theorem of the introduction.

Lemma 1.4. (i) *If* Y_1 *and* Y_2 *are* 4-*manifolds with non-empty boundary*, $\partial Y_1 = S^3$, *and* $M = Y_1 \cup D^4$, *then*

$$Y_1 \natural Y_2 = M \# Y_2 .$$

(ii) $P(\overset{1}{\bullet}) \cup D^4 = CP^2, \quad P(\overset{-1}{\bullet}) \cup D^4 = -CP^2$

$$P(\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{a}{\bullet}) \cup D^4 = S^2 \times S^2, \quad (a \text{ even}),$$
$$= CP^2 \# -CP^2, \quad (a \text{ odd}).$$

Assuming this lemma, the theorem of the introduction follows by a simple induction by Theorems 1.2, 1.3, and Lemma 1.1, which implies that one need only consider reductions RI and RII for Γ_1 one of the graphs in Lemma 1.4ii). Indeed this inductively decomposes $M = P\Gamma \cup D^4$ into a connected sum of copies of CP^2, $-CP^2$, and $S^2 \times S^2$, and the well known fact that $(S^2 \times S^2) \# \pm CP^2 = (CP^2 \# -CP^2) \# \pm CP^2$ then completes the proof.

If fact, our proof has shown somewhat more than we have claimed. Noting that throughout we have worked with M_0, rather than with M, it is easy to see that we have proven:

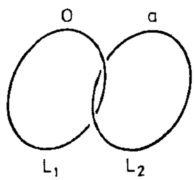
Corollary. *If* $M_0 = P\Gamma$ *is a* 4-*manifold with boundary obtained by plumbing by the graph* Γ, *and* Γ *reduces to* $\Gamma_1 \cup \ldots \cup \Gamma_s$ *(disjoint union) by a sequence of moves RI, RII, RIII, or RIV, then,* $M_0 = P\Gamma_1 \natural P\Gamma_2 \natural \ldots \natural P\Gamma_s \# M'$ *(and* $\partial M_0 = \partial P\Gamma_1 \# \partial P\Gamma_2 \# \ldots \# \partial P\Gamma_s$*) where* M' *is completely reducible.*

2. Proofs of 1.4 and 1.3

Lemma 1.4i) is trivial since $Y_1 = M \# D^4$, so $Y_1 \natural Y_2 = (M \# D^4) \natural Y^2 = M \# (D^4 \natural Y^2) = M \# Y_2$.

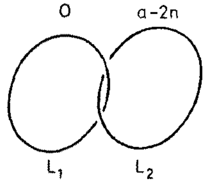
The only part of 1.4ii) that needs proof is the statement on $P(\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{a}{\bullet}) \cup D^4$. Clearly it suffices to show $P(\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{a}{\bullet}) = S^2 \times S^2 - D^4$ for a even, $= \mathrm{CP}^2 \# (-\mathrm{CP}^2) - \mathrm{D}^4$ for a odd. To do so we use Kirby's calculus of framed links [2] (only the easy half).

$P(\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{a}{\bullet})$ is given by the framed link

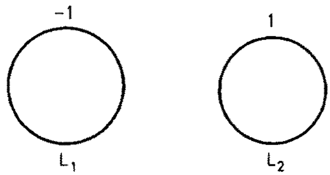


[That is, $P(\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{a}{\bullet})$ is obtained by adding two 2-handles E_1 and E_2 with core circles L_1 and L_2 in $\partial D^4 = S^3$, with the handles having the specified self-intersection numbers.]

Suppose $a=2n$ or $2n+1$. Pass handle E_2 over handle E_1 n times. We then obtain



If a is even we are done. If a is odd, pass E_1 1 time over E_2 to obtain



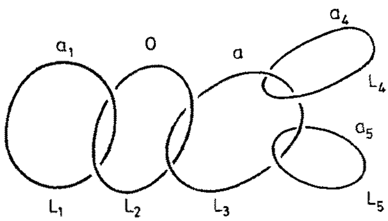
as required.

[Alternatively, Lemma 1.4ii) could be proved by exhibiting $P(\overset{0}{\bullet}\!-\!\!-\!\overset{a}{\bullet})\cup D^4$ directly as the double of the D^2 bundle over S^2 with Euler number a.]

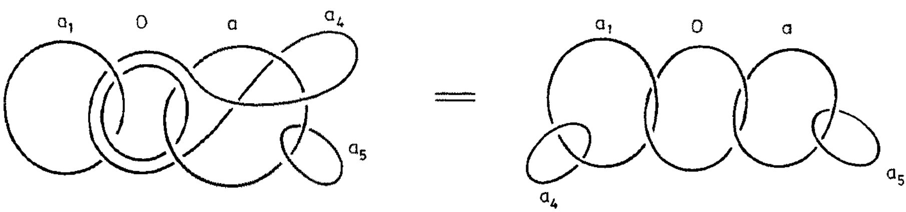
To prove Theorem 1.3 we start by discussing a simple case of RIV. Let

$$\Gamma = \overset{a_1}{\bullet} - \underbrace{\overset{0}{\bullet} - \overset{a}{\bullet}}_{\Gamma_1} \begin{matrix} \overset{a_4}{\bullet} \\ \\ \underset{a_5}{\bullet} \end{matrix}, \qquad \Gamma' = \overset{a_1+a}{\bullet} \begin{matrix} \overset{a_4}{\bullet} \\ \\ \underset{a_5}{\bullet} \end{matrix}$$

