SEIFERT MANIFOLDS, PLUMBING, μ-INVARIANT AND ORIENTATION REVERSING MAPS

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Dedicated to R.L. Wilder

Homology spheres and the μ-invariant appear to play a crucial role in many problems in low dimensional topology. For example, the existence of a 3-dimensional homology sphere Σ of μ-invariant 8 for which $\Sigma \# \Sigma$ bounds an acyclic 4-manifold would imply triangulability of all manifolds of dimension $\geqq 6$. As Casson has remarked, if Σ has μ-invariant 8 and admits an orientation reversing homeomorphism, then Σ would satisfy this criterion.

As another example, the intersection form of a simply connected almost parallelizable 4-manifold is even. No such closed 4-manifold with non-trivial definite form is known. A reasonable and popular procedure to collect empirical data towards the existence or non-existence of such manifolds is to study 4-manifolds with $(Z/2)$-homology sphere boundaries, since pasting along such boundaries preserves evenness of forms.

In this paper we have compiled some results and computations in these areas for the special case of Seifert 3-manifolds and other plumbed manifolds. For example, we classify in section 8 those Seifert manifolds which admit orientation reversing homeomorphisms. No rational homology spheres other than lens spaces are among them.

Section 1 reviews the fundamentals of Seifert manifolds in a more convenient version than the usual one.

We show in sections 2, 3, and 4 that the class of Seifert manifolds which are homology spheres coincides precisely with a natural subclass of the class of Brieskorn complete intersections (studied in section 2) and also with a natural subclass of the class of homogeneous spaces discussed in section 3. Thus Seifert homology spheres arise as links of isolated complex surface singularities with C^*-action. We show, in fact, in section 5 that almost every Seifert manifold arises

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this way, namely precisely those which do not fiber equivariantly over S^1. We do this by describing a "canonical plumbing diagram" for such Seifert manifolds. We give a table of Seifert homology spheres with small invariants which can be made to bound even definite 4-manifolds by these methods.

In sections 6 and 7 we give various useful algorithms for computing μ-invariants for Seifert $(Z/2)$-homology spheres and other plumbed manifolds. A table is included.

1. Fundamentals

In this paper, with the exception of §8, "Seifert manifold" will always mean an oriented closed connected 3-manifold admitting a fixed point free action of S^1. Such a manifold is equivariantly classified by its "Seifert invariants" [S],[OR]. We shall use non-normalized Seifert invariants in this paper (see [N1]) since they are more convenient for calculations. They are described as follows.

Let $M^3 \to M^3/S^1 = F$ be the Seifert fibration. Let $O_1, \ldots, O_s$ be a non-empty collection of disjoint orbits in M, including all singular orbits. Let $T_1, \ldots, T_s$ be disjoint invariant tubular neighborhoods of $O_1, \ldots, O_s$ and $M_0 = M - \mathrm{int}(T_1 \cup \ldots \cup T_s)$. Since $M_0 \to M_0/S^1 = F_0$ is an S^1-bundle over a connected surface with boundary, it admits a section $R \subset M_0$. Let $R_i = R \cap \partial T_i$. After choosing orientation conventions, R_i is a curve in ∂T_i which is homologous in T_i to some multiple $\beta_i O_i$ of the central curve. Let α_i be the order of the isotropy subgroup $Z/\alpha_i \subset S^1$ at the orbit O_i. Let g be the genus of the surface F. Then the unnormalized Seifert invariant is the collection of numbers

$$(g\,;\,(\alpha_1,\beta_1),\ldots,(\alpha_s,\beta_s))\,.$$

They satisfy $g \geqq 0$, $\alpha_i \geqq 1$, $\gcd(\alpha_i,\beta_i) = 1$.

The Seifert invariant is not unique: we can add or remove principal orbits from our collection of orbits O_i and we can choose different sections $R \subset M_0$. The following theorem is easily proved ([N1]).

<u>Theorem 1.1</u>. *Let M and M' be two Seifert manifolds with associated Seifert in-*

<u>variants</u> $(g; (\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s))$ <u>and</u> $(g'; (\alpha_1',\beta_1'),\dots,(\alpha_t',\beta_t'))$ <u>respectively</u>. <u>Then</u> M <u>and</u> M′ <u>are orientation preservingly homeomorphic by a fiber preserving homeomorphism if and only if, after reindexing the Seifert pairs if necessary, there exists a</u> k <u>such that</u>

(i) $\alpha_i = \alpha_i'$ <u>for</u> $i = 1, \dots, k$ <u>and</u> $\alpha_i = \alpha_j' = 1$ for $i, j > k$.

(ii) $\beta_i \equiv \beta_i' \pmod{\alpha_i}$ <u>for</u> $i = 1, \dots, k$.

(iii) $\Sigma_{i=1}^{s} (\beta_i/\alpha_i) = \Sigma_{i=1}^{t} (\beta_i'/\alpha_i')$.

<u>Remark</u>. It is easy to check that (i), (ii), (iii) above are equivalent to: $(g', (\alpha_j',\beta_j'), j=1,\dots,t)$ can be obtained from $(g,(\alpha_i,\beta_i), i = 1,\dots,s)$ by a sequence of the following moves:

a) permute the indices;

b) add or delete a Seifert pair $(1,0)$;

c) replace (α_1,β_1), (α_2,β_2) by $(\alpha_1,\beta_1+m\alpha_1)$, $(\alpha_2,\beta_2-m\alpha_2)$ for some $m \in Z$.

<u>Definition</u>. Denote the number $-\Sigma_{i=1}^{s} (\beta_i/\alpha_i)$, which is an invariant of the Seifert manifold by (iii), by $e(M)$, called the <u>Euler number</u> of M. We assume we have chosen our orientation conventions earlier, at the point where we were not specific about them, so that $e(M)$ is the usual Euler number if M is an S^1-bundle.

Note that reversing the orientation of M, either by reversing the orientation of the fibers or of the base (it does not matter which, since M admits an orientation preserving self-homeomorphism mapping fibers to fibers and reversing orientation both on fibers and base), replaces the Seifert invariant $(g,(\alpha_i,\beta_i))$ by $(g,(\alpha_i,-\beta_i))$. Hence $e(-M) = -e(M)$.

The Euler number has a delightful naturality property, which is invaluable for computations, as we shall see later.

<u>Theorem 1.2</u>. <u>Let</u> M <u>and</u> M′ <u>be Seifert manifolds with base spaces</u> F <u>and</u> F′, <u>and let</u> $f: M \to M'$ <u>be an orientation preserving fiber preserving homeomorphism</u>. <u>Let the degree of the induced map on a typical fiber be</u> n <u>and the degree of the induced map</u> $\bar{f}: F \to F'$ <u>be</u> m. <u>Then</u> $e(M) = (m/n)e(M')$.

Here is the idea behind the theorem. If $M \to F$ and $M' \to F'$ were genuine

S^1-bundles, the theorem would be an easy cohomology calculation, since $e(M) = \langle c(M),[F]\rangle$ where $c(M) \in H^2(F;Z) = [F,K(Z,2)] = [F,BS^1]$ is given by the classifying map $F \to BS^1$ of the bundle. Although $M \to F$ is not a genuine bundle, we can make it into a genuine bundle by replacing the fibers by the "rationalized circle" $S^1_{(0)}$. This replaces BS^1 by $BS^1_{(0)} = K(\mathbb{Q},2)$, so the argument sketched above then goes through. To make this argument precise, it is easiest to observe[1)] that we do not have to "localize all the way to $\mathbb{Q}$". Let a be a positive integer divisible by all the α_i's occurring in M or M'. Factor by the (Z/a)-action inside the S^1-action to get a diagram of maps

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & M' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M/(Z/a) & \xrightarrow{\quad f_a \quad} & M'/(Z/a) \end{array}$$

We need that f can be made (Z/a)-equivariant, but this is easily done. Now $M/(Z/a)$ and $M'/(Z/a)$ are both genuine S^1-bundles, so the theorem is true for f_a. If we show it is true for π and π', then it follows for f. But for π and π' it follows from the following lemma (due to Seifert [S]).

<u>Lemma 1.3</u>. <u>If</u> M <u>has Seifert invariants</u> $(g\,;(\alpha_i,\beta_i))$ <u>then</u> $M/(Z/a)$ <u>has Seifert invariants</u> $(g\,;(\bar\alpha_i,\bar\beta_i))$ <u>where</u> $\bar\beta_i/\bar\alpha_i$ is $a\beta_i/\alpha_i$ <u>expressed in lowest terms</u>.

<u>Proof</u>. The section $R \subset M_0 = M - \mathrm{int}(T_1 \cup \dots \cup T_s)$ used to compute the (α_i,β_i) projects down to a section $\bar R \subset M_0/(Z/a)$, which, when used to compute the Seifert invariant of $M/(Z/a)$, yields the lemma.