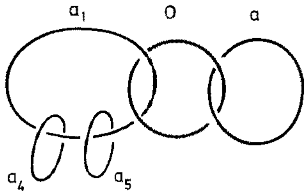
Then $P\Gamma$ is represented by the link



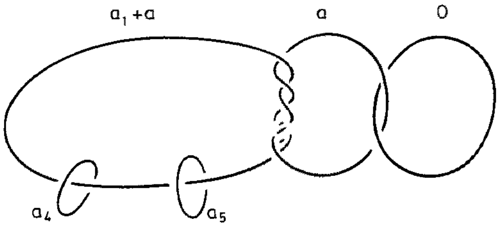
with handle E_i corresponding to link L_i. We now alter the description of $P\Gamma$ by sliding handles. Passing E_4 over E_2 yields



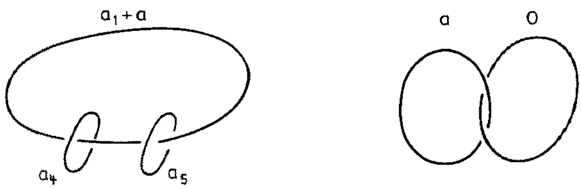
Passing E_5 over E_2 yields



Passing E_1 over E_3 now yields



and by passing the handle with framing a_1+a a times over the handle with framing 0 we get



corresponding to $\Gamma' \cup \Gamma_1$.

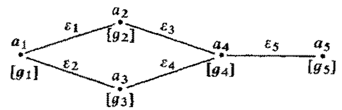
The above proof applies with no essential change to the general RIII and RIV reductions. As a special case this gives the RI and RII reductions with $\Gamma = \overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{a}{\bullet}$. The RI and RII reductions with $\Gamma = \overset{\pm 1}{\bullet}$ are proved completely analogously. The general case of RI and RII can also be proved analogously, but it also follows by induction by the remarks following the definition of RI and RII.

The above proof extends to more general plumbing, which we describe in the next section. Before we do so, we briefly describe an alternative proof of our theorem.

Lemma 1.4 shows that a regular neighborhood N of $P\Gamma_1$ in $P\Gamma$ is either $S^2 \times S^2 - D^4$ or $CP^2 \# CP^2 - D^4$ for $\Gamma_1 = \overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{a}{\bullet}$. In either case $\partial N = S^3$ and it is possible to span a $D^4 = D^2 \times D^2$ across this S^3 in such a way that we may decompose the result as a plumbing manifold (with $D^2 \times 0$ and $0 \times D^2$ intersecting as part of the cores of adjacent handles) which a homological calculation shows to be $P\Gamma'$. Then proceed inductively as above.

3. General Plumbing

If we plumb bundles over arbitrary oriented surfaces according to a weighted graph Γ which may have cycles, then Γ is weighted typically as follows

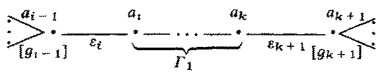


meaning that we are plumbing bundles E_i of euler number a_i over surfaces of genus g_i together in such a way that the intersection number for example of the zero sections of E_3 and E_4 is the weight $\varepsilon_4 = \pm 1$ of the adjoining edge. We omit genus weights if they are zero.

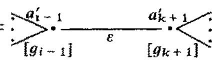
Changing the orientation of the zero section of one of the bundles does not change the euler number weight a_i but it reverses the weights of the adjoining edges in the graph. Thus the weights of edges which are not on cycles of the graph are irrelevant.

The reduction procedures generalize as follows. RI is unchanged. RII and RIV become:

RII. *Replace*

$\Gamma =$ 

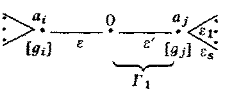
where $p(a_i, \ldots, a_k) = \pm 1$ by

$\Gamma' =$ 

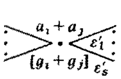
with a'_{i-1} and a'_{k+1} as before and

$$\varepsilon = (-1)^k\, p(a_i, \ldots, a_k)\, \varepsilon_i \ldots \varepsilon_{k+1}\,.$$

RIV. *Replace*

$\Gamma =$ 

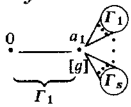
by

$\Gamma' =$ 

with $\varepsilon'_j = -\varepsilon\varepsilon'\varepsilon_j$, *for* $j = 1, \ldots, s$.

In each case $\partial P\Gamma = \partial P\Gamma'$. In each case, except RIV with $g_j \neq 0$, we also still have $P\Gamma = P\Gamma_i \natural P\Gamma'$. RIII becomes

RIII. *If*

$\Gamma =$ 

where Γ_i *is connected for* $i=1,\ldots,s$ *and is connected to the* a_1 *vertex by* d_i *edges, put* $\Gamma' =$ *(disjoint union of* $\Gamma_1,\ldots,\Gamma_s$*). Then*

$$P\Gamma = P\Gamma_1 \natural P\Gamma' \natural X,$$

where X *is the boundary connected sum of* $\sum_i (d_i - 1)$ *copies of* $S^1 \times D^3$.

The above statements are proved by similar techniques to Sect. 2.

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