Theorem 1.1 says that a Seifert manifold M is determined by knowledge of its Euler number $e(M)$ and by knowledge of the pairs $(\alpha_i,\beta_i \bmod \alpha_i)$. But $(\alpha_i, \beta_i \pmod{\alpha_i})$ is equivalent to knowing the "slice type" of the corresponding orbit, that is, the equivalence class of the representation of the isotropy subgroup in the normal plane to the orbit. Thus if $M \to M'$ is a branched covering of Seifert

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1) This observation is due to Howard Rees.

manifolds, coming maybe from factoring by a group action, then $e(M)$ and $e(M')$ determine each other by Theorem 1.2 while the slice types of M can generally be computed from those for M' and vice versa by elementary local computations.

In the following, we shall often write $M = M(g\,;\,(\alpha_i,\beta_i), i = 1,\dots,m)$ as an abbreviation for "M has Seifert invariant $(g\,;\,(\alpha_i,\beta_i), i = 1,\dots,m)$".

2. Examples; Brieskorn Complete Intersections

Let $a_1, \dots, a_n$ be integers, $a_i \geqq 1$. Then if $A = (\alpha_{ij})$ is a sufficiently general $(n-2)\times n$ - matrix of complex numbers, the variety

$$V_A(a_1,\dots,a_n) = \{z \in \mathbb{C}^n \,|\, \alpha_{i1}z_1^{a_1} + \dots + \alpha_{in}z_n^{a_n} = 0, i = 1,\dots,n-2\}$$

is a complex surface which is non-singular except perhaps at the origin and

$$\Sigma^3(a_1,\dots,a_n) = V_A(a_1,\dots,a_n) \cap S^{2n-1}$$

is a smooth 3-manifold which does not depend on A up to diffeomorphism. A is in fact sufficiently general if (and if all $a_i \geqq 2$ also only if) every $(n-2)\times(n-2)$ subdeterminant of A is nonzero, by Hamm [Ha]. We assume A satisfies this from now on.

$V_A(a_1,\dots,a_n)$ has a $\mathbb{C}^*$-action by $t(z_1,\dots,z_n) = (t^{q_1}z_1,\dots,t^{q_n}z_n)$ for $t \in \mathbb{C}^*$, where $q_j = (\operatorname{lcm} a_i)/a_j$. $S^1 \subset \mathbb{C}^*$ acts fixed point freely on $\Sigma(a_1,\dots,a_n)$, so $\Sigma(a_1,\dots,a_n)$ is a Seifert manifold.

<u>Theorem 2.1.</u> $\Sigma = \Sigma(a_1,\dots,a_n)$ <u>has Seifert invariants</u> $(g\,;\,s_1(t_1,\beta_1),\dots,s_n(t_n,\beta_n))$, <u>where</u> $s_j(t_j,\beta_j)$ <u>means</u> (t_j,β_j) <u>repeated</u> s_j <u>times and</u>

$$t_j = \operatorname{lcm}_i(a_i)/\operatorname{lcm}_{i\neq j}(a_i)$$

$$s_j = \prod_{i\neq j}(a_i)/\operatorname{lcm}_{i\neq j}(a_i)$$

$$g = \tfrac{1}{2}\Big(2 + (n-2)\prod_i(a_i)/\operatorname{lcm}(a_i) - \sum_j s_j\Big)$$

<u>and the</u> β_i <u>and the Euler number</u> $e(\Sigma)$ <u>are determined (up to equivalence of Seifert</u>

<u>invariants</u>) <u>by</u>

$$-e(\Sigma) = \sum s_j \frac{\beta_j}{t_j} = \prod (a_i)/(\mathrm{lcm}\ a_i)^2 .$$

Note that the latter equation can be rewritten (by dividing through by its right hand side)

$$\sum \beta_j q_j = 1 , \quad \text{where} \quad q_j = \mathrm{lcm}(a_i)/a_j .$$

But clearly t_i divides q_j if $i \neq j$ and is prime to q_j if $i = j$. Thus modulo t_j the equation becomes $\beta_j q_j \equiv 1 \pmod{t_j}$ and hence determines $\beta_j \pmod{t_j}$, as claimed in the theorem.

<u>Proof of theorem</u>. First note that the only points of $\Sigma = \Sigma^3(a_1,\dots,a_n)$ with non-trivial isotropy are points with some coordinate zero. The condition on the coefficient matrix A implies that $V_A \cap \{z_i = z_j = 0\} = \{0\}$ for $i \neq j$, so we need only consider $z \in \Sigma^3$ with one coordinate zero, say $z_j = 0$. At such a point the isotropy subgroup has order $\gcd_{i \neq j}(q_i) = t_j$. An easy counting argument (see [N1]) can be used to see that $\Sigma^3 \cap \{z_j = 0\}$ consists of exactly s_j orbits, but this follows also from the later discussion.

Observe that we can write

$$\Sigma^3(a_1,\dots,a_n) = (V_A(a_1,\dots,a_n) - \{0\})/\mathbb{R}_+ ,$$

where $\mathbb{R}_+ \subset \mathbb{C}^*$ is in the $\mathbb{C}^*$-action. Denote $(V_A - \{0\})/\mathbb{C}^* = \Sigma/S^1$ by $P(a_1,\dots,a_n)$. Consider the diagram

$$\begin{array}{ccc}
V_A(a_1,\dots,a_n) - \{0\} & \xrightarrow{\ \Phi\ } & V_A(1,\dots,1) - \{0\} \\
\downarrow /\mathbb{R}_+ & & \downarrow /\mathbb{R}_+ \\
\Sigma(a_1,\dots,a_n) & \xrightarrow{\ \varphi\ } & \Sigma(1,\dots,1) \\
\downarrow /S^1 & & \downarrow /S^1 \\
P(a_1,\dots,a_n) & \xrightarrow{\ \bar{\varphi}\ } & P(1,\dots,1)
\end{array}$$

with horizontal arrows induced by $\Phi(z_1,\dots,z_n) = (z_1^{a_1},\dots,z_n^{a_n})$. We intend to apply

Theorem 1.2 to the map φ to compute $e(\Sigma)$.

Note that Φ and φ are equivariant if we let C^* and S^1 act non-effectively on $V_A(1,\dots,1)$ and $\Sigma(1,\dots,1)$ by $t(z_1,\dots,z_n) = (t^a z_1,\dots,t^a z_n)$, $a = \mathrm{lcm}(a_i)$. Thus φ has degree a on a typical fiber.

To determine the degree of $\bar{\varphi}$, note that the group $H = (Z/a_1)\times\dots\times(Z/a_n)$ acts on each space on the left of the diagram by letting Z/a_j act by multiplication by $e^{2\pi i/a_j}$ in the j-th coordinate. The map Φ can be identified with the orbit map $V_A(a_1,\dots,a_n)-\{0\} \to (V_A(a_1,\dots,a_n)-\{0\})/H$, and similarly for φ and $\bar{\varphi}$. Considering S^1 and H both as subgroups of $\mathrm{Diff}(\Sigma(a_1,\dots,a_n))$ by these actions, denote $H_0 = S^1 \cap H$. Now on the one hand, H_0 is isomorphic to the non-effectivity kernel of S^1 acting on $\Sigma(a_1,\dots,a_n)/H = \Sigma(1,\dots,1)$, so $H_0 \cong Z/a$, while on the other hand H_0 is the non-effectivity kernel of H acting on $\Sigma(a_1,\dots,a_n)/S^1$ $= P(a_1,\dots,a_n)$, so the orbit map $\bar{\varphi}$ of this action has degree $|H/H_0| = \Pi(a_i)/a$.

Now $V_A(1,\dots,1) \subset C^n$ is a linear subspace and hence $\Sigma(1,\dots,1) \to P(1,\dots,1)$ is the usual Hopf map $S^3 \to CP^1$. Thus $e(\Sigma(1,\dots,1)) = -1$, so by Theorem 1.2, $e(\Sigma(a_1,\dots,a_n)) = -\Pi(a_i)/a^2$.

Finally to compute g, note that the subspace $z_j = 0$ of $P(1,\dots,1)$ is a single point and that these points are precisely the points where branching of Φ occurs. The argument we used to show $\bar{\varphi}$ itself has degree $\Pi a_i/\mathrm{lcm}\, a_i$ applies with one coordinate less to show that $\bar{\varphi}$ restricted the subspaces $z_j = 0$ of $P(a_1,\dots,a_n)$ and $P(1,\dots,1)$ has degree $\Pi_{i\neq j} a_i/\mathrm{lcm}_{i\neq j} a_i = s_j$, so $P(a_1,\dots,a_n)$ contains exactly s_j points with $z_j = 0$ (proving, by the way, that $\Sigma(a_1,\dots,a_n) \cap \{z_j = 0\}$ consists of s_j orbits, as promised earlier). The standard formula for euler characteristic of a branched cover thus gives

$$\chi(P(a_1,\dots,a_n)) = \left(\prod(a_i)/a)\chi(P(1,\dots,1)\right) + \sum_j (s_j - \prod(a_i)/a)$$

$$= (2-n)\prod_i(a_i)/a + \sum_j s_j ,$$

yielding the value for g claimed in the theorem.

Remark. One can also give a very elementary computation of the Seifert pairs

(t_j,β_j) and the Euler number $e(\Sigma)$ by observing that if the β_j are chosen to satisfy $\Sigma\beta_j q_j = 1$, then

$$R = \{z \in \Sigma \mid z_j = r_j e^{i\theta_j}, r_j > 0, \sum \beta_j\theta_j \equiv 0 \pmod{2\pi}\}$$

is a section to the S^1-action in the complement of the exceptional orbits which yields, via the definition of the Seifert invariant, the values (t_j,β_j) for the Seifert pairs. However, one still needs a computation like the above proof to determine g.

A completely analogous proof to the above shows more generally

Theorem 2.2. *Let* $a_1, \ldots, a_n$, $d_1, \ldots, d_{n-2}$ *be positive integers and* $\gcd(d_i) = 1$. *Let*

$$V = \{z \in \mathbb{C}^n \mid \alpha_{i1}z_1^{d_i a_1} + \cdots + \alpha_{in}z_n^{d_i a_n} = 0, i = 1,\ldots,n-2\}$$

with sufficiently general coefficients α_{ij}, *and let* $\Sigma = V \cap S^{2n-1}$. *Then* Σ *is a Seifert manifold with invariant* $(g; ds_j(t_j,\beta_j), j = 1,\ldots,n)$ *where* t_j *and* s_j *are as in Theorem* 2.1, $d = \Pi d_i$,

$$g = 1 - \frac{d}{2}\Big[\sum s_j - \Big(\sum d_i\Big)\prod a_j/\operatorname{lcm} a_j\Big],$$

and $e(\Sigma)$ *and the* β_j *are given by*

$$-e(\Sigma) = \sum ds_j\frac{\beta_j}{t_j} = d\prod a_i/(\operatorname{lcm} a_i)^2.$$

The only alteration necessary in the previous proof is that now $P(1,\ldots,1)$ is replaced by a complete intersection of $n-2$ hypersurfaces of degrees d_i in $\mathbb{C}P^{n-1}$, so it has Euler characteristic $d(n - \Sigma d_i)$ (by the adjunction formula for instance), and by similar reasoning $z_j = 0$ now determines exactly d points in $P(1,\ldots,1)$, instead of just 1.

Bibliographic notes: A general program for computing the Seifert invariant of the link of an isolated surface singularity with $\mathbb{C}^*$-action was given by Orlik and

Wagreich [OW],[O]. The method used here is, however, based on the case $n = 3$ of Theorem 2.1 done by Neumann [N1]. Brieskorn complete intersections of the type in Theorem 2.2 were first introduced by Randell [Ran].

3. Homogeneous Spaces

Using similar methods to the preceding section, one can describe the Seifert invariants of those Seifert manifolds which are homogeneous spaces $\Pi\backslash G$, where Π is a discrete subgroup of a Lie group G with compact quotient. This was done by Raymond and Vasquez [RV]. We describe the result for $G = PSL(2;\mathbb{R})$, the universal cover of $\widetilde{PSL}(2,\mathbb{R})$.

Theorem 3.1. $M((g),(\alpha_i,\beta_i),i=1,\dots,s)$ *has the form* $\Pi\backslash G$ *if and only if there exists a divisor* q *of* $\alpha_1 \dots \alpha_s(g+s-2-\Sigma 1/\alpha_i)$ *prime to each* α_i *such that* $\beta_i q \equiv -1 \bmod \alpha_i$ *for each* i *and*

$$e(M) = -\frac{1}{q}(g+s-2-\sum 1/\alpha_i) .$$

In this case Π *is a subgroup of index* q *in the group* $\Gamma = \pi^{-1}Q$, *where* $\pi : G \to PSL(2,\mathbb{R})$ *is the covering, and* $Q = \pi(\Gamma)$ *is a Fuchsian group with signature* $(g ; \alpha_1,\dots,\alpha_s)$.

Remark 3.2. There are precisely q^{2g} subgroups $\Pi\subset\pi^{-1}(Q)$ of index q with $\pi(\Pi) = Q$. They are all related by automorphisms of $\pi^{-1}(Q)$. This gives a classification of discrete subgroups $\Pi \subset G$ with $\Pi\backslash G$ compact.

Example 3.3. It is easy to apply this to Theorem 2.1 to see that $\Sigma(a_1,\dots,a_n)$ has the form $\Pi\backslash G$. In this case Π is the commutator subgroup of $\Gamma = \pi^{-1}(Q)$, where $Q \subset PSL(2,\mathbb{R})$ is a Fuchsian group of signature $(0 ; a_1,\dots,a_n)$. This can be shown by explicit computation of Seifert invariants, which was how we originally did it. Using automorphic forms, one can prove a stronger version of the same result ([N2]). This has been done independently by I. Dolgačev. The case $n = 3$ was done by J. Milnor [M],and F. Klein [K] for $G = SU(2)$. More generally, the manifold of Theorem 2.2 has the form $\pm\Pi\backslash G$ if and only if

$$q = \pm(1/d)\,\mathrm{lcm}\, a_i \Big[\sum d_j - d\sum 1/a_i\Big], \quad (d = \prod d_j),$$

is a positive integer and is prime to $\alpha_j = \mathrm{lcm}\, a_i/\mathrm{lcm}_{i\neq j} a_i$ for each j. This holds for, example, if d divides Σd_j.

4. Homology Spheres

Theorem 4.1. *If the Seifert manifold* $M = M(g\,;\,(\alpha_i,\beta_i), i = 1,\dots,m)$ *is a* Z-*homology sphere, then* $g = 0$ *and the* α_i *are pairwise coprime. Furthermore, to given pairwise coprime* α_i *there is exactly one* Z-*homology sphere as above, up to orientation. It is diffeomorphic to the Brieskorn complete intersection manifold* $\Sigma(\alpha_1,\dots,\alpha_m)$, *and hence also to a homogeneous space* $\Pi\backslash G$ *as in Example* 3.3.

Remark. It follows that the subgroups $\Pi \subset G$ of Example 3.3 corresponding to pairwise coprime exponents $a_1, \dots, a_n$ are the only discrete $\Pi \subset G$ with $\Pi\backslash G$ compact for which Π is perfect (i.e., $[\Pi,\Pi] = \Pi$).

Proof of theorem. The first two sentences of the theorem are due to Seifert [S]. Namely, if M is as above, then by abelianizing the standard presentation of $\pi_1(M)$, Seifert showed $H_1(M) \cong Z^{2g} \oplus \mathrm{Cok}(A)$, where $A : Z^m \to Z^m$ is a map with matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ & \ddots & & & \\ & & \ddots & & \\ 0 & 0 & \cdots & \alpha_m & \beta_m \end{pmatrix}.$$

But $\det A = \pm\sum_{i=1}^{m} \beta_i \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_m = \pm\alpha_1 \cdots \alpha_m \Sigma(\beta_i/\alpha_i)$. The condition that M is a homology sphere is thus: $g = 0$ and $\alpha_1 \cdots \alpha_m \Sigma(\beta_i/\alpha_i) = \pm 1$. By reversing orientation if necessary, we can assume

$$\alpha_1 \cdots \alpha_m \sum (\beta_i/\alpha_i) = +1 .$$

This equation implies that the α_i are pairwise coprime. Further, by considering it modulo α_i, we see it determines β_j modulo α_j for each j. It also determines $e(M)$ as

$$e(M) = -\sum \beta_i/\alpha_i = -1/\alpha_1 \cdots \alpha_m .$$

It thus determines M completely for given $\alpha_1, \ldots, \alpha_m$.

Comparing with Theorem 2.1 proves the second statement. Alternatively,a simple proof of 2.1 for this case is given by observing that by Hamm [Ha], $\Sigma(\alpha_1,\ldots,\alpha_m)$ is a homology sphere for α_i pairwise coprime and its S^1 action clearly has isotropy $Z/\alpha_1, \ldots, Z/\alpha_m$.

One can get a simple proof of Example 3.3 for this case also by applying part (iii) of the following lemma.

<u>Lemma 4.2</u>. <u>If</u> $\alpha_1, \ldots, \alpha_m$ <u>are pairwise coprime and</u> $M = M(0,(\alpha_i,\beta_i), i=1,\ldots,m)$, <u>then</u> $e(M) = c/\alpha_1 \cdot \ldots \cdot \alpha_m$ <u>for some</u> c <u>prime to</u> $\alpha_1, \ldots, \alpha_m$, <u>and</u>

(i) M <u>is uniquely determined up to orientation by</u> $|c|$; <u>denote it</u> M_c;

(ii) $H_1(M_c) = Z/|c|$, <u>generated by the class of a principal orbit</u>;

(iii) M_c <u>covers</u> M_d <u>if and only if</u> d <u>divides</u> c; <u>in particular</u> $M_1 = \Sigma(\alpha_1,\ldots,\alpha_n)$ <u>is the maximal abelian cover of</u> M_c <u>for any</u> $c > 0$.

<u>Proof</u>. Up to and including part (i), this is the same proof as the previous theorem. Parts (ii) and (iii) then follow by observing that $Z/c \subset S^1$ acts freely on M_1, so M_c must be $M_1/(Z/c)$ by Lemma 1.3.

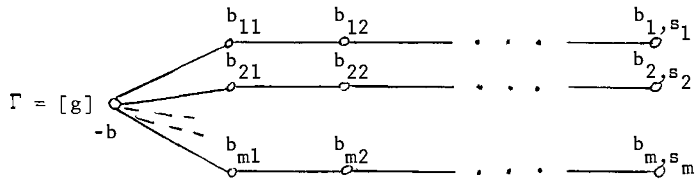
Similar statements hold for $g \neq 0$. We leave their formulation and proof to the reader.

5. <u>Seifert Manifolds via Plumbing</u>

Von Randow's algorithm ([vR], see Orlik [O] or Hirzebruch, Neumann, Koh [HNK] for a description in terms of our present orientation conventions) expressing a Seifert manifold via plumbing extends with no change to unnormalized Seifert invariants, yielding the result:

<u>Theorem 5.1</u>. <u>Let</u> $M^3 = \partial P(\Gamma)$ <u>be the result of plumbing according to the following</u>

<u>weighted tree</u>

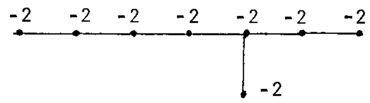


<u>Then</u> $M^3 \cong M((g),(1,b),(\alpha_i,\beta_i),i=1,\ldots,m)$, <u>where</u> α_i/β_i <u>is the continued fraction</u>

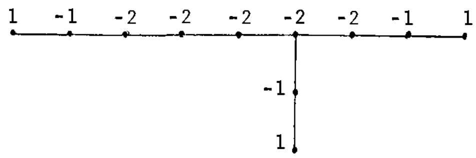
$$\alpha_i/\beta_i = b_{i1} - 1/(b_{i2} - 1/(b_{i3} - \cdots - 1/b_{is_i})\cdots)$$
$$= [b_{i1},\ldots,b_{is_i}] \qquad (\underline{\text{notation}}).$$

The [g] above means that the corresponding bundle being plumbed is the bundle of Euler number -b over a surface of genus g ; all the other bundles are bundles of Euler number b_{ij} over the sphere S^2. We omit the [g] if g = 0 . We are using the notation P(Γ) for the four-manifold obtained by plumbing disc bundles according to Γ and ∂P(Γ) for its boundary obtained by plumbing circle bundles.

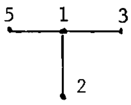
By von Randow [vR], in any plumbing graph Γ we can "blow down" vertices corresponding to a bundle of Euler number $\epsilon = \pm 1$ over S^2 having at most two adjacent vertices by removing that vertex and replacing the weights b_i of the neighboring vertices of Γ by $b_i - \epsilon$. This does not change ∂P(Γ) . For example, if we start with

E_8 = 

then we can "blow up" (reverse operation of blowing down) to get



and then by iteratively blowing down -1's we can finally get to

$\Gamma =$ 

Thus $\partial P(E_8) = \partial P(\Gamma) = \Sigma(2,3,5)$, where the last equality uses Theorem 2.1. It is not hard to show that any two plumbing graphs as in 5.1 for the same Seifert manifold are related by a sequence of blowings up and blowings down.

If Γ is an arbitrary plumbing graph with vertices $v_1, \ldots, v_r$ with Euler number weights $b_1, \ldots, b_r$ and arbitrary genus weights, then the four manifold $P(\Gamma)$ obtained by plumbing disc bundles according to Γ has intersection form (see for instance, [HNK])

$$A(\Gamma) = (\alpha_{ij}) \quad \text{with}$$

$$\begin{aligned} \alpha_{ij} &= 1 \quad \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ are connected in } \Gamma, \\ &= b_i \quad \text{if } i = j, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

We call Γ *positive definite*, *negative definite*, or *even* according to whether $A(\Gamma)$ has these properties.

Note that if Γ is positive definite, then blowing up or down $+1$ vertices does not change this property; similarly, for negative definiteness and -1 vertices.

Theorem 5.2. *Let* M *be an arbitrary Seifert manifold. Then* M *can be written as* $M \cong \partial P(\Gamma)$ *as in Theorem* 5.1 *with* Γ *definite if and only if* $e(M) \neq 0$. *In this case* Γ *is positive definite or negative definite according as* $e(M) > 0$ *or* $e(M) < 0$, *and* Γ *is unique after blowing down all* $+1$ *vertices, respectively all* -1 *vertices, which can be blown down in von Randow's sense.*

Proof. Let Γ be a graph as in Theorem 5.1. Then a simple induction shows $A(\Gamma)$ can be diagonalized as

$$\mathrm{diag}(e(M), c_{11}, \ldots, c_{1,s_1}, c_{22}, \ldots, c_{2,s_2}, \ldots, c_{m,s_m})$$

with

$$c_{ij} = [b_{ij}, b_{i,j+1}, \ldots, b_{i,s_i}] .$$

Thus Γ can be positive definite if and only if $e(M) > 0$, and similarly for negative definite. Assume now $e(M) > 0$, by reversing the orientation of M if necessary. Normalize the Seifert invariant of M to satisfy $0 < \beta_i < \alpha_i$ for $i = 1, \ldots, m$. Then α_i/β_i can be uniquely expanded as a continued fraction

$$\alpha_i/\beta_i = [b_{i1}, \ldots, b_{is}] \text{ with } b_{ij} \geq 2 .$$

A simple induction then shows $c_{ij} > 0$ for all i , j , so $A(\Gamma)$ is positive definite.

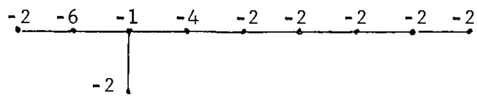
Conversely, suppose $A(\Gamma)$ is positive definite, and by blowing down if necessary, assume no b_{ij} equals 1 . Then $b_{ij} \geq 2$ for all i , j (since $b_{ij} > 0$ by positive definiteness). This forces $0 < \beta_i < \alpha_i$ for the Seifert invariants, so the Seifert invariants are in normalized form, hence unique, and the b_{ij} are then uniquely determined by the above comments.

<u>Corollary 5.3</u>. <u>Let</u> M <u>be a Seifert manifold with</u> $e(M) \neq 0$. <u>Then by reversing orientation if necessary we can assume</u> $e(M) < 0$, <u>and</u> M <u>is then the link of an isolated surface singularity with</u> C^*<u>-action, and the "canonical negative definite graph</u> Γ <u>for</u> M<u>" given by the above theorem is the dual graph of a resolution of this singularity</u>.

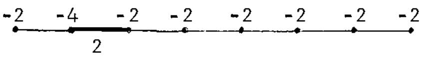
<u>Proof</u>. One can do plumbing holomorphically to obtain $P(\Gamma)$ as a complex manifold with holomorphic C^*-action and then apply Grauert [G] to blow down the central configuration of curves in $P(\Gamma)$. That one can blow down equivariantly follows by functoriality of Grauert's theorem. Since the complex structure one puts on $P(\Gamma)$ is in general far from unique, one, of course, gets a whole family of possible singularities. One can also prove this corollary directly via the injective holomorphic C^*-actions of Conner and Raymond [CR1], by showing that they can be compacti-

fied by a single singular point to give a complex affine variety, if the Euler number is negative.

Remark. The resolution given by the above corollary is the minimal "good" resolution of the singularity. In some cases, one can blow down further. For example, Theorem 2.1 shows that the link $\Sigma(2,11,19)$ of the singularity of $V(2,11,19) = \{z \in \mathbb{C}^3 / z_1^2 + z_2^{11} + z_3^{19} = 0\}$ is the Seifert manifold $M(0;(1,1),(2,-1),(11,-2),(19,-6))$ so Corollary 5.3 gives the graph



as the dual graph of the minimal good resolution. Blowing down (in the sense of complex manifolds; we cannot do it in the sense of plumbing graphs, since the (-1)-vertex has three neighbors) can be done twice, giving the result



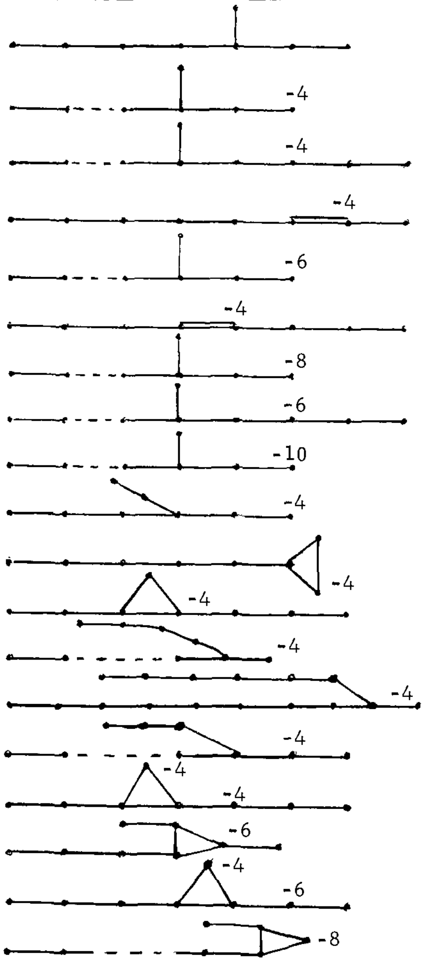
,

where the heavy line means a tangency of intersection number 2 between the corresponding curves of the resolution. Since $\Sigma(2,11,19)$ is the boundary of a regular neighborhood of the corresponding configuration of curves, this shows that $\Sigma(2,11,19)$ bounds a simply connected four manifold Y with negative definite intersection form of signature -8 (which must hence be equivalent to the standard E_8 form, by the classification of such forms, but this can easily be seen directly).

It is of interest to know which homology spheres bound simply connected manifolds with even definite intersection forms. For Seifert homology spheres, the minimal resolution of the corresponding singularity will sometimes provide a positive answer. For example, the minimal good resolution for $\Sigma(2,4k-1,8k-3)$, or what is the same, the canonical plumbing diagram, has even form of signature $-8k$. It is, in fact, the bilinear form commonly denoted Γ_{8k}. $\Sigma(2,8k-5,12k-7)$ is another example giving even forms of signature -8k .

The following table gives all $\Sigma(p,q,r)$ with $p < q < r$ pairwise coprime, $p = 2$ and $q < 20$, or $p \leqq 5$ and $q \leq 10$, for which the minimal resolution gives what we want. Omitted weights are -2 . Double lines represent curves of the resolution intersecting tangentially with intersection number 2 . Triangles represent three curves intersecting transversally in one point.

TABLE

(p,q,r)	signature	resolution graph
2 , 3 , 5	- 8	
2 , 7 , 13	-16	
2 , 11 , 17	-16	
2 , 11 , 19	- 8	
2 , 11 , 21	-24	
2 , 13 , 21	- 8	
2 , 15 , 29	-32	
2 , 19 , 29	-24	
2 , 19 , 37	-40	
3 , 4 , 7	- 8	
3 , 5 , 13	- 8	
3 , 7 , 17	- 8	
4 , 5 , 19	-24	
4 , 7 , 9	-16	
4 , 7 , 27	-32	
5 , 7 , 27	- 8	
5 , 9 , 13	- 8	
5 , 9 , 31	- 8	
5 , 9 , 43	-24	

One can simplify the algorithm provided by Theorems 2.1 and 5.1 for finding a plumbing diagram for $\Sigma(a_1,\ldots,a_n)$ by observing that if

$$[b_1,\ldots,b_s] = p/q\ ,\quad b_i \geqq 2$$

then

$$[b_s,b_{s-1},\ldots,b_1] = p/q'$$

with $0 < q \leqq p$, $0 < q' \leqq p$, $qq' \equiv 1 \pmod p$. We thus get the

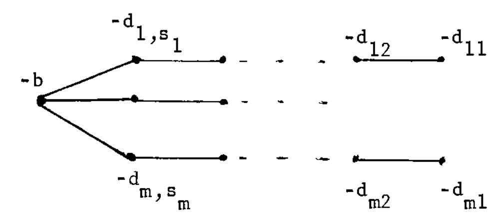
Algorithm 5.4. *If $a_1, \ldots, a_n$ are pairwise coprime integers with $a_j \geqq 2$, define $c_1, \ldots, c_n$ by*

$$c_j \equiv -a_1 \ldots \hat{a}_j \ldots a_n \pmod{a_j}\ ,\quad 0 < c_j < a_j\ .$$

Expand a_j/c_j as a continued fraction

$$a_j/c_j = [d_{j1},d_{j2},\ldots,d_{j,s_j}]\ ,\quad d_{ji} \geqq 2\ .$$

Then

$\Gamma =$ 

is the canonical (in the sense of 5.2 and 5.3) plumbing graph for $\Sigma(a_1,\ldots,a_n)$, where b is best determined in practice by estimating it via the equation

$$b = \sum_j 1/[d_{j,s_j},\ldots,d_{j2},d_{j1}] + 1/a_1 \ldots a_n$$

We leave the proof to the reader as well as the generalization to non-coprime a_i.

6. μ-Invariant for Seifert Manifolds

We first give a slightly generalized version of the usual definition of μ-invariant. If X^4 is any oriented four-manifold, a class $d \in H_2(X;Z)$ will be called a _spherical integral Wu class_ if it satisfies:

a) d can be represented by a smoothly embedded sphere;

b) The mod 2 reduction w of d satisfies $x \cdot x = x \cdot w$ for all $x \in H_2(X;Z/2)$ (dot is intersection number).

Theorem 6.1. Let M^3 be a (Z/2)-homology sphere. Let X^4 be a 4-manifold with a spherical integral Wu class d, such that $\partial X^4 = M^3$. Then

$$\mu(M) = \text{sign } X - d \cdot d \pmod{16}$$

is an invariant of M.

Proof. If X', d' is another such pair and $Y = X \cup (-X')$ pasted along the boundary M, then the Meyer Vietoris sequence shows $H_2(Y;Z/2) \cong H_2(X;Z/2) \oplus H_2(X';Z/2)$ and $d_Y = d + d'$ is a spherical integral Wu class for Y. By a theorem of Kervaire and Milnor [KM], $\text{sign}(Y) - d_Y \cdot d_Y$ is divisible by 16. Since $\text{sign}(Y) = \text{sign}(X) - \text{sign}(X')$ and $d_Y \cdot d_Y = d \cdot d - d' \cdot d'$, the theorem follows. Note that if $d = 0$ the theorem reduces to the usual definition of μ-invariant, and as is well known, such an X always exists.

Theorem 6.2. $M = M(g\,;(\alpha_i,\beta_i), i = 1,\dots,m)$ is a (Z/2)-homology sphere if and only if $g = 0$ and either:

(i) all the α_i are odd and $\Sigma\beta_i$ is odd; or

(ii) exactly one of the α_i is even, say α_1.

In these cases the μ-invariant is given respectively by

(i) $\mu(M) = \Sigma_{i=1}^{m} (c(\alpha_i,\beta_i) + \text{sign } \beta_i) + \text{sign } e(M) \pmod{16}$

(ii) $\mu(M) = \Sigma_{i=1}^{m} c(\alpha_i - \beta_i, \alpha_i) + \text{sign } e(M) \pmod{16}$

where $c(p,q)$ _is the function introduced in_ [N1] _described below,_ $e(M) = -\Sigma\beta_i/\alpha_i$, _and in case_ (ii) _we have chosen the Seifert invariant so that_ $(\alpha_i - \beta_i)$ _is odd for all_ i _(possible, by replacing_ β_i _by_ $\beta_i \pm \alpha_i$ _if necessary for each_ $i > 1$, _and then adjusting_ β_1 _so_ $e(M)$ _is unchanged)._

Here $c(p,q)$ is defined for coprime integer pairs (p,q) with p odd. It is uniquely determined by the recursions

$$\begin{aligned} c(p,\pm 1) &= 0 \\ c(p,-q) &= c(-p,q) = -c(p,q) \\ c(p,p+q) &= c(p,q) + \operatorname{sign}(q(p+q)) \\ c(p\pm 2q,q) &= c(p,q) . \end{aligned}$$

These recursion formulae, in fact, give the fastest computation of $c(p,q)$ in practice, but various other descriptions of $c(p,q)$ are known. Before proving the above theorem, we describe some of them.

<u>Proposition 6.3</u>. <u>If</u> $p, q > 0$ <u>then</u>

(i) $c(p,q) = \mu(L(q,p)) \pmod{16}$, q <u>odd</u>;

(ii) $c(p,q) = \alpha(L(q,p),T)$, <u>where</u> α <u>is the Browder Livesay invariant and</u> T <u>is the involution on</u> $L(q,p)$ <u>with orbit space</u> $L(2q,p)$;

(iii) $c(p,q) = \Sigma(-1)^i \# \{0<k<q \mid i<kp/q<i+1\}$,

$$= -\frac{1}{q}\sum_{\eta^q=-1} \frac{(\eta+1)(\eta^p+1)}{(\eta-1)(\eta^p-1)} ,$$

$$= q - 1 - 4N_{p,q} , \quad q \text{ odd} ,$$

where $N_{p,q} = \#\{1 \leq i \leq \frac{q-1}{2} \mid \frac{q-1}{2} < pi < q \pmod{q}\}$.

<u>Proof</u>. (i) is the special case of Theorem 6.2 with $M = M(0;(p,q)) = L(q,p)$ for q odd. Equation (ii) is a way $c(p,q)$ originally came up in [N1]. This function was renamed $t(q;p)$ and generalized by Hirzebruch and Zagier ([H1], [HZ], especially pp. 245-246) and (iii) is a selection of the many formulae for $t(q;p)$ given there. The last formula is especially interesting, since by Gauss, $(-1)^{N_{p,q}} = (\frac{p}{q})$ is the quadratic residue symbol, so the last formula implies

$$c(p,q) \equiv 1 - 2\left(\frac{p}{q}\right) + q \pmod 8 , \quad q \text{ odd} .$$

This was first observed by Hirzebruch [HNK].

We have appended, at the end of this section 6, a table from [N-1] for $c(p,q)$. All values for $p \leqq 27$ and $q \leqq 26$ are given.

Proof of Theorem 6.2. Firstly, as in the proof of Theorem 4.1, one sees that $M = M(g\,;(\alpha_i,\beta_i))$ is a $(Z/2)$-homology sphere if and only if $g = 0$ and $\Sigma \beta_i \alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_m$ is odd. This implies the first statement of the theorem.

To see the formulae for $\mu(M)$, we first consider case (ii). That is, we assume $M = M(0\,;(\alpha_i,\beta_i), i = 1,\ldots,m)$ with α_1 even, α_i odd for $i > 1$. As already remarked, we can also assume β_i is even for $i > 1$, by replacing β_i by $\beta_i \pm \alpha_i$ if necessary.

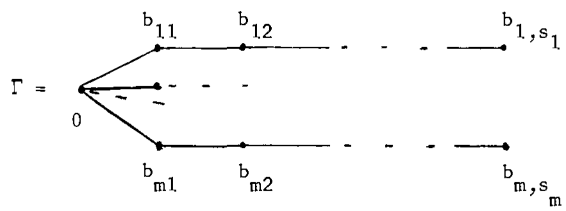
Lemma 6.3. If p and q are coprime integers, then p/q has a continued fraction expansion

$$p/q = [b_1, b_2, \ldots, b_s]$$

(see Theorem 5.1) with each b_i even if and only if exactly one of p and q is even. There is then a unique such expansion satisfying in addition: $|b_i| \geqq 2$ for $i > 1$.

The proof of this lemma is an easy induction which we omit.

Applying this lemma and Theorem 5.1, we can express $M^3 = \partial P(\Gamma)$, where Γ is a weighted tree



with all weights even, $|b_{ij}| \geqq 2$ for $j > 1$, and $\alpha_i/\beta_i = [b_{i1},\ldots,b_{i,s_i}]$ for each i. If we take $X = P(\Gamma)$, so $M = \partial X$, then the definition of $\mu(M)$ reduces to $\mu(M) = \text{sign}\,(X) \pmod{16}$. Using the diagonalization of the intersection matrix $A(\Gamma)$ of $X = P(\Gamma)$ described in the proof of Theorem 5.2, we see

$$\text{sign}\, X = \sum \mu(\alpha_i,\beta_i) + \text{sign}\, e(M)$$

where $\mu(\alpha_i,\beta_i) = \#\{j \mid 1<j\leq s_i, b_{ij}>0\} - \#\{j \mid 1<j\leq s_i, b_{ij}<0\} + \mathrm{sign}(\alpha_i/\beta_i)$. The following recursion formulae follow directly from this definition of $\mu(\alpha,\beta)$:

$$\mu(\alpha,\beta) \text{ is defined if } \alpha+\beta \text{ is odd and } \gcd(\alpha,\beta) = 1 ;$$
$$\mu(\alpha,\beta) = -\mu(\alpha,-\beta) ;$$
$$\mu(2b\alpha-\beta,\alpha) = \mu(\alpha,\beta) + \mathrm{sign}\, b \quad \text{if } |\alpha| > |\beta| ;$$
$$\mu(\beta,\alpha) = -\mu(\alpha,\beta) + \mathrm{sign}(\beta/\alpha) \quad \text{if } |\alpha| > |\beta| .$$

If we define $c'(p,q) = \mu(q,q-p)$, then it is easy to deduce that c' satisfies the recursion formulae defining c, so $c'(p,q) = c(p,q)$. Thus $\mu(q,q-p) = c(p,q)$, so $\mu(\alpha,\beta) = c(\alpha-\beta,\alpha)$, completing the proof of the formula for case (ii).

The proof in case (i) can be done similarly, although in this case M cannot be written as $\partial P\Gamma$ where Γ has only even weights, which complicates this approach slightly. A more interesting proof uses the following theorem.

Theorem 6.4. *Let M^3 be a $(Z/2)$-homology sphere and $T: M \to M$ a free orientation preserving involution. Suppose $M = \partial X^4$ and*

a) *T extends to $T': X \to X$;*

b) *T' acts trivially on $H_2(X,\mathbb{Q})$ and $H_2(X;Z/2)$;*

c) *the 2-dimensional part F of $\mathrm{Fix}(T')$ is oriented and is homologous to a smoothly embedded 2-sphere.*

Then

$$\alpha(M,T) \equiv \mu(M) \pmod{16}$$

where $\alpha(M,T)$ is the Browder Livesay invariant.

Proof. By Hirzebruch [H2] (see also [HJ] and [AS])

$$\alpha(M,T) = \mathrm{sign}(X,T') - [F]\cdot[F]$$

where $[F] \in H_2(X;Z)$ is the represented homology class. Since T' acts trivially on $H_2(X;\mathbb{Q})$, we have $\mathrm{sign}(X,T') = \mathrm{sign}\, X$. We must thus only show that $[F]_2 \in H_2(X;Z/2)$ satisfies $[F]_2\cdot x = x\cdot x$ for all $x \in H_2(X;Z/2)$. But $x = T'x$ by assumption (b), and $[F]_2 \cdot x = x \cdot T'x$, since if we represent x by a cycle C, then the intersection points of C and $T'C$ pair off under T', and thus contribute nothing to $x \cdot T'x$, unless they lie in $F \cap C$.

To apply this theorem to case (i) in Theorem 6.2, observe that in this case the involution T contained in the S^1-action on M is a free involution, and if we write $X = \partial P(T)$ as in Theorem 5.1, then T extends to $T' : X \to X$ since the whole S^1-action extends. Condition (b) is satisfied since T' is homotopic to the identity, and (c) is satisfied since $\mathrm{Fix}(T')$ is a union of disjoint spheres (the zero-sections of some of the bundles being plumbed). Thus $\mu(M) \equiv \alpha(M,T) \pmod{16}$. But $\alpha(M,T)$ was computed in [N] as

$$\alpha(M,T) = \sum (c(\alpha_i,\beta_i) + \mathrm{sign}\ \beta_i) + \mathrm{sign}\ e(M)$$

whenever $M = M(g,(\alpha_i,\beta_i))$ with all the α_i odd, so the proof is completed.

If M is a Z-homology sphere, other formulae for $\mu(M)$ are known, in view of the fact that $M \cong \Sigma^3(\alpha_1,\ldots,\alpha_n)$ up to orientation.

<u>Theorem 6.5</u>. *$M = \Sigma^3(\alpha_1,\ldots,\alpha_n)$ with α_i pairwise coprime embeds in S^5 as a fibered knot, the signature of whose fiber V is*

$$\mathrm{sign}(V) = \sum_{\substack{1 \le j < 2\alpha \\ j \text{ odd}}} \mathrm{res}_{\pi i j/\alpha}\left((\tanh \alpha z)^{n-2} \coth z \prod_{k=1}^{n} \coth \frac{\alpha z}{\alpha_k}\right),$$

$$= t(\alpha_1,\alpha_2,\alpha_3) \text{ of [HZ] if } n = 3.$$

<u>In particular</u>, $\mu(M) = \mathrm{sign}(V) \pmod{16}$.

<u>Proof</u>. If $\Sigma^5(\alpha_1,\ldots,\alpha_n)$ is defined just like $\Sigma^3(\alpha_1,\ldots,\alpha_n)$ but using $(n-3)$ instead of $(n-2)$ equations, then by Hamm [Ha], there is a "Milnor fibration" of the complement of Σ^3 in Σ^5, whose fiber V has the above signature (see also Hirzebruch [H3]). Furthermore, if $\alpha_1, \ldots, \alpha_n$ are pairwise coprime, then Σ^5 is a homotopy sphere, so $\Sigma^5 \cong S^5$. V is stably parallelizable, so its intersection form is even, so $\mu(M) = \mathrm{sign}(V) \pmod{16}$ if $\mu(M)$ is defined (e.g., α_i pair-

wise coprime).

For $n = 3$ the function in the above theorem was denoted $t(\alpha_1,\alpha_2,\alpha_3)$ and studied and tabulated by Hirzebruch and Zagier ([HZ], table on page 118).

TABLE OF $c(p,q)$

q\p	1	3	5	7	9	11	13	15	17	19	21	23	25	27
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
3	2		-2	2		-2	2		-2	2		-2	2	
4	3	1	-1	-3	3	1	-1	-3	3	1	-1	-3	3	1
5	4	0		0	-4	4	0		0	-4	4	0		0
6	5		1	-1		-5	5		1	-1		-5	5	
7	6	2	2		-2	-2	-6	6	2	2		-2	-2	-6
8	7	1	-1	1	-1	1	-1	-7	7	1	-1	1	-1	1
9	8		0	0		0	0		-8	8		0	0	
10	9	3		3	1	-1	-3		-3	-9	9	3		3
11	10	2	2	-2	2		-2	2	-2	-2	-10	10	2	2
12	11		3	1		1	-1		-1	-3		-11	11	
13	12	4	0	0	4	0		0	-4	0	0	-4	-12	12
14	13	3	1		-3	-1	1	-1	1	3		-1	-3	-13
15	14			2		2	2		-2	-2		-2		
16	15	5	3	1	-1	5	3	1	-1	-3	-5	1	-1	-3
17	16	4	4	4	0	-4	0	0		0	0	4	0	-4
18	17		1	-1		-1	1		1	-1		-1	1	
19	18	6	2	2	2	2	6	-2	2		-2	2	-6	-2
20	19	5		1	3	1	-5		-1	1	-1	1		5
21	20		4			0	0		4	0		0	-4	
22	21	7	5	3	5		1	7	1	3	1	-1	-3	-1
23	22	6	2	2	-2	2	-2	-6	2	2	2		-2	-2
24	23		3	5		1	-1		3	-3		1	-1	
25	24	8		0	0	0	0		8	0	0	0		0
26	25	7	5	3	1	3		3	-7	3	5	-1	1	-1

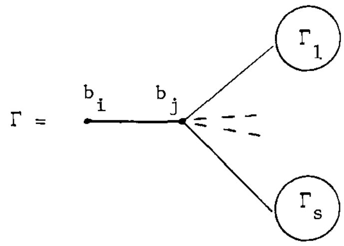
7. μ-Invariant for Plumbed Manifolds

Theorem 6.1 enables us to give an algorithm to compute $\mu(M)$ for an arbitrary (Z/2)-homology sphere obtained by plumbing. Note that a necessary condition that $M = \partial P(\Gamma)$ be a (Z/2)-sphere is that Γ be a tree and all the genus weights vanish.

Theorem 7.1.

(i) *Let* $M = \partial P(\Gamma)$ *be the result of plumbing bundles over* S^2 *according to a tree* Γ. *Then* M *is a* (Z/2)-*sphere if and only if* Γ *can be reduced to a collection of isolated points with odd weights by a sequence of moves of type* 1 *and* 2 *below.* M *is not a* (Z/2)-*sphere if and only if* Γ *can be so reduced to a collection of isolated points with at least one even weight.*

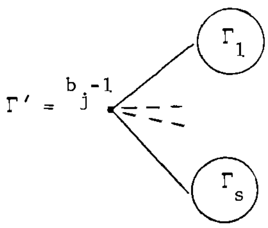
Let



where b_i *and* b_j *are the weights of vertices* i *and* j.

Move 1. *If* b_i *is even, replace* Γ *by the disjoint union* Γ' *of* $\Gamma_1, \ldots, \Gamma_s$.

Move 2. *If* b_i *is odd, replace* Γ *by*



(ii) *If* $M = \partial P(\Gamma)$ *is a* (Z/2)-*sphere, define a subset* $S(\Gamma)$ *of the vertices of* Γ *inductively as follows:*

a) *If* Γ_0 *is a set of isolated points with odd weights, put* $S(\Gamma_0)$ *equal to* the set of all these points.

b) If $S(\Gamma')$ is known and Γ reduces to Γ' by Move 1 above, put

$$S(\Gamma) = S(\Gamma') \cup \{i\} ,$$
$$= S(\Gamma') ,$$

according as the number of points in $S(\Gamma')$ adjacent to vertex j is congruent to $b_j - 1$ or b_j modulo 2.

c) If Γ reduces to Γ' by Move 2, put

$$S(\Gamma) = S(\Gamma') \cup \{i\} \quad \text{if} \quad j \notin S(\Gamma') ,$$
$$= S(\Gamma') \quad \text{if} \quad j \in S(\Gamma') .$$

Then

$$\mu(M) = \operatorname{sign} A(\Gamma) - \sum_{i \in S(\Gamma)} b_i \pmod{16} ,$$

where $A(\Gamma)$ is the matrix of the graph.

Proof. $H_1(\partial P(\Gamma)) \cong \operatorname{Cok}(A(\Gamma))$, so $M = \partial P(\Gamma)$ is a $(Z/2)$-sphere if and only if $\det A(\Gamma)$ is odd. But it is easily verified that Moves 1 and 2 do not change $\det A(\Gamma) \pmod 2$, so the first part of the theorem follows. For the second part, let $X = P(\Gamma)$ and let $\{e_i | i$ a vertex of $\Gamma\}$ be the standard basis of $H_2(P(\Gamma);Z)$ represented by the zero-sections of the plumbed bundles. Then a simple induction shows that $d = \sum_{i \in S(\Gamma)} e_i$ is a spherical integral Wu class for X and that $d \cdot d = \sum_{i \in S(\Gamma)} b_i$. Since $\operatorname{sign} X = \operatorname{sign} A(\Gamma)$, the theorem follows.

Problem 7.2. If M is a $(Z/2)$-homology 3-sphere with a free orientation preserving involution, is it true that $\mu(M) = \alpha(M,T) \pmod{16}$?

The answer is "yes" for Seifert Z-homology spheres, and more generally for Seifert $(Z/2)$-spheres $M(0;\alpha_i,\beta_i))$ with pairwise distinct Seifert pairs $(\alpha_i, \beta_i \bmod \alpha_i)$. In these cases we shall show in a later paper that any free involution must be in the S^1-action, putting us into the situation of the proof of part (i) of Theorem 6.2.

8. Orientation Reversing Maps

It turns out that many properties of a Seifert manifold are determined by its Euler number $e(M)$.

Theorem 8.1. If $e(M) \neq 0$, then $\alpha_1 \cdots \alpha_m |e(M)|$ = order of torsion of $H_1(M;Z)$. If $e(M) = 0$, then M fibers equivariantly over the circle. If M is not a principal circle bundle over a torus, then M fibers over the circle if and only if the fibering is equivariant.

This is due to Orlik, Vogt and Zieschang [OVZ] and Orlik and Raymond in certain exceptional cases. The fibering, if it exists, is far from unique. These fiberings have been constructed explicitly from the viewpoint of homologically injective actions by Conner and Raymond [CR2]. That these fiberings are S^1-equivariant follows most easily from this viewpoint. The principal circle bundles over the torus are the only Seifert fiberings which fiber over the circle but fail to fiber equivariantly. (All but the 3-torus has $e(M) \neq 0$.) The S^1-equivariant fiberings are also constructed explicitly from a plumbing viewpoint by Neumann in [N3].

The present investigation arose from the next

Theorem 8.2. If the Seifert manifold M is not a lens space, then the following statements are equivalent:

(i) *M admits a free orientation reversing involution*

(ii) *M admits an orientation reversing involution*

(iii) *M admits an orientation reversing homeomorphism*

(iv) *M admits an orientation reversing self-homotopy equivalence*

(v) *M fibers over S^1 and admits an orientation reversing free involution which commutes with the S^1-action.*

Proof. Clearly (v)$\Rightarrow$(i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv). We show (iv)$\Rightarrow$(v). We suppose first that M has infinite fundamental group. We assume M is not the 3-torus.

Then by [W] if M is sufficiently large or by [OVZ] or [CR3], in general, any homotopy equivalence $M \to -M$ is homotopic to a fiber preserving homeomorphism, so $e(M) = e(-M) = -e(M)$. Hence, $e(M) = 0$. Therefore, M fibers over S^1 equivariantly. In fact, since M has Seifert invariants $(g\,;(\alpha_i,\beta_i))$ then $-M$ has Seifert invariants $(g;(\alpha_i,-\beta_i))$. This easily yields that the Seifert invariants of M must be expressible as

$$M = (g\,;(2,b_1),\ldots,(2,b_s),(\alpha_i,\beta_i),(\alpha_i,-\beta_i))$$

for some $s, k \geq 0$, $\alpha_i > 2$, $i = 1, \ldots, k$. Now $e(M) = 0$ implies $\frac{1}{2}\Sigma b_i = 0$ and since the b_i are odd, this implies s <u>is even</u>. Thus, <u>the Seifert invariants for</u> M <u>are equivalent to</u>

(vi) $$(g\,;(\alpha_i,\beta_i),(\alpha_i,-\beta_i),i=1,\ldots,\ell).$$

Therefore, M is the orientation double covering fixed-point free S^1-manifold by Seifert [S;p. 198]. This completes the proof if $\pi_1(M)$ is infinite.

Our attack for the finite fundamental groups must be different. Each Seifert manifold with finite non-abelian fundamental group appears as S^3/G where G is a finite subgroup of SO(4) which acts freely on S^3, that is, a spherical space form. The 2-Sylow subgroups of these manifolds are either cyclic of order at least 4 or a generalized quaternionic group. The 2-Sylow subgroups are all conjugate and in the cyclic case, there is a unique subgroup of order 4. The generalized quaternionic groups have a characteristic subgroup of order 4, namely the second term of the upper central series. The quaternion group itself has a unique conjugacy class of elements of order 4. In any case, we may pass to the unique covering space corresponding to the subgroup of order 4 since this is determined up to conjugacy.

This must be a lens space $L(4,1)$ or $L(4,3)$. Whether it is $L(4,1)$ or $L(4,3)$ will be determined by the orientation of M. Now any self homotopy equivalence f of M must preserve the unique conjugacy class of our subgroup of order 4. Hence, f may be lifted to a self homotopy equivalence $\tilde{f}: L(4,a) \to L(4,a)$.

If f reverses orientation, then $\tilde{f}$ must do the same. But, $L(4,a)$, $a = 1$, or 3 admits no orientation reversing self-homotopy equivalence since -1 is not a square modulo 4 . Hence, M could not possess an orientation reversing self-homotopy equivalence. This completes the proof of Theorem 8.2.

Remarks 8.3. It has been recently shown by C.B. Thomas, [T] that if M is a closed 3-manifold whose universal covering is the 3-sphere, then $\pi_1(M)$ must be one of the fundamental groups of Seifert manifolds with finite fundamental group. Since it is also known that any free $Z/4$ action on the 3-sphere [Ri] yields a lens space, we may conclude that the argument above also shows that such manifolds admit no orientation reversing self-homotopy equivalences. Of course, no examples of closed 3-manifolds with finite fundamental group which fail to be Seifert manifolds are known at this time and so this remark may be redundant.

Our arguments extend to the other types of oriented Seifert 3-manifolds which have not been considered elsewhere in this paper. We describe this situation now.

8.4. The Closed Case

We assume that M is a closed oriented Seifert 3-manifold with a non-orientable decomposition space. The fibering mapping is not the orbit mapping of an S^1-action and its type is distinct from the Seifert manifolds considered elsewhere in this paper. With a few exceptions, none of these manifolds support an S^1-action. They do support "local $SO(2)$- actions". The Seifert invariants are written

$$(\text{On } k ; (\alpha_i, \beta_i))$$

where the On refers to orientable total space and non-orientable base. They are exactly similar to the invariants for oriented Seifert fiberings with orientable decomposition space except that k represents the non-orientable genus of the decomposition space, and so, $k \geq 1$. Just as before the unnormalized representation is not unique.

We first observe that there is a double covering M' of M which is an oriented Seifert manifold with orientable decomposition space and whose unnormalized invariants are

$$(g = k-1\ ;\ (\alpha_1,\beta_1),(\alpha_1,\beta_1),\ldots,(\alpha_s,\beta_s),(\alpha_s,\beta_s))\ .$$

One can easily deduce from this that the naturality properties of $e(M) = -\Sigma\beta_i/\alpha_i$ extend to the case of non-orientable decomposition space as long as the total space M is kept orientable. But it is also easy to reduce considerations to the case of orientable decomposition space, which is the method we shall follow.

Assume, now, that if $k = 1$, then there are at least 2 singular fibers, if $k = 2$, then there is at least one singular fiber, and if $k = 1$, $s = 2$, then $\{(\alpha_1,\beta_1),(\alpha_2,\beta_2)\} \neq \{(2,1),(2,-1)\}$. We shall treat these presently avoided cases separately.

Under thc hypothesis on the invariants, the element of the fundamental group represented by an ordinary fiber generates an infinite cyclic characteristic subgroup of $\pi_1(M)$ and $\pi_1(M')$ is the centralizer of this cyclic subgroup. It is easy to check that every automorphism of $\pi_1(M)$ induces an automorphism of the subgroup $\pi_1(M')$. Consequently, we may lift any self-homotopy equivalence h on M to a self-homotopy equivalence h' on M'. h will be orientation reversing if and only if h' is orientation reversing. Therefore, we know that M' must be of the type exhibited in (vi) of Theorem 8.2. Consequently, the Seifert invariants of M must be

$$(\mathrm{Onk}\ ;\ (\alpha_i,\beta_i),(\alpha_i,-\beta_i))\ .$$

We now wish to show that each such M actually admits an orientation reversing involution. From p. 198 of [S], observe that as long as the non-orientable genus k of M is even, M is an orientable double covering of Seifert manifolds of type (N,n,II) and (N,n,III) using Seifert's terminology. For all $k > 1$, M is also an orientable double covering of certain non-orientable 3-manifolds closely related to the classical Seifert 3-manifolds. These manifolds described by Orlik and Raymond admit "local $SO(2)$-actions". Although they are not classical Seifert 3-manifolds, they would be considered as injective Seifert 3-manifolds from the point of view of Conner and Raymond. We see from the table on page 155 of [OR] that each M (type $n_2 = On$) is a double covering of a non-orientable local SO(2)-manifold, provided that $k > 1$. For $k = 1$, none of the tabulated double coverings will work. However, there exists an involution on $\mathbb{R}P_2$ so that an isolated point and orientation reversing circle appear as the fixed point set. This involution is embedded in the effective SO(2)-action on $\mathbb{R}P_2$. With this involution, one may define an involution on M where $k = 1$ and the Seifert invariants satisfy the necessary conditions for an orientation reversing homeomorphism. First, one removes the tubular neighborhoods of the singular fibers. The resulting circle bundle with structure group O(2) is the restriction to the deleted $\mathbb{R}P_2$ of the associated sphere bundle $S(\theta \oplus 1)$ where θ is the line bundle det $(T\mathbb{R}P_2)$ and 1 denotes the trivial line bundle. It makes sense, since the bundle has a section, to flip in the 1-direction. This carries the bundle over the region away from a Möbius band into itself by a rotation in D^2 and a flip in S^1. This can be extended to the deleted tubular neighborhoods. The involution has 2 isolated fixed points and so the orbit space is not a manifold. (No free involution presumably exists in case $k = 1$. The argument to check that no free involution exists is rather complicated and the details have not been checked.)

We turn now to the omitted cases. $M = (On1 ; (2,1),(2,-1))$ has an involution as described above. $M = (On2 ; (1,\beta))$ can also be regarded as a torus bundle over the circle with monodromy $\begin{pmatrix} -1 & 0 \\ -\beta & -1 \end{pmatrix}$, $\beta \in Z$. If $\beta \neq 0$, then the fundamental group of the torus fiber is a characteristic subgroup. The outer automorphism group of $\Pi_1(M)$ is calculated in Conner and Raymond [CR4;6.14]. It is readily seen from this calculation that if $\beta \neq 0$, M admits no orientation reversing self-homotopy equivalence. For $\beta = 0$, M can be identified with $\{g = 0,(2,1)(2,-1),(2,1),(2,-1)\}$ which does admit an orientation reversing free involution.

The remaining cases to treat are $M = M(On1 ; (\alpha_1,\beta_1))$. If $(\alpha_1,\beta_1) = (1,0)$, then M is $\mathbb{R}P_3 \# \mathbb{R}P_3$ which certainly has an orientation reversing homeomorphism. If $\beta \neq 0$, then $M = M(0n1;(\alpha,\beta))$ also has a Seifert fibering with orientable base as $M = (0;(2,1)(2,-1),(\alpha,\beta))$. If $\beta = \pm 1$ this is the lens space $\pm L(4\alpha,2\alpha+1)$

and if $|\beta| > 1$ it is a nonabelian spherical space form. (This corrects a statement in [OR].) In either case it admits no orientation reversing equivalence.

We may now summarize our result for the closed case as follows:

Theorem 8.5. The following are equivalent for $M = M(Onk ; (\alpha_i, \beta_i))$ not a lens space:

(i) M admits an orientation reversing self-homotopy equivalence,

(ii) M admits an orientation reversing homeomorphism,

(iii) M admits an orientation reversing involution,

(iv) The Seifert invariants may be written as

$$(Onk ; (\alpha_j, \beta_j), (\alpha_j, -\beta_j)) .$$

Moreover, if $k > 1$, the orientation reversing involution can be chosen to be free.

8.6. M Compact But Not Closed

For this case we assume $\partial M \neq 0$. Let h denote the number of boundary components. Then the Seifert invariants for M are given by

$$((g,h) ; (\alpha_i, \beta_i))$$
$$(On(k,h) ; (\alpha_i, \beta_i)) ,$$

where we may assume that all $\alpha_i > 1$, $0 < \beta_i < \alpha_i$ and $i = 0, 1, 2, \ldots, m$. Similar to our procedure in 8.4, we assume that if $g = 0$, $h = 1$, then $m > 1$, and if $k = 1$, $h = 1$, then $m \neq 0$. We wish to consider only self-homotopy equivalences that preserve the "peripheral structure". Then, in order that $f : M \to M$ be such an orientation reversing self-homotopy equivalence, it must follow that

$$\{(\alpha_i, \beta_i)\} = \{(\alpha_1, \beta_1), (\alpha_1, \alpha_1 - \beta_1), \ldots, (\alpha_t, \beta_t), (\alpha_t, \alpha_t - \beta_t)\}$$

As before, involutions can be constructed on each of these manifolds. However, when the Euler characteristic of the decomposition space is odd, we cannot expect to find free involutions in general.

Added in Proof. The reason given in 8.2 for the quaternion group is incorrect. The result is still valid since the quaternion group can be embedded in SU(2). Consequently, the covering space associated to each subgroup of order 4 is the lens space L(4,3), or equivalently, the lens space L(4,1) if the opposite orientation of SU(2) is used. The rest of the argument proceeds as before.

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